# Using symbolic computation to prove nonexistence of distance-regular graphs

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**Abstract**

A package for the Sage computer algebra system is developed for checking feasibility of a given intersection array for a distance-regular graph. We use this tool to show that there is no distance-regular graph with intersection array

$$\{(2r+1)(4r+1)(4t-1), 8r(4rt-r+2t), (r+t)(4r+1); \\ 1, (r+t)(4r+1), 4r(2r+1)(4t-1)\} \quad (r, t \geqslant 1),$$

$\{135,128,16; 1,16,120\}$, $\{234,165,12; 1,30,198\}$ or $\{55,54,50,35,10; 1,5,20,45,55\}$. In all cases, the proofs rely on equality in the Krein condition, from which triple intersection numbers are determined. Further combinatorial arguments are then used to derive nonexistence.

**Mathematics Subject Classifications:** 05E30

## 1 Introduction

Distance-regular graphs were introduced around 1970 by N. Biggs [1]. As they are intimately linked to many other combinatorial objects, such as finite simple groups, finite geometries, and codes, a natural goal is trying to classify them.

Many distance-regular graphs are known, however constructing new ones has proved to be a difficult task. A number of feasibility conditions for distance-regular graphs have been found, which allows us to compile a list of feasible intersection arrays for small distance-regular graphs (or related structures, such as $Q$-polynomial association schemes), see Brouwer et al. [2, 3, 4] and Williford [22]. However, feasibility is no guarantee for

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existence, so proofs of nonexistence of distance-regular graphs with feasible intersection arrays are also a contribution to the classification. In certain cases, single intersection arrays have been ruled out [12, 13], while other proofs may show nonexistence for a whole infinite family of feasible intersection arrays [6, 9, 19]. In this paper we give proofs of nonexistence for distance-regular graphs belonging to a two-parameter infinite family, as well as for graphs with intersection arrays

$$\{135, 128, 16;\ 1, 16, 120\},$$
$$\{234, 165, 12;\ 1, 30, 198\},$$
$$\{55, 54, 50, 35, 10;\ 1, 5, 20, 45, 55\}.$$

We develop a package called `sage-drg` [21] for the Sage computer algebra system [18]. Sage is free open-source software written in the Python programming language [17], with many functionalities deriving from other free open-source software, such as Maxima [16], which Sage uses for symbolic computation. The `sage-drg` package is thus also free open-source software available under the MIT license, written in the Python programming language, making use of the Sage library. The package can be used to check for feasibility of a given intersection array against known feasibility conditions, see Van Dam, Koolen and Tanaka for an up-to-date survey [7]. Furthermore, using equality in the Krein condition (see Theorem 1), restrictions on triple intersection numbers can be derived. In this paper, we use them to derive some nonexistence results. The `sage-drg` package also includes Jupyter notebooks demonstrating its use to obtain these results, as well as the notebook `jupyter/Demo.ipynb` giving some general examples of use of the package. A more detailed description of the `sage-drg` package is given in the supplementary files[1].

The results from Sections 3, 4 and 6 appeared in the author's PhD thesis [20], where computation was done using a Mathematica [23] notebook originally developed by M. Urlep. Thus, the `sage-drg` package can be seen as a move from closed-source proprietary software to free open-source software, which allows one to check all code for correctness, thus making the results verifiable.

## 2 Preliminaries

In this section we review some basic definitions and concepts. See Brouwer, Cohen and Neumaier [3] for further details.

Let $\Gamma$ be a connected graph with diameter $d$ and $n$ vertices, and let $\partial(u, v)$ denote the distance between the vertices $u$ and $v$ of $\Gamma$. The graph $\Gamma$ is *distance-regular* if there exist constants $p^h_{ij}$ $(0 \leqslant h, i, j \leqslant d)$, called the *intersection numbers*, such that for any pair of vertices $u, v$ at distance $h$, there are precisely $p^h_{ij}$ vertices at distances $i$ and $j$ from $u$ and $v$, respectively. In fact, all intersection numbers can be computed given only the intersection numbers $b_i = p^i_{1,i+1}$ and $c_{i+1} = p^{i+1}_{1,i}$ $(0 \leqslant i \leqslant d-1)$ [3, §4.1A]. These intersection numbers are usually gathered in the *intersection array* $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$. We also

