Reflexive graphs with Near-Unanimity but no Semilattice Polymorphisms

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Abstract

We show that every generator, in a certain set of generators for the variety of reflexive near unanimity graphs, admits a semilattice polymorphism. We then find a retract of a product of such graphs (paths, in fact) that has no semilattice polymorphism. This verifies for reflexive graphs that the variety of graphs with semilattice polymorpisms does not contain the variety of graphs with near-unanimity, or even 3-ary near-unanimity polymorphisms.

Mathematics Subject Classifications: 05C75, 08B05

1 Introduction

For relational structures such as graphs, the existence of relation preserving operations, or polymorphisms, satisfying various identities has been of great interest recently due to its relation to the complexity of the problem of deciding whether or not there is a homomorphism between given structures. We refer the reader to [6] for a general discussion of such topics, to [9] for a discussion of the results on general digraphs, or to [5] and [8] for more concise discussion directly related to the present paper.

In this paper we look at near-unanimity (NU), and semilattice (SL) polymorphisms on reflexive graphs. For context, we also talk of totally symmetric idempotent (TSI) polymorphisms. The necessary definitions of these are given in the next section.

It is a trivial fact that any structure with an SL polymorphism has a TSI polymorphism, and it is known, see [11], that the converse is not generally true. Moreover, there

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are structures admitting SL (and so TSI) polymorphisms, but not NU polymorphisms, and vice versa.

When one restricts ones scope to reflexive graphs though, things change. It is known, see for example [13], that any graph having an NU polymorphism also has TSI polymorphisms of all arities. Moreover it was shown in [5] that any reflexive graph with an NU polymorphism has a symmetric NU polymorphism. It is natural to ask if the existence of an NU polymorphism on a reflexive graph might imply the existence of an SL polymorphism, or vice-versa. Indeed, it was asked in [11] and again in [9] if there are posets (another type of reflexive digraph) that admit NU polymorphisms but no SL polymorphisms.

For the rest of the paper, all graphs are reflexive and symmetric. A graph is an *NU graph* (*SL graph*), if it admits an NU polymorphism (resp. SL polymorphism).

The class, **NU**, of graphs admitting NU polymorphisms has been well studied; see, for example, [1], [3], [5], [10], and [12]. In [3] for example, it was verified that **NU** is a variety, i.e., is closed under categorical products and retractions. In fact it was verified that for each $k \geq 3$, the class $\mathbf{NU}_k$ of graphs admitting k-ary NU polymorphisms, is a variety. The variety $\mathbf{NU}_{k+1}$ contains $\mathbf{NU}_k$ for all k. It was also shown that every chordal graph is in $\mathbf{NU}_k$ for some k, but also that for every k there chordal graphs in $\mathbf{NU}_{k+1} \setminus \mathbf{NU}_k$. In [5] an explicit description of the generators of the variety $\mathbf{NU}_k$ was given for all $k \geq 3$.

The class **SL** of graphs admitting SL polymorphisms, on the other hand, has not been so extensively studied. It has only been looked at recently in [8] and in a more specialised context in [14]. In [8], we showed that **SL** contains all chordal graphs. We also verified that **SL** is not closed under retraction, so though it is closed under products, it is not a variety. This shows that it is different from the classes of graphs admitting TSI or NU polymorphisms. We also found graphs in $\mathbf{SL} \setminus \mathbf{NU}$, and asked, as was asked about posets in [9], whether or not $\mathbf{NU} \subset \mathbf{SL}$. In this paper, we answer this question in the negative.

In Proposition 7, we observe that the generators of $\mathbf{NU}_k$ found in [5], are in **SL**. This is, of course, a first step towards showing $\mathbf{NU}_k \subset \mathbf{SL}$. Our second result however, Theorem 9, answering the question above, shows that this is not true. We find a retraction of a product of paths (the generators of $\mathbf{NU}_3$) which is not in **SL**. This shows that $\mathbf{NU}_3$, and so **NU**, is not contained in **SL**.

Proposition 7 is not simply a ploy for building tension before surprising the reader with Theorem 9. It also yields alternate proofs of known facts, which we discuss briefly now, and raises some questions that we talk of in Section 5.

It was observed in [4] that every NU structure H is the retraction of some universal structure $U_{\mathrm{TSI}}(H)$ which can easily be shown to admit an SL polymorphism. So every NU graph is a retract of an SL graph. This follows also from Proposition 7. The graph $U_{\mathrm{TSI}}(H)$ is large as its vertex set is the set of subsets of vertices of H; our result generally embeds an NU graph as a retract of a much smaller SL graph.

