Cohomology classes of interval positroid varieties and a conjecture of Liu

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Abstract

To each finite subset of $\mathbb{Z}^2$ (a diagram), one can associate a subvariety of a complex Grassmannian (a diagram variety), and a representation of a symmetric group (a Specht module). Liu has conjectured that the cohomology class of a diagram variety is represented by the Frobenius characteristic of the corresponding Specht module. We give a counterexample to this conjecture.

However, we show that for the diagram variety of a permutation diagram, Liu's conjectured cohomology class σ is at least an upper bound on the actual class τ, in the sense that $\sigma - \tau$ is a nonnegative linear combination of Schubert classes. To do this, we exhibit the appropriate diagram variety as a component in a degeneration of one of Knutson's interval positroid varieties (up to Grassmann duality). A priori, the cohomology classes of these interval positroid varieties are represented by affine Stanley symmetric functions. We give a different formula for these classes as ordinary Stanley symmetric functions, one with the advantage of being Schur-positive and compatible with inclusions between Grassmannians.

Mathematics Subject Classifications: 05E05, 05E10, 14N15

1 Introduction

1.1 Diagram varieties

A *diagram* is a finite subset D of $\mathbb{Z}^2$. Write $[n]$ for $\{1, 2, \ldots, n\}$. Given a diagram contained in $[k] \times [n-k]$, define a subvariety X_D of the Grassmannian $\mathrm{Gr}_k(n)$ of k-planes in $\mathbb{C}^n$ as the Zariski closure of

$$\{\mathrm{rowspan}\, [A \mid I_k] : A \in M_{k,n-k} \text{ with } A_{ij} = 0 \text{ when } (i,j) \in D\}.$$

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Here $M_{k,n-k}$ is the set of $k \times (n-k)$ complex matrices, and I_k is the $k \times k$ identity matrix. Call this variety X_D a *diagram variety.* For example, if $D = \{(1,1),(1,2),(2,1)\}$, $k=2$, $n=4$, then X_D is the closure of the set of 2-planes in $\mathbb{C}^4$ which are the rowspans of matrices of the form

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & * & 0 & 1 \end{bmatrix}.$$

Let S_D denote the set of permutations of D. One can associate a (complex) representation Sp_D of the symmetric group S_D to a diagram D, called the *Specht module of D*. These generalize the usual irreducible Specht modules, which occur when D is the Young diagram of a partition; the definition for general diagrams is due to James and Peel [8].

Each of these objects, diagram variety and Specht module, naturally leads to a class in the cohomology ring $H^* \operatorname{Gr}_k(n) := H^*(\operatorname{Gr}_k(n), \mathbb{Z})$. For the diagram variety, we take the Chow ring class of X_D and use the natural isomorphism between $H^*(\operatorname{Gr}_k(n), \mathbb{Z})$ and the Chow ring of $\operatorname{Gr}_k(n)$ to obtain a cohomology class $[X_D] \in H^{2\#D}(\operatorname{Gr}_k(n), \mathbb{Z})$.

As for the Specht module, let s_D be the Frobenius characteristic of S^D, meaning $s_D = \sum_\lambda a_\lambda s_\lambda$ if $S^D \simeq \bigoplus_\lambda a_\lambda S^\lambda$, where s_λ is a Schur function. Here λ runs over partitions, and S^λ is an irreducible Specht module. There is a surjective ring homomorphism ϕ from the ring of symmetric functions to $H^*(\operatorname{Gr}_k(n), \mathbb{Z})$, sending the Schur function s_λ to the Schubert class $\sigma_\lambda := [X_\lambda]$, or to 0 if $\lambda \not\subseteq (k^{n-k})$ [6]. Hence we can consider the cohomology class $\phi(s_D)$.

Conjecture (Liu [14], Conjecture 5 below)*.* For any diagram D, the cohomology classes $[X_D]$ and $\phi(s_D)$ are equal.

Liu proved Conjecture 5, or the weaker variant claiming equality of degrees, for various classes of diagrams [14]. However, it turns out that this conjecture fails in general, as we show in Section 2.

Theorem. Conjecture 5 fails for $X_D \subseteq \operatorname{Gr}_4(8)$ where $D = \{(1,1),(2,2),(3,3),(4,4)\}$.

Let $D(w)$ denote the *Rothe diagram* of $w \in S_n$: the diagram with a cell $(i, w(j))$ for each inversion $i < j$, $w(i) > w(j)$ of w. Work of Kraśkiewicz and Pragacz [11] and of Reiner and Shimozono [17] shows that $s_{D(w)}$ is the Stanley symmetric function F_w [21]. Thus, if Conjecture 5 were to hold for $D(w)$, we would have $[X_{D(w)}] = \phi(F_w)$.

Building on work of Postnikov [16], Knutson, Lam, and Speyer [10] have defined a collection of subvarieties Π_f of Grassmannians called *positroid varieties*, indexed by certain affine permutations f. A positroid variety is defined by imposing some rank conditions on cyclic intervals of columns of matrices representing points in $\operatorname{Gr}_k(n)$, and any irreducible variety defined by such rank conditions is a positroid variety. They show that the positroid variety Π_f has cohomology class $\phi(\tilde{F}_f)$, where $\tilde{F}_f$ is the affine Stanley symmetric function of f, as defined in [12]. Given an ordinary permutation $w \in S_n$, define $f_w : \mathbb{Z} \to \mathbb{Z}$ by

$$f_w(i) = \begin{cases} i+n & \text{if } 1 \leqslant i \leqslant n \\ w(i)+2n & \text{if } n \leqslant i \leqslant 2n \end{cases}$$

and $f(i+2n) = f(i) + 2n$. One can show that $\tilde{F}_{f_w} = F_w$.

