On the Schur Positivity of $\Delta_{e_2} e_n[X]$

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Submitted: Nov 24, 2017; Accepted: May 21, 2018; Published: Oct 19, 2018

Abstract

Let $\mathbb{N}$ denote the set of non-negative integers. Haglund, Wilson, and the second author have conjectured that the coefficient of any Schur function $s_\lambda[X]$ in $\Delta_{e_k} e_n[X]$ is a polynomial in $\mathbb{N}[q,t]$. We present four proofs of a stronger statement in the case $k=2$; we show that the coefficient of any Schur function $s_\lambda[X]$ in $\Delta_{e_2} e_n[X]$ has a positive expansion in terms of q,t-analogs.

Mathematics Subject Classifications: 05E05, 05E10

1 Introduction

Let Λ denote the ring of symmetric functions with coefficients in $\mathbb{Q}(q,t)$. If μ is a partition of n, we shall write $\mu \vdash n$. Let $X = x_1 + \cdots + x_N$. The sets $\{e_\mu[X] : \mu \vdash n\}$, $\{s_\mu[X] : \mu \vdash n\}$ and $\{\tilde{H}_\mu[X;q,t] : \mu \vdash n\}$ are the elementary, the Schur, and the (modified) Macdonald symmetric function bases for $\Lambda^{(n)}$, the elements of Λ that are homogeneous of degree n. Given a partition $\mu \vdash n$ and a cell c in the Young diagram of μ (drawn in French notation), we set $a'(c)$ and $\ell'(c)$ to be the number of cells in μ that are strictly to the left and strictly below c in μ, respectively. For example, if $\mu = (3,4,4,5)$ and c is the

*Partially supported by NSF grant DMS-1603681.

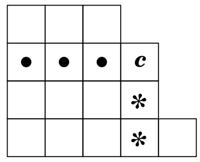


Figure 1: $a'(c)$ and $\ell'(c)$.

cell pictured in Figure 1, then $a'(c) = 3$ is represented by the cells containing dots and $\ell'(c) = 2$ is represented by the cells containing stars.

We set

$$B_\mu(q,t) = \sum_{c\in\mu} q^{a'(c)}t^{\ell'(c)}, \qquad\qquad T_\mu(q,t) = \prod_{c\in\mu} q^{a'(c)}t^{\ell'(c)}.$$

Given any symmetric function $f \in \Lambda$, we define operators Δ_f and Δ'_f on Λ by their action on the Macdonald basis:

$$\Delta_f \tilde{H}_\mu[X;q,t] = f[B_\mu(q,t)]\tilde{H}_\mu[X;q,t], \quad \Delta'_f \tilde{H}_\mu[X;q,t] = f[B_\mu(q,t)-1]\tilde{H}_\mu[X;q,t].$$

Here, we have used the notation that, for a symmetric function f and a sum $A = a_1 + \ldots + a_N$ of monic monomials, $f[A]$ is equal to the specialization of f at $x_1 = a_1, \ldots, x_N = a_N$, where the remaining variables are set equal to zero. We also set $\nabla = \Delta_{e_n}$ as an operator on $\Lambda^{(n)}$. Note that, by definition, for any $1 \leqslant k \leqslant n$,

$$\Delta_{e_k} e_n[X] = \Delta'_{e_k + e_{k-1}} e_n[X] = \Delta'_{e_k} e_n[X] + \Delta'_{e_{k-1}} e_n[X]. \tag{1}$$

Furthermore, for any $k > n$, $\Delta_{e_k} e_n[X] = \Delta'_{e_{k-1}} e_n[X] = 0$. Therefore $\Delta_{e_n} e_n[X] = \Delta'_{e_{n-1}} e_n[X]$.

In [5], Haglund, Remmel, and Wilson conjectured a combinatorial interpretation of the coefficients that appear in the expansion of $\Delta_{e_k} e_n[X]$ in terms of the fundamental quasi-symmetric functions. Their conjecture is now referred to as the Δ-conjecture. They also conjectured that coefficients in the Schur function expansion of $\Delta_{e_k} e_n[X]$ are polynomials in q and t with non-negative integer coefficients. There are two cases that are known. Namely, when $k = n$, then Haiman [6] proved that $\Delta_{e_n} e_n[X] = \nabla e_n[X]$ is the Frobenius image of the character generating function of the ring of diagonal co-invariants. Thus in this case, repesentation theory tells us that the coefficient of the Schur function $s_\lambda[X]$, $\langle \nabla e_n[X], s_\lambda[X]\rangle$, is a polynomial in q and t with non-negative integer coefficients. Also in this case, the so-called "Shuffle conjecture" of Haglund, Haiman, Loehr, Remmel, and Ulyanov [4] gives a combinatorial interpretation of the coefficients that arise in the expansion of $\nabla e_n[X]$ in terms of fundamental quasi-symmetric functions. The Shuffle conjecture was recently proved by Carlsson and Mellit [2].

The other known case is when $k = 1$. In [5], the authors proved that

$$\Delta_{e_1} e_n[X] = \sum_{m=0}^{\lfloor n/2\rfloor} s_{2^m,1^{n-2m}}[X] \sum_{p=m}^{n-m} [p]_{q,t} \tag{2}$$

where $[n]_{q,t} = \frac{q^n - t^n}{q-t} = q^{n-1} + q^{n-2}t + \cdots + qt^{n-2} + t^{n-1}$ for $n \geqslant 0$.

The main goal of this paper is to give four different proofs of the fact that $\Delta_{e_2} e_n[X]$ is Schur positive, i.e. for all $\lambda \vdash n$, $\langle \Delta_{e_2} e_n[X], s_\lambda[X] \rangle \in \mathbb{N}[q,t]$, in hopes that some of the ideas in those proofs can be adapted to prove the Schur positivity of $\Delta_{e_k} e_n[X]$ for $k \geqslant 3$.

All of our proofs start with the following result of Haglund [3].

Lemma 1. *For all n, d, and symmetric functions $f[X]$,*

$$\langle \Delta_{e_{d-1}} e_n[X],\ f[X] \rangle = \langle \Delta_{\omega f}\, e_d[X],\ s_d[X] \rangle. \tag{3}$$

Let λ be any partition of n. By setting $f = s_\lambda$, we have

$$\langle \Delta_{e_{d-1}} e_n[X],\ s_\lambda \rangle = \langle \Delta_{s_{\lambda'}} e_d,\ s_d \rangle. \tag{4}$$

The formula works nicely when d is small, since we have explicit expansion of e_d in terms of Macdonald polynomials. In the case $d = 2$ we have

$$e_2[X] = \frac{1}{t-q} \tilde{H}_{1,1}[X;q,t] - \frac{1}{t-q} \tilde{H}_2[X;q,t].$$

This leads to

$$\begin{aligned}
\langle \Delta_{e_1} e_n[X],\ s_\lambda[X] \rangle &= \langle \Delta_{s_{\lambda'}} e_2[X],\ s_2[X] \rangle \\
&= \Big\langle \frac{1}{t-q} s_{\lambda'}[1+t] \tilde{H}_{1,1}[X;q,t] - \frac{1}{t-q} s_{\lambda'}[1+q] \tilde{H}_2[X;q,t],\ s_2[X] \Big\rangle \\
&= \frac{1}{t-q} s_{\lambda'}[1+t] - \frac{1}{t-q} s_{\lambda'}[1+q],
\end{aligned}$$

which is easily seen to be an element of $\mathbb{N}(q,t)$.

In the case $d = 3$, the expansion of e_3 leads to the following formula.

$$g_\lambda := \langle \Delta_{e_2} e_n[X],\ s_\lambda[X] \rangle = \frac{(t-q^2) s_{\lambda'}[1+t+t^2] - (q+t+1)(t-q) s_{\lambda'}[1+q+t] + (t^2-q) s_{\lambda'}[1+q+q^2]}{(t-q)(t^2-q)(t-q^2)} \tag{5}$$

At first glance, this formula does not seem to be useful. Indeed, it is not immediately obvious that this quotient is a polynomial.

Our (chronologically) first approach to proving that g_λ is in $\mathbb{N}[q,t]$ is based on the following observations.

i) If λ' has more than three parts, then $g_\lambda = 0$;

ii) If we expand $s_{a,b,c}[x + y + z]$ as a quotient of alternates, then from the view of MacMahon partition analysis, one can easily see that the generating function

$$\sum_{a \geqslant b \geqslant c \geqslant 0} g_{(a,b,c)'} u_1^a u_2^b u_3^c$$

is a rational function.

iii) Hence, it might be easier to show that this generating function has only nonnegative coefficients.

We succeeded in this approach by finding a proof that can be easily verified by computer, but it is too long to be printed. We will explain this approach in Section 4, but we will not include full details.

Our other approaches rely on the following alternative representation of g_λ.

Lemma 2. *Let τ be the operation which switches t and q. Then*

$$g_\lambda = \langle \Delta_{e_2} e_n[X],\ s_\lambda[X] \rangle = \frac{F_{\lambda'} - \tau F_{\lambda'}}{t - q} = \frac{id - \tau}{t - q} F_{\lambda'}, \tag{6}$$

where $\tau F = F\big|_{q=t,t=q}$ and

$$F_{\lambda'} = \frac{s_{\lambda'}[1+t+t^2] - s_{\lambda'}[1+t+q]}{t^2 - q}. \tag{7}$$

Proof. By using the formula

$$(t-q)(1+q+t) \ = \ (t-q^2) - (q-t^2) \ = \ (id-\tau)(t-q^2),$$

equation (5) becomes

$$\begin{aligned} \langle \Delta_{e_2} e_n[X],\ s_\lambda[X] \rangle &= \frac{(id-\tau)(t-q^2) s_{\lambda'}[1+t+t^2] - (id-\tau)(t-q^2) s_{\lambda'}[1+q+t]}{(t-q)(t^2-q)(t-q^2)} \\ &= \frac{1}{t-q}(id-\tau)\left(\frac{s_{\lambda'}[1+t+t^2] - s_{\lambda'}[1+t+q]}{t^2-q} \right). \end{aligned}$$

This is just the desired (6). ∎

We will show that $F_{\lambda'}$ is a polynomial that can be interpreted as a sum over semi-standard Young tableaux filled with numbers $0, 1, 2$. From this formula, it is clear that g_λ is in $\mathbb{Z}[q, t]$ where $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is the set of integers.

