On upper transversals in 3-uniform hypergraphs

Michael A. Henning[1,*] Anders Yeo[1,2]

[1]Department of Pure and Applied Mathematics
University of Johannesburg
Auckland Park, 2006 South Africa
mahenning@uj.ac.za

[2]Department of Mathematics and Computer Science
University of Southern Denmark
Campusvej 55, 5230 Odense M, Denmark
andersyeo@gmail.com

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Abstract

A set S of vertices in a hypergraph H is a transversal if it has a nonempty intersection with every edge of H. The upper transversal number $\Upsilon(H)$ of H is the maximum cardinality of a minimal transversal in H. We show that if H is a connected 3-uniform hypergraph of order n, then $\Upsilon(H) > 1.4855\sqrt[3]{n} - 2$. For n sufficiently large, we construct infinitely many connected 3-uniform hypergraphs, H, of order n satisfying $\Upsilon(H) < 2.5199\sqrt[3]{n}$. We conjecture that $\sup_{n\to\infty}\left(\inf \frac{\Upsilon(H)}{\sqrt[3]{n}}\right) = \sqrt[3]{16}$, where the infimum is taken over all connected 3-uniform hypergraphs H of order n.

Mathematics Subject Classifications: 05C88, 05C89

1 Introduction

In this paper, we continue the study of transversals in hypergraphs. Hypergraphs are systems of sets which are conceived as natural extensions of graphs. A *hypergraph* $H = (V(H), E(H))$ is a finite set $V(H)$ of elements, called *vertices*, together with a finite multiset $E(H)$ of nonempty subsets of $V(H)$, called *hyperedges* or simply *edges*. A k*-edge* in H is an edge of size k. The hypergraph H is k*-uniform* if every edge of H is a k-edge. Every 2-uniform hypergraph is a graph. Thus graphs are special hypergraphs. The *degree* of a vertex v in H, denoted by $d_H(v)$, is the number of edges of H which contain v. The

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minimum and maximum degrees among the vertices of H is denoted by $\delta(H)$ and $\Delta(H)$, respectively.

A subset T of vertices in a hypergraph H is a *transversal* (also called *hitting set* or *vertex cover* or *blocking set* in many papers) if T has a nonempty intersection with every edge of H. A vertex *hits* or *covers* an edge if it belongs to that edge. The *transversal number* $\tau(H)$ of H is the minimum size of a transversal in H, while the *upper transversal number* $\Upsilon(H)$ of H is the maximum cardinality of a minimal transversal in H. In hypergraph theory the concept of transversal is fundamental and well studied. The major monograph [1] of hypergraph theory gives a detailed introduction to this topic. Transversals in hypergraphs are well studied in the literature (see, for example, [3, 4, 11, 12, 13] for recent results and further references).

A set S of vertices in a graph G is a *dominating set* of G if each vertex in $V(G) \setminus S$ has a neighbor in S. A set is *independent* if no two vertices in it are adjacent. An *independent dominating set* of G is a set that is both dominating and independent in G. The *independent domination number* of G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set. Domination is well studied in graph theory and we refer the reader to the monographs [9, 10] which detail and survey many results on the topic. A survey of known results on independent domination in graphs can be found in [8].

2 Main Results

We have two immediate aims in this paper. First to provide a sharp lower bound on the upper transversal number of graphs. Secondly to present a lower bound on the upper transversal number of 3-uniform hypergraphs, and to show that this bound is a sense asymptotically best possible. More precisely, we prove the following results, where we use the notation $n_H = |V(H)|$ to denote the *order* of H. Proofs of Theorem 1 and Theorem 2 are given in Section 3 and Section 4, respectively.

Theorem 1. *If H is a connected graph with $\delta(H) \geqslant \delta$, then*

$$\Upsilon(H) \geqslant 2\sqrt{\delta n_H} - 2\delta,$$

and this bound is sharp.

