λ-Euler's difference table for colored permutations

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Abstract

Motivated by the λ-Euler's difference table of Eriksen et al. and colored Euler's difference table of Faliharimalala and Zeng, we study the λ-analogue of colored Euler's difference table and generalize their results. We generalize the number of permutations with k-excedances studied by Liese and Remmel in colored permutations. We also extend Wang et al.'s recent results about r-derangements by relating with the sequences arising from the difference table.

Mathematics Subject Classifications: 05A18, 05A15

1 Introduction

Euler [4] studied the difference table $(g_n^m)_{0\leqslant m\leqslant n}$, where the coefficients are defined by $g_n^n = n!$ and

$$g_n^m = g_n^{m+1} - g_{n-1}^m, \tag{1}$$

for $0 \leqslant m \leqslant n-1$. Dumont and Randrianarivony [4] studied the combinatorial interpretation of g_n^m in the symmetric group S_n, which consists of permutations of $[n] = \{1, \ldots, n\}$. In particular, they showed that the sequence $\{g_n^0\}_{n\geqslant 0}$ is the number of derangements, i.e., the fixed point free permutations in S_n. Then Rakotondrajao [11] developed further combinatorial interpretations. The reader is referred to [4, 11, 12, 7, 3, 5, 10, 2], where several generalizations of Euler's difference table with combinatorial meanings were studied.

Definition 1. For fixed integer $\ell \geqslant 1$, we define λ-Euler's difference table $(g_{\ell,n}^m(\lambda))_{0\leqslant m\leqslant n}$ for $C_\ell \wr S_n$, where the coefficients are defined by

$$\begin{cases} g_{\ell,n}^n(\lambda) = \ell^n\, n! & (m = n); \\ g_{\ell,n}^m(\lambda) = g_{\ell,n}^{m+1}(\lambda) + (\lambda - 1) g_{\ell,n-1}^m(\lambda) & (0 \leqslant m \leqslant n-1). \end{cases} \tag{2}$$

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From the above definition, it is easy to see the coefficients $g_{\ell,n}^m(\lambda)$ are polynomials in λ. Faliharimalala and Zeng [7] studied the combinatorial interpretation of $g_{\ell,n}^m(0)$ in terms of k-circular successions in $C_\ell \wr S_n$. Eriksen et al. [5] gave a combinatorial interpretation for the coefficients $g_{1,n}^m(\lambda)$ by assuming that λ is a non-negative integer. They showed that $g_{1,n}^m(\lambda)$ count the number of permutations of $[n]$ such that fixed points on the last $n-k$ positions may be colored in any one of λ colors. Liese and Remmel [10] interpreted the coefficients of polynomial $g_{1,n}^m(\lambda)$ by counting certain rook placements in the $[n]\times[n]$ board.

It is not hard to see that the coefficient $g_{\ell,n}^m(\lambda)$ is divisible by $\ell^m m!$. This prompted us to introduce $d_{\ell,n}^m(\lambda) = g_{\ell,n}^m(\lambda)/\ell^m m!$. Then we derive the following allied array $(d_{\ell,n}^m(\lambda))_{0\leqslant m\leqslant n}$ from (1.2).

Definition 2. For a fixed integer $\ell \geqslant 1$, the coefficients of the λ-difference table

$$(d_{\ell,n}^m(\lambda))_{0\leqslant m\leqslant n}$$

are defined by

$$\begin{cases} d_{\ell,n}^n(\lambda) = 1 & (m=n); \\ d_{\ell,n}^m(\lambda) = \ell(m+1)d_{\ell,n}^{m+1}(\lambda) + (\lambda-1)d_{\ell,n-1}^m(\lambda) & (0\leqslant m\leqslant n-1). \end{cases} \tag{3}$$

The first terms of these coefficients for $\ell = 1, 2$ are given in Tables 1 and 2.

$n\backslash m$	0	1	2	3	4
0	1				
1	λ	1			
2	λ^2+1	$\lambda+1$	1		
3	$\lambda^3+3\lambda+2$	$\lambda^2+2\lambda+3$	$\lambda+2$	1	
4	$\lambda^4+6\lambda^2+8\lambda+9$	$\lambda^3+3\lambda^2+9\lambda+11$	$\lambda^2+4\lambda+7$	$\lambda+3$	1

Table 1: Values of $d_{\ell,n}^m(\lambda)$ for $0\leqslant m\leqslant n\leqslant 4$ and $\ell=1$.

$n\backslash m$	0	1	2	3	4
0	1				
1	$\lambda+1$	1			
2	$\lambda^2+2\lambda+5$	$\lambda+3$	1		
3	$\lambda^3+3\lambda^2+15\lambda+29$	$\lambda^2+6\lambda+17$	$\lambda+5$	1	
4	$\lambda^4+4\lambda^3+30\lambda^2+116\lambda+233$	$\lambda^3+9\lambda^2+51\lambda+131$	$\lambda^2+10\lambda+37$	$\lambda+7$	1

Table 2: Values of $d_{\ell,n}^m(\lambda)$ for $0\leqslant m\leqslant n\leqslant 4$ and $\ell=2$.

Two combinatorial interpretations of $d_{\ell,n}^m(0)$ were given in [7]. When λ is a non-negative integer, Eriksen et al. [5] gave a combinatorial interpretation for the coefficients

$d^m_{1,n}(\lambda)$ in the symmetric group. Wang et al.[14] introduced the r-derangement number, which counts the derangements of $[n]$ with the first r elements appear in distinct cycles.

Motivated by [7, 5, 10, 14], we study the combinatorial interpretation of $g^m_{\ell,n}(\lambda)$ and $d^m_{\ell,n}(\lambda)$ in the colored symmetric group $G_{\ell,n}$, i.e., the wreath product of a cyclic group and a symmetric group. The paper is organized as follows. In Sections 3 and 4, we interpret the polynomial $g^m_{\ell,n}(\lambda)$ and the coefficients in $g^m_{\ell,n}(\lambda)$, respectively. In Sections 5 and 6, we prove the linear combinatorial interpretation and cyclic combinatorial interpretation of $d^m_{\ell,n}(\lambda)$, respectively. In Section 7, we obtain the generating functions and recurrence relations of $d^m_{\ell,n}(\lambda)$. In Section 8, we generalize r-derangement number by relating with $d^m_{\ell,n}(\lambda)$. In Section 9, we give a combinatorial proof of recurrence relation of $d^m_{\ell,n}(\lambda)$.

2 Definitions and main results

For positive integers $\ell, n \geqslant 1$, the group of *colored permutations* of n elements with ℓ colors is the wreath product $G_{\ell,n} := C_\ell \wr S_n = C^n_\ell \rtimes S_n$, where C_ℓ is the ℓ-cyclic group generated by $\zeta = e^{2i\pi/\ell}(i^2 = -1)$. From definition, it is obvious to see the elements in $G_{\ell,n}$ are pairs $(\epsilon, \sigma) \in C^n_\ell \times S_n$.

And $G_{\ell,n}$ can also be seen as a permutation group on the colored set:

$$\Sigma_{\ell,n} := C_\ell \times [n] = \{\zeta^j i \mid i \in [n], 0 \leqslant j \leqslant \ell - 1\}.$$

Clearly there are $\ell^n n!$ signed permutations in the group $G_{\ell,n}$. For more details, see [6].

A signed permutation $\pi \in G_{\ell,n}$ can be written in two-line form. For example, if $\pi = (\epsilon, \sigma) \in G_{4,11}$, where $\epsilon = (1, \zeta^3, 1, \zeta, 1, 1, \zeta^2, \zeta, 1, \zeta, 1)$ and

$$\sigma = 7 \quad 5 \quad 3 \quad 1 \quad 2 \quad 6 \quad 8 \quad 9 \quad 4 \quad 10 \quad 11,$$

we write

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \zeta^2 7 & 5 & 3 & 1 & \zeta^3 2 & 6 & \zeta 8 & 9 & \zeta 4 & \zeta 10 & 11 \end{pmatrix}.$$

To be convenient, we write j bars over i instead of $\zeta^j i$. Thus, we rewrite the above permutation in linear form as $\pi = \bar{\bar{7}}\ 5\ 3\ 1\ \bar{\bar{\bar{2}}}\ 6\ \bar{8}\ 9\ \bar{4}\ \overline{10}\ 11$, or in disjoint cyclic form as

$$\pi = (1,\ \bar{\bar{\bar{7}}},\ \bar{8},\ 9,\ \bar{4})\ (\bar{\bar{\bar{2}}},\ 5)\ (3)\ (6)\ (\overline{10})\ (11).$$

That is, when using disjoint cyclic notation to determine the image of a number, we ignore the sign on that number and only consider the sign on the number to which it is mapped. Thus, in the above example, we ignore the sign ζ^2 on the 7 and 7 maps to $\zeta 8$ since the sign on 8 is ζ. Moreover, let $[m+1, n]$ denote the interval {m+1, . . . , n}, and we give the following conventions:

i) If $\pi = (\epsilon, \sigma) \in G_{\ell,n}$, we define $|\pi| = \sigma$ and $\mathrm{sign}_\pi(i) = \epsilon_i$ for $i \in [n]$. For example, if $\pi = 1\,\bar{4}\,3\,\bar{\bar{2}}$ then $\epsilon = (1, \zeta^2, 1, \zeta)$ and $\mathrm{sign}_\pi(4) = \zeta$.

ii) For $i \in [n]$ and $j \in \{0, 1, \ldots, \ell-1\}$, we define $\zeta^j i + k = \zeta^j(i+k)$ for $0 \leqslant k \leqslant n-i$, and $\zeta^j i - k = \zeta^j(i-k)$ for $0 \leqslant k \leqslant i$. For example, $\bar{\bar{2}} + 1 = \bar{\bar{3}}$ in $G_{4,11}$.

iii) We define the total ordering on $\Sigma_{\ell,n}$ as follows. For $i, j \in \{0, \ldots, \ell-1\}$ and $a, b \in [n]$,

$$\zeta^i a < \zeta^j b \Longleftrightarrow i > j \quad \text{or} \quad i = j \text{ and } a < b.$$

In $G_{\ell,n}$, Faliharimalala and Zeng [7] introduced the k-successions as follows.

Definition 3. Given a permutation $\pi \in G_{\ell,n}$ and an integer $0 \leqslant k \leqslant n-1$, $\pi(i)$ is a k-succession at position $i \in [n-k]$ if $\pi(i) = i + k$. In particular, the 0-succession is also called fixed point.

Note that the above k-succession $\pi(i)$ needs to be uncolored, that is, $\mathrm{sign}_\pi(\pi(i)) = 1$.

To obtain the combinatorial interpretation of $g^m_{\ell,n}(\lambda)$, we introduce the following definition.

Definition 4. For any integer $0 \leqslant k \leqslant n-1$, let $SUC_k(\pi)$ denote the set of k-successions in $\pi \in G_{\ell,n}$, i.e.,

$$SUC_k(\pi) = \{\pi(i) | \pi(i) = i + k, i \in [n-k], \pi \in G_{\ell,n}\}.$$

For an integer $0 \leqslant m \leqslant n$, we define the statistic $\mathrm{suc}^{(k)}_{>m}(\pi)$ as the number of k-successions included in $[m+1, n]$ for $\pi \in G_{\ell,n}$, i.e.,

$$\mathrm{suc}^{(k)}_{>m}(\pi) = \#\{\pi(i) \in [m+1, n] | \pi(i) \in SUC_k(\pi)\}.$$

In particular, for $\pi \in G_{\ell,n}$, by taking $k = 0$ and $k = m$, $\mathrm{suc}^{(k)}_{>m}$ is the number of fixed points and m-successions concerning $\pi \in G_{\ell,n}$, respectively, which are included in $[m+1, n]$.