In [3] it was shown that there are chordal graphs in $\mathbf{NU}_k \setminus \mathbf{NU}_{k-1}$ for all $k \geq 4$. As chordal graphs were shown in [8] to be in **SL**, it follows that there are SL graphs in $\mathbf{NU}_k \setminus \mathbf{NU}_{k-1}$. Corollary 8 points out how this also follows from Proposition 7, but the examples it provides are far from chordal, and the proof is much different.

In Section 2 we introduce the required definitions. In Section 3 we introduce the generators of $\mathbf{NU}_k$ from [5] and prove Proposition 7. In Section 4 we prove Theorem 9. Finally, in Section 5 we ask some questions.

2 Basics

2.1 Semilattices

In this subsection we recall some standard definitions related to semilattices.

A *semilattice* on a set V can alternately be described as an ordering $\leq$ such that for every pair $u, v \in V$ there is a unique greatest lower bound denoted $u \wedge v$; or as a 2-ary function $\wedge : V \times V \to V : (u, v) \mapsto u \wedge v$ on V that is idempotent (i.e., $u \wedge u = u$), symmetric, and associative. We thus use '$\wedge$' and '$\leq$' interchangeably, and may refer to either of them as a semilattice ordering. We use the common variant $\geq$ of the symbol $\leq$ where it is convenient to do so.

It is well known, and easily verified, that the two definition of a semilattice are related through the identity

$$u \leq v \iff u \wedge v = u.$$

As our semilattices are finite, the existence of a lower bound for every pair of elements extends by

$$\bigwedge S = s_1 \wedge \cdots \wedge s_d.$$

to subsets $S \subset V$.

The *width* of a semilattice is the maximum number of pairwise incomparable elements. An element v *covers* or *is a cover of* an element u if $u \leq v$ and if $u \leq x \leq v$ for some x implies that $x \in \{u, v\}$. Given a semilattice $\wedge_1$ on V_1 and a semilattice $\wedge_2$ on V_1, the *product semilattice* $\wedge = \wedge_1 \times \wedge_2$ defined by

$$(u_1, u_2) \wedge (v_1, v_2) = (u_1 \wedge_1 v_1, u_2 \wedge_2 v_2)$$

is a semilattice on $V_1 \times V_2$.

2.2 Semilattice polymorphisms

We denote the adjacency of two vertices u and v of a graph by $u \sim v$. A *(k-ary) polymorphism* $F : G^k \to G$ of a graph G is function $f : V(G)^k \to V(G)$, on the vertex set $V(G)$, which satisfies the following for all choices of $u_i, v_i \in V(G)$.

$$u_i \sim v_i \text{ for all } i \in \{1, \ldots, k\} \Rightarrow f(u_1, \ldots, u_k) \sim f(v_1, \ldots, v_k)$$

A semilattice $\wedge : V(G) \times V(G) \to V(G)$ is *compatible with* G, or is a *semilattice (SL) polymorphism on* G, if it is a polymorphism. It is easily seen that for a reflexive graph G, a semilattice $\wedge$ on $V(G)$ is a polymorphism of G if and only if it satisfies the following:

$$a \sim a',\, b \sim b' \Rightarrow (a \wedge b) \sim (a' \wedge b') \tag{1}$$

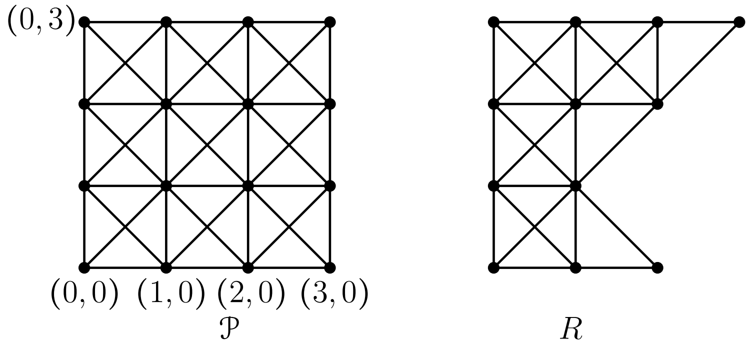

Figure 1: The product $\mathcal{P}$ of 3-paths and a retract R. (Loops omitted.)