We present our second proof in Section 2. We introduce new combinatorial objects, called "enriched" semi-standard Young tableaux, to interpret the coefficients of g_λ. We then define an injection on these enriched tableaux which will allow us to prove that g_λ is in $\mathbb{N}[q, t]$.

In Section 3, we present our third proof that g_λ is in $\mathbb{N}[q,t]$. The proof in this section is a direct computation of g_λ carried out by breaking g_λ into a sum of terms where each term is easily seen to be a polynomial in q and t with non-negative coefficients. The advantage of this proof is that we can recursively produce explicit formulas for g_λ.

In Section 4, we shall expand our discussion of the generating function approach described above and describe an alternate way to analyze the resulting generating functions which is our fourth proof.

Finally in Section 5, we give a formula of $\Delta_{e_3} e_n[X]$. However it is not clear how we can split up this formula into pieces which are easily seen to be polynomials in q and t with non-negative coefficients. Thus, the general problem of establishing the Schur-positivity of $\Delta_{e_d} e_n[X]$ seems to require new ideas.

2 Combinatorial Proof

The idea is based on the following observation:

$$\frac{(id-\tau)t^j q^i}{t-q} = -\frac{(id-\tau)t^i q^j}{t-q} \text{ and } \frac{(id-\tau)t^i q^j}{t-q} = (tq)^j [i-j]_{q,t}, \quad \text{if } i \geqslant j.$$

Thus if $F_{\lambda'} = \sum_{i,j} a_{i,j} q^i t^j$, we have

$$g_\lambda = \frac{(id-\tau)F_{\lambda'}}{t-q} = \sum_{i>j} (a_{i,j}-a_{j,i})(tq)^j [i-j]_{q,t}.$$

To show that $g_\lambda \in \mathbb{N}[q,t]$, it is sufficient to show that $a_{i,j}-a_{j,i} > 0$ for every $i > j$. Note that this condition indeed shows the Schur-positivity of g_λ in q,t-analogs, a stronger condition than $g_\lambda \in \mathbb{N}[q,t]$: for instance, $q^2+t^2 \in \mathbb{N}[q,t]$ but $q^2+t^2 = [3]_{q,t} - qt\,[1]_{q,t}$.

Now we have a combinatorial interpretation of $F_{\lambda'}$ using formula (7) of Section 1. Firstly,

$$s_{\lambda'}[x_0+x_1+x_2] = \sum_T x_0^{\kappa_0(T)} x_1^{\kappa_1(T)} x_2^{\kappa_2(T)},$$

where the sum is over all semi-standard Young tableaux T of shape λ' filled with numbers $0,1,2$, and $\kappa_i(T)$ is the number of i's in T. Generic semi-standard Young tableaux T of shape λ' are pictured in Figure 6. For any given tableau T, we see that contribution to

$$\frac{s_{\lambda'}[1+t+t^2] - s_{\lambda'}[1+t+q]}{t^2-q}$$

is

$$t^{\kappa_1(T)} \frac{t^{2\kappa_2(T)} - q^{\kappa_2(T)}}{t^2-q} = t^{\kappa_1(T)} [\kappa_2(T)]_{t^2,q}.$$

Thus

$$F_{\lambda'} = \sum_T t^{\kappa_1(T)} [\kappa_2(T)]_{t^2,q}.$$

This can be interpreted as

$$F_{\lambda'} = \sum_{T'} t^{\kappa_1(T')} q^{\kappa_2(T')} t^{2\kappa_{\bar{2}}(T')},$$

where T' ranges over the following objects, which we call enriched tableaux: T' consists of a semi-standard Young tableau T filled with $0, 1, 2$ and additional markings on some 2's. When reading the 2's from left to right in T, the corresponding cells in T' contain some undecorated 2's (weighted by q), followed by some $\bar{2}$'s (weighted by t^2), followed by a single $\hat{2}$ (weighted by 1). The remaining entries, 0's and 1's, get weights 1 and t respectively. See the figures below for examples. For each character $x \in \{0, 1, 2, \bar{2}, \hat{2}\}$, $\kappa_x(T')$ denotes the number of times x occurs in T'.

Theorem 3. *For any shape λ,*

$$g_\lambda = \sum_{T'} (tq)^{\kappa_2(T')} [\kappa_1(T') + 2\kappa_{\bar{2}}(T') - \kappa_2(T')]_{q,t}$$

where the sum ranges over enriched tableaux T' of shape λ which satisfy at least one of the following conditions:

1. *T' has a $\bar{2}$ or $\hat{2}$ in the third row,*
2. *T' has a $\hat{2}$ in the second row* and *fewer than $\kappa_1(T') + 2\kappa_{\bar{2}}(T') - \kappa_2(T')$ 1's at the top of columns of height 2, or*
3. *T' has a $\hat{2}$ in the bottom row, fewer than $2\kappa_{\bar{2}}$-many 2's in the bottom row* and *fewer than $\kappa_1(T') + 2\kappa_{\bar{2}}(T') - \kappa_2(T')$ 1's at the top of columns of height 2.*

Proof. Following the remarks above, for each $i > j$, we will give an injection from enriched tableaux of weight $t^j q^i$ (which are counted by $a_{j,i}$) into those of weight $t^i q^j$ (counted by $a_{i,j}$). The enriched tableaux which are not in the image of this injection will be precisely those enumerated above.

Let $i > j$ and let T' be an enriched filling of the (french) Young diagram of λ with weight $t^j q^i$. Note that $\kappa_2 > 2\kappa_{\bar{2}}$ since undecorated 2's are the only entries contributing q's to the weight of T'. Note also that T' cannot have a $\bar{2}$ or $\hat{2}$ in the third row. This is because all 2's in the third row are "balanced" by the 1's which must lie beneath them, and the presence of a $\bar{2}$ or $\hat{2}$ in the third row makes it impossible to gain any more powers of q later in T'. Hence we can safely ignore (freeze) all columns of height 3.

Case 1: Suppose that the single $\hat{2}$ lies in the bottom row. Further suppose that there are at least $2\kappa_{\bar{2}}$-many 2's in the bottom row. (Recall that if there are *any* 2's in the bottom row, then *all* $\bar{2}$'s are also in the bottom row.) Construct T'' as follows: Freeze $2\kappa_{\bar{2}}$-many

2's in the bottom row as well as all 1's in the bottom row which are under a 2. Then exchange the number of unfrozen 1's and unfrozen 2's in the bottom row, and also the number of 1's and 2's above 0's (in the second row). Then reorder cells within these rows to make them weakly increasing.

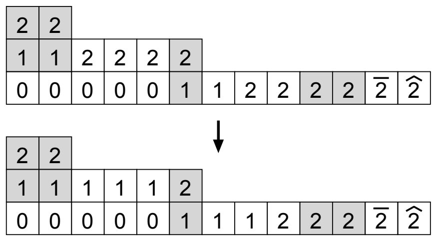


Figure 2: An illustration of Case 1 with $i = 10$ and $j = 6$. Grey columns are fixed.

Case 2: Suppose that the single $\hat{2}$ is not in the bottom row or that there are fewer than $2\kappa_{\bar{2}}$-many 2's in the bottom row. Note that in the former situation, there are no 2's in the bottom row. Hence, either way, the total weight of all cells in columns of height 1 has a (weakly) larger power of t than q. Furthermore, we noted above that the weight of the columns of height 3 has equal powers of t and q. Hence the total weight of the columns of height 2 must be $t^b q^a$ for some a, b with $a - b \geqslant i - j$. In particular, the number of 2's above 0's must be at least $i - j$ (since 2's above 1's are "balanced"). To construct T'', simply change the leftmost $i - j$ of these to 1's (still above 0's).

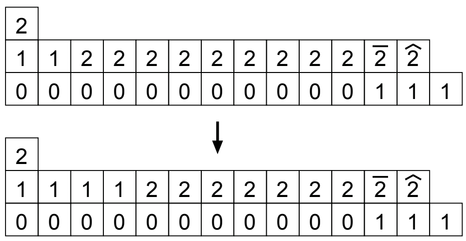


Figure 3: An illustration of Case 2 with $i = 10$ and $j = 8$.

It is easy to see that each of these maps alone is injective. If you know a particular enriched tableaux T'' is the image of a Case 1 tableau T', you can simply freeze $2\kappa_{\bar{2}}$-many 2's in the bottom row and then switch the roles of 1's and 2's back to reconstruct T'. If you know T'' is a Case 2 image, you just swap as $(i-j)$-many 1's for 2's at the tops of columns of height 2.

Furthermore, these images don't intersect: the image of a Case 1 tableau always has the $\hat{2}$ and at least $2\kappa_{\bar{2}}$-many 2's in the bottom row and the image of a Case 2 tableau never does. Hence for any shape λ and any $i > j$, these maps together form an injection from enriched tableaux of shape λ of weight $t^j q^i$ into those of weight $t^i q^j$.

Using the partial inverses mentioned above, we can see that the enriched tableaux of weight $t^i q^j$ not in the image of our injection are those which 1) have a $\bar{2}$ or $\hat{2}$ in the third row; 2) have a $\hat{2}$ in the second row *and* fewer than $i-j$ 1's at the top of columns of height 2; or 3) have a $\hat{2}$ in the bottom row, fewer than $2\kappa_{\bar{2}}$-many 2's in the bottom row *and* fewer than $i-j$ 1's at the top of columns of height 2. This gives the desired combinatorial interpretation of $\sum_{i>j}(a_{i,j}-a_{j,i})t^i q^j$ as an enumeration of certain enriched tableaux. $\square$

For example, consider the case $\lambda = (3^1, 2^4, 1^5)$. Figure 4 shows all enriched tableaux of shape λ' and weight $t^5 q^7$ together with their images under the injection above. For the tableaux belonging to Case 1, the frozen cells are shaded. Only the last falls into Case 2 from the proof of the theorem. Then in Figure 5 we give all the remaining enriched tableaux of weight $t^7 q^5$.

Note that in this example there are no such enriched tableaux which have a $\bar{2}$ or $\hat{2}$ in the third row. This can only happen when all 2's are at the tops of columns of height 3, that is, when the power of q is less than the number of parts of size 3 in λ. Similarly, there are no enriched tableaux which have a $\hat{2}$ in the second row. This can only happen when the power of q is less than the number of parts of size 2 or 3.