[1] Available in the 'Appendix' at http://www.combinatorics.org/ojs/index.php/eljc/article/view/v25i4p21

define the *valency* $k = b_0$ and $a_i = k - b_i - c_i$ $(0 \leqslant i \leqslant d)$, where $b_d = c_0 = 0$. A connected noncomplete *strongly regular* graph with parameters $(v, k, \lambda, \mu)$ is a distance-regular graph of diameter 2 with $v$ vertices, valency $k$ and intersection numbers $a_1 = \lambda$, $c_2 = \mu$.

Let $A_i$ $(0 \leqslant i \leqslant d)$ be a binary square matrix indexed with the vertices of a graph $\Gamma$ of diameter $d$, with entry corresponding to vertices $u$ and $v$ equal to 1 precisely when $\partial(u, v) = i$. The matrix $A = A_1$ is the *adjacency matrix* of $\Gamma$. The graph $\Gamma$ is called *primitive* if all $A_i$ $(1 \leqslant i \leqslant d)$ are adjacency matrices of connected graphs. A distance-regular graph of valency $k \geqslant 3$ that is not primitive is bipartite or antipodal (or both) [3, Thm. 4.2.1]. The spectrum of $\Gamma$ is defined to be the spectrum of $A$ (i.e., eigenvalues with multiplicities) and can be computed directly from the intersection array of $\Gamma$ [3, §4.1B].

Suppose that $\Gamma$ is distance-regular. Let $\mathcal{M}$ be the *Bose-Mesner* algebra, i.e., the algebra generated by $A$. The matrices $\{A_i\}_{i=0}^d$ form a basis of $\mathcal{M}$, which also has a second basis $\{E_i\}_{i=0}^d$ consisting of projectors to the eigenspaces of $A$ [3, §2.2]. Note that the indexing in this second basis depends on the ordering of eigenvalues. The descending ordering of eigenvalues is known as the *natural ordering*. We define the *eigenmatrix* $P$ and *dual eigenmatrix* $Q$ as $(d + 1) \times (d + 1)$ matrices such that $A_j = \sum_{i=0}^d P_{ij} E_i$ and $E_j = n^{-1} \sum_{i=0}^d Q_{ij} A_i$. The graph $\Gamma$ is called *formally self-dual* [3, p. 49] if $P = Q$ holds for some ordering of eigenvalues. The graph $\Gamma$ is called *Q-polynomial* [3, §2.7] with respect to some ordering of eigenvalues if there exist real numbers $z_0, \dots, z_d$ and polynomials $q_j$ of degree $j$ such that $Q_{ij} = q_j(z_i)$ $(0 \leqslant i, j \leqslant d)$. Finally, we define the *Krein parameters* $q_{ij}^h$ [3, §2.3] as such numbers that $E_i \circ E_j = n^{-1} \sum_{h=0}^d q_{ij}^h E_h$, where $\circ$ represents entrywise multiplication of matrices. A formally self-dual distance-regular graph is also $Q$-polynomial with respect to the corresponding ordering of eigenvalues and has $p_{ij}^h = q_{ij}^h$ $(0 \leqslant i, j, h \leqslant d)$. In this paper, we will use the natural ordering for indexing, noting when a graph is $Q$-polynomial or formally self-dual for some other ordering.

For vertices $u$, $v$, $w$ of the distance-regular graph $\Gamma$ and integers $i$, $j$, $h$ $(0 \leqslant i, j, h \leqslant d)$ we denote by $\begin{bmatrix} u & v & w \\ i & j & h \end{bmatrix}$ (or simply $[i\ j\ h]$ when it is clear which triple $(u, v, w)$ we have in mind) the number of vertices of $\Gamma$ that are at distances $i$, $j$, $h$ from $u$, $v$, $w$, respectively. We call these numbers *triple intersection numbers*. They have first been studied in the case of strongly regular graphs [5], and later also for distance-regular graphs, see for example [6, 8, 9, 10, 19]. Unlike the intersection numbers, these numbers may depend on the particular choice of vertices $u, v, w$ and not only on their pairwise distances. We may however write down a system of $3d^2$ linear Diophantine equations with $d^3$ triple intersection numbers as variables, thus relating them to the intersection numbers, cf. [9]:

$$\sum_{\ell=1}^d [\ell\ j\ h] = p_{jh}^U - [0\ j\ h], \qquad \sum_{\ell=1}^d [i\ \ell\ h] = p_{ih}^V - [i\ 0\ h], \qquad \sum_{\ell=1}^d [i\ j\ \ell] = p_{ij}^W - [i\ j\ 0], \qquad (1)$$

where $U = \partial(v, w)$, $V = \partial(u, w)$, $W = \partial(u, v)$, and

$$[0\ j\ h] = \delta_{jW}\delta_{hV}, \qquad [i\ 0\ h] = \delta_{iW}\delta_{hU}, \qquad [i\ j\ 0] = \delta_{iV}\delta_{jU}.$$

Furthermore, we can use the triangle inequality to conclude that certain triple intersection numbers must be zero. Moreover, the following theorem sometimes gives additional equations.

**Theorem 1.** ([6, Theorem 3], cf. [3, Theorem 2.3.2]) *Let $\Gamma$ be a distance-regular graph with diameter $d$, dual eigenmatrix $Q$ and Krein parameters $q^h_{ij}$ $(0 \leqslant i, j, h \leqslant d)$. Then,*

$$q^h_{ij} = 0 \quad \iff \quad \sum_{r,s,t=0}^{d} Q_{ri} Q_{sj} Q_{th} \begin{bmatrix} u & v & w \\ r & s & t \end{bmatrix} = 0 \quad \textit{for all} \ \ u, v, w \in V\Gamma. \qquad \square$$

Together with integrality and nonnegativity of triple intersection numbers, we can use all of the above to either derive that the system of equations has no solution, or arrive at a small number of solutions, which gives us new information on the structure of the graph and may lead to proving its nonexistence.

## 3 A two-parameter family of primitive graphs of diameter 3

In [9], graphs meeting necessary conditions for the existence of extremal codes were studied. One of the families of primitive graphs of diameter 3 for which these conditions were met was

$$\{a(p+1), (a+1)p, c; 1, c, ap\}, \tag{2}$$

where $a = a_3$, $c = c_2$ and $p = p^3_{33}$. Graphs belonging to this family are $Q$-polynomial with respect to the natural ordering of eigenvalues precisely when the Krein parameter $q^3_{11}$ is zero, which is equivalent to

$$c = \frac{1}{4}\left((p+1)^2 + \frac{2a(p+1)}{p+2}\right). \tag{3}$$

Hence, $p+2$ must divide $2a$ for $c$ to be integral. If $p = 2r - 1$, then $a = t(2r+1)$ and $c = r(r+t)$ for some positive integers $r, t$, which gives us the two-parameter family

$$\{2rt(2r+1), (2r-1)(2rt+t+1), r(r+t); 1, r(r+t), t(4r^2-1)\}.$$

In [9], nonexistence was shown for a feasible subfamily with $r = t \geqslant 2$. If, on the other hand, $p$ is even, integrality of the multiplicity of the second largest eigenvalue implies that we must have $p = 4r$, $a = (2r+1)(4t-1)$ and $c = (r+t)(4r+1)$ for some positive integers $r, t$, giving the family

$$\begin{split}\{(2r+1)(4r+1)(4t-1), 8r(4rt-r+2t), (r+t)(4r+1); \\ 1, (r+t)(4r+1), 4r(2r+1)(4t-1)\}.\end{split} \tag{4}$$

We find two one-parameter infinite subfamilies of feasible intersection arrays by setting $t = 4r^2$ or $t = 4r^2 + 2r$:

$$\begin{split}\{(2r+1)(4r+1)(16r^2-1), 8r^2(16r^2+8r-1), r(4r+1)^2; \\ 1, r(4r+1)^2, 4r(2r+1)(16r^2-1)\}, \\ \{(2r+1)(4r+1)(16r^2+8r-1), 8r^2(4r+1)(4r+3), r(4r+1)(4r+3); \\ 1, r(4r+1)(4r+3), 4r(2r+1)(16r^2+8r-1)\}.\end{split}$$

There are also other feasible cases – for instance, when $r = 2$, we have, besides the cases from the two subfamilies above, feasible examples when $t \in \{4, 7, 196\}$. The case with $r = 1$ and $t = 4$ belonging to the first subfamily above is also listed in the list of feasible parameter sets for 3-class $Q$-polynomial association schemes by J. S. Williford [22].