The *(categorical) product* of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ with vertices $V(G_1) \times V(G_2)$ such that $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 \sim a_2$ and $b_1 \sim b_2$. A *retraction* of a graph G is a homomorphism $r : G \to G'$ to a subgraph G', that is the identity on G'.

The following standard fact from [8] verifies that **SL** is closed under taking products. It was also shown in [8] that **SL** is not closed under retractions.

Fact 1. *If $\wedge_1$ is a SL polymorphism of G_1 and $\wedge_2$ is an SL polymorphism of G_2 then the product semilattice $\wedge$ of $\wedge_1$ and $\wedge_2$ is an SL polymorphism of $G_1 \times G_2$.*

We will frequently use products of paths. Let P_ℓ denote the path of length ℓ having vertex set $[0, \ell]$, where $a \sim b$ if $|a - b| \leq 1$. Figure 1 shows the product $\mathcal{P} = P_3 \times P_3$ of two 3-paths, and a typical retract R of $\mathcal{P}$. Though the graph is reflexive, we have omitted all loops from the figure. We will do so on all figures.

Given a semilattice $(V, \wedge)$, a sub-semilattice consists of a subset $V' \subset V$ that is closed under $\wedge$:

$$a, b \in V' \Rightarrow a \wedge b \in V'.$$

A subset S of the vertices of a graph G is *conservative* (sometimes called a subalgebra) if for every idempotent polymorphism $\phi : G^d \to G$ of G, $s_1, \ldots, s_d \in S$ implies that $\phi(s_1, \ldots, s_d) \in S$. In particular, an SL polymorphism of a graph G induces a sub-semilattice on any conservative set. So the subgraph of any SL graph induced by any conservative set is also an SL graph. It is well known that the i^{th} *distance neighbourhood* $N^i(v)$ of a reflexive graph G, consisting of all vertices that are distance at most i from a vertex v, are conservative. It is also known that the intersection of conservative sets is conservative.

That is to say, we have the following.

Fact 2. *Let G be a reflexive SL graph, then following sets induce SL subgraphs of G.*

i. The set $N^i(v)$ for any vertex $v \in G$, and any integer $i \geq 0$.

ii. Intersections of the above sets.

As the only semilattice on a two element set is a totally ordered set, the following useful fact is immediate from the above fact.

Fact 3. *If an edge (u, v) of an SL graph G is the intersection of distance neighbourhoods of vertices of G, then either $u \leq v$ or $v \leq u$ with respect to any compatible semilattice.*

2.3 NU polymorphisms

A k-ary polymorphism $f : G^k \to G$ is *near-unanimity (NU)* if

$$f(v_1, \ldots, v_k) = a$$

whenever at least $k-1$ of the v_i are a. Specifying the arity, we often refer to such a polymorphism f as being k-NU. A 3-NU polymorphism is also known as a *majority polymorphism.*

There are many characterisations graphs with *NU* polymorphisms. We introduce here the description from [5].

Definition 4. Let T be a tree with k leaves and m edges $e_1, \ldots, e_m$. Let U and D be the partition of its vertices into two independent set, and let U^* and D^* be the subsets of U and D respectively, of vertices of degree at least 2. Define a graph $\mathbb{K}^0(T)$ as follows: its vertices are the tuples $(x_1, \ldots, x_m)$ such that

i. $x_i \in \{0, 1, 2\}$ for every $1 \leq i \leq m$;

ii. for each $u \in U^*$, $x_i = 2$ for at least one edge e_i incident with u; and

iii. for each $d \in D^*$, $x_i = 0$ for at least one edge e_i incident with d.

Tuples $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ are adjacent if $|x_i - y_i| \leq 1$ for all i.

For an example of the construction $\mathbb{K}^0(T)$, remove the orientation from the edges of the graphs shown in Figure 2. The vertices of the tree $T = K_{1,3}$ are partitioned into the set U of three leaves, and set D containing the last vertex. Though this is not the case with our later figures, all the vertices of $\mathbb{K}^0(T)$ are shown in this figure: U^* is empty, but D^* contains the one vertex of D, so every vertex has exactly one coordinate equal to 0. Not all edges are shown, however; two vertices are adjacent if they are in the same unit cube.