For example, when $\pi \in G_{4,11}$, if

$$\pi = 5\ 3\ 1\ \bar{\bar{\bar{2}}}\ 6\ \bar{8}\ 9\ \bar{4}\ \overline{10}\ 11\ \bar{\bar{\bar{7}}}$$

and

$$\pi' = 3\ 1\ \bar{\bar{\bar{2}}}\ 6\ \bar{8}\ 9\ \bar{4}\ \overline{10}\ 11\ \bar{\bar{\bar{7}}}\ 5,$$

we have $SUC_1(\pi) = SUC_2(\pi') = \{3, 6, 11\}$ and $\mathrm{suc}^{(1)}_{>4}(\pi) = \mathrm{suc}^{(2)}_{>4}(\pi') = 2$.

Theorem 5. *For fixed integers ℓ, k, m and n, let $\ell \geqslant 1$ and $0 \leqslant k \leqslant m \leqslant n$, we have*

$$g^m_{\ell,n}(\lambda) = \sum_{\pi \in G_{\ell,n}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)}. \tag{4}$$

Remark 6. We recover Faliharimalala and Zeng's result [7, Theorem 3] about the combinatorial interpretation of $g^m_{\ell,n}(0)$ in $G_{\ell,n}$. And we prove Theorem 5 in Section 3.

We give an example to illustrate the above theorem. For $\ell = 2, n = 2$ and $m = 1$, the permutations in $G_{2,2}$ are

$$1\ 2,\ \bar{1}\ 2,\ 1\ \bar{2},\ \bar{1}\ \bar{2},\ 2\ 1,\ \bar{2}\ 1,\ 2\ \bar{1},\ \bar{2}\ \bar{1}.$$

For $k = 0$, $\sum_{\pi \in G_{2,2}} \lambda^{\mathrm{suc}^{(0)}_{>1}(\pi)} = 2\lambda + 6$. For $k = 1$, $\sum_{\pi \in G_{2,2}} \lambda^{\mathrm{suc}^{(1)}_{>1}(\pi)} = 2\lambda + 6$.

For $n, m, s \geqslant 0$, Rakotondrajao [12] also studied the number of permutations in S_n having exactly s m-successions. Similarly, we define that $c^m_{\ell,n,s}$ is the number of permutations $\pi \in G_{\ell,n}$ having s m-successions. In other words,

$$c^m_{\ell,n,s} = |\{\pi \in G_{\ell,n}|\ |SUC_m(\pi)| = s\}|, \quad for\ \ n, s, m \geqslant 0.$$

With Theorem 5 and above definition, we state an expression of $g^m_{\ell,n}(\lambda)$ as follows.

Corollary 7. *For $\ell \geqslant 1$, $0 \leqslant m \leqslant n$ and $0 \leqslant s \leqslant n - m$, we have*

$$g^m_{\ell,n}(\lambda) = \sum_{s \geqslant 0} c^m_{\ell,n,s} \lambda^s. \tag{5}$$

Remark 8. With the equations (2) and (5), we obtain that

$$c^{m+1}_{\ell,n,s} = c^m_{\ell,n,s} + c^m_{\ell,n-1,s} - c^m_{\ell,n-1,s-1},$$

which is the result of [7, Theorem 4].

To show the combinatorial interpretations and recursions of $c^m_{\ell,n,s}$, we review the generalized rook theory model in [1].

Let B^ℓ_n be the $n \times \ell n$ array of squares, we label the n columns from left to right by $1, 2, \ldots, n$ and the ℓn rows from bottom to top by

$$1, \zeta 1, \ldots, \zeta^{\ell-1} 1, 2, \zeta 2, \ldots, \zeta^{\ell-1} 2, \ldots, n, \zeta n, \ldots, \zeta^{\ell-1} n,$$

respectively. For instance, the board B^3_n is pictured in Figure 1. The square in the column labeled with i and the row labeled with $\zeta^r j$ is denoted by $(i, \zeta^r j)$. Each such square is called a *cell* and the rows labeled by $j, \zeta j, \ldots, \zeta^{\ell-1} j$ are called *level* j.

Given a board $B \subseteq B^\ell_n$, we let $R^\ell_{k,n}(B)$ denote the set of k element subsets $\mathbb{P}$ of B such that no two elements lie in the same level or column for non-negative integers k. We call the subset $\mathbb{P}$ a placement of non-attacking ℓ-rooks in B. Since the cells in the placement are considered to contain ℓ-rooks, we define the kth *ℓ-rook number* of B by $r^\ell_{k,n}(B) = |R^\ell_{k,n}(B)|$.

Given a permutation $\pi \in G_{\ell,n}$, we can identify π with a placement $\mathbb{P}_\pi$ of n ℓ-rooks in B^ℓ_n. In other word, $\mathbb{P}_\pi = \{(i, \zeta^r j) : \pi(i) = \zeta^r j \text{ for} 1 \leqslant i \leqslant n\}$, then we define the kth *ℓ-hit number* of B denoted by $h^\ell_{k,n}(B)$, which is the number of $\pi \in G_{\ell,n}$ such that the placement $\mathbb{P}_\pi$ intersects the board B in exactly k cells, i.e.,

$$h^\ell_{k,n}(B) = |\{\mathbb{P}_\pi | \pi \in G_{\ell,n} \text{ and} |\mathbb{P}_\pi \cap B| = k\}|.$$

Briggs and Remmel [1, Theorem 1] found the following relationship between the ℓ-hit numbers and the ℓ-rook numbers.

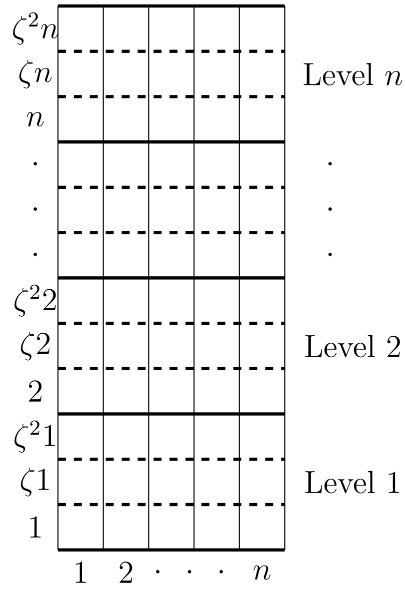

Figure 1: A board B_n^3.

Theorem 9 (Briggs-Remmel)**.** *Let B be a board contained in B_n^ℓ. Then*

$$\sum_{k=0}^{n} h_{k,n}^{\ell}(B)x^k = \sum_{k=0}^{n} r_{k,n}^{\ell}(B)\ell^{n-k}(n-k)!(x-1)^k.$$

By interpreting $c_{\ell,n,s}^m$ in terms of ℓ-hit numbers for a certain board, we obtain the following formula.

Theorem 10. *For $\ell, n \geqslant 1$, $0 \leqslant m \leqslant n$ and $s \geqslant 0$, we have*

$$c_{\ell,n,s}^m = \sum_{t=s}^{n-m} (-1)^{t-s}\ell^{n-t}(n-t)!\binom{t}{s}\binom{n-m}{t}. \tag{6}$$

Remark 11. When $\ell = 1$, (6) reduce to the result of [10, Theorem 2.2]. And we prove Theorem 10 in Section 4.

To make our arguments of interpreting the coefficients $d_{\ell,n}^m(\lambda)$ clear, we only consider the case $k = 0$ in statistics $\mathrm{suc}_{>m}^{(k)}(\pi)$. We define $FIX(\pi) := SUC_0(\pi)$, which denote the set of fixed points in $\pi \in G_{\ell,n}$, i.e., $FIX(\pi) = \{\pi(i) | \pi(i) = i, i \in [n], \pi \in G_{\ell,n}\}$. Define $\mathrm{fix}_{>m}(\pi) := \mathrm{suc}_{>m}^{(0)}(\pi)$, i.e.,

$$\mathrm{fix}_{>m}(\pi) := \#\{\pi(i) \in [m+1, n] | \pi(i) \in FIX(\pi)\}.$$

For example, when $\pi \in G_{4,11}$, if $\pi = \bar{\bar{7}}\ 5\ 3\ 1\ \bar{\bar{\bar{2}}}\ 6\ \bar{8}\ 9\ \bar{4}\ \overline{10}\ 11$, we have $FIX(\pi) = \{3, 6, 11\}$ and $\mathrm{fix}_{>4}(\pi) = 2$.

To give the linear interpretation of $d_{\ell,n}^m(\lambda)$, we give the following definition.

Definition 12. For $0 \leqslant m \leqslant n$, a permutation $\pi = (\epsilon, \sigma) \in G_{\ell,n}$ is called an m-decreasing permutation if satisfies the following conditions:

i) $\mathrm{sign}_\pi(\pi(i)) = 1(i \in [m])$;

ii) $\pi(1) > \pi(2) > \cdots > \pi(m)$.

Let $L^m_{\ell,n}$ be the set of m-decreasing permutations in $G_{\ell,n}$. For example, when $\ell = 2, n = 3$ and $m = 2$,

$$L^2_{2,3} = \{213, 21\bar{3}, 312, 31\bar{2}, 321, 32\bar{1}\}, \quad \textit{and} \quad \sum_{\pi \in L^2_{2,3}} \lambda^{\mathrm{fix}_{>2}(\pi)} = \lambda + 5.$$

Theorem 13. *For $0 \leqslant m \leqslant n$, we have*

$$d^m_{\ell,n}(\lambda) = \sum_{\pi \in L^m_{\ell,n}} \lambda^{\mathrm{fix}_{>m}(\pi)}.$$

Remark 14. When $\lambda = 0$, Theorem 13 reduce to the result of [7, Theorem 10], we prove above theorem in Section 5.

To give the cyclic interpretation of $d^m_{\ell,n}(\lambda)$, we give the following definition.

Definition 15. For $0 \leqslant m \leqslant n$, a permutation $\pi = (\epsilon, \sigma) \in G_{\ell,n}$ is called m-separated permutation if satisfies the following conditions:

i) $\mathrm{sign}_\pi(i) = 1(i \in [m])$;

ii) the first m elements belong into distinct cycles.

Let $C^m_{\ell,n}$ be the set of m-separated permutations in $G_{\ell,n}$. For example, when $\ell = 2, n = 3$ and $m = 2$,

$$C^2_{2,3} = \{(13)(2), (1\bar{3})(2), (1)(23), (1)(2\bar{3}), (1)(2)(3), (1)(2)(\bar{3})\}, \quad \textit{and} \sum_{\pi \in C^2_{2,3}} \lambda^{\mathrm{fix}_{>2}(\pi)} = \lambda + 5.$$

Theorem 16. *For $0 \leqslant m \leqslant n$, we have*

$$d^m_{\ell,n}(\lambda) = \sum_{\pi \in C^m_{\ell,n}} \lambda^{\mathrm{fix}_{>m}(\pi)}.$$

Remark 17. When $\lambda = 0$, Theorem 16 reduce to the result of [7, Theorem 12], we prove above theorem in Section 6.