By the previous two paragraphs, Conjecture 5 would imply equality of the classes $[\Pi_{f_w}]$ and $[X_{D(w)}]$. As we will see, Conjecture 5 can fail for permutation diagrams $D = D(w)$, and in general $[\Pi_{f_w}]$ and $[X_{D(w)}]$ need not be equal. Nevertheless, we will give a degeneration of Π_{f_w} to a (possibly reducible) variety containing $X_{D(w)}$ as a component, which implies the following upper bound on $[X_{D(w)}]$.

Theorem (Theorem 31). The cohomology class $\phi(F_w) - [X_{D(w)}]$ is a nonnegative integer combination of Schubert classes.

1.2 Limits of classes of interval positroid varieties

The positroid varieties defined by rank conditions only involving honest intervals of columns (as opposed to cyclic intervals) are called *interval positroid varieties* [9]. For $w \in S_n$, the Grassmann duals of the varieties Π_{f_w} described above are examples of interval positroid varieties. There are several ways to compute the class $[\Sigma]$ of an interval positroid variety Σ. First, $[\Sigma] = \phi(\tilde{F}_f)$ for some affine permutation f by the work of Knutson-Lam-Speyer described above. Second, Coskun [3] gives a recursive rule for computing $[\Sigma]$ by degenerating Σ to a union of Schubert varieties, and in [9], Knutson computes the more general torus-equivariant K-theory class of Σ in this way.

We give a different formula for $[\Sigma]$ which is *stable* in the following sense. Given a list $M = (S_1, \ldots, S_m)$ of intervals all contained in $[n]$ and a vector $r = (r_1, \ldots, r_m)$ of nonnegative integers, define $\Sigma_{M,r,n}$ to be

$$\{\text{rowspan}(A) \in \mathrm{Gr}_k(n) : \text{the submatrix of } A \text{ in columns } S_i \text{ has rank} \leqslant r_i \text{ for all } i\}.$$

If $\Sigma_{M,r,n}$ is irreducible, then it is an interval positroid variety.

The standard inclusion $\mathbb{C}^n \hookrightarrow \mathbb{C}^{n+1}$ defines an inclusion $\mathrm{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathrm{Gr}_k(\mathbb{C}^{n+1})$, hence a pullback map $H^* \mathrm{Gr}_k(n+1) \twoheadrightarrow H^* \mathrm{Gr}_k(n)$, and this pullback sends $[\Sigma_{M,r,n+1}]$ to $[\Sigma_{M,r,n}]$. We can therefore ask for a formula for a class α in the inverse limit $\varprojlim_N H^* \mathrm{Gr}_k(N)$ which represents the classes $[\Sigma_{M,r,n}]$ for every n, in the sense that for every n the map $\varprojlim H^* \mathrm{Gr}_k(N) \to H^* \mathrm{Gr}_k(n)$ sends α to $[\Sigma_{M,r,n}]$.

Theorem (Theorem 26). If $\Sigma_{M,r,n} \subseteq \mathrm{Gr}_k(n)$ is an interval positroid variety, there is an ordinary permutation w such that the ordinary Stanley symmetric function F_w represents the class $\Sigma_{M,r,n}$ for all n.

2 A counterexample to Liu's conjecture

Definition 1. A *diagram* is a finite subset of $\mathbb{Z}^2$.

Given a diagram D contained in $[k] \times [n-k]$, define an open subset

$$X_D^\circ = \{\text{rowspan}\,[A \,|\, I_k] : A \in M_{k,n-k} \text{ such that } A_{ij} = 0 \text{ whenever } (i,j) \in D\}$$

of the complex Grassmannian $\mathrm{Gr}_k(n)$. For example, if $D = \{(1,1),(1,2),(2,2),(2,3)\}$, $k = 2$, and $n = 5$, then

$$X_D^\circ = \left\{ \text{rowspan} \begin{pmatrix} 0 & 0 & * & 1 & 0 \\ * & 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Definition 2. The *diagram variety* X_D of D is $\overline{X_D^\circ}$, the closure being in the Zariski topology.

Notice that X_D° is an open dense subset of X_D isomorphic to $\mathbb{C}^{k(n-k)-\#D}$. In particular, it is irreducible, so X_D is also irreducible and has codimension $\#D$.

We now describe a representation of S_D associated to each diagram D. Let $R(D)$ denote the group of permutations $\sigma \in S_D$ for which b and $\sigma(b)$ are in the same row for any $b \in D$. Let $C(D)$ be the analogous subgroup with "row" replaced by "column".

Definition 3. The *Specht module* of D is the left ideal

$$Sp_D = \mathbb{C}[S_D] \sum_{p \in R(D)} \sum_{q \in C(D)} \operatorname{sgn}(q) qp$$

of $\mathbb{C}[S_D]$, viewed as an S_D-module.

The Specht modules associated to general diagrams were studied by James and Peel [8]. As D runs over (Ferrers diagrams of) partitions of m, the Specht modules provide a complete, irredundant set of complex irreducibles for S_m (see [6, 20]). The isomorphism type of Sp_D is unaltered by permuting the rows or the columns of D. If the rows and columns of D cannot be permuted to obtain a partition—equivalently, the rows of D are not totally ordered under inclusion—then Sp_D will not be irreducible. For example, if $\lambda \setminus \mu$ is a skew shape, then

$$Sp_{\lambda \setminus \mu} \simeq \bigoplus_\nu c_{\mu\nu}^\lambda Sp_\nu,$$

where $c_{\mu\nu}^\lambda$ is a Littlewood-Richardson coefficient.