The following was a main result of [5]

Theorem 5. *A reflexive graph G admits a k-NU polymorphism if and only if it is a retract of the product of the graphs $\mathbb{K}^0(T_i)$, for a finite family of trees $T_1, \ldots, T_d$ each having at most $k-1$ leaves.*

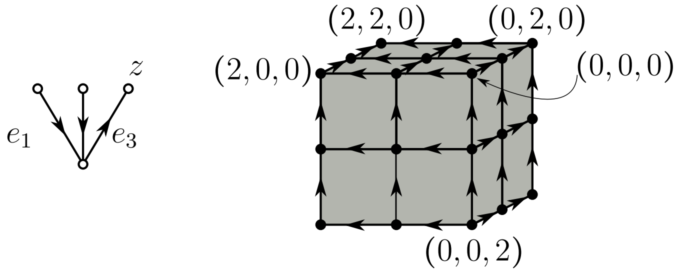

Figure 2: An orientation of the rooted tree $T = K_{1,3}$ and the corresponding semilattice $\wedge_z$, on the graph $\mathbb{K}^0(T)$. The semilattice $\wedge_z$ is the transitive closure of the shown digraph on $\mathbb{K}^0(T)$. (Unlike similar figures later in the paper, all vertices are shown here.)

3 SL polymorphisms on the generators of $\mathbf{NU}_k$

We define an SL polymorphism on $\mathbb{K}^0(\mathbb{T})$.

Definition 6. Let T be a tree with k leaves and m edges $e_1, \ldots, e_m$, having vertex partition U and D as in Definition 4. Choose a root z of T, and orient the edges of T towards z, (so that any vertex of T has at most one incident edge oriented away from it).

Define $\wedge = \wedge_z : \mathbb{K}^0(T) \times \mathbb{K}^0(T) \to \mathbb{K}^0(T)$ as follows: for $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m) \in \mathbb{K}^0(T)$, let

$$(x_1, \ldots, x_m) \wedge (y_1, \ldots, y_m) = (z_1, \ldots, z_m)$$

where z_i is $\max(x_i, y_i)$ if e_i is oriented towards U, and is $\min(x_i, y_i)$ if e_i is oriented towards D. (See Figure 2 for an example.)

Proposition 7. *The map $\wedge_z : \mathbb{K}^0(T) \times \mathbb{K}^0(T) \to \mathbb{K}^0(T)$ defined in Definition 6 is an SL polymorphism on $\mathbb{K}^0(T)$.*

Proof. To prove that this function is onto $\mathbb{K}^0(\mathbb{T})$ we must show for $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ in $\mathbb{K}^0(T)$ that

$$(z_1, \ldots, z_m) = (x_1, \ldots, x_m) \wedge_z (y_1, \ldots, y_m)$$

is also in $\mathbb{K}^0(T)$. This requires showing that for any $d \in D^*$, there is at least one each edge e_i incident to d such that $z_i = 0$; and for any $u \in U^*$ there is an incident edge e_i such that $z_i = 2$. We show the former, the proof of the latter is essentially the same.

Let $d \in D^*$ have incident edges $e_1, \ldots, e_c$. At most one, say e_1, is directed away from d. If there is some $i \neq 1$ such that $x_i = 0$ or $y_i = 0$, then $z_i = \min(x_i, y_i) = 0$. Otherwise, both $x_1 = 0$ and $y_1 = 0$ and so $z_1 = \max(x_1, y_1) = 0$. So $(z_1, \ldots, z_m) \in \mathbb{K}^0(T)$, as needed.

To see that $\wedge_z$ it is a homomorphism, assume that $x_i \sim x_i'$ and $y_i \sim y_i'$ for all i, where x, x', y and y' are in $\mathbb{K}^0(T)$. Then $|x_i - x_i'| \leq 1$ and $|y_i - y_i'| \leq 1$. As both $|\min(x_i, y_i) - \min(x_i', y_i')|$ and $|\max(x_i, y_i) - \max(x_i', y_i')|$ are clearly at most 1, we get that $|z_i - z_i'| \leq 1$ for all i, and so $(z_1, \ldots, z_m) \sim (z_1', \ldots, z_m')$.

The homomorphism is symmetric and associative, as it is in each coordinate, and is clearly idempotent by definition. Thus it is an SL polymorphism, as needed. □

In [5] it was shown that for a tree T with $k-1$ leaves, $\mathbb{K}^0(T) \in \mathbf{NU}_k \smallsetminus \mathbf{NU}_{k-1}$. Thus we get the following.

Corollary 8. *For all $k \geq 4$ there are graphs in $\mathbf{SL} \cap \mathbf{NU}_k$ that are not in $\mathbf{NU}_{k-1}$.*

This is already known, it follows from [3] and [8]; but the examples one gets from these papers are chordal or products of chordal graphs. The above examples are far from this.