Theorem 2. *If H is a connected 3-uniform hypergraph, then*

$$\Upsilon(H) > \sqrt[3]{\frac{n_H}{0.305}} - 2 > 1.4855\sqrt[3]{n_H} - 2.$$

Further, there exist infinitely many connected 3-uniform hypergraphs H of sufficiently large order n_H satisfying

$$\Upsilon(H) < \sqrt[3]{16 \cdot n_H} < 2.52\sqrt[3]{n_H}.$$

3 Proof of Theorem 1

Recall that a transversal in a graph is a set of vertices covering all the edges of the graph, where a vertex covers an edge if it is incident with it. Theorem 1 can be restated as follows.

Theorem 1. *If G is a connected graph of order n with $\delta(G) \geqslant \delta$, then $\Upsilon(G) \geqslant 2\sqrt{\delta n} - 2\delta$, and this bound is sharp.*

In order to prove Theorem 1, we first establish a relationship between the upper transversal number and independent domination number of a graph.

Theorem 3. *If G is an isolate-free graph on n vertices, then $i(G) + \Upsilon(G) = n$.*

Proof. Let G be an isolate-free graph. Let S be an independent dominating set in G of minimum cardinality, and so $|S| = i(G)$. Let $T = V(G) \setminus S$ and note that T is a transversal in G as S is an independent set. Since every vertex in T has a neighbor in S, we furthermore note that T is a minimal transversal, which implies that $\Upsilon(G) \geqslant |T| = n - |S| = n - i(G)$.

Conversely, let T be a minimal transversal in G of maximum cardinality, and so $|T| = \Upsilon(G)$. Let $S = V(G) \setminus T$ and note that S is an independent set as T is a transversal. Since T is a minimal transversal, every vertex in T has a neighbor in S, implying that S is an independent dominating set in G. Therefore, $i(G) \leqslant |S| = n - |T| = n - \Upsilon(G)$. Consequently, $i(G) + \Upsilon(G) = n$. □

Favaron [5] was the first to prove the following upper bound on the independent domination of a graph with no isolated vertex: If G is an isolate-free graph of order n, then $i(G) \leqslant n + 2 - 2\sqrt{n}$. We remark that this result also follows from a result due to Bollobás and Cockayne [2] (and was also proved in [6]). Sun and Wang [14] proved the following more general result, which was originally posed as a conjecture by Favaron [5] and was proved for $\delta = 2$ by Glebov and Kostochka [7].

Theorem 4. ([14]) *If G is a graph of order n with $\delta(G) \geqslant \delta$, then $i(G) \leqslant n + 2\delta - 2\sqrt{\delta n}$.*

Theorem 1 is an immediate consequence of Theorem 3 and Theorem 4. That this bound is sharp, follows from a result of Favaron [5] who showed that for every positive integer δ, the bound in Theorem 4 is attained for infinitely many graphs. The same graphs achieve equality in the bound for Theorem 1. For example, for $c \geqslant 2$, let G_c be the connected graph constructed as follows. Let F_c be the complete graph of order c, and so $F_c \cong K_c$. For every vertex v of F_c, add $c-1$ new vertices $v_1, \ldots, v_{c-1}$ and add the $c-1$ edges vv_i for all $i \in [c-1]$. Let $G = G_c$ denote the resulting graph of order $n = c^2$. We note that all the new vertices added to F_c have degree 1 in G_c. Every transversal in G_c must contain all except possibly one vertex of F_c in order to cover all the edges of F_c. If a minimal transversal in G_c contains exactly $c-1$ vertices of F_c, say all vertices of F_c except for v, then the transversal contains exactly $c-1$ vertices not in F_c, namely $v_1, \ldots, v_{c-1}$, in order to cover the edges vv_i for all $i \in [c-1]$. Such a minimal transversal therefore has size exactly $2(c-1)$. If a minimal transversal in G_c contains all c vertices of F_c, then it contains no other vertex of G_c and therefore has size exactly c. Therefore, the connected graph G of order $n = c^2$ satisfies $\Upsilon(G) = \max\{2(c-1), c\} = 2(c-1) = 2\sqrt{n} - 2$, noting that $c \geqslant 2$. Thus, the bound of Theorem 1 when $\delta = 1$ is sharp. For every $\delta \geqslant 2$ one can similarly show that Theorem 1 is tight.