We now prove that a graph $\Delta$ with intersection array (4) does not exist. The proof parallels that of [9, Lems. 1, 3] – in fact, a significant part of the proof may be extended to the entire family (3), as it has been done in [20]. The computation needed to obtain the results in this section is illustrated in the `jupyter/DRG-d3-2param.ipynb` notebook included in the `sage-drg` package [21].

**Lemma 2.** *Let $\Delta$ be a distance-regular graph with intersection array (4), and $u', v, w$ be vertices of $\Delta$ with $\partial(u', v) = 1$, $\partial(u', w) = 2$ and $\partial(v, w) = 3$. Then* $\begin{bmatrix} u' & v & w \\ 3 & 3 & 3 \end{bmatrix} = 1$.

*Proof.* Let $u$ be a vertex of $\Delta$ at distance 3 from both $v$ and $w$ (such a vertex exists since $p^3_{33} = 4r > 0$). We consider the triple intersection numbers $[i\ j\ h]$ that correspond to $(u, v, w)$. As $q^3_{11} = q^1_{13} = q^1_{31} = 0$, Theorem 1 gives three additional equations to the system (1), allowing us to express its solution in terms of a single parameter $\alpha = [3\ 3\ 3]$. Let us express the counts of vertices at distance 1 or 2 from one of $u, v, w$ and at distance 3 from the other two vertices:

$$[3\ 3\ 1] = [3\ 1\ 3] = [1\ 3\ 3] = \frac{(\alpha - 4r + 1)(4r + 1)}{4r - 1},$$
$$[3\ 3\ 2] = [3\ 2\ 3] = [2\ 3\ 3] = \frac{8r(4r - 1 - \alpha)}{4r - 1}.$$

For the values above to be nonnegative, we must have $\alpha = 4r - 1$, which means that they are all zero. As the choice of $u, v, w$ was arbitrary, this implies that any pair of vertices at distance 3 induces a set of $4r + 2$ vertices pairwise at distance 3 – in the terminology of [9], this is a maximal 1-code in $\Delta$. Since we have $a_3 p^3_{33} = 4r(2r + 1)(4t - 1) = c_3$, it follows by [9, Prop. 2] that $\begin{bmatrix} u' & v & w \\ 3 & 3 & 3 \end{bmatrix} = 1$ holds. □

**Theorem 3.** *A distance-regular graph $\Delta$ with intersection array (4) does not exist.*

*Proof.* Let $u', v, w$ be vertices of $\Delta$ with $\partial(u', v) = 1$, $\partial(u', w) = 2$ and $\partial(v, w) = 3$ (such vertices exist, since we have $p^2_{13} = b_2 = (r + t)(4r + 1) > 0$). We consider the triple intersection numbers $[i\ j\ h]$ that correspond to $(u', v, w)$. By Lemma 2, we have $[3\ 3\ 3] = 1$. Using $q^3_{11} = 0$, Theorem 1 gives an additional equation which allows us to obtain a unique solution to the system (1). However, we obtain $[1\ 1\ 3] = 2t - 1/2$, which is nonintegral for all integers $t$. Therefore, the graph $\Delta$ does not exist. □

## 4 A primitive graph with diameter 3 and 1360 vertices

Let $\Lambda$ be a distance-regular graph with intersection array

$$\{135, 128, 16; 1, 16, 120\}. \tag{5}$$

This intersection array can be obtained from (2) by setting $a = 15$, $c = 16$ and $p = 8$. The graph $\Lambda$ has diameter 3 and 1360 vertices. It is not $Q$-polynomial, however its Krein parameter $q^3_{33}$ is zero. We show that such a graph does not exist. The computation needed to prove Theorem 4 is illustrated in the `jupyter/DRG-135-128-16-1-16-120.ipynb` notebook included in the `sage-drg` package [21].