To generalize the definition of r-derangement number, we give the following definition.

Definition 18. For $0 \leqslant m \leqslant n$, a permutation $\pi \in G_{\ell,n}$ is called m-fixed point-free colored permutation if satisfies the following conditions:

i) For $i \in [m]$, let $\pi(i) \in [m+1, n]$ and $\mathrm{sign}_\pi(i) = \mathrm{sign}_\pi(\pi(i)) = 1$;

ii) no two elements of $[m]$ are in the same cycle.

Let $F^m_{\ell,n+m}$ be the set of m-fixed point-free colored permutations in $G_{\ell,n+2m}$, we define

$$f^m_{\ell,n+m}(\lambda) = \sum_{\pi \in F^m_{\ell,n+m}} \lambda^{\mathrm{fix}_{>m}(\pi)}. \tag{7}$$

For example, when $\ell = 2$, $n = 1$ and $m = 1$,

$$F^1_{2,2} = \{(12)(3), (12)(\bar{3}), (13)(2), (13)(\bar{2}), (123), (12\bar{3}), (132), (13\bar{2})\}$$

and $f^1_{2,2}(\lambda) = 2\lambda + 6$.

Remark 19. When $(\ell, \lambda) = (1, 0)$, the equation (7) reduce to the sum over

$$\{\pi \in F^m_{\ell,n+m} | \, \mathrm{fix}_{>m}(\pi) = 0\},$$

then the polynomial $f^m_{\ell,n+m}(\lambda)$ reduces to the r-derangement number, see [14, Definition 1]. By the above definition, we generalize the generating functions and recurrence relations of Wang et al. [14].

By observing the above definitions, we prove the following combinatorial relation between the $f^m_{\ell,n+m}(\lambda)$ and $d^m_{\ell,n+m}(\lambda)$ in Section 8.

Theorem 20. *For $\ell \geqslant 1$ and $m, n \geqslant 0$, we have*

$$f^m_{\ell,n+m}(\lambda) = \frac{(n+m)!}{n!} d^m_{\ell,n+m}(\lambda). \tag{8}$$

3 Proof of Theorem 5

In the section, to prove Theorem 5, we prove the following equations,

$$\begin{cases} \displaystyle\sum_{\pi \in G_{\ell,n}} \lambda^{\mathrm{suc}^{(k)}_{>n}(\pi)} = \ell^n \, n! & (m = n); \\ \displaystyle\sum_{\pi \in G_{\ell,n}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)} = \sum_{\pi \in G_{\ell,n}} \lambda^{\mathrm{suc}^{(k)}_{>m+1}(\pi)} + (\lambda - 1) \sum_{\pi \in G_{\ell,n-1}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)} & (0 \leqslant m \leqslant n-1). \end{cases} \tag{9}$$

Lemma 21. *For any integer k such that $0 \leqslant k \leqslant m$ and $0 \leqslant m \leqslant n$, there holds*

$$\sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \in SUC_k(\pi)}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)} = \lambda \sum_{\pi \in G_{\ell,n-1}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)}. \tag{10}$$

Proof. Let us define the bijection ψ: $G_{\ell,n} \mapsto G_{\ell,n-1}$, For $\pi \in G_{\ell,n}$, we delete the $m+1$ at position $m+1-k$ and define the $\psi(\pi) = \widehat{\pi}_1\widehat{\pi}_2 \dots \widehat{\pi}_{m-k}\widehat{\pi}_{m-k+2} \dots \widehat{\pi}_{n-1} \in G_{\ell,n-1}$ where

$$\widehat{\pi}_i = \begin{cases} \pi_i, & if \quad |\pi|_i < m+1; \\ \pi_i - 1, & if \quad |\pi|_i > m+1. \end{cases}$$

Conversely, starting from $\psi(\pi) = \widehat{\pi}_1\widehat{\pi}_2 \dots \widehat{\pi}_{m-k}\widehat{\pi}_{m-k+2} \dots \widehat{\pi}_n \in G_{\ell,n-1}$, we define $\pi = \pi_1\pi_2 \dots \pi_n \in G_{\ell,n}$ where

$$\pi_i = \begin{cases} \widehat{\pi}_i, & if \quad |\widehat{\pi}|_i < m+1; \\ \widehat{\pi}_i + 1, & if \quad |\widehat{\pi}|_i \geqslant m+1. \end{cases}$$

Then we put $m+1$ at the position $m+1-k$, from the map, we can easily see $\mathrm{suc}^{(k)}_{>m}(\pi) = \mathrm{suc}^{(k)}_{>m}(\psi(\pi)) + 1$. □

For example $\ell = 4, n = 9, m = 4, k = 1$, $\pi = \overline{\overline{\overline{7}}}\ 3\ 4\ 5\ \overline{\overline{\overline{2}}}\ \overline{1}\ 8\ 9\ \overline{6}$, $\psi(\pi) = \overline{\overline{\overline{6}}}\ 3\ 4\ \overline{\overline{\overline{2}}}\ \overline{1}\ 7\ 8\ \overline{5}$, and $\mathrm{suc}^{(1)}_{>4}(\pi) = \mathrm{suc}^{(1)}_{>4}(\psi(\pi)) + 1$.

Lemma 22. *For $0 \leqslant m \leqslant n$, there holds*

$$\sum_{\pi \in G_{\ell,n-1}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)} = \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \in SUC_k(\pi)}} \lambda^{\mathrm{suc}^{(k)}_{>m+1}(\pi)}. \tag{11}$$

Proof. It follows similar arguments as in the proof of Lemma 21. □

Proof of Theorem 5. First we check the initial condition in (9), when $m = n$, $\mathrm{suc}^{(k)}_{>n}(\pi) = 0$, $\sum\limits_{\pi \in G_{\ell,n}} \lambda^{\mathrm{suc}^{(k)}_{>n}(\pi)} = \ell^n n!$.

We start to prove the recurrence in (9). Then, by considering the following equation,

$$\sum_{\pi \in G_{\ell,n}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)} = \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \notin SUC_k(\pi)}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)} + \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \in SUC_k(\pi)}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)}. \tag{12}$$

Because for $\pi \in G_{\ell,n}$ with $m+1 \notin SUC_k(\pi)$, we have $\mathrm{suc}^{(k)}_{>m}(\pi) = \mathrm{suc}^{(k)}_{>m+1}(\pi)$, then (12) is equivalent to

$$\sum_{\pi \in G_{\ell,n}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)} = \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \notin SUC_k(\pi)}} \lambda^{\mathrm{suc}^{(k)}_{>m+1}(\pi)} + \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \in SUC_k(\pi)}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)}. \tag{13}$$

By equations (10) and (13), we obtain that

$$\sum_{\pi \in G_{\ell,n}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)} = \sum_{\substack{\pi \in G_{\ell,n} \\ m+1 \notin SUC_k(\pi)}} \lambda^{\mathrm{suc}^{(k)}_{>m+1}(\pi)} + \lambda \sum_{\pi \in G_{\ell,n-1}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)}. \tag{14}$$

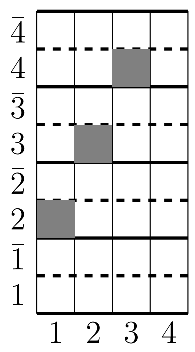

Figure 2: The board $B^2_{4,1}$ corresponds to the shaded cells.

By combining the equations (11) and (14), we obtain

$$\sum_{\pi\in G_{\ell,n}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)} = (\lambda-1)\sum_{\pi\in G_{\ell,n-1}} \lambda^{\mathrm{suc}^{(k)}_{>m}(\pi)} + \sum_{\substack{\pi\in G_{\ell,n}\\ m+1\notin SUC_k(\pi)}} \lambda^{\mathrm{suc}^{(k)}_{>m+1}(\pi)} + \sum_{\substack{\pi\in G_{\ell,n}\\ m+1\in SUC_k(\pi)}} \lambda^{\mathrm{suc}^{(k)}_{>m+1}(\pi)}.$$

With (12), it is easy to see that the above equation is equivalent to the recurrence relation in (9), this completes the proof of Theorem 5. □

Remark 23. Since why $g^m_{\ell,n}(\lambda)$ is independent from k ($0 \leqslant k \leqslant m$) in the above proof is not mentioned, we state an argument as follows. By considering the bijection d which transforms $\pi = \pi_1\pi_2\pi_3\cdots\pi_n$ into $d(\pi) = \pi' = \pi_2\pi_3\cdots\pi_n\pi_1$. It is easy to see that the k-successions of π are in $[m+1,n]$ if and only if the $(k+1)$-successions of π' are in $[m+1,n]$. Hence, let the composition of j times of d is denoted by d^j, the application of $d^{k_2-k_1}$ permits to transfer the k_1-successions to k_2-successions if $k_1 < k_2$. In particular if we apply d^m to a permutation whose fixed points are in $[m+1,n]$, then we obtain a permutation whose m-succession are in $[m+1,n]$ and vice versa.

4 Proof of Theorem 10

In this section, first we prove the following expressions of $c^m_{\ell,n,s}$ in Theorem 10,

$$c^m_{\ell,n,s} = \sum_{t=s}^{n-m}(-1)^{t-s}\ell^{n-t}(n-t)!\binom{t}{s}\binom{n-m}{t}. \tag{15}$$

Then we derive several recurrence relations of $c^m_{\ell,n,s}$.

Proof of Theorem 10. First, we give the combinatorial interpretation of $c^m_{\ell,n,s}$ as follows. Let $B^\ell_{n,m}$ be the board contained in B^ℓ_n consisting of the cells $(1,1+m)$, $(2,2+m)$, $(3,3+m)$, …, $(n-m,n)$. For example, the board $B^2_{4,1}$ is pictured in Figure 2. Then the number of $\pi \in G_{\ell,n}$ with s m-successions is the s-th ℓ-hit number of $B^\ell_{n,m}$, i.e.,

$$c^m_{\ell,n,s} = h^\ell_{s,n}(B^\ell_{n,m}). \tag{16}$$

With the definitions of $B^{\ell}_{n,m}$, we have $r^{\ell}_{s,n}(B^{\ell}_{n,m}) = \binom{n-m}{s}$. By Theorem 9,

$$
\begin{aligned}
g^m_{\ell,n}(\lambda) &= \sum_{s=0}^{n} c^m_{\ell,n,s}\lambda^s = \sum_{s=0}^{n} h^{\ell}_{s,n}(B^{\ell}_{n,m})\lambda^s \\
&= \sum_{s=0}^{n} r^{\ell}_{s,n}(B^{\ell}_{n,m})\ell^{n-s}(n-s)!(\lambda-1)^s \qquad (17)\\
&= \sum_{s=0}^{n} \binom{n-m}{s}\ell^{n-s}(n-s)!(\lambda-1)^s.
\end{aligned}
$$

Equating the coefficients of λ^s yields (15) immediately. □

Remark 24. We also obtain the above expression (17) of $g^m_{\ell,n}(\lambda)$ by generating function, see Proposition 41.

Let $s = n - m$ in (15), we obtain the following corollary.

Corollary 25. *For all $\ell \geqslant 1$ and $n \geqslant m \geqslant 0$,*

$$c^m_{\ell,n,n-m} = \ell^m m!$$

Next we show the recurrence relations of $c^m_{\ell,n,s}$ in colored symmetric group.