In general it is an open problem to give a combinatorial rule for decomposing Sp_D into irreducibles. The widest class of diagrams for which such a rule is known are the *percent-avoiding* diagrams, studied by Reiner and Shimozono [19]; see also [13] and [18].

Given a diagram $D \subset [k] \times [n-k]$, let $D^\vee$ be the complement of D in $[k] \times [n-k]$ rotated by 180°. For example, if $\mu \subseteq \lambda \subseteq [k] \times [n-k]$ are partitions, then $X_{\lambda^\vee}^\circ \cap X_\mu^\circ = X_{(\lambda/\mu)^\vee}^\circ$. This intersection is transverse on the dense open subset $X_{(\lambda/\mu)^\vee}^\circ$ of $X_{(\lambda/\mu)^\vee}$, and indeed one can show that $[X_{(\lambda/\mu)^\vee}] = \sum_\nu c_{\mu\nu}^\lambda \sigma_{\nu^\vee}$ [14, Proposition 5.5.3].

Magyar has shown that Specht module decompositions behave as nicely as possible with respect to the box complement operation.

Theorem 4 (Magyar [15])**.** *For any diagram D contained in $[k] \times [n-k]$, $Sp_D \simeq \bigoplus_\lambda a_\lambda Sp_\lambda$ if and only if $Sp_{D^\vee} \simeq \bigoplus_\lambda a_\lambda Sp_{\lambda^\vee}$.*

In particular, $Sp_{(\lambda/\mu)^\vee} \simeq \bigoplus_\nu c_{\mu\nu}^\lambda Sp_{\nu^\vee}$. Comparing this decomposition of $Sp_{(\lambda/\mu)^\vee}$ to the expansion $[X_{(\lambda/\mu)^\vee}] = \sum_\nu c_{\mu\nu}^\lambda \sigma_{\nu^\vee}$ discussed above suggests the next conjecture (and proves it when $D = (\lambda/\mu)^\vee$).

Conjecture 5 (Liu [14])**.** For any diagram D, the cohomology classes $[X_D]$ and $\phi(s_D)$ are equal.

Liu proved Conjecture 5 in the case above where $D^\vee$ is a skew shape, or when it corresponds to a forest [14] in the sense that one can represent a diagram $D \subset [k] \times [n-k]$ as the bipartite graph with white vertices $[k]$, black vertices $[n-k]$, and an edge between a white i and black j whenever $(i,j) \in D$. In [2], we proved Conjecture 5 when $D^\vee$ is a permutation diagram and Sp_D is multiplicity-free.

One gets a weaker version of Conjecture 5 by comparing degrees. The *degree* of a codimension d subvariety X of $\mathrm{Gr}_k(n)$ is the integer $\deg(X)$ defined by $[X]\sigma_1^{k(n-k)-d} = \deg(X)\sigma_{(k^{n-k})}$. Under the Plücker embedding, this gives the usual notion of the degree of a subvariety of projective space, namely the number of points in the intersection of X with a generic d-dimensional linear subspace. One can check using Pieri's rule that $\deg(\sigma_\lambda) = f^{\lambda^\vee}$, the number of standard Young tableaux of shape $\lambda^\vee$. This is also $\dim Sp_{\lambda^\vee}$. Since degree is additive on cohomology classes, Conjecture 5 predicts the following.

Conjecture 6 (Liu)**.** The degree of X_D is $\dim Sp_{D^\vee}$.

Liu proved Conjecture 6 when $D^\vee$ is a permutation diagram, and when $D^\vee$ has the property that if $(i,j_1),(i,j_2) \in D$ and $j_1 < j < j_2$, then $(i,j) \in D$. In light of the assertion of Theorem 4 that taking complements in the decomposition of Sp_D gives the decomposition of $Sp_{D^\vee}$, it is tempting to gloss over the distinction between D and $D^\vee$. In fact, the analogue of Theorem 4 fails for the classes $[X_D]$, and Conjecture 5 can fail for D while holding for $D^\vee$.

Suppose $D = \{(1,1),(2,2),(3,3),(4,4)\}$, with $k=4$ and $n=8$. This is the skew shape $4321/321$. The Specht module Sp_D is simply the regular representation of S_4, with

$$Sp_D \simeq Sp_{1111} \oplus 3Sp_{211} \oplus 2Sp_{22} \oplus 3Sp_{31} \oplus Sp_4.$$

Theorem 4 then says

$$Sp_{D^\vee} \simeq Sp_{3333} \oplus 3Sp_{4332} \oplus 2Sp_{4422} \oplus 3Sp_{4431} \oplus Sp_{444},$$

so $\dim Sp_{D^\vee} = f^{3333} + 3f^{4332} + 2f^{4422} + 3f^{4431} + f^{444} = 24024$.

On the other hand, an explicit calculation in Macaulay2 shows $\deg X_D = 21384$. Therefore Conjectures 6 and 5 both fail for D. (One may wonder how such a seemingly small counterexample remained undetected. It is perhaps more natural to index diagram varieties by $D^\vee$ than D—notice that the cases mentioned above for which Conjecture 5 has been established all have the property that $D^\vee$, rather than D, falls into some nice class of diagrams—and from this point of view the counterexample is no longer so small.)

The discrepancy in degrees is $24024 - 21384 = 2640 = f^{4422}$, which hints at how to see this discrepancy more explicitly. Given a k-subset I of $[n]$, write p_I for the corresponding Plücker coordinate on $\mathrm{Gr}_k(n)$, so $p_I(A)$ is the maximal minor of A in columns I. Let Y be the subscheme determined by the vanishing of the Plücker coordinates $p_{1678}, p_{2578}, p_{3568}, p_{4567}$. These are exactly the Plücker coordinates which vanish on X_D. One can check by computer that Y is a complete intersection, so that $[Y] = \sigma_1^4 = \sigma_{1111} + 3\sigma_{211} + 2\sigma_{22} + 3\sigma_{31} + \sigma_4$.