**Theorem 4.** *A distance-regular graph* $\Lambda$ *with intersection array* (5) *does not exist.*

*Proof.* Let $u, v, w$ be three pairwise adjacent vertices of $\Lambda$ (such vertices exist, since we have $p^1_{11} = 6 > 0$). We consider triple intersection numbers $[i\ j\ h]$ that correspond to $(u, v, w)$. As $q^3_{33} = 0$, Theorem 1 gives an additional equation to the system (1), allowing us to express its solution in terms of a single parameter $\alpha = [1\ 1\ 1]$. In particular, we obtain

$$[3\ 3\ 3] = \frac{71 - 27\alpha}{8}.$$

Clearly, $\alpha$ must be a nonnegative integer. For $[3\ 3\ 3]$ to be nonnegative, we must have $\alpha \in \{0, 1, 2\}$. However, $[3\ 3\ 3]$ is still nonintegral in these cases, showing that the graph $\Lambda$ does not exist. □

## 5 A primitive graph with diameter 3 and 1600 vertices

Let $\Xi$ be a distance-regular graph with intersection array

$$\{234, 165, 12; 1, 30, 198\}. \tag{6}$$

The graph $\Xi$ has diameter 3 and 1600 vertices. The intersection array (6) has been found as an example of a feasible parameter set for a distance-regular graph which is formally self-dual for an ordering of eigenvalues distinct from the natural ordering – in fact, $\Xi$ is $Q$-polynomial for the ordering $0, 2, 3, 1$, so its Krein parameters $q^1_{22}$, $q^2_{12}$ and $q^2_{21}$ are zero. The intersection array (6) is also listed in the list of feasible parameter sets for 3-class $Q$-polynomial association schemes by J. S. Williford [22]. We show that such a graph does not exist. The computation needed to prove Theorem 5 is illustrated in the `jupyter/DRG-234-165-12-1-30-198.ipynb` notebook included in the `sage-drg` package [21].

**Theorem 5.** *A distance-regular graph* $\Xi$ *with intersection array* (6) *does not exist.*

*Proof.* Let $u, v, w$ be three vertices of $\Xi$ that are pairwise at distance 3 (such vertices exist, since we have $p^3_{33} = 8 > 0$). We consider triple intersection numbers $[i\ j\ h]$ that correspond to $(u, v, w)$. As $q^1_{22} = q^2_{12} = q^2_{21} = 0$, Theorem 1 gives three additional equations to the system (1), allowing us to express its solution in terms of a single parameter $\alpha = [3\ 3\ 3]$. In particular, we obtain

$$[3\ 3\ 2] = [3\ 2\ 3] = [2\ 3\ 3] = -17 - 4\alpha.$$

Clearly, $\alpha$ must be nonnegative, but then we have $[3\ 3\ 2] = [3\ 2\ 3] = [2\ 3\ 3] < 0$, a contradiction. We conclude that the graph $\Xi$ does not exist. □

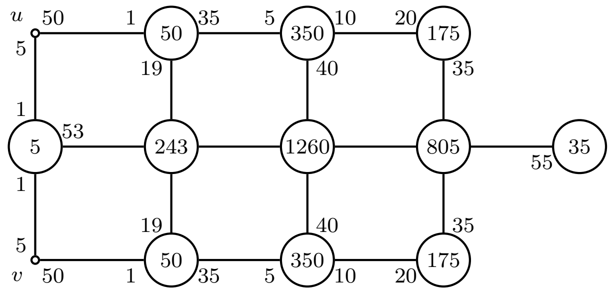


Figure 1: The partition of vertices of $\Sigma$ by distance from a pair of vertices $u, v$ at distance 2. The part that is at distance $i$ from $u$ and distance $j$ from $v$ has size $p^2_{ij}$. As the graph is bipartite, the intersection number $p^2_{ij}$ is nonzero only when $i + j$ is even. Moreover, there are no edges within each part. It is natural to consider [1 1 1] for $w$ at distance 2 from both $u$ and $v$, see Lemma 6.