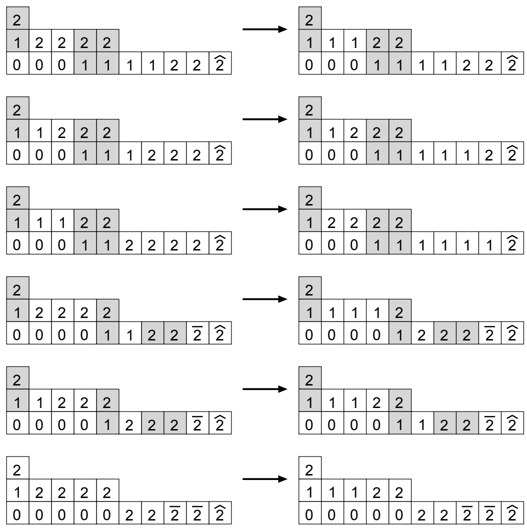


Figure 4: All enriched tableaux of shape $(10, 5, 1)$ and weight $t^5 q^7$ along with their images under the injection from the proof of Theorem 3.

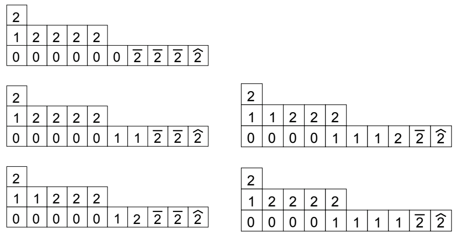


Figure 5: All enriched tableaux of shape $(10, 5, 1)$ and weight $t^7 q^5$ not included in Figure 4.

3 Proof by Direct Computation

3.1 Preliminaries

In this section, we shall show the computation of an explicit formula for g_λ.

We let

(1) $[n]_q = q^{n-1}+q^{n-2}+\cdots+1 = \frac{q^n-1}{q-1}$,

(2) $[n]_{q,t} = q^{n-1}+q^{n-2}t+\cdots+t^{n-1} = \frac{q^n-t^n}{q-t}$ for $n \geqslant 0$,

(3) $[-n]_{q,t} = \frac{q^{-n}-t^{-n}}{q-t} = \frac{-[n]_{q,t}}{(qt)^n}$ for $n>0$, and

(4) $[n \to m]_{q,t} = \sum_{i=n}^m [i]_{q,t} = \frac{\sum_{i=n}^m t^i - \sum_{i=n}^m q^i}{t-q} = \frac{t^n[m-n+1]_t - q^n[m-n+1]_q}{t-q}$ or alternatively $\frac{(q-1)(t^{m+1}-t^n)-(t-1)(q^{m+1}-q^n)}{(t-1)(q-1)(t-q)}$.

We know that $g_\lambda = 0$ if λ' has more than 3 rows. Thus we can assume that λ' has 3 or fewer rows. We let $\mathrm{SSYT}(\lambda', 012)$ denote the set of all semi-standard Young tableaux T of shape λ' with cells filled by $\{0,1,2\}$. Given a semi-standard Young tableau $T \in \mathrm{SSYT}(\lambda', 012)$, the contribution of T to g_λ is denoted as g_T. This is also known as T's weight. We can write

$$g_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda', 012)} g_T. \tag{8}$$

Since we are only considering the weight of $T \in \mathrm{SSYT}(\lambda', 012)$, we can write T in 4 parts as shown in Figure 6: a_1 – the part with 3 rows, k_1 – the part with two rows and the bottom row is filled with 0's, a_2 – the part with two rows and the bottom row is filled with 1's, k_2 – the part with one row and the fillings are not 0. If there is no a_2 part, there can be a part called a_0 at the same place which consists of one row filled with 0's. In our weighting scheme for $T \in \mathrm{SSYT}(\lambda', 012)$ given below, the weight of any 0 will be 1. Hence a_0 won't contribute anything to g_T so that we will not consider a_0 in our formulas. We define the set $S[a_1, k_1, a_2, k_2]$ to be the collections of T's having the part composition $[a_1, k_1, a_2, k_2]$. Since a_1 and a_2 have exactly the same kind of contribution to the formula, we can define

$$g[a_1+a_2, k_1, k_2] = \sum_{T \in S[a_1,k_1,a_2,k_2]} g_T. \tag{9}$$

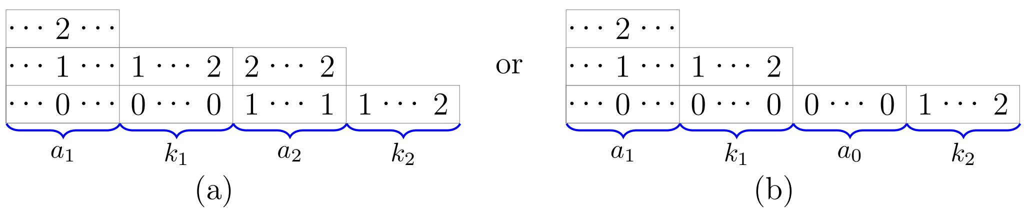


Figure 6: $T \in \mathrm{SSYT}(\lambda', 012)$

By Lemma 2, we can simplify the formula for g_λ as:

$$\begin{aligned}
g_\lambda &= \langle \Delta_{e_2} e_n[X], s_\lambda\rangle \\
&= \frac{id-\tau}{t-q} F_{\lambda'} \\
&= \frac{id-\tau}{t-q}\,\frac{s_{\lambda'}[1+t+t^2]-s_{\lambda'}[1+t+q]}{t^2-q} \\
&= \frac{s_{\lambda'}[1+t+t^2]-s_{\lambda'}[1+t+q]}{(t-q)(t^2-q)}+\frac{s_{\lambda'}[1+q+q^2]-s_{\lambda'}[1+q+t]}{(t-q)(t-q^2)}. \qquad (10)
\end{aligned}$$

Suppose a Young tableau $T\in \mathrm{SSYT}(\lambda',012)$ has ω_1 1's and ω_2 2's, then it has weight

$$\begin{aligned}
g_T &= \frac{t^{\omega_1+2\omega_2}-t^{\omega_1}q^{\omega_2}}{(t-q)(t^2-q)}+\frac{q^{\omega_1+2\omega_2}-q^{\omega_1}t^{\omega_2}}{(t-q)(t-q^2)} \\
&= \frac{t^{\omega_1}[\omega_2]_{t^2,q}-q^{\omega_1}[\omega_2]_{q^2,t}}{t-q}. \qquad (11)
\end{aligned}$$

Now we define

$$w(\omega_1,\omega_2)=\frac{t^{\omega_1}[\omega_2]_{t^2,q}-q^{\omega_1}[\omega_2]_{q^2,t}}{t-q} \qquad (12)$$

and

$$W(T)=w(\omega_1,\omega_2)=\frac{t^{\omega_1}[\omega_2]_{t^2,q}-q^{\omega_1}[\omega_2]_{q^2,t}}{t-q}. \qquad (13)$$

Then it is clear that

$$g[a_1+a_2,k_1,k_2]=\sum_{T\in S[a_1,k_1,a_2,k_2]} g_T = \sum_{T\in S[a_1,k_1,a_2,k_2]} W(T). \qquad (14)$$

We will use the new weight $W(T)$ to deduce a formula for $g[a_1+a_2,k_1,k_2]$ which will, in turn, allow us to compute an explicit formula for g_λ.

3.2 The computation of $g[a_1+a_2,k_1,k_2]$

3.2.1 A formula for $g[0,0,k]$

The set $S[0,0,0,k]$ contains the tableaux T of shape $\underbrace{\boxed{1\cdots 2}}_{k}$. If there are i 1's, then there will be $k-i$ 2's. For any statement A, we let $\chi(A)=1$ if A is true and $\chi(A)=0$ if A is false. Then we have the following theorem.

Theorem 4. *We have* $g[0,0,1]=0$ *and, for* $k\geqslant 2$,

$$g[0,0,k]=\sum_{i=0}^{\lfloor(2k-2)/3\rfloor-\chi(k\,\equiv\,1 \mod 3)} (qt)^i\left[k-i-\lfloor \tfrac{i+1}{2}\rfloor \to 2k-2-3i\right]_{q,t}. \qquad (15)$$

Proof. It is easy to see by direct calculation that $g[0,0,1]=0$. Next observe that for any $r \geqslant 1$, $\omega(r,0)=0$. Thus we need only consider the cases where there is at least one 2 in the tableau. It follows that

$$\begin{aligned} g[0,0,k] &= \sum_{T \in S[0,0,0,k]} g_T = \sum_{i=0}^{k-1} w(i,k-i) \\ &= \sum_{i=0}^{k-1} \frac{t^i[k-i]_{t^2,q} - q^i[k-i]_{q^2,t}}{t-q} = \sum_{i=0}^{k-1} \frac{\sum_{j=0}^{k-1-i} t^{2k-2j-i-2}q^j - q^{2k-2j-i-2}t^j}{t-q} \\ &= \sum_{i=0}^{k-1} \sum_{j=0}^{k-1-i} (qt)^j [2k-3j-i-2]_{q,t}. \end{aligned}$$

Now let $A_{i,j}^{(k)} = (qt)^j[2k-3j-i-2]_{q,t}$. In Figure 7, we have pictured the array $\{A_{i,j}^{(8)} : 0 \leqslant i \leqslant 7 \ \& \ 0 \leqslant j \leqslant i\}$. In general, if one looks at the first row of the $A_{i,j}^{(k)} = (qt)^j[2k-3j-i-2]_{q,t}$, which is the sequence $((qt)^j[2k-3j-2]_{q,t})$, the terms will be non-negative if $2k-2 \geqslant 3j$, or, equivalently, if $j \leqslant \lfloor (2k-2)/3 \rfloor$. We shall show that for any negative terms in the first row of the form $(qt)^j[-k]_{q,t}$, the first $k+1$ terms along the anti-diagonal starting at that position will sum to 0. This will leave us only with positive terms corresponding to sum stated in the theorem. For example, in Figure 7, one can easily compute that the sum of the first two terms of the anti-diagonal starting at the term $(qt)^5[-1]_{q,t}$ equals 0, the sum of the first five terms of the anti-diagonal starting at the term $(qt)^6[-4]_{q,t}$ equals 0, and the sum of the first eight terms of the anti-diagonal starting at the term $(qt)^7[-7]_{q,t}$ equals 0. These are the terms corresponding to the green, blue, and red diagonals respectively. In this case, we see that $g[0,0,8]$ equals

$$[8 \to 14]_{q,t} + qt[6 \to 11]_{q,t} + (qt)^2[5 \to 8]_{q,t} + (qt)^3[3 \to 5]_{q,t} + (qt)^4[2 \to 2]_{q,t}$$

which is exactly the formula predicted by the theorem.