Since the four Plücker coordinates cutting out Y vanish on X_D°, the diagram variety X_D is contained in Y. However, Y has another component, namely the Schubert variety which is the closure of

$$\left\{ \text{rowspan} \begin{bmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & 1 & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & * & * & 1 & 0 \\ * & * & 0 & 0 & * & * & 0 & 1 \end{bmatrix} \right\}.$$

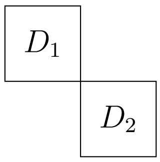
This Schubert variety has degree $\dim Sp_{(22)^\vee} = f^{4422} = 2640$, which is $\deg Y - \deg X_D$. Therefore

$$[X_D] = [Y] - \sigma_{22} = \sigma_{1111} + 3\sigma_{211} + \sigma_{22} + 3\sigma_{31} + \sigma_4.$$

Larger counterexamples to Conjecture 5 can be easily manufactured from this one. For two diagrams D_1 and D_2 where $D_1 \subseteq [a] \times [b]$, define

$$D_1 \cdot D_2 = D_1 \cup \{(i + a, j + b) : (i, j) \in D_2\}.$$

Graphically, $D_1 \cdot D_2$ is the diagram



.

One can show that $[X_{D_1 \cdot D_2}] = [X_{D_1}][X_{D_2}]$ and similarly that $s_{D_1 \cdot D_2} = s_{D_1} s_{D_2}$. Therefore if Conjecture 5 holds for D_1 but not D_2, then it will fail for $D_1 \cdot D_2$.

Remark 7. It is natural to wonder about the diagram

$$D' = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\},$$

and whether Conjecture 5 fails for D'. Trying to repeat the analysis above runs into an immediate problem, however (I thank Ricky Liu for pointing this out). Namely, the analogue of Y, which is the scheme Z cut out by

$$p_{1789(10)}, p_{2689(10)}, p_{3679(10)}, p_{4678(10)}, p_{56789}$$

no longer even has the same codimension as X_D. Indeed, X_D has codimension 5 but Z contains the codimension 4 Schubert cell

$$\left\{ \text{rowspan} \begin{bmatrix} * & * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & * & * & 1 & 0 & 0 \\ * & * & * & 0 & 0 & * & * & 0 & 1 & 0 \\ * & * & * & 0 & 0 & * & * & 0 & 0 & 1 \end{bmatrix} \right\}.$$

3 Cohomology classes of interval positroid varieties

3.1 Positroid varieties

Definition 8. An *affine permutation* of quasi-period n is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(i+n) = f(i) + n$ for all i. Write $\tilde{S}_n$ for the set of affine permutations of quasi-period n.

Note that an $f \in \tilde{S}_n$ is completely determined by any sequence $f(a), f(a+1), \dots, f(a+n-1)$, which we call a *window*. We will usually specify an affine permutation $f \in \tilde{S}_n$ by giving the sequence $f(1), \dots, f(n)$, so that $14825 \in \tilde{S}_5$ fixes 1, sends 3 to 8, 7 to 9, etc. Members of any window are all distinct modulo n, so $\sum_{i=1}^{n} f(i) \equiv n(n+1)/2 \pmod{n}$. Let $\mathrm{av}(f)$ be the integer $\frac{1}{n}\sum_{i=1}^{n}(f(i) - i)$.

Write $\tilde{S}_n^k$ for the set of affine permutations with $\mathrm{av}(f) = k$. In particular, $\tilde{S}_n^0$ is a Coxeter group with simple generators $s_0, \dots, s_{n-1}$, where s_i interchanges $i + np$ and $i + 1 + np$ for every p. The groups $\tilde{S}_n^0$ are the affine Weyl groups of type A, and one should beware that affine permutations are frequently defined to be members of $\tilde{S}_n^0$ rather than by our broader definition. The shift map $\tau : \mathbb{Z} \to \mathbb{Z}$, $\tau(i) = i + 1$ yields a bijection $\tilde{S}_n^0 \to \tilde{S}_n^k$ for each k, namely $f \mapsto \tau^k f$, and we will use these bijections to transport Coxeter structure from $\tilde{S}_n^0$ to any $\tilde{S}_n^k$. For instance, we define the reduced words of $f \in \tilde{S}_n^k$ to be the reduced words of $\tau^{-k} f \in \tilde{S}_n^0$. The next definition provides another example.

Definition 9. The *length* $\ell(f)$ of an affine permutation f is the number of inversions $i < j$, $f(i) > f(j)$, provided that we regard any two inversions $i < j$ and $i + pn < j + pn$ as equivalent.

Clearly $\ell(\tau f) = \ell(f)$, and one checks that $\ell(f)$ agrees with the usual Coxeter length when $f \in \tilde{S}_n^0$.

Definition 10. An affine permutation $f \in \tilde{S}_n$ is *bounded* if $i \leqslant f(i) \leqslant i + n$ for all i. Let $\mathrm{Bound}(k, n)$ denote the set of bounded affine permutations in $\tilde{S}_n^k$.

The next proposition makes it easy to identify members of $\mathrm{Bound}(k, n)$.