## 6 A bipartite graph with diameter 5

Let $\Sigma$ be a distance-regular graph with intersection array

$$\{55, 54, 50, 35, 10; 1, 5, 20, 45, 55\}. \tag{7}$$

This intersection array appears in the list of feasible intersection arrays for bipartite non-antipodal distance-regular graphs of diameter 5 by Brouwer et. al. [3, p. 418] as an open case. The existence of such a graph would give a counterexample to a conjecture by MacLean and Terwilliger [15], cf. Lang [14]. The computation needed to obtain the results in this section is illustrated in the `jupyter/DRG-55-54-50-35-10-bipartite.ipynb` notebook included in the `sage-drg` package [21].

The graph $\Sigma$ has diameter 5 and 3500 vertices. The partition of $\Sigma$ corresponding to two vertices at distance 2 is shown in Figure 1. The graph is $Q$-polynomial for the natural ordering of eigenvalues, see for example [3, p. 418]. Moreover, as the graph is bipartite, it is also $Q$-antipodal [3, Thm. 8.2.1]. Many Krein parameters are zero, in particular $q^3_{11}$ and $q^4_{11}$ due to the triangle inequality. We use this fact in the proof of the following statement.

**Lemma 6.** *Let $\Sigma$ be a distance-regular graph with intersection array* (7), *and $u, v, w$ be vertices of $\Sigma$ that are pairwise at distance* 2*. Then* $\begin{bmatrix} u & v & w \\ 1 & 1 & 1 \end{bmatrix} \leqslant 1$.

*Proof.* We consider the triple intersection numbers $[i\ j\ h]$ that correspond to $(u, v, w)$. Since the graph $\Sigma$ is bipartite, we have $[i\ j\ h] = 0$ whenever any of the sums $i + j$, $j + h$, $h + i$ is odd. As $q^3_{11} = q^4_{11} = 0$, Theorem 1 gives us two additional equations to the system (1), thus allowing us to express the solution of the system in terms of a single parameter $\alpha = [1\ 1\ 1]$. In particular, we obtain

$$[5\ 5\ 5] = 20 - 12\alpha.$$

The integrality and nonnegativity of [5 5 5] now gives $\alpha \leqslant \lfloor 5/3 \rfloor = 1$. ∎

**Note.** It can also be shown with a method similar to the one used in Lemma 6 that the graph $[\Sigma_5(u)]_2$ for a vertex $u \in V\Sigma$ (i.e., the graph of vertices at distance 5 from a vertex $u$, with adjacency corresponding to distance 2 in $\Sigma$) is strongly regular with parameters $(v, k, \lambda, \mu) = (210, 99, 48, 45)$. A strongly regular graph with such parameters has been constructed by M. Klin [11].

**Theorem 7.** *A distance-regular graph* $\Sigma$ *with intersection array* (7) *does not exist.*

*Proof.* Let $u$ and $v$ be vertices of $\Sigma$ at distance 2, see Figure 1, and let $\{i\ j\}$ denote the set of vertices at distances $i$ and $j$ from $u$ and $v$, respectively. There are $p^2_{11}(k-2) = 5 \cdot 53 = 265$ edges between the sets $\{1\ 1\}$ and $\{2\ 2\}$. However, the cardinality of the latter set is $p^2_{22} = 243 < 265$, so there is a vertex $w \in \{2\ 2\}$ that has at least two neighbours in $\{1\ 1\}$, i.e., $\begin{bmatrix} u & v & w \\ 1 & 1 & 1 \end{bmatrix} \geqslant 2$, which is in contradiction with Lemma 6. Hence, the graph $\Sigma$ does not exist. □

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## References

[1] N. L. Biggs. Intersection matrices for linear graphs. In D. J. A. Welsh, editor, *Combinatorial Mathematics and its applications*, pages 15–23. Academic Press, London, 1971.

[2] A. E. Brouwer. Parameters of distance-regular graphs, 2011. http://www.win.tue.nl/~aeb/drg/drgtables.html.