In [5] we defined retracts $\mathbb{K}(T)$ of the $\mathbb{K}^0(T)$ which also served as generators of the variety $\mathbf{NU}_k$. One can show that the SL polymorphism defined above survives the retraction from $\mathbb{K}^0(T)$ to $\mathbb{K}(T)$, so these smaller generators are also in $\mathbf{SL}$. The proof of this is basic, but is too messy for what it gains us; we chose to omit it from the paper.

4 A graph in $\mathbf{NU_3} \smallsetminus \mathbf{SL}$

In this section we prove the following theorem.

Theorem 9. *There exists a reflexive graph admitting a 3-NU polymorphism, but no SL polymorphism.*

Our proof is constructive. For a semilattice $\leq$ defined on the vertex set of a graph G, we say an edge $u \sim v$ is *oriented* $u \to v$ by $\leq$ if $u \leq v$, and $u \neq v$. A *bad* 2*-path* is an induced 2-path in G such that the edges are oriented towards each other: $u \to v \leftarrow w$. The following simple observation is key.

Lemma 10. *Given an SL polymorphism on a graph, the graph contains no bad 2-paths.*

Proof. Indeed, if $u \to v \leftarrow w$, then $u = u \wedge v \sim v \wedge w = w$. This contradicts the fact that the 2-path $u \sim v \sim w$ is induced. □

The main idea of our proof of Theorem 9, is that in light of the above observation, there are very few compatible semilattices for a product $\mathcal{P}$ of paths. The only compatible semilattices are, up to some skewing, the product of semilattices that are compatible with the component paths; and each of these can be 'killed' by making a retraction in the right place. The 'right place' is not very specific, but does require that the 'killing' retraction is well inside one of the orthants of $\mathcal{P}$ one gets by viewing it as a three dimensional cube with the minimum element of the lattice as its origin.

We thus start with a product of long paths, and make several killing retractions in such a way that wherever the minimum vertex is for a semilattice defined on $V(\mathcal{P})$, one of the killing retractions is contained well inside an orthant, and so kills the compatibility of the semilattice with the graph.

Before we get to the actually constructive proof, we set up for it by defining our 'killing' retractions.

4.1 Setup for the proof

Let $\mathcal{P} = P_\ell^3$ be a product of three paths of length ℓ; so vertices of $\mathcal{P}$ are triples $v = (v_1, v_2, v_3) \in [0, \ell]^3$. For $i \in [3]$, let e_i denote the vector with a 1 in the i^{th} coordinate, and 0 elsewhere, so that for a vertex v, $v + e_i$ is the vertex we get from it by increasing the i-coordinate by one.

An edge of $\mathcal{P}$ of the form $\{v, v+e_i\}$ is *i-square*. It is *inner i-square* if $v_j \notin \{0, \ell\}$ for any $j \neq i$. An edge is *(inner) square* if it is (inner) i-square for some i. These edges will be important because of the following simple observation.

Lemma 11. *Under any SL polymorphism of $\mathcal{P}$, all inner square edges are oriented.*

Proof. Clearly we may assume our inner square edge is of the form $\{u, u+e_1\}$. In this case, as it is an inner square edge, the set

$$S = \{u, u+e_1, u \pm e_2, u \pm e_3\}$$

is contained in $\mathcal{P}$. As $\bigcap_{s \in S} N(s) = \{u, u+e_1\}$, we have by Fact 3 that $\{u, u+e_1\}$ is oriented. □

For a subgraph G of $\mathcal{P}$, the square edges are *consistently oriented* under a semilattice on $V(G)$, if for each i, there exists $d \in \{-1, 1\}$ such that all i-square edges of G are oriented $v \to v + de_i$. If $d = 1$ they are *positively oriented*, if $d = -1$ they are negatively oriented.

For a square edge $\{v, v+e_i\}$ let $C(v; i^+)$ be the *cone* of vertices that are closer (graph distance) to $v + e_i$ than to v. That is, let

$$C(v; i^+) = \left\{ v + \sum_{j=1}^{3} a_i e_i \in V(\mathcal{P}) : \forall j \neq i, 0 \leq |a_j| < a_i \right\}.$$

Define $C(v, i^-)$ analogously as the cone of vertices closer to $v - e_i$ that to v. The graph $\mathcal{P} \smallsetminus C(v, i^+)$ is in fact a retract of $\mathcal{P}$, as one can easily check that the following map r, which 'pushes $C(v; i^+)$ in the i direction', is a homomorphism: for $x \in \mathcal{P} \smallsetminus C(v; i^+)$, let $r(x) = x$; and for $x \in C(v; i^+)$ let $r(x) = x - m_x e_i$ for the smallest $m_x > 0$ such that $x - m_x e_i$ is not in $C(v; i^+)$.