Proposition 11. *An affine permutation f is in* $\mathrm{Bound}(k, n)$ *if and only if it is bounded and exactly k of $f(1), \dots, f(n)$ exceed n.*

Any affine permutation f has a permutation matrix, the $\mathbb{Z} \times \mathbb{Z}$ matrix A with $A_{i,f(i)} = 1$ and all other entries 0. For any $i, j \in \mathbb{Z}$, define

$$[i, j](f) = \{p < i : f(p) > j\}.$$

That is, $\#[i, j](f)$ is the number of 1's strictly northeast of (i, j) in the permutation matrix of f, in matrix coordinates.

Fix a basis $e_1, \dots, e_n$ of $\mathbb{C}^n$. With this choice in mind, we adopt the following abuse of notation: if $X \subseteq \mathbb{C}^n$, $\langle X \rangle$ will mean the span of X, while if $X \subseteq [n]$, $\langle X \rangle$ will mean the span of $\{e_i : i \in X\}$. For $X \subseteq [n]$, let $\mathrm{Prj}_X : \mathbb{C}^n \to \langle X \rangle$ be the projection which fixes those basis vectors e_i with $i \in X$ and sends the rest to 0. For integers $i \leqslant j$, write $[i, j]$ for $\{i, i+1, \dots, j\}$. We interpret indices of basis vectors modulo n, so that $\langle [i, j] \rangle \subseteq \mathbb{C}^n$ even if i, j fail to lie in $[1, n]$.

Definition 12 ([10])**.** Given a bounded affine permutation $f \in \mathrm{Bound}(k,n)$, the *positroid variety* $\Pi_f \subseteq \mathrm{Gr}_k(n)$ is

$$\{V \in \mathrm{Gr}_k(n) : \dim \mathrm{Prj}_{[i,j]} V \leqslant k - \#[i,j](f) \text{ for all } i \leqslant j\}.$$

Theorem 13 ([10], Theorem 5.9)**.** *The positroid variety $\Pi_f \subseteq \mathrm{Gr}_k(n)$ is irreducible of codimension $\ell(f)$.*

Knutson–Lam–Speyer also computed the cohomology class of Π_f in terms of affine Stanley symmetric functions. These are a class of symmetric functions indexed by affine permutations introduced by Lam in [12], which we now define.

A *reduced word* for $f \in \tilde{S}_n^0$ is a word $a_1 \cdots a_\ell$ in the alphabet $[0, n-1]$ with $s_{a_1} \cdots s_{a_\ell} = f$ and such that ℓ is minimal with this property. Let $\mathrm{Red}(f)$ denote the set of reduced words for f. A reduced word $a = a_1 \cdots a_\ell$ is *cyclically decreasing* if all the a_i are distinct, and if whenever some j and $j+1$ appear in a (modulo n), $j+1$ precedes j. An affine permutation is cyclically decreasing if it has a cyclically decreasing reduced word. For a partition λ, let m_λ be the monomial symmetric function indexed by λ.

Definition 14. The *affine Stanley symmetric function* of $f \in \tilde{S}_n^0$ is

$$\tilde{F}_f = \sum_{(f^1,\dots,f^p)} x_1^{\ell(f^1)} \cdots x_p^{\ell(f^p)},$$

where $(f^1, \dots, f^p)$ runs over all factorizations $f = f^1 \cdots f^p$ with each f_i cyclically decreasing.

As above, we extend this definition to $f \in \tilde{S}_n^k$ for arbitrary k by defining $\tilde{F}_f$ as $\tilde{F}_{\tau^{-k} f}$.

Theorem 15 ([10], Theorem 7.1)**.** *For $f \in \mathrm{Bound}(k,n)$, the cohomology class $[\Pi_f]$ is $\phi(\tilde{F}_f)$.*

The ordinary Stanley symmetric functions indexed by members of S_n, introduced by Stanley in [21], are examples of affine Stanley symmetric functions. To be precise, we can view $w \in S_n$ as the affine permutation in $\tilde{S}_n^0$ sending $i + pn$ to $w(i) + pn$ for $1 \leqslant i \leqslant n$. Then the Stanley symmetric function F_w of w is $\tilde{F}_w$. This is Proposition 5 in [12], but we will simply take it as a definition of F_w. One should be aware, however, that the F_w defined in [21] is our $F_{w^{-1}}$.

3.2 Grassmann duality

Let $\mathrm{Gr}^k(n)$ be the Grassmannian of k-planes in $(\mathbb{C}^n)^*$. The *annihilator* of $V \in \mathrm{Gr}_k(n)$ is

$$\mathrm{ann}(V) = \{\alpha \in (\mathbb{C}^n)^* : \alpha|_V = 0\} \in \mathrm{Gr}^{n-k}(n).$$

The map $\mathrm{Gr}_k(n) \to \mathrm{Gr}^{n-k}(n)$ sending V to $\mathrm{ann}(V)$ is an isomorphism, and we refer to a pair of closed subvarieties which correspond under this isomorphism as *Grassmann duals.*