As a special case of Theorem 1, if H is a connected 2-uniform hypergraph of order $n \geqslant 2$, then $\Upsilon(H) \geqslant 2\sqrt{n} - 2$. When $n \geqslant 3$, we observe that $2\sqrt{n} - 2 \geqslant \sqrt{\frac{1}{2}n}$. Further, we observe that when $n = 2$, $\Upsilon(H) = 1 = \sqrt{\frac{1}{2}n}$. Thus, as an immediate consequence of Theorem 1 we observe that if H is a connected graph, then $\Upsilon(H) \geqslant \sqrt{\frac{1}{2}n_H}$.

4 Proof of Theorem 2

We first prove the lower bound in Theorem 2.

Theorem 5. *If H is a connected 3-uniform hypergraph, then* $\Upsilon(H) > \sqrt[3]{\frac{n_H}{0.305}} - 2$.

Proof. Let H be a connected 3-uniform hypergraph of order n_H and let T be a minimal transversal of maximum size. Let $T = \{t_1, t_2, \ldots, t_c\}$, and so $\Upsilon(H) = |T| = c$. For all i and j where $1 \leqslant i < j \leqslant c$ and for all $k \in [c]$, define $Z_{i,j}$, E_k and Y_k as follows.

$$
\begin{aligned}
Z_{i,j} &= \{v \in V(H) \setminus T \mid \{t_i, t_j, v\} \in E(H)\} \\
E_k &= \{e \in E(H) \mid V(e) \cap T = \{t_k\}\} \\
Y_k &= V(E_k) \setminus \{t_k\}.
\end{aligned}
$$

We note that $Y_k \subseteq V(H) \setminus T$ for each $k \in [c]$. Let $Q \subseteq V(T)$ be a minimum set of vertices in T that covers all edges that are completely within T (i.e., all edges e with $V(e) \subseteq V(T)$). Possibly, $Q = \emptyset$. Let $q = |Q|$. Renaming vertices of T if necessary, we may assume that $Q = \{t_1, \ldots, t_q\}$. Let

$$
\begin{aligned}
\mathcal{I} &= \{(i,j) \mid 1 \leqslant i < j \leqslant c\} \\
\mathcal{I}_q &= \{(i,j) \mid 1 \leqslant i \leqslant q \text{ and } i < j \leqslant c\} \\
\mathcal{I}_{>q} &= \{(i,j) \mid q+1 \leqslant i < j \leqslant c\}.
\end{aligned}
$$

We note that $\mathcal{I} = \mathcal{I}_q \cup \mathcal{I}_{>q}$. We proceed further with the following claims.

Claim 6. $|Z_{i,j}| \leqslant c - q$ *for all* $(i,j) \in \mathcal{I}$.

Proof. Suppose, to the contrary, that $|Z_{i,j}| > c - q$ for some i and j, where $1 \leqslant i < j \leqslant c$. Let $R = V(H) \setminus \{t_i, t_j\}$. Clearly, R is a transversal in H as it contains all vertices in H except two and H is 3-uniform. Let R' be obtained from R by removing vertices until we get a minimal transversal in H. We note that $Z_{i,j} \subseteq R'$ since each vertex $z \in Z_{i,j}$ is needed in order to cover the edge $\{t_i, t_j, z\}$. Further, R' contains at least q vertices from T in order to cover the edges that are contained entirely within T. Hence, $\Upsilon(H) \geqslant |R'| \geqslant |Z_{i,j}| + q > c$, contradicting the fact that $\Upsilon(H) = c$. □

Claim 7. $\left|\bigcup_{(i,j)\in\mathcal{I}_{>q}} Z_{i,j}\right| \leqslant c-q.$