The retract R in Figure 1 can be viewed as a 2-dimensional version of the construction $\mathcal{P} \smallsetminus C(v; i^+)$. Specifically, it would be $\mathcal{P} \smallsetminus C((1,1); 1^+))$, as we have removed the cone of vertices closer to $(2,1)$ than to $(1,1)$.

Figure 3 gives two different depictions of $B = P_2^3 \smallsetminus C((1,1,1); 1^+)$. The first depiction shows the subgraphs induced by the 1-layers of P_2^3, the i^{th} *1-layer* being the set of vertices v such that $v_1 = i$. We have only shown the edges between 1-layers that involve the vertex $(1,2,2)$. The second depiction, in which the 1^{st} coordinate increases toward the reader, is more suggestive of how B can be viewed as subgraph of a product of paths achieved by removing a cone. Many edges are hidden, but any 'unit cube' in this picture induces a clique of B.

It is easy to show that removing a vertex in the i direction, as in Figure 3 for $i = 1$, will 'kill' SL polymorphisms with consistently oriented square edges that are directed, in

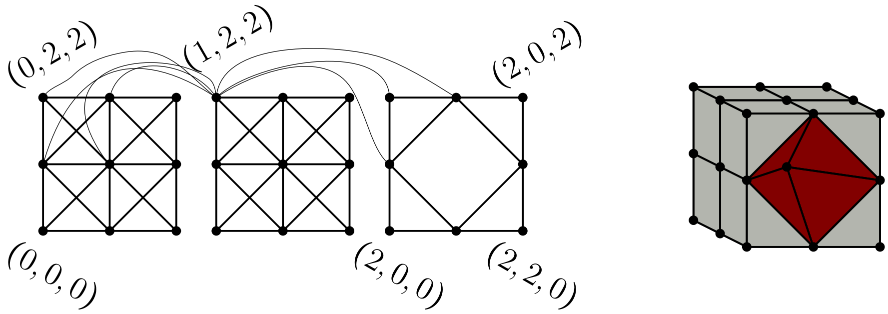

Figure 3: Two depictions of graph $B = \mathcal{P} \smallsetminus C((1,1,1);1^+)$

the i direction, away from the missing vertex. The useful property for us though, is the less intuitive fact that it kills such polymorphisms directed towards the missing vertex.

Lemma 12. *Let $\mathcal{P} = P_2^3$, and let $B = \mathcal{P} \smallsetminus C((1,1,1);i^+)$ for some $i \in [3]$. (See Figure 3 for the case $i = 1$.) There is no SL polymorphism on B with consistently oriented square edges in which the i-square edges are positively oriented. The same holds when replacing '$B = \mathcal{P} \smallsetminus C((1,1,1);i^+)$' with '$B = \mathcal{P} \smallsetminus C((1,1,1);i^-)$' and 'positively' with 'negatively'.*

Proof. We prove only the first statement, the second follows by relabeling vertices. We further assume that $i = 1$. Towards contradiction, assume that $\wedge$ is an SL polymorphism of B. Without loss of generality we may assume that the j-square edges are positively oriented for all $j \in [3]$; so $x \to y$ if $x_j \leq y_j$ for all j.

We now show that there is no value viable value for $x = (2,1,2) \wedge (2,2,1)$. Indeed, as $(2,1,2) \sim (2,2,1)$ we have that x is adjacent to both of these vertices, so must be in $\{1,2\}^3 \smallsetminus (2,1,1)$. Further, it is below both these vertices. So its second co-ordinate cannot be 2, or else $x \to (2,1,2) \leftarrow (2,0,2)$ would be a bad 2-path, which is impossible by Lemma 10. Similarily its third co-ordinate cannot be 2 or $x \to (2,2,1) \leftarrow (2,2,0)$ would be a bad 2-path. Thus x can only be $(1,1,1)$. But as x is $(2,1,2) \wedge (2,2,1)$ it is then above $(2,0,0)$, and so $(0,1,1) \to x \leftarrow (2,0,0)$ is a bad 2-path. $\square$