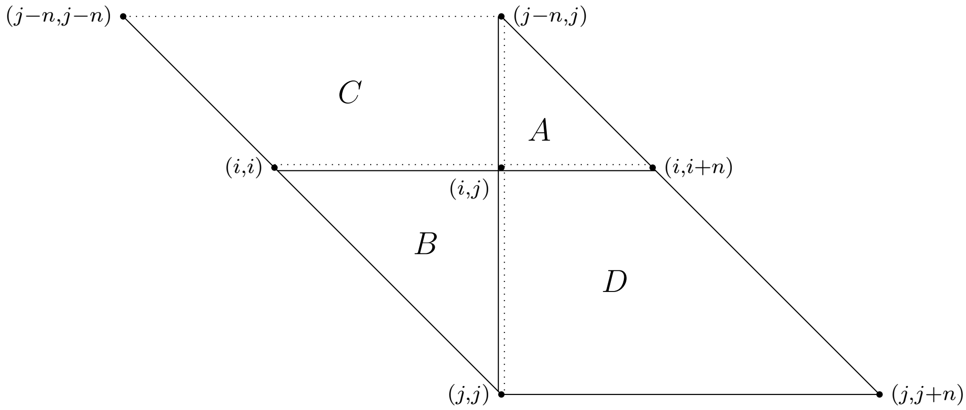
Let $\varepsilon_1, \dots, \varepsilon_n$ denote the dual basis of $e_1, \dots, e_n$. For $S \subseteq [n]$, we write $\bar{S}$ for $[n] \setminus S$ and $\langle S^* \rangle$ for $\langle \varepsilon_i : i \in S \rangle$. Observe that if $f \in \mathrm{Bound}(k,n)$, then $\tau^n f^{-1} \in \mathrm{Bound}(n-k, n)$.

Lemma 16 ([9], Proposition 2.1). *For $f \in \mathrm{Bound}(k,n)$, the positroid varieties $\Pi_f \subseteq \mathrm{Gr}_k(n)$ and $\Pi_{\tau^n f^{-1}} \subseteq \mathrm{Gr}^{n-k}(n)$ are Grassmann dual.*

Lemma 16 is straightforward given the following technical lemma, which will also be useful later on.

Lemma 17. *For $f \in \mathrm{Bound}(k,n)$ and $i \leqslant j \leqslant i+n$, let $a = \#[i,j](f)$ and b be the number of 1's in the permutation matrix of f which are strictly northeast and weakly southwest of (i,j), respectively. Then $\#[i,j] + a = k + b$.*

Proof. Consider the following part of the permutation matrix of f, divided into four regions:



Here a line segment on the boundary of a region is included in the region if the segment is solid, and not included if it is dotted. For instance, $C = \{(p,q) : j{-}n < p < i, p \leqslant q \leqslant j\}$. Let a,b,c,d denote the number of 1's in the regions A,B,C,D. Boundedness of f implies that all the 1's in its permutation matrix lie (weakly) between the two diagonal lines in this picture, so since $B \cup D$ contains $\#[i,j]$ rows we have $b+d = \#[i,j]$. Since $f \in \mathrm{Bound}(k,n)$, exactly k of $f(1),\ldots,f(n)$ exceed n, and by quasi-periodicity this says $a+d=k$. But now $\#[i,j] + a = b+d+a = k+b$. $\square$

Proof of Lemma 16. Take $V \in \mathrm{Gr}_k(n)$. We claim that for any cyclic interval $[i,j]$ in $[n]$,

$$\dim \mathrm{Prj}_{[i,j]} V \leqslant k - \#[i,j](f) \quad \Longleftrightarrow \quad \dim \mathrm{Prj}_{\overline{[i,j]}^*} \mathrm{ann}(V) \leqslant (n-k) - \#\overline{[i,j]}(\tau^n f^{-1}),$$

which will prove the lemma according to Definition 12. For any $S \subseteq [n]$, the rank of the composite $V \hookrightarrow \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n/\langle \bar{S} \rangle$ is $\dim \mathrm{Prj}_S V$, and by dualizing one sees that this is the same as $\#S - (n{-}k) + \dim \mathrm{Prj}_{\bar{S}^*} \mathrm{ann}(V)$. Taking $S = [i,j]$,

$$\dim \mathrm{Prj}_{[i,j]} V \leqslant k - \#[i,j](f) \quad \Longleftrightarrow \quad \dim \mathrm{Prj}_{\overline{[i,j]}^*} \mathrm{ann}(V) \leqslant n - \#[i,j] - \#[i,j](f).$$

Thus, to prove the claim we must show that

$$\#[i,j] + \#[i,j](f) = k + \#\overline{[i,j]}(\tau^n f^{-1}). \tag{1}$$

The permutation matrix of f is the permutation matrix of $\tau^n f^{-1}$ shifted left n units and reflected across the diagonal of $\mathbb{Z} \times \mathbb{Z}$, and so $\#\overline{[i,j]}(\tau^n f^{-1}) = \#[j{+}1, n{+}i{-}1](\tau^n f^{-1})$ is the number of 1's weakly southwest of (i,j) in the permutation matrix of f. Lemma 17 now implies equation (1). $\square$

3.3 Interval positroid varieties

An *interval positroid variety* is one for which all rank conditions in Definition 12 are implied by conditions involving actual intervals in $[n]$.

Theorem 18 ([9]). *For $f \in \mathrm{Bound}(k,n)$, Π_f is an interval positroid variety if and only if the subsequence of $f(1),\dots,f(n)$ consisting of the entries exceeding n is increasing.*

Any f as in the preceding theorem is determined by the subsequence of $f(1),\dots,f(n)$ of entries not exceeding n, which is a *partial permutation*, i.e. an injection from a subset of $[n]$ into $[n]$. Let $\bar{f}$ denote the partial permutation associated to $f \in \mathrm{Bound}(k,n)$ in this way. For instance, if $f = 15748$ then $\bar{f} = 15\_4\_$, where a _ in position i indicates that i is not in the domain of $\bar{f}$. Conversely, if the domain $\mathrm{dom}(\bar{f})$ has size $n-k$ and $\bar{f}(i) \geqslant i$ for $i \in \mathrm{dom}(\bar{f})$, then $\bar{f}$ labels an interval positroid variety. We now describe a different way to index interval positroid varieties, following [1] (up to Grassmann duality).