Proof. Suppose, to the contrary, that

$$|\bigcup_{(i,j)\in\mathcal{I}_{>q}} Z_{i,j}| > c-q.$$

Let $R = V(H)\setminus\{t_{q+1},\ldots,t_c\}$. By definition of the set Q, every edge of H intersects $\{t_{q+1},\ldots,t_c\}$ in at most two vertices, implying that R is a transversal in H. Let R' be obtained from R by removing vertices from R until we get a minimal transversal in H. We note that

$$\bigcup_{(i,j)\in\mathcal{I}_{>q}} Z_{i,j} \subseteq R'$$

since each vertex $z\in Z_{i,j}$ where $q+1\leqslant i<j\leqslant c$ is needed in order to cover the edge $\{t_i,t_j,z\}$. Further, R' contains at least q vertices from Q in order to cover the edges that are contained entirely within T. Hence,

$$\Upsilon(H)\geqslant |R'|\geqslant |\bigcup_{(i,j)\in\mathcal{I}_{>q}} Z_{i,j}|+q>c,$$

contradicting the fact that $\Upsilon(H)=c$. □

Claim 8. $\left|\bigcup_{(i,j)\in\mathcal{I}} Z_{i,j}\right| \leqslant \left(\binom{c}{2}-\binom{c-q}{2}\right)(c-q)+(c-q).$

Proof. As observed earlier, $\mathcal{I}=\mathcal{I}_q\cup\mathcal{I}_{>q}$. By Claim 6, $|Z_{i,j}|\leqslant c-q$ for all $(i,j)\in\mathcal{I}_q$. Since there are $\binom{c}{2}-\binom{c-q}{2}$ pairs $(i,j)\in\mathcal{I}_q$ where $1\leqslant i\leqslant q$ and $i<j\leqslant c$, we note by Claim 6 and Claim 7 that

$$\left|\bigcup_{(i,j)\in\mathcal{I}} Z_{i,j}\right|\leqslant\left|\bigcup_{(i,j)\in\mathcal{I}_q} Z_{i,j}\right|+\left|\bigcup_{(i,j)\in\mathcal{I}_{>q}} Z_{i,j}\right|\leqslant\left(\binom{c}{2}-\binom{c-q}{2}\right)(c-q)+(c-q).$$ □

Claim 9. $|Y_i|\leqslant(\frac{c-q}{2}+1)^2$ *for all* $i\in[c]$.

Proof. Suppose, to the contrary, that $|Y_i|>((c-q)/2+1)^2$ for some $i\in[c]$. Let H' be the graph with vertex set $V(H')=Y_i$ and with edge set $E(H')=\{e\setminus\{t_i\}\mid e\in E_i\}$. By Theorem 1, there is a minimal transversal T' in H', such that $|T'|\geqslant 2(\sqrt{|Y_i|}-1)$. Let $R'=T'\cup(T\setminus\{t_i\})$ and note that R' is a transversal in H. Let R'' be obtained from R' by removing vertices from R' until we get a minimal transversal in H. In order to cover the edges E_i, we must have $T'\subseteq R''$, noting that T' is a minimal transversal in H'. Further, R' contains at least q vertices from $T\setminus\{t_i\}$ in order to cover the edges that are contained entirely within T. Therefore,

$$\Upsilon(H)\geqslant|R''|\geqslant|T'|+q\geqslant 2(\sqrt{|Y_i|}-1)+q>2\left(\sqrt{\left(\frac{c-q}{2}+1\right)^2}-1\right)+q=c$$

contradicting the fact that $\Upsilon(H) = c$. $\square$

Since T is a transversal in H, we note that

$$V(H) = T \cup \left(\bigcup_{(i,j)\in\mathcal{I}} Z_{i,j} \right) \cup \left(\bigcup_{i=1}^{c} Y_i \right). \tag{1}$$

Let β be defined such that $(c-q) = \beta c$. We note that $0 \leqslant \beta \leqslant 1$. By Equation (1) and by Claim 8 and 9, we therefore get the following.