4.2 Graph with 3-NU but no SL

Let $\mathcal{P} = P_{17}^3$. Consider the sets

$$\mathcal{V}_i^- = \{v \in \mathcal{P} \mid v_i = 2 \text{ and } v_j \in \{4,9,13\} \text{ for } j \neq i\}$$
$$\mathcal{V}_i^+ = \{v \in \mathcal{P} \mid v_i = 15 \text{ and } v_j \in \{4,9,13\} \text{ for } j \neq i\}$$

So $\mathcal{V} = \mathcal{V}^- \cup \mathcal{V}^+$ contains 54 vertices, each two layers in from an outside layer of $\mathcal{P}$. For each v in $\mathcal{V}_i^-$ let $C(v) = C(v;i^-)$ and for each v in $\mathcal{V}_i^+$ let $C(v) = C(v;i^+)$, and let

$$R = \mathcal{P} \smallsetminus \Big(\bigcup_{v \in \mathcal{V}} C(v)\Big).$$

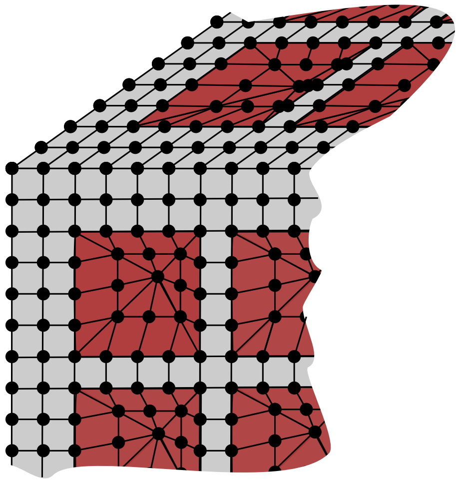

Figure 4: The retract $R = \mathcal{P} \setminus (\bigcup C(v))$ from Lemma 13

As we observed following the definition of the notation $C(v, i^-)$, R can be viewed as a retract $r(\mathcal{P})$ where r is the retraction that 'pushes in' each of the cones $C(v)$. Figure 4 shows part of the graph R. The dimples in the face of the cube are some of the 54 different cones $C(v)$ removed from $\mathcal{P}$. As with previous figures, not all edges are drawn; vertices in the same unit cube, even in these dimples, are adjacent.

Lemma 13. *The retract R of $\mathcal{P} = P^3_{17}$ defined above has no SL polymorphisms.*

Proof. For each $v \in \mathcal{V}$ let $B(v)$ be the neighbourhood of v in R. It is easy to check that the vertices in $\mathcal{V}$ are far enough apart that each subgraph $B(v)$ is isomorphic to the graph B from Lemma 12.

Assume, towards contradiction, that R has a SL polymorphism $\wedge$. We will show that $\wedge$ must be consistently ordered on one of the subgraphs $B(v)$ in such a way as to contradict Lemma 12.

The first step is the following claim, which says that Lemma 11 holds not only on $\mathcal{P}$, but on the retract R

Claim 14. *All edges of R that are inner i-square edges of $\mathcal{P}$, are oriented.*

Proof. By Fact 3 it is enough to show that any inner i-square edge, which we may assume to be $\{x, x + e_1\}$, is the intersection of distance neighbourhoods.

This is easy if all neighbours of x and $x+e_1$ in $\mathcal{P}$ are in R, as then $\{x, x+e_1\} = \bigcap_{s \in S} N(s)$ where $S = \{x, x + e_1, x \pm e_2, x \pm e_3\}$ as in the proof of Lemma 11. If $x + e_j$ or $x - e_j$ is not in R for $j = 2$ or 3, then we can replace it in the set S. Indeed, it is enough to consider the case that $x + e_2$ is not in R. In this case, the proof above works by replacing $x + e_2$ in S with any vertex in

$$\{x + e_1 + e_2, x + e_2 \pm e_3, x + e_1 + e_2 \pm e_3\}.$$

So we must show that at least one of these vertices are in R. If none of them are, then as the subgraphs $B(v)$ of $\mathcal{P}$ are distance at least 3 apart, all of these vertices are in $C(v)$ for some single $v \in \mathcal{V}$.