Definition 19 ([1]). A *rank set* in $[n]$ is a finite set of intervals $M = \{[a_1,b_1],\dots,[a_m,b_m]\}$ with $a_i \leqslant b_i \leqslant n$ positive integers, where all a_i are distinct and all b_i are distinct. For $S \subseteq [n]$, let $S(M)$ denote the set of intervals $S' \in M$ such that $S' \subseteq S$.

To a rank set M in $[n]$ with $n-k$ intervals we associate the variety

$$\Pi_M = \{V \in \mathrm{Gr}_k(n) : \dim \mathrm{Prj}_S V \leqslant \#S - \#S(M) \text{ for all intervals } S \subseteq [n]\}.$$

This is in fact an interval positroid variety, labelled by the affine permutation constructed as follows. Say $M = \{[a_1,b_1],\dots,[a_{n-k},b_{n-k}]\}$ is a rank set with $a_1 < \cdots < a_{n-k} \leqslant n$. Define

$$\{c_1 < \cdots < c_k\} = [n] \setminus \{a_1,\dots,a_{n-k}\} \quad \text{and} \quad \{d_1 < \cdots < d_k\} = [n] \setminus \{b_1,\dots,b_{n-k}\}.$$

Let $f_M \in \tilde{S}_n$ be the affine permutation which maps a_i to b_i and c_i to $d_i + n$. Then f_M is bounded because $a_i \leqslant b_i$, which implies $d_i \leqslant c_i$.

Example 20. Take $M = \{[1,1],[2,5],[4,4]\}$ and $n = 5$. Then $f_M = 15748$ and $\bar{f}_M = 15\_4\_$.

Lemma 21. *For a rank set M in $[n]$ we have $\Pi_M = \Pi_{f_M}$.*

Proof. By construction, the entries of $f_M(1),\dots,f_M(n)$ exceeding n appear in increasing order, so Π_M is an interval positroid variety by Theorem 18. Therefore it suffices to show that $\#[i,j] - \#[i,j](M) = k - \#[i,j](f_M)$ for all intervals $[i,j]$ in $[n]$.

Let $B = \{q \in \mathbb{Z} : (q, f_M(q)) \text{ is weakly southwest of } (i,j)\}$. We claim that $\#B = \#[i,j](M)$, in which case we are done by Lemma 17. Clearly $[a_p,b_p] = [a_p, f_M(a_p)] \subseteq [i,j]$ if and only if $a_p \in B$, so $\#[i,j](M) = B \cap \{a_1,\dots,a_{n-k}\}$. But in fact every $q \in B$ is some a_p, because $f_M(q) \leqslant j \leqslant n$ and $1 \leqslant i \leqslant q$ force $q \in \{a_1,\dots,a_{n-k}\}$. $\square$

It follows from Theorem 13 that Π_M is irreducible of dimension $k(n-k) - \ell(f_M)$. The next lemma gives a formula for this dimension more directly in terms of M (cf. [3, Lemma 3.29]).

Lemma 22. *For any rank set M, $\dim \Pi_M = \sum_{S \in M}(\#S - \#S(M))$.*

Proof. As before, write $M = \{[a_1, b_1], \ldots, [a_{n-k}, b_{n-k}]\}$ where $a_1 < \cdots < a_{n-k}$. Also, write $\dim(M)$ for $\sum_{S \in M}(\#S - \#S(M))$, so we want to prove that $\dim(M) = \dim \Pi_M$. Let $i(M)$ be the maximal $i \in [n-k]$ such that $a_i < k + i$; if no such i exists, set $i(M) = -\infty$. When $i(M)$ is finite, we will define a new rank set M' with the property that either $\dim(M') < \dim(M)$, or $\dim(M') = \dim(M)$ and $i(M') < i(M)$. Thus, after finitely many operations of the form $M \mapsto M'$ we obtain an M'' with $i(M'') = -\infty$, which must be $M'' = \{\{k+1\}, \{k+2\}, \ldots, \{n\}\}$. In this case $f_{M''} = (n+1) \cdots (n+k)(k+1) \cdots n$ has length $k(n-k)$, so $\dim \Pi_{M''} = 0$ and the lemma holds. It therefore suffices to show that $\dim(M) - \dim(M') = \dim \Pi_M - \dim \Pi_{M'}$.

(a) First suppose $a_i < b_i$. Let M' be M with $S = [a_i, b_i]$ replaced by $S' = [a_i + 1, b_i]$. The choice of i implies that $a_i + 1$ remains in $[n]$ and is not the left endpoint of an interval in M, so M' is a valid rank set. Moreover, the multiset of numbers $\#T(M)$ for $T \in M$ is the same as the multiset of numbers $\#T'(M')$ for $T' \in M'$, so $\dim(M) - \dim(M') = 1$. On the other hand, f_M and $f_{M'}$ agree except in positions a_i and $a_i + 1$, where

$$\begin{aligned} f_M(a_i) &= b_i, & f_M(a_i+1) &= d_j + n \text{ (for some } j) \\ f_{M'}(a_i) &= d_j + n, & f_{M'}(a_i+1) &= b_i. \end{aligned}$$

In particular, $f_{M'} = f_M s_{a_i} > f_M$ in weak Bruhat order, so

$$\dim \Pi_M - \dim \Pi_{M'} = \ell(f_{M'}) - \ell(f_M) = 1 = \dim(M) - \dim(M').$$

(b) Now suppose $a_i = b_i$.

(i) Suppose $a_i + 1$ is not the *right* endpoint of an interval. Define M' to be M with $[a_i, a_i]$ replaced by $[a_i + 1, a_i + 1]$. Then M' is a valid rank set with $\dim M' = \dim M$, On the other hand, $\Pi_{M'}$ is the image of Π_M under the invertible linear map switching e_{a_i} with e_{a_i+1} and fixing all other e_j, and so $\dim \Pi_{M'} = \dim \Pi_M$.