$$\begin{aligned}
n_H &\leqslant |T| + \sum_{i,j} |Z_{i,j}| + \sum_{i} |Y_i| \\
&\leqslant c + \left(\binom{c}{2} - \binom{c-q}{2}\right)(c-q) + (c-q) + c(\tfrac{c-q}{2}+1)^2 \\
&\leqslant c + \left(\tfrac{c(c-1)}{2} - \tfrac{\beta c(\beta c-1)}{2}\right)\beta c + \beta c + c(\tfrac{\beta c}{2}+1)^2 \\
&= c + \tfrac{1}{2}\left(c^2 - c - \beta^2 c^2 + \beta c\right)\beta c + \beta c + c(\tfrac{\beta^2 c^2}{4} + \beta c + 1) \\
&= c^3\left(\tfrac{\beta}{2} - \tfrac{\beta^3}{2} + \tfrac{\beta^2}{4}\right) + c^2\left(\tfrac{-\beta}{2} + \tfrac{\beta^2}{2} + \beta\right) + c(2+\beta) \\
&= \tfrac{c^3}{4}\left(-2\beta^3 + \beta^2 + 2\beta\right) + c^2\left(\tfrac{\beta}{2} + \tfrac{\beta^2}{2}\right) + c(2+\beta).
\end{aligned}$$

Let

$$f(\beta) = -2\beta^3 + \beta^2 + 2\beta.$$

The maximum value of $f(\beta)$ when $0 \leqslant \beta \leqslant 1$ is obtained when $\beta = (1+\sqrt{13})/6$, noting that $0 = f'(\beta) = -6\beta^2 + 2\beta + 2$ implies $\beta = (1 \pm \sqrt{13})/6$. Therefore, $f(\beta) \leqslant f\left(\frac{1+\sqrt{13}}{6}\right) < 1.22$ for all $0 \leqslant \beta \leqslant 1$, implying by our earlier observations that

$$\begin{aligned}
n_H &\leqslant \tfrac{c^3}{4}\left(-2\beta^3 + \beta^2 + 2\beta\right) + c^2\left(\tfrac{\beta}{2} + \tfrac{\beta^2}{2}\right) + c(2+\beta) \\
&< \tfrac{c^3}{4}(1.22) + c^2\left(\tfrac{1}{2} + \tfrac{1}{2}\right) + c(2+1) \\
&= 0.305c^3 + c^2 + 3c \\
&< 0.305(c+2)^3,
\end{aligned}$$

and so $\Upsilon(H) = c > \sqrt[3]{\frac{n_H}{0.305}} - 2$. This completes the proof of Theorem 5. $\square$

We remark that $\sqrt[3]{\frac{1}{0.305}} > 1.48559$, and so as a consequence of Theorem 5, if H is a connected 3-uniform hypergraph of order $n \geqslant 3$, then $\Upsilon(H) > 1.4855\sqrt[3]{n} - 2$. When $n \geqslant 17$, we observe that $1.4855\sqrt[3]{n} - 2 > \sqrt[3]{\frac{1}{3}n}$. Further, we observe that when $n = 3$, $\Upsilon(H) = 1 = \sqrt[3]{\frac{1}{3}n}$, while for $4 \leqslant n \leqslant 16$, $\Upsilon(H) \geqslant 2 > \sqrt[3]{\frac{1}{3}n}$. Thus, as an immediate consequence of Theorem 5, we observe that if H is a connected 3-uniform hypergraph, then $\Upsilon(H) \geqslant \sqrt[3]{\frac{n_H}{3}}$. We show next that the lower bound in Theorem 5 is asymptotically best possible.

Proposition 10. *For all $n \geqslant 3$, there exists a connected 3-uniform hypergraph $H = H_n$ of order $n_H = \frac{1}{2}(n^3 - n^2 + 2n)$ such that*

$$\Upsilon(H) = \sqrt[3]{16(1-\epsilon_n)\cdot n_H} \quad \textit{where} \quad \epsilon_n = \frac{2n^2 - n + 1}{n^3 - n^2 + 2n}.$$