Proposition 26. *For all $\ell \geqslant 1, n \geqslant 2$, $0 \leqslant m < n$, and $s \geqslant 1$,*

$$c^m_{\ell,n,s} = (\ell(n-s-1) + (\ell-1))c^m_{\ell,n-1,s} + \ell(s+1)c^m_{\ell,n-1,s+1} + c^m_{\ell,n-1,s-1}. \qquad (18)$$

Proof. Let us consider the map from $\pi = \pi_1 \ldots \pi_{n-1} \in G_{\ell,n-1}$ to $\bar{\pi} \in G_{\ell,n}$ such that $\bar{\pi}$ has s m-successions, we consider the following three cases.

- If $\pi \in G_{\ell,n-1}$ has s m-successions.

  1. Let $\bar{\pi} = \pi_1\pi_{n-m-1}(\zeta^j n)\pi_{n-m+1}\ldots\pi_{n-1}\pi_{n-m}$ and $1 \leqslant j \leqslant \ell - 1$, the number of m-successions of $\bar{\pi}$ and π are the same, so there are $(\ell-1)c^m_{\ell,n-1,s}$ permutations in this case.
  2. Let $\bar{\pi} = \pi_1\pi_{i-1}(\zeta^j n)\pi_{i+1}\ldots\pi_{n-1}\pi_i$, where $i \neq n-m$ and i is a position without m-successions, the number of m-successions of $\bar{\pi}$ and π are the same. Since we have $n-s-1$ choices for position i and $0 \leqslant j \leqslant \ell-1$, there are $\ell(n-s-1)c^m_{\ell,n-1,s}$ permutations in this case.

- If $\pi \in G_{\ell,n-1}$ has $s+1$ m-successions. Let $\bar{\pi} = \pi_1\pi_{i-1}(\zeta^j n)\pi_{i+1}\ldots\pi_{n-1}\pi_i$, where i is a position with m-succession, the number of m-successions of $\bar{\pi}$ is the number of m-successions of π minus one. Since we have $s+1$ choices for position i and $0 \leqslant j \leqslant \ell - 1$, there are $\ell(s+1)c^m_{\ell,n-1,s+1}$ permutations in this case.

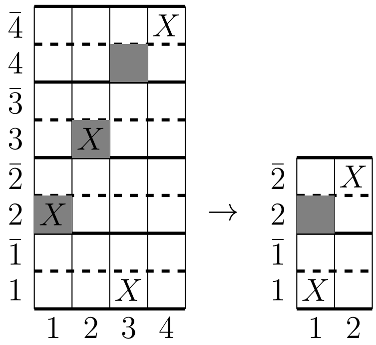

Figure 3: From Board B_4^2 to Board B_2^2.

- If $\pi \in G_{\ell,n-1}$ has $s-1$ m-successions. Let $\bar{\pi} = \pi_1 \pi_{n-m-1}(\zeta^j n)\pi_{n-m+1} \dots \pi_{n-1}\pi_{n-m}$ and $j=0$, the number of m-successions of $\bar{\pi}$ is the number of m-successions of π plus one, so there are $c^m_{\ell,n-1,s-1}$ permutations in this case. ☐

Proposition 27. *For all $\ell, n \geqslant 1$, $0 \leqslant m < n$, and $s \geqslant 0$,*

$$c^m_{\ell,n,s} = \binom{n-m}{s} c^m_{\ell,n-s,0}. \tag{19}$$

Proof. Note that in (16), $c^m_{\ell,n,s}$ is the number of placements of n non-attacking ℓ-rooks in B_n^ℓ that intersect $B^\ell_{n,m}$ in exactly s squares. By removing the level $i+m$ and column i of these ℓ-rooks which lie in the cell $(i, i+m)(1 \leqslant i \leqslant n-m)$, we obtain these placements of $n-s$ non-attacking ℓ-rooks in B^ℓ_{n-s} that intersect $B^\ell_{n-s,m}$ in exactly 0 squares, which is counted by $c^m_{\ell,n-s,0}$. The process is pictured in Figure 3. ☐

Remark 28. Faliharimalala and Zeng [7, Lemma 14] proved the above (19) directly by interpreting $c^m_{\ell,n,s}$ as the number of permutation in $G_{\ell,n}$ with s m-successions. However, we give a trivial proof by interpreting $c^m_{\ell,n,s}$ as the number of placements of n non-attacking ℓ-rooks in B_n^ℓ that intersect $B^\ell_{n,m}$ in exactly s squares.

Proposition 29. *For all $\ell \geqslant 1$, $n \geqslant 2$ and $0 \leqslant m < n$,*

$$c^m_{\ell,n,0} = (\ell n - 1)c^m_{\ell,n-1,0} + \ell(n-m-1)c^m_{\ell,n-2,0}. \tag{20}$$

Proof. Let us consider the map from $\pi = \pi_1 \dots \pi_n \in G_{\ell,n}$ to $\pi' \in G_{\ell,n-1}$, starting from π without m-successions, we define

$$\pi' = \begin{cases} \pi_1 \dots \pi_{i-1}\pi_n\pi_{i+1} \dots \pi_{n-1}, & \text{if } \pi_i = \zeta^j n (0 \leqslant j \leqslant \ell-1) \text{ for } 1 \leqslant i < n; \\ \pi_1 \dots \pi_{n-1}, & \text{if } \pi_n = \zeta^j n (0 \leqslant j \leqslant \ell-1). \end{cases}$$

1. $\pi' \in G_{\ell,n-1}$ has no m-successions.

   Either $\pi_n = \zeta^j n(0 \leqslant j \leqslant \ell - 1)$ or $\pi_i = \zeta^j n(1 \leqslant i < n, 0 \leqslant j \leqslant \ell - 1)$ and $\pi_n \neq i + m$, π' has no m-successions. Conversely, for $\pi' \in G_{\ell,n-1}$ without m-successions, by inserting $\zeta^j n(0 \leqslant j \leqslant \ell - 1)$ into π' in every position except putting n in to position $n - m$, we obtain the permutation in $G_{\ell,n}$ without m-successions. Since $\zeta^j n(0 \leqslant j \leqslant \ell - 1)$ can be in any position except $\pi_{n-m} = n$, there are $(\ell n - 1)c^m_{n-1,0}$ permutations.

2. $\pi' \in G_{\ell,n-1}$ has 1 m-succession.

   When $\pi_i = \zeta^j n(1 \leqslant i \leqslant n - 1 - m, 0 \leqslant j \leqslant \ell - 1)$ and $\pi_n = i + m$, π' has 1 m-succession, then the 1 m-succession of π' corresponds to the ℓ-rook $(i, i + m)$ of the rook placement in the board B^ℓ_{n-1}. For the rook placement corresponds to π' in B^ℓ_{n-1}, removing the column i and level $i + m$ from the board B^ℓ_{n-1}, we obtain the rook placement in B^ℓ_{n-2} without intersecting $B^\ell_{n-2,m}$, which corresponding to the permutation denoted by $\widetilde{\pi} \in G_{\ell,n-2}$, and $\widetilde{\pi}$ has no m-successions.

   Conversely, let $\widetilde{\pi}$ be a permutation in $G_{\ell,n-2}$ without m-successions, we obtain $\pi \in G_{\ell,n}$ in two steps.

   Step 1. For $1 \leqslant i \leqslant n-1-m$, by adding the column i and level $i+m$ to the boards B^ℓ_{n-2}, we choose $(i, i + m)$ as the new ℓ-rook and take the same rook placement corresponds to $\widetilde{\pi}$, then we obtain the rook placement in B^ℓ_{n-1} corresponding to $\pi' \in G_{\ell,n-1}$ with 1 m-succession.

   Step 2. Adding the column n and level n in the board B^ℓ_{n-1}, by taking away the ℓ-rook $(i, i+m)$ and putting ℓ-rooks at $(i, \zeta^j n)(0 \leqslant j \leqslant \ell-1)$ and $(n, i+m)$, we obtain the rook placement without intersecting $B^\ell_{n,m}$, which corresponds to the permutation $\pi \in G_{\ell,n}$ without m-successions. Since $1 \leqslant i \leqslant n - 1 - m$ and $0 \leqslant j \leqslant \ell - 1$, there are $\ell(n - 1 - m)c^m_{\ell,n-2,0}$ permutations. □

Remark 30. When $m = 0$, we define that

$$D^\ell_n := c^0_{\ell,n,0}, \tag{21}$$

which counts the number of derangements in $G_{\ell,n}$. It is easy to see (20) reduce to

$$D^\ell_n = (\ell n - 1)D^\ell_{n-1} + \ell(n - 1)D^\ell_{n-2}.$$

Proposition 31. *For all $\ell \geqslant 1$, $n \geqslant 2$, and $1 \leqslant m < n$,*

$$c^m_{\ell,n,0} = \ell m c^{m-1}_{\ell,n-1,0} + \ell(n - m)c^m_{\ell,n-1,0}. \tag{22}$$

Proof. We prove the above equation by considering level 1 in the rook placement corresponding to the permutation $\pi \in G_{\ell,n}$ without m-successions.

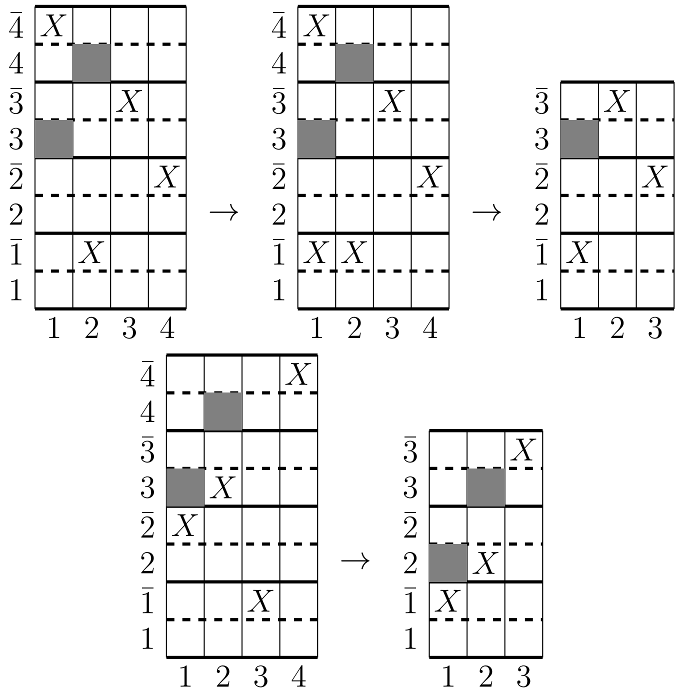

Figure 4: Reducing rook placements from $B^2_{4,2}$ by the rook position in *level* 1.

1. When the ℓ-rook of level 1 is in column $i\,(1 \leqslant i \leqslant n-m)$, if the ℓ-rook of level $i+m$ lies at the position $(k, \zeta^j(i+m))(k \neq i \,\text{and}\, 0 \leqslant j \leqslant \ell-1)$, by adding a ℓ-rook at the position $(k, \zeta^j 1)$ in the level 1, we obtain a placement of $n+1$ ℓ-rooks without intersecting $B^\ell_{n,m}$. Then removing column i and level $i+m$ will result in a placement of $n-1$ non-attacking ℓ-rooks without intersecting $B^\ell_{n-1,m}$. Since the rook placement corresponding to π has ℓ different positions in level 1 and column $1 \leqslant i \leqslant n-m$, thus there are $\ell(n-m)c^m_{\ell,n-1,0}$ placements. The process is illustrated in top of the Figure 4.