$i \backslash j$	0	1	2	3	4	5	6	7
0	$[14]_{q,t}$	$(qt)^1[11]_{q,t}$	$(qt)^2[8]_{q,t}$	$(qt)^3[5]_{q,t}$	$(qt)^4[2]_{q,t}$	$(qt)^5[-1]_{q,t}$	$(qt)^6[-4]_{q,t}$	$(qt)^7[-7]_{q,t}$
1	$[13]_{q,t}$	$(qt)^1[10]_{q,t}$	$(qt)^2[7]_{q,t}$	$(qt)^3[4]_{q,t}$	$(qt)^4[1]_{q,t}$	$(qt)^5[-2]_{q,t}$	$(qt)^6[-5]_{q,t}$	
2	$[12]_{q,t}$	$(qt)^1[9]_{q,t}$	$(qt)^2[6]_{q,t}$	$(qt)^3[3]_{q,t}$	$(qt)^4[0]_{q,t}$	$(qt)^5[-3]_{q,t}$		
3	$[11]_{q,t}$	$(qt)^1[8]_{q,t}$	$(qt)^2[5]_{q,t}$	$(qt)^3[2]_{q,t}$	$(qt)^4[-1]_{q,t}$			
4	$[10]_{q,t}$	$(qt)^1[7]_{q,t}$	$(qt)^2[4]_{q,t}$	$(qt)^3[1]_{q,t}$				
5	$[9]_{q,t}$	$(qt)^1[6]_{q,t}$	$(qt)^2[3]_{q,t}$					
6	$[8]_{q,t}$	$(qt)^1[5]_{q,t}$						
7	$[7]_{q,t}$							

Figure 7: The table of $A_{i,j}^{(8)}$.

The proof requires a careful case by case analysis by considering the value of k modulo 3. Note that

1. if $k=3t$, then $\lfloor (2k-2)/3\rfloor = 2t-1$,
2. if $k=3t+1$, then $\lfloor (2k-2)/3\rfloor = 2t$, and
3. if $k=3t+2$, then $\lfloor (2k-2)/3\rfloor = 2t$.

Case 1. $k=3t$.

The negative terms in the first row are

$$(qt)^{2t-1+s}[6t-2-3(2t-1+s)]_{q,t} = (qt)^{2t-1+s}[-3s+1]_{q,t}$$

for $s=1,\ldots,t$. In particular, the last term in the first row equals $(qt)^{3t-1}[-3t+1]_{q,t}$ and the first negative term is $A^{(3t)}_{0,2t} = (qt)^{2t}[-2]_{q,t}$.

Then we have two subcases depending on whether s is even or odd.

Subcase 1.1. $s=2r$.

In this case, $A^{(3t)}_{0,2t-1+2r} = q^{2t+2r-1}[-6r+1]_{q,t}$. We claim that $\sum_{a=0}^{6r-1} A^{(3t)}_{a,2t-1+2r-a} = 0$. We shall prove this by showing that for all $0 \leqslant a \leqslant 3r-1$,

$$A^{(3t)}_{a,2t-1+2r-a} \;=\; -A^{(3t)}_{6r-1-a,2t-1+2r-(6r-1-a)} \;=\; -A^{(3t)}_{6r-1-a,2t-4r+a}.$$

Note that

$$\begin{aligned} A^{(3t)}_{a,2t-1+2r-a} &= (qt)^{2t-1+2r-a}[6t-2-a-3(2t-1+2r-a)]_{q,t} \\ &= (qt)^{2t-1+2r-a}[-6r+1+2a]_{q,t} \\ &= -(qt)^{2t-1+2r-a-(6r-1-2a)}[6r-1+2a]_{q,t} \\ &= -(qt)^{2t-4r+a}[6r-1+2a]_{q,t}. \end{aligned}$$

On the other hand,

$$A^{(3t)}_{6r-1-a,2t-4r+a} = \\ (qt)^{2t-4r+a}[6t-2-(6r-1-a)-3(2t-4r+a)]_{q,t} = (qt)^{2t-4r+a}[6t-1+2a]_{q,t}$$

as desired.

Subcase 1.2. $s=2r+1$.

In this case, $A^{(3t)}_{0,2t-1+2r+1} = q^{2t+2r-1}[-6r-2]_{q,t}$. We claim that $\sum_{a=0}^{6r+2} A^{(3t)}_{a,2t+2r-a} = 0$. First note that

$$A^{(3t)}_{3r+1,2t+2r-(3r+1)} = A^{(3t)}_{3r+1,2t-r-1} = \\ (qt)^{2t-r-1}[6t-2-(3r+1)-3(2t-r-1)]_{q,t} = (qt)^{2t-r-1}[0]_{q,t} = 0.$$

Thus we can prove our claim if we show that $0 \leqslant a \leqslant 3r$,

$$A^{(3t)}_{a,2t+2r-a} = -A^{(3t)}_{6r+2-a,2t+2r-(6r+2-a)} = -A^{(3t)}_{6r+2-a,2t-4r-2+a}.$$

Note that

$$\begin{aligned} A^{(3t)}_{a,2t+2r-a} &= (qt)^{2t+2r-a}[6t-2-a-3(2t+2r-a)]_{q,t} \\ &= (qt)^{2t+2r-a}[-6r-2+2a]_{q,t} \\ &= -(qt)^{2t+2r-a-(6r+2-2a)}[6r+2-2a]_{q,t} \\ &= -(qt)^{2t-4r-2+a}[6r+2-2a]_{q,t}. \end{aligned}$$

On the other hand,

$$\begin{aligned} A^{(3t)}_{6r+2-a,2t-4r-2+a} &= (qt)^{2t-4r-2+a}[6t-2-(6r+2-a)-3(2t-4r-2+a)]_{q,t} \\ &= (qt)^{2t-4r+a}[6r+2-2a]_{q,t}. \end{aligned}$$

Observe that the bottom term of the r-th column of the array $\{A^{(3t)}_{i,j}\}_{i=0,\ldots,3t-1\,\&\,0\leqslant j\leqslant i}$ is $A_{3t-1-r,r}$. Our computations above show that in the array $\{A^{(3t)}_{i,j}\}_{i=0,\ldots,3t-1\,\&\,0\leqslant j\leqslant i}$, the first $3s$ terms of any anti-diagonal starting at $A_{0,2t-1+s}$ sum to 0 for $s = 1, \ldots, t$. This means that the corresponding terms in the array make no contribution to $g[0, 0, k]$. It follows that we can ignore all the terms in columns $2t, \ldots, 3t-1$. Note that the first $3t$ terms of the anti-diagonal starting at $A^{(3t)}_{0,3t-1}$ cancel out the bottom term in each column. Next the first $3t - 3$ terms of the anti-diagonal starting at $A^{(3t)}_{0,3t-2}$ reach only to column 2 so they will cancel out the next to last term in columns $2, \ldots, 2t-1$. Then the first $3t-6$ terms of the anti-diagonal starting at $A^{(3t)}_{0,3t-3}$ reach only to column 4 so they will cancel out the second to last terms in columns $4, \ldots, 2t-1$. Continuing on in this way, we finally see that the 3 anti-diagonal terms starting at $A^{(3t)}_{0,2t}$ will only cancel out terms in columns $2t-2$ and $2t-1$. It follows that for $r = 0, \ldots, t-1$, we can ignore that last $r+1$ terms in columns $2r$ and $2r+1$. This means that if $0 \leqslant r \leqslant t-1$, the lowest term that can contribute to $g[0, 0, k]$ in column $2r$ is

$$\begin{aligned} A^{(3t)}_{3t-1-2r-(r+1),2r} &= A_{3t-3r-2,2r} = (qt)^{2r}[6t-2-(3t-3r-2)-3(2r)]_{q,t} \\ &= (qt)^{2r}[3t-3r]_{q,t} = [3t-(2r)-\lfloor(2r+1)/2\rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r$ is $A^{(3t)}_{0,2r} = (qt)^{2r}[3t-2-3(2r)]_{q,t}$. Since the q,t-numbers of the terms in column $2r$ increase by 1 as one moves up, it follows that the contribution of column $2r$ to $g[0, 0, k]$ is $(qt)^{2r}[k-(2r)-\lfloor(2r+1)/2\rfloor \to 2k-2-3(2r)]_{q,t}$ as predicted by our formula.

Similarly, if $0 \leqslant r \leqslant t - 1$, the lowest term that can contribute to $g[0, 0, k]$ in column $2r+1$ is

$$\begin{aligned} A^{(3t)}_{3t-1-(2r+1)-(r+1),2r+1} &= A_{3t-3r-3,2r+1} = (qt)^{2r+1}[6t-2-(3t-3r-3)-3(2r+1)]_{q,t} \\ &= (qt)^{2r+1}[3t-3r-2]_{q,t} = [3t-(2r+1)-\lfloor(2r+2)/2\rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r+1$ is $A^{(3t)}_{0,2r+1} = (qt)^{2r+1}[3t-2-3(2r+1)]_{q,t}$. Since the q,t-numbers in the terms in column $2r+1$ increase by 1 as one moves up, it follows that the contribution of column $2r+1$ to $g[0,0,k]$ is $(qt)^{2r+1}[k-(2r+1)-\lfloor(2r+1)/2\rfloor \to 2k-2-3(2r+1)]_{q,t}$ as predicted by our formula.

Thus our formula holds in this case.

Case 2. $k = 3t+1$.

The negative terms in the first row are

$$(qt)^{2t+s}[6t+2-2-3(2t+s)]_{q,t} = (qt)^{2t+s}[-3s]_{q,t}$$

for $s = 1,\ldots,t$. In particular, the last term in the first row equals $(qt)^{3t}[-3t]_{q,t}$ and the first negative term is $A^{(3t+1)}_{0,2t+1} = (qt)^{2t+1}[-3]_{q,t}$.

Then as in Case 1, we have two subcases depending on whether s is even or odd.

Subcase 2.1. $s = 2r$.