[3] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-regular graphs*, volume 18 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1989. doi:10.1007/978-3-642-74341-2.

[4] A. E. Brouwer, A. M. Cohen, and A. Neumaier. Corrections and additions to the book 'Distance-regular graphs', 1994. http://www.win.tue.nl/~aeb/drg/.

[5] P. J. Cameron, J.-M. Goethals, and J. J. Seidel. Strongly regular graphs having strongly regular subconstituents. *J. Algebra*, 55(2):257–280, 1978. doi:10.1016/0021-8693(78)90220-X.

[6] K. Coolsaet and A. Jurišić. Using equality in the Krein conditions to prove nonexistence of certain distance-regular graphs. *J. Combin. Theory Ser. A*, 115(6):1086–1095, 2008. doi:10.1016/j.jcta.2007.12.001.

[7] E. R. van Dam, J. H. Koolen, and H. Tanaka. Distance-regular graphs. *Electron. J. Combin.*, DS:22, 2016. http://www.combinatorics.org/ojs/index.php/eljc/article/view/DS22.

[8] A. Jurišić, J. Koolen, and P. Terwilliger. Tight distance-regular graphs. *J. Algebraic Combin.*, 12(2):163–197, 2000. doi:10.1023/A:1026544111089.

[9] A. Jurišić and J. Vidali. Extremal 1-codes in distance-regular graphs of diameter 3. *Des. Codes Cryptogr.*, 65(1–2):29–47, 2012. doi:10.1007/s10623-012-9651-0.

[10] A. Jurišić and J. Vidali. Restrictions on classical distance-regular graphs. *J. Algebraic Combin.*, 46(3–4):571–588, 2017. doi:10.1007/s10801-017-0765-3.

[11] M. Klin, C. Pech, S. Reichard, A. Woldar, and M. Ziv-Av. Examples of computer experimentation in algebraic combinatorics. *Ars Math. Contemp.*, 3(2):238–248, 2010. http://amc-journal.eu/index.php/amc/article/view/119.

[12] J. H. Koolen. A new condition for distance-regular graphs. *European J. Combin.*, 13(1):63–64, 1992. doi:10.1016/0195-6698(92)90068-B.

[13] E. Lambeck. Some elementary inequalities for distance-regular graphs. *European J. Combin.*, 14(1):53–54, 1993. doi:10.1006/eujc.1993.1008.

[14] M. S. Lang. Bipartite distance-regular graphs: the $Q$-polynomial property and pseudo primitive idempotents. *Discrete Math.*, 331:27–35, 2014. doi:10.1016/j.disc.2014.04.025.

[15] M. S. MacLean and P. Terwilliger. Taut distance-regular graphs and the subconstituent algebra. *Discrete Math.*, 306(15):1694–1721, 2006. doi:10.1016/j.disc.2006.03.046.

[16] Maxima. Maxima, a computer algebra system. version 5.39.0, 2017. http://maxima.sourceforge.net/.

[17] Python Software Foundation. *Python Language Reference, version 2.7.13*, 2017. http://www.python.org.

[18] The Sage Developers. *SageMath, the Sage Mathematics Software System (Version 7.6)*, 2017. http://www.sagemath.org.

[19] M. Urlep. Triple intersection numbers of $Q$-polynomial distance-regular graphs. *European J. Combin.*, 33(6):1246–1252, 2012. doi:10.1016/j.ejc.2012.02.005.

[20] J. Vidali. *Codes in distance-regular graphs*. PhD thesis, University of Ljubljana, 2013. http://eprints.fri.uni-lj.si/2167/ (in Slovene).

[21] J. Vidali. `jaanos/sage-drg`: `sage-drg` Sage package v0.8, 2018. https://github.com/jaanos/sage-drg/, doi:10.5281/zenodo.1418410.

[22] J. S. Williford. Primitive 3-class Q-polynomial association schemes, 2017. Mirrored version available at https://jaanos.github.io/tables/qpoly/qprim3_table.html.

[23] Wolfram Research, Inc. *Mathematica, Version 8.0*. Champaign, Illinois, 2010. http://www.wolfram.com/mathematica/.