2. When the ℓ-rook of level 1 is in column $i(n-m < i \leqslant n)$, by removing column i and level 1, we obtain a placement of $n-1$ non-attacking ℓ-rooks without intersecting $B^\ell_{n-1,m-1}$. Thus there are $\ell m c^{m-1}_{\ell,n-1,0}$ placements. The process is illustrated in the bottom of Figure 4. □

Proposition 32. *For all $\ell, n \geqslant 1$ and $0 \leqslant m < n$,*

$$c^m_{\ell,n,0} = \ell^m m! \sum_{r=0}^{m} \binom{m}{r} \binom{n-m}{m-r} c^{m-r}_{\ell,n-m,0}. \tag{23}$$

Proof. To obtain a placement of n non-attacking ℓ-rooks without intersecting $B^\ell_{n,m}$, starting from the lightly shaded cells in the lower right corner of the board in Figure 5, we see that 0 to m ℓ-rooks can be placed in this area. Suppose that we choose r levels in

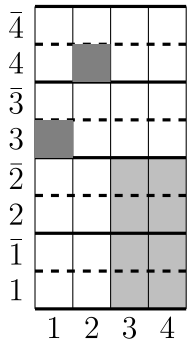

Figure 5: Board $B_{4,2}^{2}$ with some lightly shaded cells.

this area, there are $\binom{m}{r}$ ways. Since there should be m ℓ-rooks in the last m columns, we choose $m-r$ ℓ-rooks above the lower right corner of the board. Thus we choose $m-r$ levels from the $n-m$ levels, there are $\binom{n-m}{m-r}$ ways.

After picking the m levels that contain the ℓ-rooks in the last m columns, there are $\ell^m m!$ ways to place the ℓ-rooks in the last m columns.

Let P denote the non-attacking rook placement in the last m columns with r ℓ-rooks falling in the lightly shaded area, we extend P to a non-attacking rook placement Q with n ℓ-rooks, where there is no intersection with $B_{n,m}^{\ell}$. By removing the levels and columns of rook placement P, we obtain the non-attacking rook placement without intersecting $B_{n-m,m-r}^{\ell}$, which is counted by $c_{\ell,n-m,0}^{m-r}$. Summing over all possible values of r yields the desired result. □

Proposition 33. *For all $\ell \geqslant 1$, $n \geqslant 2$ and $0 \leqslant m < n$,*

$$c_{\ell,n,0}^{m} = \ell c_{\ell,n,1}^{m+1} + (\ell m + \ell - 1)c_{\ell,n-1,0}^{m}. \tag{24}$$

Proof. Let us consider the rook position of level n in the rook placement which corresponding to the permutation $\pi \in G_{\ell,n}$ without m-successions.

1. When the ℓ-rook of level n is in column $i(1 \leqslant i \leqslant n-m-1)$. If the ℓ-rook of column i is in row n, we keep it unchanged. If the ℓ-rook of column i is in row $\zeta^j n(1 \leqslant j \leqslant \ell - 1)$, we exchange the row $\zeta^j n$ with row n. Then we move the level n to the bottom level of the board, which is denoted by level $1'$, other levels are increased by one such as level $2'$, $\ldots$, level n'. By exchanging the level $1'$ and level $(i+m+1)'$, we obtain a non-attacking rook placement that intersect $B_{n,m+1}^{\ell}$ one rook $(i, i+m+1)$. Since the ℓ-rook can be in the row $\zeta^j n(0 \leqslant j \leqslant \ell-1)$, there are $\ell c_{n,1}^{m+1}$ permutations in this case. This process is shown in top of Figure 6.

2. When the ℓ-rook of level n is in column $i(n-m \leqslant i \leqslant n)$, the ℓ-rook can be in the position $(i, \zeta^j n)(n-m \leqslant i \leqslant n, 0 \leqslant j \leqslant \ell-1)$, since π has no m-successions, the ℓ-rook can not be in the square $(n-m, n)$, so there are $\ell m + \ell - 1$ choices in level n.

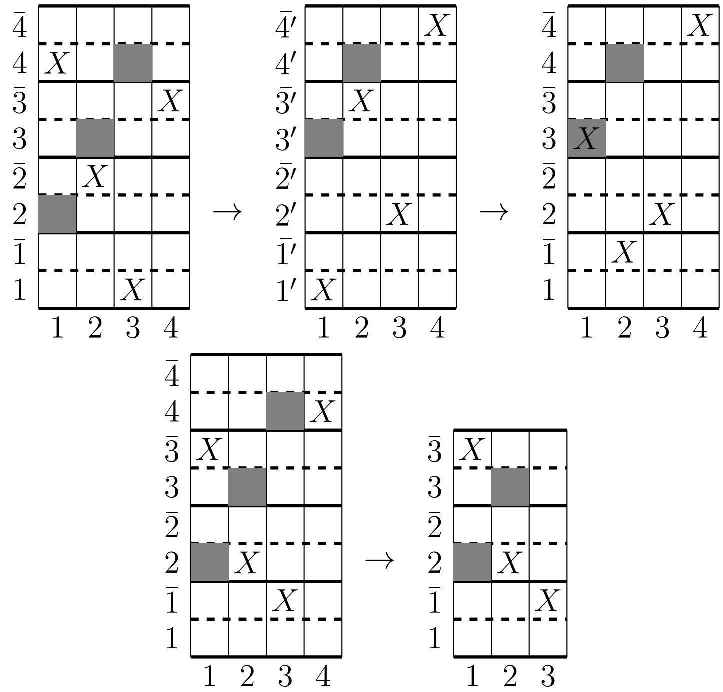

Figure 6: Reducing rook placements from $B^2_{4,1}$ by the rook position in top *level*.

Removing the level n and column i will result in a non-attacking rook placement without intersecting $B^{\ell}_{n-1,m}$. Then there are $(\ell m+\ell-1)B^{\ell}_{n-1,m}$ permutations in this kind. This process is shown in bottom of Figure 6. □

Proposition 34. *For $\ell \geqslant 1, n \geqslant 2$ and $1 \leqslant m < n$,*

$$c^{m}_{\ell,n,0} = c^{m-1}_{\ell,n,0} + c^{m-1}_{\ell,n-1,0}. \tag{25}$$

Proof. Let us consider the non-attacking rook placement corresponding to the permutation $\pi \in G_{\ell,n}$ without m-successions. We move the bottom level to the top level and all other levels reduced by one, which is shown in Figure 7.

1. When the bottom ℓ-rook is not in the position $(n-m+1,1)$, the process is shown in the top of Figure 7. After the movement of ℓ-rooks in the board, we obtain the non-attacking rook placement without intersecting $B^{\ell}_{n,m-1}$. Thus there are $c^{m-1}_{\ell,n,0}$ permutations in this case.

2. When the bottom ℓ-rook is in the position $(n-m+1,1)$, the process is shown in the bottom of Figure 7. The resulting rook placement intersect $B^{\ell}_{n,m-1}$ in the position $(n-m+1,n)$. By removing the column $n-m+1$ and level n, we get the non-attacking rook placement without intersecting $B^{\ell}_{n-1,m-1}$. Thus there are $c^{m-1}_{\ell,n-1,0}$ permutations in this case. □

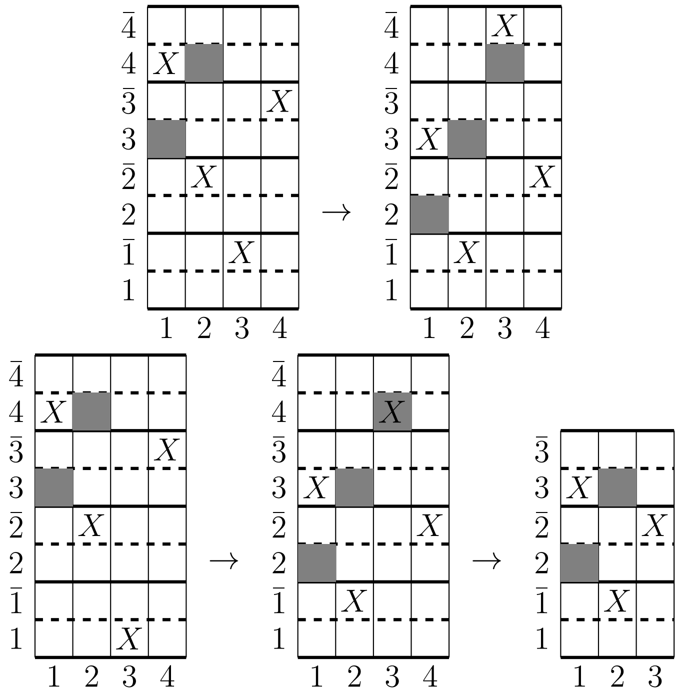

Figure 7: Moving the bottom level to top level in $B^2_{4,2}$.

Proposition 35. *For* $\ell, n \geqslant 1$ *and* $0 \leqslant m < n$,

$$c^m_{\ell,n,0} = \sum_{r=0}^{m} \binom{m}{r} D^\ell_{n-m+r}. \tag{26}$$

Proof. We prove this theorem by inductions on m. If $m = 0$, we have $c^0_{\ell,n,0} = D^\ell_n$ by equation (21). Suppose that $c^i_{\ell,n,0} = \sum_{r=0}^{i} \binom{i}{r} D^\ell_{n-i+r}$ is satisfied for $i \leqslant m-1$, then

$$c^{m-1}_{\ell,n,0} + c^{m-1}_{\ell,n-1,0} = \sum_{r=0}^{m-1} \binom{m-1}{r} D^\ell_{n-m+1+r} + \sum_{r=0}^{m-1} \binom{m-1}{r} D^\ell_{n-m+r}. \tag{27}$$

By separating out the $m-1$ term of the first sum and the 0 term of the second sum in (27), which is equivalent to

$$\binom{m-1}{m-1} D^\ell_n + \sum_{r=0}^{m-2} \binom{m-1}{r} D^\ell_{n-m+1+r} + \binom{m-1}{0} D^\ell_{n-m} + \sum_{r=1}^{m-1} \binom{m-1}{r} D^\ell_{n-m+r}.$$

By transforming r to $r-1$ in the first sum and using $\binom{m-1}{r-1} + \binom{m-1}{r} = \binom{m}{r}$, we have

$$c^{m-1}_{\ell,n,0} + c^{m-1}_{\ell,n-1,0} = \sum_{r=0}^{m} \binom{m}{r} D^\ell_{n-m+r}.$$

With the recurrence (25),

$$c^m_{\ell,n,0} = c^{m-1}_{\ell,n,0} + c^{m-1}_{\ell,n-1,0},$$

we obtain

$$c^m_{\ell,n,0} = \sum_{r=0}^{m} \binom{m}{r} D^{\ell}_{n-m+r}.$$

The proof is thus completed. □

With equations (19) and (26), we give the relation between $c^m_{\ell,n,s}$ and D^{ℓ}_n directly.