In this case, $A^{(3t+1)}_{0,2t+2r} = q^{2t+2r}[-6r]_{q,t}$. We claim that $\sum_{a=0}^{6r} A^{(3t)}_{a,2t-1+2r-a} = 0$. First observe that

$$A^{(3t+1)}_{3r,2t+2r-(3r)} = q^{2t-r}[6t+2-2-3r-3(2t-r)]_{q,t} = q^{2t-r}[0]_{q,t}.$$

Thus we can prove our claim by showing that for $0 \leqslant a \leqslant 3r-1$,

$$A^{(3t+1)}_{a,2t+2r-a} = -A^{(3t+1)}_{6r-a,2t+2r-(6r-a)}.$$

This is a straightforward calculation so we will not include the details here.

Subcase 2.2. $s = 2r+1$.

In this case, $A^{(3t+1)}_{0,2t+2r+1} = q^{2t+2r+1}[-6r-3]_{q,t}$. We claim that $\sum_{a=0}^{6r+3} A^{(3t+1)}_{a,2t+2r+1-a} = 0$. In this case, one can easily check that for $0 \leqslant a \leqslant 3r+1$,

$$A^{(3t+1)}_{a,2t+2r+1-a} = -A^{(3t+1)}_{6r+3-a,2t+2r+1-(6r+3-a)},$$

so we shall not include the details here.

Next observe that the bottom term of the array $\{A^{(3t+1)}_{i,j}\}_{i=0,\ldots,3t\ \&\ 0\leqslant j\leqslant i}$ in the r-th column is $A_{3t-r,r}$. Our computations above show that in the array $\{A^{(3t+1)}_{i,j}\}_{i=0,\ldots,3t\ \&\ 0\leqslant j\leqslant i}$, the first $3s+1$ terms of any anti-diagonal terms starting at $A^{(3t+1)}_{0,2t+s}$ sum to 0 for $s = 1,\ldots,t$. This means that the corresponding terms in the array make no contribution to $g[0,0,k]$. It follows that we can ignore all the terms in columns $2t+1,\ldots,3t$. One can use a similar reasoning as we used in Case 1 to show that for $r = 0,\ldots,t-1$, we can ignore the bottom $r+1$ terms in columns $2r$ and $2r+1$. Moreover, we can ignore the bottom t terms in column $2t$. This is because $A^{(3t+1)}_{0,2t+1} = [-3]_{q,t}$, which means that the first four terms of the anti-diagonal starting at $A^{(3t+1)}_{0,2t+1}$ will cancel terms in columns $2t{-}2$, $2t{-}1$, and $2t$. It follows that if $0 \leqslant r \leqslant t{-}1$, the lowest term that can contribute to $g[0,0,k]$ in column $2r$ is

$$\begin{aligned} A^{(3t+1)}_{3t-2r-(r+1),2r} &= A^{(3t+1)}_{3t-3r-1,2r} = (qt)^{2r}[6t{+}2{-}2{-}(3t{-}3r{-}1){-}3(2r)]_{q,t} \\ &= (qt)^{2r}[3t{-}3r{+}1]_{q,t} = [3t{+}1{-}(2r){-}\lfloor(2r{+}1)/2\rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r$ is $A^{(3t+1)}_{0,2r} = (qt)^{2r}[2(3t{+}1){-}2{-}3(2r)]_{q,t}$. Since the q,t-numbers in the terms in column $2r$ increase by 1 as one moves up, it follows that the contribution of column $2r$ to $g[0,0,k]$ is $(qt)^{2r}[k{-}(2r){-}\lfloor(2r{+}1)/2\rfloor \to 2k{-}2{-}3(2r)]_{q,t}$ as predicted by our formula.

Similarly, if $0 \leqslant r \leqslant t{-}1$, the lowest term that can contribute to $g[0,0,k]$ in column $2r+1$ is

$$\begin{aligned} A^{(3t+1)}_{3t-(2r+1)-(r+1),2r+1} &= A^{(3t+1)}_{3t-3r-2,2r+1} = (qt)^{2r+1}[6t{+}2{-}2{-}(3t{-}3r{-}2){-}3(2r{+}1)]_{q,t} \\ &= (qt)^{2r+1}[3t{-}3r{-}1]_{q,t} = [(3t{+}1){-}(2r{+}1){-}\lfloor(2r{+}2)/2\rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r{+}1$ is $A^{(3t+1)}_{0,2r+1} = (qt)^{2r+1}[2k{-}2{-}3(2r{+}1)]_{q,t}$. Since the q,t-numbers in the terms in column $2r{+}1$ increase by 1 as one moves up, it follows that the contribution of column $2r{+}1$ to $g[0,0,k]$ is $(qt)^{2r+1}[k{-}(2r{+}1){-}\lfloor(2r{+}1)/2\rfloor \to 2k{-}2{-}3(2r{+}1)]_{q,t}$ as predicted by our formula.

Finally in column $2t$, the lowest term that can contribute to $g[0,0,k]$ is

$$\begin{aligned} A^{(3t+1)}_{3t-(2t)-(t),2r+1} &= A^{(3t+1)}_{0,2t} = (qt)^{2t}[6t{+}2{-}2{-}3(2t)]_{q,t} \\ &= (qt)^{2t}[0]_{q,t}. \end{aligned}$$

Thus this column makes no contribution which is why we exclude this term from the sum. Note that in this case $3t{+}1{-}2t{-}\lfloor(2t{+}1)/2\rfloor = 1$ while $2k{-}2{-}3(2t) = 6t{+}2{-}2{-}6t = 0$ so that $[k{-}2t{-}\lfloor(2t{+}1)/2\rfloor \to 2k{-}2{-}3(6t)]_{q,t} = [1 \to 0]_{q,t}$ which is an empty sum.

Thus our formula holds in this case.

Case 3. $k = 3t{+}2$.

Then the negative terms in the first row are

$$(qt)^{2t+s}[6t{+}4{-}2{-}3(2t{+}s)]_{q,t} = (qt)^{2t+s}[-3s{+}2]_{q,t}$$

for $s = 1,\ldots,t{+}1$. In particular, the last term in the first row equals $(qt)^{3t+1}[-3t{-}1]_{q,t}$ and the first negative term is $A^{(3t+2)}_{0,2t+1} = (qt)^{2t+1}[-1]_{q,t}$.

Then as before, we have two subcases depending on whether s is even or odd.

Subcase 3.1. $s = 2r$.

In this case, $A^{(3t+2)}_{0,2t+2r} = q^{2t+2r}[-6r+2]_{q,t}$. We claim that $\sum_{a=0}^{6r-2} A^{(3t)}_{a,2t-1+2r-a} = 0$. First observe that

$$A^{(3t+2)}_{3r-1,2t+2r-(3r-1)} = q^{2t-r+1}[6t+4-2-(3r-1)-3(2t-r+1)]_{q,t} = q^{2t-r+1}[0]_{q,t}.$$

Thus we can prove our claim by showing that for $0 \leqslant a \leqslant 3r-2$,

$$A^{(3t+2)}_{a,2t+2r-a} = -A^{(3t+2)}_{6r-2-a,2t+2r-(6r-2-a)}.$$

This is a straightforward calculation so we will not include the details here.

Subcase 3.2. $s = 2r+1$.

In this case, $A^{(3t+2)}_{0,2t+2r+1} = q^{2t+2r+1}[-6r-1]_{q,t}$. We claim that $\sum_{a=0}^{6r+1} A^{(3t+2)}_{a,2t+2r+1-a} = 0$. In this case, one can easily check that for $0 \leqslant a \leqslant 3r$,

$$A^{(3t+1)}_{a,2t+2r+1-a} = -A^{(3t+1)}_{6r+1-a,2t+2r+1-(6r+1-a)},$$

so we shall not include the details here.

Next we observe that the bottom term of the array $\{A^{(3t+2)}_{i,j}\}_{i=0,\ldots,3t \ \& \ 0\leqslant j\leqslant i}$ in the r-th column is $A_{3t+1-r,r}$. Our computations above have shown that in the array $\{A^{(3t+2)}_{i,j}\}_{i=0,\ldots,3t+1 \ \& \ 0\leqslant j\leqslant i}$, the first $3s-1$ terms of any anti-diagonal starting at $A^{(3t+2)}_{0,2t+s}$ sum to 0 for $s = 1,\ldots,t+1$. This means that the corresponding terms in the array make no contribution to $g[0,0,k]$. It follows that we can ignore all the terms in columns $2t+1,\ldots,3t+1$. One can use a similar reasoning as we used in Case 1 to show that for $r = 0,\ldots,t-1$, we can ignore the bottom $r+1$ terms in columns $2r$ and $2r+1$. We can also ignore the bottom $t+1$ terms in column $2t$. This is because $A^{(3t+2)}_{0,2t+1} = (qt)^{2t+1}[-1]_{q,t}$ so that the sum of first two anti-diagonal terms starting at $A^{(3t+2)}_{0,2t+1}$ will only cancel elements in columns $2t$ and $2t+1$.

This means that if $0 \leqslant r \leqslant t-1$, the lowest term that can contribute to $g[0,0,k]$ in column $2r$ is

$$\begin{aligned} A^{(3t+2)}_{3t+1-2r-(r+1),2r} &= A^{(3t+2)}_{3t-3r,2r} = (qt)^{2r}[6t+4-2-(3t-3r)-3(2r)]_{q,t} \\ &= (qt)^{2r}[3t-3r+2]_{q,t} = [3t+2-(2r)-\lfloor(2r+1)/2\rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r$ is $A^{(3t+2)}_{0,2r} = (qt)^{2r}[2(3t+2)-2-3(2r)]_{q,t}$. Since the q,t-numbers in the terms in column $2r$ increase by 1 as one moves up, it follows that the contribution of column $2r$ to $g[0,0,k]$ is $(qt)^{2r}[k-(2r)-\lfloor(2r+1)/2\rfloor \to 2k-2-3(2r)]_{q,t}$ as predicted by our formula.

Similarly, if $0 \leqslant r \leqslant t-1$, the lowest term that can contribute to $g[0,0,k]$ in column $2r+1$ is

$$\begin{aligned} A^{(3t+2)}_{3t+1-(2r+1)-(r+1),2r+1} &= A^{(3t+2)}_{3t-3r-1,2r+1} = (qt)^{2r+1}[6t+4-2-(3t-3r-1)-3(2r+1)]_{q,t} \\ &= (qt)^{2r+1}[3t-3r]_{q,t} = [(3t+2)-(2r+1)-\lfloor (2r+2)/2 \rfloor]_{q,t}. \end{aligned}$$

Note that the top element in column $2r+1$ is $A^{(3t+2)}_{0,2r+1} = (qt)^{2r+1}[2(3t+2)-2-3(2r+1)]_{q,t}$. Since the q,t-numbers in the terms in column $2r+1$ increase by 1 as one moves up, it follows that the contribution of column $2r+1$ to $g[0,0,k]$ is $(qt)^{2r+1}[k-(2r+1)-\lfloor (2r+1)/2 \rfloor \to 2k-2-3(2r+1)]_{q,t}$ as predicted by our formula.