Now, these five vertices, along with $x + e_2$, make up a copy of $P_2 \times P_3$ and all have the same 2 coordinate $x_2 + 1$. The only way they can fit into $C(v)$ is if v is in $\mathcal{V}_2^+$, and if v is one of x and $x + e_1$. But then $v \notin R$ contradicts the fact that both x and $x + e_1$ are in R. ◇

Now, for a vertex $v \in R$ let the *i-line* of v be the set

$$L_i(v) = \{u \in V(\mathcal{P}) \mid v_j = u_j \text{ for } j \neq i\}$$

of vertices of $\mathcal{P}$ that differ from it in at most the i^{th} coordinate. Observe that for an i-line $L = L_i(v)$ of inner square edges, which is fully contained in R, the edges are oriented by the previous claim, so we have from Lemma 10 that L contains a *center vertex* c_L away from which all edges must be oriented. If an i-line $L = L_i(v)$ is not contained in R, then the retraction $r : \mathcal{P} \to R$ maps a unique vertex c of L onto center of some other i-line, and we define this c as the center of L.

Claim 15. *Let u and v be adjacent, then for any i, the centers of the i-lines $L_i(u)$ and $L_i(v)$ are adjacent.*

Proof. It is enough to prove the claim for u and v whose i-lines are in R. Assume that the centers $a = c_{L_i(u)}$ and $b = c_{L_i(v)}$ are not adjacent. Then as they are in the i-lines of adjacent vertices, their i-coordinates must differ by more than one. We may assume that $a_i \leq b_i - 2$. Let a' be the vertex in $L_i(u)$ with $a'_i = b_i - 1$. Then $a' \sim b$ and $a' - e_i \sim b - e_i$, but

$$a' - e_i = a' \wedge a' - e_i \sim b \wedge b - e_i = b,$$

which contradicts the fact that $a' - e_i \not\sim b$. ◇

The *inner i-floor* is the graph induced by the set of centers of the inner i-lines. By the previous claim, we have that it is a product of two paths. A simple consequence of this is that if x and x' are in the i-floor then

$$|x_i - x'_i| \leq |x_j - x'_j| \text{ for all } j \neq i \tag{2}$$

Equation (2) now implies that the inner 1-floor cannot intersect $B(v)$ and $B(v')$ for v in $\mathcal{V}_1^+$ and v' in $\mathcal{V}_1^-$. Indeed the maximum distance $|x_j - x'_j|$ between the $j \neq 1$ coordinates of vertices $x \in B(v)$ and $x' \in B(v')$ is $13 - 4 + 2 = 12$ while the distance $|x_1 - x'_1|$ between their 1 coordinates is at least $17 - 4 = 13$. So we may assume that the 1-floor does not intersect $B(v)$ for any $v \in \mathcal{V}_1^+$.

Similarly, equation (2) gives us that the inner 2-floor can intersect $B(v)$ for at most three $v \in \mathcal{V}_1^+$. We show that for any $m \in \{4, 9, 13\}$, it can intersect $B(v)$ for at most one v with $v_1 = 15$ and $v_3 = m$. Indeed, if it were to contain $x \in B(v)$ and $x' \in B(v')$ for $v \neq v'$ where $v_1 = 15 = v_1'$ and $v_3 = m = v_3$, then $|x_3 - x_3'| \leq 2$ and as $v \neq v'$, $|x_2 - x_2'| \geq (9-1)-(4+1) = 3$.

So there is some $v \in \mathcal{V}_1^+$ such that $B(v)$ is not intersected by any i-floor. This means that its square edges are consistently oriented, and in particular its 1-square edges are positively oriented. This is the contradiction of Lemma 12 we were looking for, and so completes the proof of Lemma 13. □

As the graph R is a retract of a product of paths, it follows by [7] that it admits a 3-NU polymorphism, so this completes the proof of Theorem 9.

5 Questions and Discussion

The construction for Theorem 9 is certainly bigger than it needs to be, having $18^3 - 540$ vertices. It is probably not hard to refine the construction to make it a little smaller, but a much smaller example would be interesting. In particular, our construction is a retraction of a product of three paths. We spent considerable time trying to prove the results with a retract of a product of two paths, but it proved to be quite a stubborn problem.

Question 16. Does every retract of a product of two paths admit an SL polymorphism?

As there are generators of $\mathbf{NU}_k$ for $k \geq 5$ that retract to a product of three paths, there are retracts of such generators that omit SL polymorphisms. However we wonder about the following; a positive answer to this question would imply a positive answer to the previous question.

Question 17. Does every retract of $\mathbb{K}^0(T)$, where T is a tree with three leaves, admit an SL polymorphism?

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