Corollary 36. *For $\ell, n \geqslant 1$, $0 \leqslant m < n$ and $s \geqslant 0$,*

$$c^m_{\ell,n,s} = \binom{n-m}{s} \sum_{r=0}^{m} \binom{m}{r} D^{\ell}_{n-s-m+r}.$$

By observing the coefficients of polynomial $g^m_{\ell,n}(\lambda)$, we find $c^m_{\ell,n,s}$ decreases as s increases.

Proposition 37. *For $\ell, n \geqslant 1$, $1 \leqslant m < n$ and $s \geqslant 1$,*

$$c^m_{\ell,n,s-1} \geqslant c^m_{\ell,n,s}. \tag{28}$$

Proof. With recursion (19) and (22), we have

$$\begin{aligned} &c^m_{\ell,n,s-1} - c^m_{\ell,n,s} \\ =&\binom{n-m}{s-1} c^m_{\ell,n-s+1,0} - \binom{n-m}{s} c^m_{\ell,n-s,0} \\ =&(\ell(n-m-s+1)\binom{n-m}{s-1} - \binom{n-m}{s})c^m_{\ell,n-s,0} + \ell m \binom{n-m}{s-1} c^{m-1}_{\ell,n-s,0}. \end{aligned}$$

Since

$$\ell(n-m-s+1)\binom{n-m}{s-1} - \binom{n-m}{s} = \frac{(n-m)!}{(n-m-s)!s!}(\ell s - 1),$$

we obtain (28) immediately. □

Remark 38. In particular, when $m = 0$, we have the similar result for $c^0_{\ell,n,s}$. By using $D^{\ell}_n = \ell n D^{\ell}_{n-1} + (-1)^n$ [7, equation (2.8)] and similar arguments above, we have $c^0_{\ell,n,s-1} \geqslant c^0_{\ell,n,s}$ for $2 \leqslant s \leqslant n-2$ and $n \geqslant 3$.

Remark 39. When $\ell = 1$, the above expressions and relations of $c^m_{\ell,n,s}$ in this section reduce to Liese and Remmel's results [10, Sections 2 and 3].

5 Proof of Theorem 13

In this section, to prove Theorem 13, we prove the following equations,

$$\sum_{\pi\in L_{\ell,n}^{m}} \lambda^{\mathrm{fix}_{>m}(\pi)} = \sum_{\pi\in G_{\ell,n}} \lambda^{\mathrm{fix}_{>m}(\pi)}/\ell^m m!, \quad \text{for } 0 \leqslant m \leqslant n. \tag{29}$$

Proof of Theorem 13. For $1 \leqslant k \leqslant n$, if $\pi = \pi(1)\pi(2)\dots\pi(k-1)\pi(k)\pi(k+1)\dots\pi(n) \in G_{\ell,n}$, let $T(\pi)$ be the vector that record the numbers of the last $n-k$ positions in π, i.e., $T(\pi) = (\pi(k+1), \pi(k+2), \dots, \pi(n))$. For example, if $n = 12$, $k = 4$, $\pi = \bar{9}\ 5\ \bar{\bar{4}}\ 1\ 3\ \bar{\bar{\bar{8}}}\ 2\ 6\ \bar{7}\ 10\ \overline{12}\ 11 \in G_{4,12}$, then $T(\pi) = (3, \bar{\bar{\bar{8}}}, \dots, 11)$. We define the relation $\sim$ on $G_{\ell,n}$ by

$$\pi \sim \pi' \Leftrightarrow T(\pi) = T(\pi'),$$

it is easy to see this is an equivalence relation. Let us consider the map $\delta : (\eta, \pi) \to \delta(\eta, \pi)$ from $G_{\ell,m} \times G_{\ell,n}$ to $G_{\ell,n}$, where $G_{\ell,m}$ can be seen as a permutation group of colored set $C_\ell \times \{|\pi|(1), |\pi|(2), \dots, |\pi|(m)\}$. Define the permutation $\delta(\eta, \pi)$ such that $\delta(\eta, \pi)(i) = \eta(i)(i \leqslant m)$, and $\delta(\eta, \pi)(i) = \pi(i)(i > m)$. For example, if $\pi = \bar{9}\ 5\ \bar{\bar{4}}\ 1\ 3\ \bar{\bar{\bar{8}}}\ 2\ 6\ \bar{7}\ 10\ \overline{12}\ 11 \in G_{4,12}$, and $\eta = 5\ \bar{4}\ 1\ \bar{9} \in G_{4,4'}$, then

$$\delta(\eta, \pi) = 5\ \bar{4}\ 1\ \bar{9}\ 3\ \bar{\bar{8}}\ 2\ 6\ \bar{7}\ 10\ \overline{12}\ 11.$$

So the equivalence class of $\pi \in G_{\ell,n}$ is $\{\delta(\eta, \pi) | \eta \in G_{\ell,m}\}$, it's easy to see the cardinality of each equivalence class is $\ell^m m!$, choosing the representative of the equivalence class $\delta(\iota, \pi)$ such that

$$\mathrm{sign}(|\iota|(i)) = 1 \quad \textit{and} \quad \iota(1) > \iota(2) \cdots > \iota(m).$$

Since the fix points of π and $\delta(\iota, \pi)$ on $[m+1, n]$ keep unaltered. By Theorem 5, we obtain that the number of equivalence class is $g_{\ell,n}^{m}(\lambda)/\ell^m m!$, which yields the equation (29). ☐

Remark 40. As in the proof of Theorem 5, we can also prove the recurrence relations of (3) by constructing bijections directly, the proof is left to the interested reader.

6 Proof of Theorem 16

In this section, we give two proofs of Theorem 16. In the first proof, we give a bijection from $C_{\ell,n}^{m}$ to $L_{\ell,n}^{m}$, that is,

$$\sum_{\pi\in L_{\ell,n}^{m}} \lambda^{\mathrm{fix}_{>m}(\pi)} = \sum_{\pi\in C_{\ell,n}^{m}} \lambda^{\mathrm{fix}_{>m}(\pi)} \quad \text{for } 0 \leqslant m \leqslant n. \tag{30}$$

In the second proof, we prove this cyclic result by constructing a equivalence relation on $G_{\ell,n}$, that is,

$$\sum_{\pi\in C_{\ell,n}^{m}} \lambda^{\mathrm{fix}_{>m}(\pi)} = \sum_{\pi\in G_{\ell,n}} \lambda^{\mathrm{fix}_{>m}(\pi)}/\ell^m m!, \quad \text{for } 0 \leqslant m \leqslant n. \tag{31}$$

6.1 First Proof

we will give a bijection $\rho : \pi \to \pi'$ from $C^m_{\ell,n}$ to $L^m_{\ell,n}$ such that $\mathrm{fix}_{>m}(\pi) = \mathrm{fix}_{>m}(\pi')$. First we give the map $|\pi| \to' |\pi'|$ and then construct the sign transformation.

- Let $|\pi'| = |\pi'|(1), \ldots, |\pi'|(m), \ldots, |\pi'(n)|$, where $|\pi'|(1), \ldots, |\pi'|(m)$ in decreasing rearrangement of $|\pi|(1), \ldots, |\pi|(m)$ and $|\pi'|(m+i) = |\pi|(m+i)(1 \leqslant i \leqslant n-m)$. Conversely, we give the reverse map by $\pi' \to \pi$ from $L^m_{\ell,n}$ to $C^m_{\ell,n}$. For $\pi' \in L^m_{\ell,n}$, we define $P := \{|\pi'|(i), i \in [m]\}$. let

  $$(|\pi'|^{-s}(i), \ldots, |\pi'|^{-2}(i), |\pi'|^{-1}(i), i)$$

  be the cycle of $|\pi|$ containing $i(i \in [m])$, where s is the least non-negative number such that $|\pi'|^{-s}(i) \in P$ and if $|\pi'|(j) = i(j \in [n])$, then $|\pi'|^{-1}(i) := j$. And setting $|\pi'|^0(i) = i$, that is, if $i \in P \cap [m]$, then $s = 0$, and i is a fixed point of $|\pi|$. The other cycles keep in accordance with $|\pi'|$.

- We define the sign transformation as follows. Since each element $i \in [m]$ in π and $\pi(i)(i \in [m])$ in π' are uncolored, we exchange the sign of $|\pi|(i) \in [m]$ in π and $i \in [m]$ in π'. In other words,

  $$\mathrm{sign}_\pi(i) = \mathrm{sign}_{\pi'}(|\pi|(i)) = 1 \quad \textit{and} \quad \mathrm{sign}_\pi(|\pi|(i)) = \mathrm{sign}_{\pi'}(i), \quad i \in [m].$$

  The signs of other elements remain unchanged, i.e.,

  $$\mathrm{sign}_{\pi'}(i) = \mathrm{sign}_\pi(i), \quad i \notin [m] \cup \{|\pi|(i) | i \in [m]\}.$$

For example: For $\ell = 4, n = 12, m = 4$, $\pi = (1\ \bar{\bar{9}})\ (2\ \bar{7})\ (3\ 5)\ (4)\ (6\ \bar{8})\ (10)\ (11\ \bar{\bar{\bar{12}}}) \in C^4_{4,12}$,

$$\mathrm{sign}_{\pi'}(1) = \mathrm{sign}_\pi(|\pi|(1)) = \zeta^2, \mathrm{sign}_{\pi'}(2) = \mathrm{sign}_\pi(|\pi|(2)) = \zeta,$$

we have

$$\pi' = 9\ 7\ 5\ 4\ 3\ \bar{8}\ \bar{2}\ 6\ \bar{\bar{1}}\ 10\ \bar{\bar{\bar{12}}}\ 11 \in L^4_{4,12} \quad \textit{and} \quad \mathrm{fix}_{>4}(\pi) = \mathrm{fix}_{>4}(\pi') = 1.$$

6.2 Second Proof

We decompose a permutation $\pi \in G_{\ell,n}$ as a product of disjoint cycles. For each $i \in [m]$, we define $\omega_\pi(i) = \pi(i)\pi^2(i) \ldots \pi^{s-1}(i)$ where $s \geqslant 1$ is the least integer such that $|\pi|^s(i) \in [m]$. Obviously $\omega_\pi(i) = \varnothing$ if $s = 1$. Let $\Omega(\pi)$ be the product of cycles of π which have no common elements with $\{\zeta^j i | i \in [m], 0 \leqslant j \leqslant \ell - 1\}$, let $\pi_m \in G_{\ell,m}$ be the permutation obtained from π by deleting elements in $\omega_\pi(i)$ and the cycles in $\Omega(\pi)$ for $i \in [m]$.

For example, if $\ell = 4, n = 12, m = 4$ and $\pi = (\bar{1}\ 9\ \bar{7}\ 2\ 5\ \bar{\bar{3}}\ 4)\ (6\ \bar{8})\ (10)\ (11\ \bar{\bar{\bar{12}}})$, then $\pi_4 = (\bar{1}2\bar{\bar{3}}4)$ and

$$\omega_\pi(1) = 9\bar{7}, \quad \omega_\pi(2) = 5, \quad \omega_\pi(3) = \varnothing, \quad \omega_\pi(4) = \varnothing, \quad \textit{and} \quad \Omega(\pi) = (6\ \bar{8})\ (10)\ (11\ \bar{\bar{\bar{12}}}).$$

Setting $E(\pi) = (\omega_\pi(1), \omega_\pi(2), \cdots, \omega_\pi(k), \Omega(\pi))$, we define the relation $\sim$ on $G_{\ell,n}$ by

$$\pi_1 \sim \pi_2 \Leftrightarrow E(\pi_1) = E(\pi_2),$$

it is easy to see that this is an equivalence relation. Then we define the mapping $\theta : (\tau, \pi) \mapsto \theta(\tau, \pi)$ from $G_{\ell,m} \times G_{\ell,n}$ to $G_{\ell,n}$. We obtain the permutation $\theta(\tau, \pi)$ by inserting the elements $\omega_i(\pi)$ after the elements $\zeta^j i (i \in [m], 0 \leqslant j \leqslant \ell - 1)$ of τ and adding the cycles of $\Omega(\pi)$.