Finally for column $2t$, the lowest term that can contribute to $g[0,0,k]$ in column $2t$ is

$$\begin{aligned} A^{(3t+2)}_{3t+1-(2t)-(t+1),2t} &= A^{(3t+2)}_{0,2t} = (qt)^{2t}[6t+4-2-3(2t)]_{q,t} \\ &= (qt)^{2t}[2]_{q,t} = [(3t+2)-(2t)-\lfloor (2t+1)/2 \rfloor]_{q,t}. \end{aligned}$$

It follows that the contribution of column $2t$ to $g[0,0,k]$ is $(qt)^{2t}[k-(2t)-\lfloor (2t+1)/2 \rfloor \to 2k-2-3(2t)]_{q,t} = (qt)^{2t}[2]_{q,t}$ as predicted by our formula.

Thus our formula holds in this case which completes our proof. $\square$

For example, we have

$$\begin{aligned} g[0,0,12] &= \sum_{i=0}^{7} (qt)^i \left[12-i-\lfloor \tfrac{i+1}{2} \rfloor \to 22-3i \right]_{q,t} \\ &= [12 \to 22]_{q,t} + (qt)[10 \to 19]_{q,t} + (qt)^2[9 \to 16]_{q,t} + (qt)^3[7 \to 13]_{q,t} \\ &\quad + (qt)^4[6 \to 10]_{q,t} + (qt)^5[4 \to 7]_{q,t} + (qt)^6[3 \to 4]_{q,t} + (qt)^7[1]_{q,t}. \end{aligned}$$

3.2.2 A formula for $g[a,0,k]$

Theorem 5.

$$g[a,0,k] = (qt)^a g[0,0,k] + \sum_{i=1}^{a} (qt)^{a-i}[k+3i \to 2k+3i]_{q,t}. \tag{16}$$

Proof. We have

$$\begin{aligned} g[a,0,k] &= \sum_{T \in S[a,0,0,k]} g_T \\ &= \sum_{i=0}^{k} w(a+i, a+k-i) \\ &= \sum_{i=0}^{k} \frac{t^{a+i}[a+k-i]_{t^2,q} - q^{a+i}[a+k-i]_{q^2,t}}{t-q}. \end{aligned}$$

Notice that

$$[a+k-i]_{t^2,q} = q^a[k-i]_{t^2,q} + \sum_{j=0}^{a-1} t^{2(k-i+a-j-1)}q^j$$

and

$$[a+k-i]_{q^2,t} = t^a[k-i]_{q^2,t} + \sum_{j=0}^{a-1} q^{2(k-i+a-j-1)}t^j.$$

By plugging in these new equations we can get

$$\begin{aligned}
g[a,0,k] =& t^a q^a \sum_{i=0}^{k} \frac{t^i[k-i]_{t^2,q} - q^i[k-i]_{q^2,t}}{t-q} \\
&+ \sum_{i=0}^{k}\sum_{j=0}^{a-1}(qt)^j \frac{t^{2k-i+3a-3j-2} - q^{2k-i+3a-3j-2}}{t-q} \\
=&(qt)^a g[0,0,k] + \sum_{j=0}^{a-1}(qt)^j \sum_{i=0}^{k}[2k-i+3a-3j-2]_{q,t} \\
=&(qt)^a g[0,0,k] + \sum_{i=0}^{a-1}(qt)^i[k+3a-3i-2 \to 2k+3a-3i-2]_{q,t} \\
=&(qt)^a g[0,0,k] + \sum_{i=1}^{a}(qt)^{a-i}[k+3i-2 \to 2k+3i-2]_{q,t}.
\end{aligned}$$

□

3.2.3 The computation of $g[a, k_1, k_2]$

We still need to add k_1 to complete the formula. Since the function $g[a, k_1, k_2]$ is equal to $g[a, k_2, k_1]$, we suppose $k_1 \leqslant k_2$ without loss of generality.

Theorem 6. *For $k_1 \leqslant k_2$, we have*

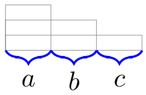
$$g[a, k_1, k_2] = \sum_{i=0}^{k_1} g[a+i, 0, k_1+k_2-2i]. \tag{17}$$

Proof.

$$\begin{aligned}
g[a, k_1, k_2] &= \sum_{T\in S[a,k_1,0,k_2]} g_T \\
&= \sum_{j=0}^{k_1}\sum_{i=0}^{k_2} w(a+i+j, a+k_1+k_2-i-j) \\
&= \sum_{i=0}^{k_1}\sum_{j=0}^{k_1+k_2-2i} w(a+i+j, a+k_1+k_2-i-j) \\
&= \sum_{i=0}^{k_1} g[a+i, 0, k_1+k_2-2i].
\end{aligned}$$

□

3.3 Formula for g_λ

For any $\lambda = (3^a 2^b 1^c)$, λ' has the shape  We can then write the formula of g_λ in terms of $g[x, y, z]$.

Theorem 7. *Let* $\lambda = (3^a 2^b 1^c)$. *Then*

$$g_\lambda = \sum_{i=0}^{b} g[a+i, b-i, c] + \sum_{i=1}^{c} g[a, b, c-i]. \tag{18}$$

Proof. The first term $\sum_{i=0}^{b} g[a+i, b-i, c]$ sums over all the cases in Figure 6(a) and the second term $\sum_{i=0}^{c} g[a, b, c-i]$ sums over all the cases in Figure 6(b). $\square$

Thus, we have a formula for g_λ which is based on recursively compute functions $g[a, b, c]$. The recursive formula for $g[a, b, c]$ not only shows that g_λ is Schur-positive in q, t-analogs, but also gives us a way of writing g_λ into q, t-analogs and powers of (qt). For example, suppose $\lambda = 1^4$. Then $\lambda' = (4)$ so that taking into account the possible numbers of 0's in a tableau $T \in SSYT((4), 012)$, we see that

$$g_{(1^4)} = g[0,0,0] + g[0,0,1] + g[0,0,2] + g[0,0,3] + g[0,0,4].$$

$g[0,0,0] = g[0,0,1] = 0$, and we can apply Theorem 4 to compute

$$g[0,0,2] = \sum_{i=0}^{0} (qt)^i [2-i-\lfloor (i+1)/2 \rfloor \to 4-2-3i]_{q,t} = [2]_{q,t},$$

$$\begin{aligned} g[0,0,3] &= \sum_{i=0}^{1} (qt)^i [3-i-\lfloor (i+1)/2 \rfloor \to 6-2-3i]_{q,t} \\ &= [3 \to 4]_{q,t} + (qt)[1 \to 1]_{q,t} = [3]_{q,t} + [4]_{q,t} + qt, \end{aligned}$$

and

$$\begin{aligned} g[0,0,4] &= \sum_{i=0}^{1} (qt)^i [4-i-\lfloor (i+1)/2 \rfloor \to 8-2-3i]_{q,t} \\ &= [4 \to 6]_{q,t} + (qt)[2 \to 3]_{q,t} \\ &= [4]_{q,t} + [5]_{q,t} + [6]_{q,t} + qt([2]_{q,t} + [3]_{q,t}), \end{aligned}$$

Thus

$$g_{(1^4)} = [2]_{q,t} + [3]_{q,t} + 2[4]_{q,t} + [5]_{q,t} + [6]_{q,t} + qt(1 + [2]_{q,t} + [3]_{q,t}).$$

In general, we see that

$$\langle \Delta_{e_2} e_n[X], e_n[X] \rangle = \sum_{s=2}^{n} g[0,0,s]. \tag{19}$$

We claim that $g[0,0,n]$ is a q,t-analogue of $2\binom{n+1}{3}$. To see this, we shall use a formula of [5] to show that

$$\langle \Delta_{e_2} e_n[X], e_n[X]\rangle|_{q=t=1} = 2\binom{n+2}{4} \tag{20}$$

from which it follows that

$$\begin{aligned} g[0,0,n]|_{q=t=1} &= \langle \Delta_{e_2} e_n[X], e_n[X]\rangle|_{q=t=1} - \langle \Delta_{e_2} e_{n-1}[X], e_{n-1}[X]\rangle|_{q=t=1} \\ &= 2\binom{n+2}{4} - 2\binom{n+1}{4} = 2\binom{n+1}{3}. \end{aligned} \tag{21}$$

It is proved in [5] that

$$\Delta_{e_k} e_n[X]|_{t=1/q} = \frac{q^{\binom{k}{2}-k(n-1)}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q e_n[X(1+q+\cdots+q^k)]. \tag{22}$$

Repeatedly applying the sum rule that

$$s_\lambda[X+Y] = \sum_{\mu\subseteq\lambda} s_\mu[X] s_{\lambda/\mu}[Y],$$

we see that

$$\begin{aligned} \Delta_{e_k} e_n[X]|_{t=1/q} &= \frac{q^{\binom{k}{2}-k(n-1)}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\substack{i_s\geqslant 0 \\ i_0+i_1+\cdots+i_k=n}} \prod_{s=0}^{k} e_{i_s}[q^s X] \\ &= \frac{q^{\binom{k}{2}-k(n-1)}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\substack{i_s\geqslant 0 \\ i_0+i_1+\cdots+i_k=n}} \prod_{s=0}^{k} q^{si_s} e_{i_s}[X]. \end{aligned} \tag{23}$$

It follows that

$$\langle \Delta_{e_k} e_n[X], e_n[X]\rangle|_{t=1/q} = \frac{q^{\binom{k}{2}-k(n-1)}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{\substack{i_s\geqslant 0 \\ i_0+i_1+\cdots+i_k=n}} \prod_{s=0}^{k} q^{si_s}. \tag{24}$$