Proof. For each $n \geqslant 3$, let H_n be the connected 3-uniform hypergraph constructed as follows. Let F_n be the complete 3-uniform hypergraph on n vertices, and so F_n has $\binom{n}{3}$ hyperedges corresponding to the 3-element subsets of $V(F_n)$. Thus, every set of three vertices in F_n belongs to a 3-edge of F_n. Let $S = V(F_n)$. For every pair of vertices, $\{x, y\}$, in S add n new vertices, $v_1^{xy}, v_2^{xy}, \dots, v_n^{xy}$ to F_n and add the n hyperedges $\{x, y, v_1^{xy}\}, \{x, y, v_2^{xy}\}, \dots, \{x, y, v_n^{xy}\}$. Let $H = H_n$ denote the resulting hypergraph of order

$$n_H = \binom{n}{2} n + |V(F_n)| = \frac{n^2(n-1)}{2} + n = \frac{1}{2}(n^3 - n^2 + 2n).$$

We note that all the new vertices added to F_n have degree 1 in H_n. Every transversal in H_n must contain all n vertices in F_n, except for possibly two vertices in order to cover all the edges in F_n. Every minimal transversal in H_n contains either exactly $n-1$ vertices of S (and no other vertex in H_n) or exactly $n-2$ vertices of S, say all vertices of S except for the vertices x and y, and exactly n vertices not in S, namely $v_1^{xy}, v_2^{xy}, \dots, v_n^{xy}$, in order to cover the edges $\{x, y, v_1^{xy}\}, \{x, y, v_2^{xy}\}, \dots, \{x, y, v_n^{xy}\}$, implying that

$$\Upsilon(H_n) = (|S| - 2) + n = 2(n-1).$$

Therefore, letting $\epsilon_n = \frac{2n^2-n+1}{n^3-n^2+2n}$, we note that the connected 3-uniform hypergraph $H = H_n$ satisfies

$$\begin{aligned}
\Upsilon(H) &= 2(n-1) \\
&= \sqrt[3]{8\cdot(n-1)^3} \\
&= \sqrt[3]{16\cdot\frac{(n-1)^3}{2n_H}\cdot n_H} \\
&= \sqrt[3]{16\left(\frac{n^3-3n^2+3n-1}{n^3-n^2+2n}\right)\cdot n_H} \\
&= \sqrt[3]{16\left(1-\frac{2n^2-n+1}{n^3-n^2+2n}\right)\cdot n_H} \\
&= \sqrt[3]{16(1-\epsilon_n)\cdot n_H}.
\end{aligned}$$

□

Using the notation introduced in the statement of Proposition 10, we note that $\frac{2}{3} = \epsilon_3 > \epsilon_4 > \epsilon_5 > \cdots$ and that $\epsilon_n \to 0$ as $n \to \infty$. In particular, we note that given any

$\epsilon > 0$, we can choose n sufficiently large so that $\epsilon_n < \epsilon$, implying by Proposition 10 that the connected 3-uniform hypergraph $H = H_n$ satisfies

$$\sqrt[3]{16(1-\epsilon) \cdot n_{_H}} < \Upsilon(H) < \sqrt[3]{16 \cdot n_{_H}}$$

5 Closing Conjectures

We pose the following conjecture. As observed earlier, Conjecture 11 is true for $k \in \{2, 3\}$. However, we have yet to settle the conjecture for $k \geqslant 4$.

Conjecture 11. For $k \geqslant 2$, if H is a connected k-uniform hypergraph then $\Upsilon(H) \geqslant \sqrt[k]{\dfrac{n_{_H}}{k}}$.

Let $\mathcal{H}_n$ denote the class of all connected 3-uniform hypergraphs of order n. As observed earlier, Proposition 10 implies that for n sufficiently large there exist hypergraphs $H \in \mathcal{H}_n$ such that

$$\frac{\Upsilon(H)}{\sqrt[3]{n}} > \sqrt[3]{16(1-\epsilon)}$$

for any given $\epsilon > 0$. We close with the following conjecture that we have yet to settle.

Conjecture 12. $\displaystyle \sup_{n \to \infty} \left(\inf_{H \in \mathcal{H}_n} \frac{\Upsilon(H)}{\sqrt[3]{n}} \right) = \sqrt[3]{16}.$

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