For example, if $\pi = (\bar{1}\ 9\ \bar{\bar{7}}\ 2\ 5\ \bar{\bar{3}}\ 4)\ (6\ \bar{8})\ (10)\ (11\ \overline{\overline{\overline{12}}})$ and $\tau = (1\ \bar{2})\ (\bar{3})\ (4)$ then

$$\theta(\tau, \pi) = (1\ 9\ \bar{\bar{7}}\ \bar{2}\ 5)\ (\bar{3})\ (4)\ (6\ \bar{8})\ (10)\ (11\ \overline{\overline{\overline{12}}}).$$

Obviously $\{\theta(\tau, \pi) | \tau \in G_{\ell,m}\}$ is the equivalence class of $\pi \in G_{\ell,n}$. From the construction of $\theta(\tau, \pi)$, for $\tau \in G_{\ell,m}$ and $\pi \in G^m_{\ell,n}$, we have $\theta(\tau, \pi) \sim \pi$. Conversely, if $\pi' \sim \pi$, then $\pi' = \theta(\pi'_m, \pi)$, and if $\theta(\tau, \pi) = \theta(\tau', \pi) = \pi'$ for $\tau, \tau' \in G_{\ell,m}$, then $\tau = \tau' = \pi'_m$. Hence the cardinality of each equivalence class is $\ell^m m!$. Let η be the identity permutation of $G_{\ell,m}$, then we choose $\theta(\eta, \pi)$ as the representative of each equivalence class $\{\theta(\tau, \pi) | \tau \in G_{\ell,m}\}$, that is, $\theta(\eta, \pi)$ represents the the permutation $\pi \in G_{\ell,n}$ where $\mathrm{sign}_\pi(i) = 1 (i \in [m])$ with the first m elements belong into distinct cycles. It is obvious to see $\mathrm{fix}_{>m}(\pi) = \mathrm{fix}_{>m}(\theta(\eta, \pi))$. By Theorem 5, the number of equivalence classes is $g^m_{\ell,n}(\lambda)/\ell^m m!$, as desired.

7 Generating functions and further recurrence relations

In this section, by using the recurrence relation (2), we obtain the generating functions and further recurrence relations of $g^m_{\ell,n}(\lambda)$ and $d^m_{\ell,n}(\lambda)$.

Proposition 41. *For $m \geqslant 0$ we have the following identities:*

$$g^m_{\ell,n+m}(\lambda) = \sum_{i=0}^{n} (\lambda - 1)^{n-i} \binom{n}{i} \ell^{m+i} (m+i)!, \tag{32}$$

$$\sum_{n \geqslant 0} g^m_{\ell,n+m}(\lambda) \frac{u^n}{n!} = \frac{\ell^m m! \exp((\lambda - 1)u)}{(1 - \ell u)^{m+1}}, \tag{33}$$

$$\sum_{m,n \geqslant 0} g^m_{\ell,n+m}(\lambda) \frac{x^m}{m!} \frac{u^n}{n!} = \frac{\exp((\lambda - 1)u)}{1 - \ell x - \ell u}. \tag{34}$$

Proof. For any function $f(k)(k \geqslant 0)$: $\mathbb{Z}[\lambda] \to \mathbb{C}[\lambda]$, we define the operator $\Delta f(n)(\lambda) = f(n)(\lambda) + (\lambda - 1) f(n-1)(\lambda)$. By inductions on $N \geqslant 0$, we have

$$\Delta^N f(n)(\lambda) = \sum_{i=0}^{N} (\lambda - 1)^i \binom{N}{i} f(n-i)(\lambda) = \sum_{i=0}^{N} (\lambda - 1)^{N-i} \binom{N}{i} f(n - N + i)(\lambda). \tag{35}$$

If $f(n)(\lambda) = g^n_{\ell,n}(\lambda)$, thus $g^{n+m-i}_{\ell,n+m}(\lambda) = \Delta^i f(n+m)(\lambda)$ for $i \geqslant 0$. From (35), we obtain

$$g^m_{\ell,n+m}(\lambda) = \Delta^n f(n+m)(\lambda) = \sum_{i=0}^{n} (\lambda-1)^{n-i} \binom{n}{i} \ell^{m+i}(m+i)!. \tag{36}$$

For the above identity, multiplying both sides by $u^n/n!$ and summing over $n \geqslant 0$, we obtain

$$\sum_{n\geqslant 0} g^m_{\ell,n+m}(\lambda)\frac{u^n}{n!} = \ell^m m! \sum_{n,i\geqslant 0} (\lambda-1)^{n-i}\binom{m+i}{i}\frac{\ell^i u^n}{(n-i)!}.$$

By shifting n to $n+i$, we have

$$\sum_{n\geqslant 0} g^m_{\ell,n+m}(\lambda)\frac{u^n}{n!} = \ell^m m! \left(\sum_{n\geqslant 0}(\lambda-1)^n\frac{u^n}{n!}\right)\cdot\left(\sum_{i\geqslant 0}\binom{m+i}{i}(\ell u)^i\right).$$

Clearly the above equation implies (33) immediately. Finally multiplying both sides of (33) by $x^m/m!$ and summing over $m \geqslant 0$ yields (34). □

Remark 42. Setting $m = 0$ in (32), we obtain

$$d^0_{\ell,n}(\lambda) = g^0_{\ell,n}(\lambda) = n! \sum_{i=0}^{n} \frac{(\lambda-1)^i}{i!}\ell^{n-i}, \tag{37}$$

which implies immediately the following recurrence relation,

$$d^0_{\ell,n}(\lambda) = \ell n d^0_{\ell,n-1}(\lambda) + (\lambda-1)^n \qquad (n \geqslant 1). \tag{38}$$

Proposition 43. *For $\ell \geqslant 1$ and $0 \leqslant m \leqslant n-2$ we have*

$$g^m_{\ell,n}(\lambda) = (\ell n + \lambda - 1)g^m_{\ell,n-1}(\lambda) - \ell(n-m-1)(\lambda-1)g^m_{\ell,n-2}(\lambda) \qquad (n \geqslant 2); \tag{39}$$

$$g^m_{\ell,n}(\lambda) = \ell(n-m)g^m_{\ell,n-1}(\lambda) + \ell m g^{m-1}_{\ell,n-1}(\lambda) \qquad (m \geqslant 1,\ n \geqslant 1); \tag{40}$$

$$g^m_{\ell,n}(\lambda) = \ell n g^m_{\ell,n-1}(\lambda) + \ell m(\lambda-1)g^{m-1}_{\ell,n-2}(\lambda) \qquad (m \geqslant 1,\ n \geqslant 2), \tag{41}$$

where $g^0_{\ell,0}(\lambda) = 1$, $g^0_{\ell,1}(\lambda) = \lambda + \ell - 1$ and $g^1_{\ell,1}(\lambda) = \ell$.

Proof. Let $F(u)$ denote the left-hand side of (33). By using the differentiation of $F(u)$ and (33), we obtain

$$(1-\ell u)F'(u) = [\ell(m+1) + (\lambda-1)(1-\ell u)]F(u). \tag{42}$$

By equating the coefficients of $u^n/n!$ in (42), we have

$$g^m_{\ell,n+m+1}(\lambda) = [\ell(m+n+1) + \lambda - 1]g^m_{\ell,n+m}(\lambda) - \ell n(\lambda-1)g^m_{\ell,n+m-1}(\lambda),$$

shifting $n+m+1$ to n yields (39) immediately.

Then multiplying both sides of (33) by $1-\ell u$, we have

$$(1-\ell u)\sum_{n\geqslant 0} g^m_{\ell,n+m}(\lambda)\frac{u^n}{n!} = \frac{\ell^m m!\exp((\lambda-1)u)}{(1-\ell u)^m} = \ell m\sum_{n\geqslant 0} g^{m-1}_{\ell,n+m-1}(\lambda)\frac{u^n}{n!}. \tag{43}$$

By equating the coefficients of $u^n/n!$, we have

$$g^m_{\ell,n+m}(\lambda) - \ell n g^m_{\ell,n+m-1}(\lambda) = \ell m g^{m-1}_{\ell,n+m-1}(\lambda), \tag{44}$$

shifting $n+m$ to n yields (40).

Finally, from (40) and (2), we have

$$\begin{aligned} g^m_{\ell,n}(\lambda) &= \ell n g^m_{\ell,n-1}(\lambda) - \ell m(g^m_{\ell,n-1}(\lambda) - g^{m-1}_{\ell,n-1}(\lambda)) \\ &= \ell n g^m_{\ell,n-1}(\lambda) + \ell m(\lambda-1) g^{m-1}_{\ell,n-2}(\lambda), \end{aligned}$$

which yields (41), the proof is completed. □

With the above Proposition 43, we derive the following propositions immediately.

Proposition 44. *For $\ell \geqslant 1$ and $0 \leqslant m \leqslant n-2$ we have*

$$d^m_{\ell,n}(\lambda) = (\ell n+\lambda-1)d^m_{\ell,n-1}(\lambda) - \ell(n-m-1)(\lambda-1)d^m_{\ell,n-2}(\lambda) \qquad (n\geqslant 2); \tag{45}$$

$$d^m_{\ell,n}(\lambda) = \ell(n-m)d^m_{\ell,n-1}(\lambda) + d^{m-1}_{\ell,n-1}(\lambda) \qquad (m\geqslant 1,\ n\geqslant 1); \tag{46}$$

$$d^m_{\ell,n}(\lambda) - (\lambda-1)d^{m-1}_{\ell,n-2}(\lambda) = \ell n d^m_{\ell,n-1}(\lambda) \qquad (m\geqslant 1,\ n\geqslant 2), \tag{47}$$

where $d^0_{\ell,0}(\lambda)=1$, $d^0_{\ell,1}(\lambda)=\lambda+\ell-1$ and $d^1_{\ell,1}(\lambda)=1$.

Proof. With Proposition 43, we can get these equations (45), (46) and (47) directly. □

Remark 45. Setting $\ell=1$, (3), (45), and (47) reduce to the result of Eriksen et al. [5, Propositions 8.1, 8.3 and 8.2]. In this case, (33) and (34) recover the result of Rakotondrajao [11, Theorem 6.7 and Theorem 6.8]. Setting $\lambda=0$, Propositions 41, 43 and 44 reduce to the result of Faliharimalala and Zeng [7, Propositions 17, 18, and 19].