It is easy to see that

$$\sum_{\substack{i_s\geqslant 0 \\ i_0+i_1+\cdots+i_k=n}} \prod_{s=0}^{k} q^{si_s} = \begin{bmatrix} n+k \\ k \end{bmatrix}_q \tag{25}$$

since the LHS is just the sum of $q^{|\lambda|}$ over all partitions λ contained in the $n\times k$ rectangle. Thus

$$\langle \Delta_{e_k} e_n[X], e_n[X]\rangle|_{t=1/q} = \frac{q^{\binom{k}{2}-k(n-1)}}{[k+1]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q. \tag{26}$$

Setting $q = 1$ and $k = 2$ in (26), we see that

$$\begin{aligned}
\langle \Delta_{e_2} e_n[X], e_n[X] \rangle|_{q=t=1} &= \frac{1}{3} \frac{n!}{2!(n-2)!} \frac{(n+2)!}{2!n!} \\
&= \frac{4}{2} \frac{(n+2)!}{4!(n-2)!} = 2\binom{n+2}{4}.
\end{aligned} \tag{27}$$

For another example, we compute g_λ for $\lambda = (1^2, 2)$, and the conjugate $\lambda' = (1, 3)$. In this case, we can classify the tableau $T \in SSYT((1,3), 012)$ by whether the bottom corner square contains a 1, in which case we get a term $g[1, 0, 2]$, or the bottom corner square contains a 0, in which case we get a contribution of $g[0, 1, 2]$, $g[0, 1, 1]$, or $g[0, 1, 0]$, depending on the number of 0's in the first row. Now $g[0, 1, 0] = g[0, 0, 1] = 0$, and by Theorem 6,

$$\begin{aligned}
g[0,1,1] &= \sum_{i=0}^{1} g[i, 0, 2{-}2i] \\
&= g[0,0,2] + g[1,0,0] \\
&= [2]_{q,t} + g[1,0,0],
\end{aligned}$$

and

$$\begin{aligned}
g[0,1,2] &= \sum_{i=0}^{1} g[i, 0, 3{-}2i] \\
&= g[0,0,3] + g[1,0,1] \\
&= [3]_{q,t} + [4]_{q,t} + qt + g[1,0,1].
\end{aligned}$$

By Theorem 5

$$\begin{aligned}
g[1,0,0] &= (qt)g[0,0,0] + [3 \to 3]_{q,t} = [1]_{q,t}, \\
g[1,0,1] &= (qt)g[0,0,1] + [4 \to 5]_{q,t} = [2]_{q,t} + [3]_{q,t}, \text{ and} \\
g[1,0,2] &= (qt)g[0,0,2] + [3 \to 5]_{q,t} \\
&= (qt)[2]_{q,t} + [3]_{q,t} + [4]_{q,t} + [5]_{q,t}.
\end{aligned}$$

It follows that

$$g_{1^2,2} = [1]_{q,t} + 2[2]_{q,t} + 3[3]_{q,t} + 2[4]_{q,t} + [5]_{q,t} + (qt)(1 + [2]_{q,t}).$$

3.4 The relation between the combinatorial proof and direct computation

We show in this subsection that the combinatorial involution of the enriched tableaux implies the cancellation step of the computation of $g[a, k_1, k_2]$.

Firstly, we present the case $g[0,0,k]$. We illustrate the relation by an example of $k=5$. Let $\lambda=(5)$ and suppose there are no 0's in the filling. Then the contribution of all tableaux of this form is $g[0,0,5]$.

If there are i 1's in the filled Young diagram, then there are $5-i$ 2's. Theorem 4 shows that when we sum over all cases classified by number of 1's, we have

$$\begin{aligned}
g[0,0,5] &= \sum_{i=0}^{k} w(i,k-i)\\
&= -[4]_{q,t}-(qt)^2[1]_{q,t}+(qt)^2[2]_{q,t}+(qt)[5]_{q,t}+[8]_{q,t}\\
&\quad -(qt)[2]_{q,t}+(qt)^2[1]_{q,t}+(qt)[4]_{q,t}+[7]_{q,t}\\
&\quad +0+(qt)[3]_{q,t}+[6]_{q,t}\\
&\quad +(qt)[2]_{q,t}+[5]_{q,t}\\
&\quad +[4]_{q,t}\\
&= [5\to 8]_{q,t}+(qt)[3\to 5]_{q,t}+(qt)^2[2]_{q,t}.
\end{aligned}$$

Notice that there is a cancellation in the last step of the equation. The cancellation cancels terms of different signs in the last steps, which follows the same idea of the combinatorial proof. The injection of the combinatorial proof maps the negative terms into the positive terms, giving this cancellation. Table 1 shows all enriched tableaux of shape $\lambda=(5)$ with no 0's and their corresponding weight. From this, we can see that the first column and the first two rows of the second column are canceled, leaving only the red terms. These give $g[0,0,5]=[5\to 8]_{q,t}+(qt)[3\to 5]_{q,t}+(qt)^2[2]_{q,t}$.

# of 1's	no $\overline{2}$	one $\overline{2}$	two $\overline{2}$	three $\overline{2}$	four $\overline{2}$
0	$\boxed{2}\boxed{2}\boxed{2}\boxed{2}\boxed{\widehat{2}}$ $-[4]_{q,t}$	$\boxed{2}\boxed{2}\boxed{2}\boxed{\overline{2}}\boxed{\widehat{2}}$ $-(qt)^2[1]_{q,t}$	$\boxed{2}\boxed{2}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\widehat{2}}$ $(qt)^2[2]_{q,t}$	$\boxed{2}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\widehat{2}}$ $(qt)[5]_{q,t}$	$\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\widehat{2}}$ $[8]_{q,t}$
1	$\boxed{1}\boxed{2}\boxed{2}\boxed{2}\boxed{\widehat{2}}$ $-(qt)[2]_{q,t}$	$\boxed{1}\boxed{2}\boxed{2}\boxed{\overline{2}}\boxed{\widehat{2}}$ $(qt)^2[1]_{q,t}$	$\boxed{1}\boxed{2}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\widehat{2}}$ $(qt)[4]_{q,t}$	$\boxed{1}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\widehat{2}}$ $[7]_{q,t}$	
2	$\boxed{1}\boxed{1}\boxed{2}\boxed{2}\boxed{\widehat{2}}$ 0	$\boxed{1}\boxed{1}\boxed{2}\boxed{\overline{2}}\boxed{\widehat{2}}$ $(qt)[3]_{q,t}$	$\boxed{1}\boxed{1}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\widehat{2}}$ $[6]_{q,t}$		
3	$\boxed{1}\boxed{1}\boxed{1}\boxed{2}\boxed{\widehat{2}}$ $(qt)[2]_{q,t}$	$\boxed{1}\boxed{1}\boxed{1}\boxed{\overline{2}}\boxed{\widehat{2}}$ $[5]_{q,t}$			
4	$\boxed{1}\boxed{1}\boxed{1}\boxed{1}\boxed{\widehat{2}}$ $[4]_{q,t}$				

Table 1: Contribution of tableaux of shape $\lambda=(5)$

Next, we show this relation for the case $g[a,0,k]$. The semi-standard Young tableaux contributing to such g_λ contain 2 parts – the first part has a columns of 2 rows of which the bottom row is filled with 1's, and the second part has k columns of 1 row filled with some 1's and 2's, looking like $T = \begin{array}{l} \boxed{2\cdots 2} \\ \underbrace{\boxed{1\cdots 1}}_{a}\underbrace{\boxed{1\cdots 2}}_{k} \end{array}$.

Theorem 5 gives that $g[a,0,k] = (qt)^a g[0,0,k] + \sum_{j=0}^{a-1}(qt)^j \sum_{i=0}^{k}[2k-i+3a-3j-2]_{q,t}$. We want to show this formula combinatorially. If we fill all cells of the top row of the first a columns of T with 2, then the contribution of the first a columns is $(qt)^a$, and the contribution of the last k columns is $g[0,0,k]$ for the same reason as the first case of $g[0,0,k]$, except that the last k columns cannot be all 1's as there should be a $\hat{2}$ in these columns. Actually this exceptional restriction about filling does not affect $g[0,0,k]$ as an all 1's filling case contributes 0 to $g[0,0,k]$. Otherwise, if there is at least one $\bar{2}$ or $\hat{2}$ in the first a columns, then there must be no 2 in the last k columns. Suppose there are i 1's in the last k columns and j 2's in the first a columns. Then the contribution is $(qt)^j[2k-i+3a-3j-2]_{q,t}$. These are all fixed points in the involution described in the combinatorial proof, and we now have found the implication of combinatorial involution in the cancellation of $g[a,0,k]$.

Finally, for the case $g[a,k_1,k_2]$, the combinatorics is straightforward in the recursion in Theorem 7. Thus we see the connection of the combinatorial proof and the direct computation.

4 Proofs by Generating Functions

Here we illustrate two proofs using generating functions. They are not different in nature.

First generating function proof.

It is clear that $g_\lambda = 0$ unless λ' has at most 3 parts, i.e., $\lambda' = (a+b+c, b+c, c)$ for $a,b,c \geqslant 0$. The idea is to show the generating function

$$G(u_1,u_2,u_3) = \sum_{a,b,c\geqslant 0} g_{(a+b+c,b+c,c)'} u_1^a u_2^b u_3^c$$

has only nonnegative coefficients.

Firstly, we use the quotient formula for Schur functions:

$$s_{a+b+c,b+c,c}[x+y+z] = \frac{1}{(x-y)(y-z)(x-z)} \det \begin{pmatrix} x^{a+b+c+2} & x^{b+c+1} & x^c \\ y^{a+b+c+2} & y^{b+c+1} & y^c \\ z^{a+b+c+2} & z^{b+c+1} & z^c \end{pmatrix}.$$

Next, from the view of MacMahon partition analysis (see, e.g., [1], [7]), G is easily seen to be a rational power series. Here we only need the following fact:

If $\gamma_{ij} \geqslant 0$ for all i,j, then

$$\sum_{a,b,c\geqslant 0} x^{\gamma_{11}a+\gamma_{12}b+\gamma_{13}c} y^{\gamma_{21}a+\gamma_{22}b+\gamma_{23}c} z^{\gamma_{31}a+\gamma_{32}b+\gamma_{33}c} u_1^a u_2^b u_3^c$$
$$= \frac{1}{(1-x^{\gamma_{11}}y^{\gamma_{21}}z^{\gamma_{31}}u_1)(1-x^{\gamma_{12}}y^{\gamma_{22}}z^{\gamma_{32}}u_2)(1-x^{\gamma_{13}}y^{\gamma_{23}}z^{\gamma_{33}}u_3)}.$$

One simple case will illustrate the idea. By the quotient formula,

$$s_{a+b+c,b+c,c}[1+q+t] = \frac{1}{(1-q)(1-t)(q-t)}(t^{a+b+c+2}q^{b+c+1} + \text{other terms}),$$

where the "other terms" are the five terms of similar type obtained by expanding the determinant. Now we have

$$\sum_{a,b,c\geqslant 0} R(q,t)t^{a+b+c}q^{b+c}u_1^a u_2^b u_3^c = R(q,t)\frac{1}{(1-tu_1)(1-u_2qt)(1-u_3qt)},$$

where $R(q,t) = t^2q/((1-q)(1-t)(q-t))$ is a rational function independent of a,b,c.