8 Proof of Theorem 20

In this section, to prove Theorem 20, we prove the following equation,

$$\sum_{\pi\in F^m_{\ell,n+m}} \lambda^{\mathrm{fix}_{>m}(\pi)} = \frac{(n+m)!}{n!}\sum_{\pi\in C^m_{\ell,n+m}} \lambda^{\mathrm{fix}_{>m}(\pi)} \text{ for } m,n\geqslant 0. \tag{48}$$

And with the generating functions of $d^m_{\ell,n}(\lambda)$, we obtain the generating functions and recurrence relations of $f^m_{\ell,n}(\lambda)$.

Proof of Theorem 20. For $\ell \geqslant 1$, we construct such a permutation $\pi \in F^m_{\ell,n+m}$ in following way, see Definition 18.

Starting from the set $[n+2m]$, we take m elements from the set $[m+1, 2m]$ as the image of $[1, m]$, which is labeled as $\pi(i)(i \in [m])$. Clearly there are $\binom{n+m}{m} m!$ ways to choose. Let $i'(i \in [m])$ represent the two element set $\{i, \pi(i)\}$, and let $i'(i \in [m+1, n+m])$ denote the remaining element $[n+2m] \setminus \{i, \pi(i)\}$. Let π' denote the permutation on the colored set $C_\ell \times \{1', 2', 3', \cdots, (m+n)'\}$ such that $\mathrm{sign}_{\pi'}(i') = 1(i' \in [m])$ and $i'(i' \in [m])$ belong into distinct cycles, by transforming the i' into $\{i, \pi(i)\}$, we obtain the desired permutation in $F^m_{\ell,n+m}$ and vice versa. From this construction, we have $\mathrm{fix}_{>m}(\pi) = \mathrm{fix}_{>m}(\pi')$. This completes the proof. □

Theorem 46. *For $\ell \geqslant 1$ and $0 \leqslant m \leqslant n$, we have*

$$\sum_{n \geqslant 0} f^m_{\ell,n}(\lambda) \frac{u^n}{n!} = \frac{u^m \exp(\lambda - 1)u}{(1 - \ell u)^{m+1}}. \tag{49}$$

Proof. According to the generating function (33) of $g^m_{\ell,n}(\lambda)$, it is clear to see that

$$\sum_{n \geqslant 0} d^m_{\ell,n+m}(\lambda) \frac{u^n}{n!} = \frac{\exp{(\lambda - 1)u}}{(1 - \ell u)^{m+1}}.$$

For the above identity, multiplying both sides by u^m, we obtain

$$\sum_{n \geqslant 0} \frac{(n+m)!}{n!} d^m_{\ell,n+m}(\lambda) \frac{u^{n+m}}{(n+m)!} = \frac{u^m \exp{(\lambda - 1)u}}{(1 - \ell u)^{m+1}}.$$

With Theorem 20,

$$\sum_{n \geqslant 0} f^m_{\ell,n+m} \frac{u^{n+m}}{(n+m)!} = \frac{u^m \exp{(\lambda - 1)u}}{(1 - \ell u)^{m+1}},$$

which is (49) by shifting $n + m$ to n. □

With (3), (45), (46), (47) and Theorem 20, we obtain the following corollary.

Corollary 47. *For $\ell \geqslant 1, 1 \leqslant m \leqslant n - 2$, we have*

$$(n - m + 1) f^{m-1}_{\ell,n}(\lambda) = \ell m f^m_{\ell,n}(\lambda) + (\lambda - 1) n f^{m-1}_{\ell,n-1}(\lambda); \tag{50}$$

$$(n - m) f^m_{\ell,n}(\lambda) = n(\ell n - 1 + \lambda) f^m_{\ell,n-1}(\lambda) - \ell(\lambda - 1) n(n-1) f^m_{\ell,n-2}(\lambda); \tag{51}$$

$$f^m_{\ell,n}(\lambda) = \ell n f^m_{\ell,n-1}(\lambda) + n f^{m-1}_{\ell,n-1}(\lambda); \tag{52}$$

$$(n - m) f^m_{\ell,n}(\lambda) = (\lambda - 1) n(n-1) f^{m-1}_{\ell,n-2}(\lambda) + \ell n^2 f^m_{\ell,n-1}(\lambda); \tag{53}$$

where $f^0_{\ell,0}(\lambda) = 1$, $f^0_{\ell,1}(\lambda) = \lambda + \ell - 1$ and $f^1_{\ell,1}(\lambda) = 1$.

With (51) and (52), we have the following corollary.

Corollary 48. *For $\ell \geqslant 1$ and $1 \leqslant m \leqslant n - 2$, we have*

$$f^m_{\ell,n}(\lambda) = m f^{m-1}_{\ell,n-1}(\lambda) - \ell(\lambda - 1)(n-1) f^m_{\ell,n-2}(\lambda) + (\ell m + \ell n - 1 + \lambda) f^m_{\ell,n-1}(\lambda). \tag{54}$$

Remark 49. When $(\ell, \lambda) = (1, 0)$, (49) and (54) reduce to the results of [14, Theorem 3 and Theorem 2].

9 Combinatorial proof of recurrence relation (46)

In this section, we give the combinatorial proof of recurrence (46), that is,

$$\sum_{\pi \in C_{\ell,n}^{m}} \lambda^{\mathrm{fix}_{>m}(\pi)} = \ell(n-m) \sum_{\pi \in C_{\ell,n-1}^{m}} \lambda^{\mathrm{fix}_{>m}(\pi)} + \sum_{\pi \in C_{\ell,n-1}^{m-1}} \lambda^{\mathrm{fix}_{>m-1}(\pi)}, \tag{55}$$

other recurrences (45) and (47) can be proved in similar ways.

Lemma 50. *For* $0 \leqslant m \leqslant n$*,*

$$\sum_{\substack{\pi \in C_{\ell,n}^{m} \\ m \in FIX(\pi)}} \lambda^{\mathrm{fix}_{>m}(\pi)} = \sum_{\pi \in C_{\ell,n-1}^{m-1}} \lambda^{\mathrm{fix}_{>m-1}(\pi)}. \tag{56}$$

Proof. It follows similar arguments as in the proof of Lemma 21. □

Lemma 51. *For* $0 \leqslant m \leqslant n$*,*

$$\sum_{\substack{\pi \in C_{\ell,n}^{m} \\ m \notin FIX(\pi)}} \lambda^{\mathrm{fix}_{>m}(\pi)} = \ell(n-m) \sum_{\pi \in C_{\ell,n-1}^{m}} \lambda^{\mathrm{fix}_{>m}(\pi)}. \tag{57}$$

Proof. Let us consider the map $\chi : \pi \to (\epsilon, \beta, \pi')$ from $C_{\ell,n}^{m} \cap \{\pi \in G_{\ell,n} | m \notin FIX(\pi)\}$ to $C_\ell \times [n-m] \times C_{\ell,n-1}^{m}$ such that $\mathrm{fix}_{>m}(\pi) = \mathrm{fix}_{>m}(\pi')$.

For $\pi \in C_{\ell,n}^{m} \cap \{\pi \in G_{\ell,n} | m \notin FIX(\pi)\}$, we decompose π as the product of disjoint cycles. Let $\pi(m) = \beta$, it is easy to see $|\beta| \in [m+1, n]$ and $\mathrm{sign}_{\pi}(\beta) = \epsilon$.

For the element $i \in \pi$, we delete the element β and define the element $i' \in [n-1]$ in π' by

$$i' = \begin{cases} i, & if \quad |i| < |\beta|; \\ i-1, & if \quad |i| > |\beta|. \end{cases}$$

Conversely, starting from $(\epsilon, \beta, \pi') \in C_\ell \times [n-k] \times C_{\ell,n-1}^{m}$, for the element $i' \in \pi'$, we define the element $i \in [n]$ in π by

$$i = \begin{cases} i', & if \quad |i'| < |\beta|; \\ i'+1, & if \quad |i'| \geqslant |\beta|, \end{cases}$$

and let $\pi(m) = (\epsilon, \beta)$. □

For example, let $\ell = 4, n = 9, k = 4$, if $\pi = (1\ \bar{\bar{\bar{7}}})\ (2\ \bar{\bar{5}})\ (3\ \bar{\bar{\bar{8}}})\ (4\ \bar{9})\ (6) \in C_{4,9}^{4}$,

$$\epsilon = \zeta, \beta = 9, \quad \pi' = (1\ \bar{\bar{\bar{7}}})\ (2\ \bar{\bar{5}})\ (3\ \bar{\bar{\bar{8}}})\ (4)\ (6) \in C_{4,8}^{4} \quad and \quad \mathrm{fix}_{>4}(\pi) = \mathrm{fix}_{>4}(\pi').$$

for $\pi = (1\ \bar{\bar{\bar{7}}})\ (2)\ (3\ \bar{8})\ (4\ \bar{\bar{5}})\ (6)\ (9) \in C_{4,9}^{4}$,

$$\epsilon = \zeta^2, \beta = 5, \quad \pi' = (1\ \bar{\bar{\bar{6}}})\ (2)\ (3\ \bar{7})\ (4)\ (5)\ (8) \in C_{4,8}^{4} \quad and \quad \mathrm{fix}_{>4}(\pi) = \mathrm{fix}_{>4}(\pi').$$

Proof of equation (55). By considering the following equation,

$$\sum_{\pi\in C_{\ell,n}^m} \lambda^{\mathrm{fix}_{>m}(\pi)} = \sum_{\substack{\pi\in C_{\ell,n}^m\\ m\notin FIX(\pi)}} \lambda^{\mathrm{fix}_{>m}(\pi)} + \sum_{\substack{\pi\in C_{\ell,n}^m\\ m\in FIX(\pi)}} \lambda^{\mathrm{fix}_{>m}(\pi)}, \tag{58}$$

by Lemma 50, the (58) is equivalent to

$$\sum_{\pi\in C_{\ell,n}^m} \lambda^{\mathrm{fix}_{>m}(\pi)} = \sum_{\substack{\pi\in C_{\ell,n}^m\\ m\notin FIX(\pi)}} \lambda^{\mathrm{fix}_{>m}(\pi)} + \sum_{\pi\in C_{\ell,n-1}^{m-1}} \lambda^{\mathrm{fix}_{>m-1}(\pi)}. \tag{59}$$

By Lemma 51, we obtain (55) immediately. This completes the proof. □

10 Final remarks

Faliharimalala and Zeng [8, eq. (1.2)] studied the wreath product analogue of Euler's q-difference table $\{g_{\ell,n}^m(q)\}_{0\leqslant m\leqslant n}$ as follows.

Definition 52 (Faliharimalala-Zeng)**.** For fixed integer $\ell \geqslant 1$, the coefficients of Euler's q-difference table $(g_{\ell,n}^m(q))_{0\leqslant m\leqslant n}$ for $C_\ell \wr S_n$ are defined by

$$\begin{cases} g_{\ell,n}^n(q) = [\ell]_q[2\ell]_q\dots[n\ell]_q, & (m=n);\\ g_{\ell,n}^m(q) = g_{\ell,n}^{m+1}(q) - q^{\ell(n-m-1)}g_{\ell,n-1}^m(q) & (0\leqslant m\leqslant n-1). \end{cases} \tag{60}$$

Faliharimalala and Zeng found a combinatorial interpretation of $(g_{\ell,n}^m(q))_{0\leqslant m\leqslant n}$ by introducing a new Mahonian statistic *fmaf* on the wreath products. So the natural question is to find a q-λ-Euler's difference table for λ-Euler's difference table in Definition 1, it seems the statistic *fmaf* cannot help directly.

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