Thus we can write G as a sum of $6\times 3 = 18$ rational functions. This can be carried out by Maple and we normalize to obtain

$$G = \frac{P}{(1-m_1)(1-m_2)\cdots(1-m_{15})},$$

where m_i are monomials, and P is a polynomial with 1023 terms. Through a complicated search procedure, we found a decomposition $G = \sum_{i=1}^{27} Q_i$ where each Q_i is easily seen to have only nonnegative coefficients. For instance, one of the terms is

$$Q_1 = \frac{u_1u_2q\,(q^3u_2+t)}{(qtu_2-1)\,(t^2u_1-1)\,(u_1-1)\,(qu_2-1)\,(qtu_3-1)\,(qu_1-1)\,(q^2u_2-1)\,(q^3u_2-1)}.$$

This proves that G has only nonnegative coefficients and hence $g_\lambda \in \mathbb{Z}_{\geqslant 0}[q,t]$. As a proof, we only need to verify that these Q_i's sum to G (which is routine by computer) but not how to find them. We are not going to explain in detail how to decompose G since the idea is not mature.

Second generating function proof.

After the first proof was obtained, Professor Adriano Garsia investigated some data of g_λ and conjectured that g_λ is indeed also Schur positive in q,t-analogs. More precisely, we have

$$g_\lambda = \sum_{i>j\geqslant 0} b_{i,j}s_{(i-1,j)}[q+t] = \sum_{i>j\geqslant 0} b_{i,j}q^jt^j[i-j-1]_{q,t},$$

where $b_{i,j}$ are nonnegative for all $i>j\geqslant 0$.

This is equivalent to writing $(t-q)G = F - \tau F$, where

$$F = \sum_{i>j\geqslant 0} b_{i,j}(u_1,u_2,u_3)t^i q^j,$$

and showing the nonnegativity of $b_{i,j}(u_1,u_2,u_3)$. To obtain an explicit formula of F from G, it is better to make the change of variable by $q = \bar{q}/t$. Then

$$\begin{aligned} F &= \sum_{i>j\geqslant 0} b_{i,j}(u_1,u_2,u_3)t^{i-j}\bar{q}^j, \\ \tau F &= \sum_{i>j\geqslant 0} b_{i,j}(u_1,u_2,u_3)q^i t^j = \sum_{i>j\geqslant 0} b_{i,j}(u_1,u_2,u_3)t^{-(i-j)}\bar{q}^i. \end{aligned}$$

It follows that F consists of all terms in the series expansion of $(t-\bar{q}/t)G$ with positive powers in t. This can be realized by the following constant term

$$F = (z-\bar{q}/z)G\big|_{t=z} \sum_{k\geqslant 1}(t/z)^k\Big|_{z^0} = (z-\bar{q}/z)G\big|_{t=z} \frac{t/z}{1-t/z}\Big|_{z^0}.$$

Thus F can be calculated by MacMahon's partition analysis techniques.

The complexity of G suggests that this approach does not work for $\Delta_{e_3}e_n[X]$, so we go over the computation and the use of Lemma 2 which is the point of departure for the other proofs.

Using the explicit formula of F, which has 132 terms in the numerator and 11 factors in the denominator, we are able to decompose F as a sum of 7 rational functions that are easily seen to have nonnegative coefficients:

$$\begin{aligned} F = &-\frac{\left(qtu_1{}^3+qtu_1{}^2u_2+qtu_1u_2{}^2+tu_1{}^2+tu_1u_2+tu_2{}^2\right)t}{\left(u_1-1\right)\left(tu_2-1\right)\left(t^3u_3-1\right)\left(t^2u_1-1\right)\left(t^3u_2-1\right)\left(q^2t^2u_1{}^3-1\right)\left(q^2t^2u_2{}^3-1\right)} \\ &-\frac{u_2{}^3t\left(t^3+qt\right)}{\left(u_1-1\right)\left(tu_2-1\right)\left(t^3u_3-1\right)\left(t^2u_1-1\right)\left(t^3u_2-1\right)\left(q^2t^2u_2{}^3-1\right)\left(t^2u_2-1\right)} \\ &-\frac{u_2t\left(q^2t^4u_1{}^2u_2{}^2+q^2t^2u_1{}^2u_2+q^2t^2u_1u_2{}^2+qt^2u_1{}^2+qt^2u_1u_2+qt^2u_2{}^2+t^2u_1+t^2u_2+1\right)}{\left(u_1-1\right)\left(tu_2-1\right)\left(t^3u_3-1\right)\left(t^3u_2-1\right)\left(q^2t^2u_1{}^3-1\right)\left(q^2t^2u_2{}^3-1\right)\left(qtu_2-1\right)} \\ &-\frac{\left(qtu_1+t\right)t^2u_1{}^3}{\left(q^2t^2u_1{}^3-1\right)\left(t^2u_1-1\right)\left(t^3u_3-1\right)\left(tu_2-1\right)\left(qtu_2-1\right)\left(u_1-1\right)\left(tu_1-1\right)} \\ &-\frac{u_1u_2t^2}{\left(t^2u_1-1\right)\left(t^3u_3-1\right)\left(tu_2-1\right)\left(qtu_2-1\right)\left(u_1-1\right)\left(tu_1-1\right)\left(t^3u_2-1\right)} \\ &-\frac{u_1t^2u_3\left(qt^2u_1{}^2+t^2u_1+1\right)}{\left(u_1-1\right)\left(qtu_2-1\right)\left(tu_2-1\right)\left(t^3u_3-1\right)\left(q^2t^2u_1{}^3-1\right)\left(qtu_3-1\right)\left(tu_1-1\right)} \\ &-\frac{\left(q^2t^4u_1{}^2u_2{}^2+q^2t^2u_1{}^2u_2+q^2t^2u_1u_2{}^2+qt^2u_1{}^2+qt^2u_1u_2+qt^2u_2{}^2+t^2u_1+t^2u_2+1\right)tu_3}{\left(u_1-1\right)\left(qtu_2-1\right)\left(tu_2-1\right)\left(t^3u_3-1\right)\left(q^2t^2u_1{}^3-1\right)\left(q^2t^2u_2{}^3-1\right)\left(qtu_3-1\right)}. \end{aligned}$$

This may be treated as our fourth proof, but in the same vein of our first proof.

5 The $\boldsymbol{\Delta}_{e_3}$ Case

For the $\Delta_{e_3} e_n[X]$ case, we have a similar formula

$$e_4[X] = \frac{\tilde{H}_4[X;q,t]}{(q-t)\,(q^2-t)\,(q^3-t)} - \frac{(q^2+q+t+1)\,\tilde{H}_{3,1[X;q,t]}}{(q+t)\,(q^3-t)\,(q-t)^2} - \frac{(qt-1)\,\tilde{H}_{2,2}[X;q,t]}{(-t^2+q)\,(q^2-t)\,(q-t)^2} + \frac{(t^2+q+t+1)\,\tilde{H}_{2,1,1}[X;q,t]}{(q+t)\,(-t^3+q)\,(q-t)^2} - \frac{\tilde{H}_{1,1,1,1}[X;q,t]}{(q-t)\,(-t^3+q)\,(-t^2+q)}.$$

This gives

$$\langle \Delta_{e_3} e_n[X],\ s_{\lambda'}\rangle = \frac{s_\lambda[B_4]}{(q-t)\,(q^2-t)\,(q^3-t)} - \frac{(q^2+q+t+1)\,s_\lambda[B_{3,1}]}{(q+t)\,(q^3-t)\,(q-t)^2} - \frac{(qt-1)\,s_\lambda[B_{2,2}]}{(-t^2+q)\,(q^2-t)\,(q-t)^2} + \frac{(t^2+q+t+1)\,s_\lambda[B_{2,1,1}]}{(q+t)\,(-t^3+q)\,(q-t)^2} - \frac{s_\lambda[B_{1,1,1,1}]}{(q-t)\,(-t^3+q)\,(-t^2+q)}.$$

By playing with partial fraction decompositions, the best formula we have is

$$\langle \Delta_{e_3} e_n[X],\ s_{\lambda'}\rangle = \frac{F_\lambda(q,t) - F_\lambda(t,q)}{q-t} - \frac{s_\lambda[1+q+t+q^2]/q^2 - s_\lambda[1+q+t+t^2]/t^2}{2\,(q^2-t^2)}, \tag{28}$$

where $F_\lambda = F_\lambda(q,t)$ is given by

$$F_\lambda = \frac{s_\lambda[1+q+q^2+q^3] - s_\lambda[1+q+t+qt]}{(q-1)\,q^2\,(q^2-t)} - \frac{s_\lambda[1+q+q^2+q^3] - s_\lambda[1+q+t+q^2]}{q^2\,(q-1)\,(q^3-t)} - \frac{(q+1)(s_\lambda[1+q+t+q^2] - s_\lambda[1+q+t+qt])}{2\,(q-t)\,q^2\,(q-1)} + \frac{s_\lambda[1+q+t+qt]}{2q^2t}.$$

One can use this formula to prove that g_λ is a polynomial divided by $(1-q)$. Nevertheless, it is clear that this approach becomes more and more complicated so that the proof of the general $\Delta_{e_d} e_n[X]$ case seems to require a new idea.

Acknowledgements

The authors would like to thank Professor Adriano Garsia for his invaluable contributions, including many productive discussions. The first, third, and fourth authors would also like to dedicate this paper to the memory of our second author, Professor Jeff Remmel, who passed away recently. His mentorship, collaboration, and friendship will be greatly missed. The authors would also like to thank the anonymous referee for careful reading and valuable comments.

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