(ii) Suppose $a_i + 1 = b_h$ for some h. Define M' to be M with $[a_i, a_i]$ replaced by $[a_i + 1, a_i + 1]$ and $[a_h, b_h]$ replaced by $[a_h, b_h - 1] = [a_h, a_i]$. This is a valid rank set, and one checks that $\dim(M) = \dim(M')$ again. The affine permutations f_M and $f_{M'}$ agree except that

$$\begin{aligned} f_M(a_h) &= a_i + 1, & f_M(a_i) &= a_i, & f_M(a_i+1) &= d_j + n \text{ (for some } j) \\ f_{M'}(a_h) &= a_i, & f_{M'}(a_i) &= d_j + n, & f_{M'}(a_i+1) &= a_i + 1 \end{aligned}$$

Hence, $f_{M'} = s_{a_i} f_M s_{a_i}$ with $f_M < f_M s_{a_i} > s_{a_i} f_M s_{a_i}$ in weak Bruhat order. In particular, $\ell(f_M) = \ell(f_{M'})$ so that $\dim \Pi_M = \dim \Pi_{M'}$.

In either case, $\dim(M) = \dim(M')$ and $\dim \Pi_M = \dim \Pi_{M'}$. If $a_i + 1 < k + i$, then $i(M') = i(M)$, but after $k + i - a_i$ steps of type (b) the statistic i must decrease.

∎

3.4 Stability

Fix inclusions $\mathbb{C} \subseteq \mathbb{C}^2 \subseteq \cdots$ and linearly independent vectors $e_1, e_2, \dots$ with $e_i \in \mathbb{C}^i$ for all i. Let $R_{k,n}$ denote the homogeneous coordinate ring of $\mathrm{Gr}_k(n)$ under the Plücker embedding, so $R_{k,n}$ is generated by Plücker coordinates p_I for $I \in \binom{[n]}{k}$. Any Plücker relation in $R_{k,n}$ is still a Plücker relation in $R_{k,n+1}$, so there are injective ring homomorphisms $R_{k,n} \hookrightarrow R_{k,n+1} \hookrightarrow \cdots$ sending p_I to p_I, which we view as inclusions. Given a subscheme $Z \subseteq \mathrm{Gr}_k(n)$ determined by a homogeneous ideal $J \subseteq R_{k,n}$, let Z^+ be the subscheme of $\mathrm{Gr}_k(n+1)$ determined by the ideal $R_{k,n+1}J$. That is, Z^+ is cut out by the same equations as Z, but now inside $\mathrm{Gr}_k(n+1)$.

Proposition 23. *Let* $\iota : \mathrm{Gr}_k(n) \to \mathrm{Gr}_k(n{+}1)$ *be the inclusion, inducing a pullback* $\iota^* : H^* \mathrm{Gr}_k(n{+}1) \to H^* \mathrm{Gr}_k(n)$. *Then* $\iota^*[Z^+] = [Z]$.

Proof. Whenever $Y \subseteq \mathrm{Gr}_k(n+1)$ intersects $\iota\,\mathrm{Gr}_k(n)$ transversely it holds that $\iota^*[Y] = [Y \cap \iota\,\mathrm{Gr}_k(n)]$ with $[Y \cap \iota\,\mathrm{Gr}_k(n)]$ viewed as a cycle on $\mathrm{Gr}_k(n)$, and one can verify that Z^+ intersects $\iota\,\mathrm{Gr}_k(n)$ transversely by working in charts. $\square$

Let Λ_k be the ring of symmetric polynomials over $\mathbb{Z}$ in $x_1, \dots, x_k$. Then $H^* \mathrm{Gr}_k(n) \simeq \Lambda_k/(s_\lambda : \lambda \not\subseteq [k] \times [n-k])$, and these isomorphisms induce an isomorphism of the inverse limit $\varprojlim_N H^* \mathrm{Gr}_k(N)$ with Λ_k. Here, we take the inverse limit with respect to the maps

$$\cdots \xrightarrow{\iota^*} H^* \mathrm{Gr}_k(k+1) \xrightarrow{\iota^*} H^* \mathrm{Gr}_k(k).$$

Proposition 23 shows that the classes $[Z], [Z^+], [Z^{++}], \dots$ define an element α of the inverse limit $\varprojlim H^* \mathrm{Gr}_k(N)$; we say $F \in \Lambda_k$ is a *stable* representative for $[Z]$ if it represents α.

Now suppose M is a rank set for $\mathrm{Gr}_k(n)$. Define M^+ to be $M \cup \{[a, n+1]\}$ where a is the minimal member of $[n+1]$ which is not a left endpoint in $[n]$. Evidently M^+ is a rank set for $\mathrm{Gr}_k(n+1)$.

Lemma 24. $\Pi_{M^+} = \Pi_M^+$.

Proof. Let $S \subseteq [n+1]$ be an interval, and consider a rank condition

$$\dim \mathrm{Prj}_S V \leqslant \#S - \#S(M^+) \tag{2}$$

for Π_{M^+}. We must see that (2) follows from the rank conditions defining Π_M. Consider three cases.

(a) If $n+1 \notin S$, then $S(M^+) = S(M)$, and (2) is itself a rank condition defining Π_M.

(b) Suppose $S = [i, n+1]$ with $i > a$, and set $S' = [i, n]$. Then $\#S - \#S(M^+) = \#S' - \#S'(M) + 1$, so (2) follows from the rank condition $\dim \mathrm{Prj}_{S'} V \leqslant \#S' - \#S'(M)$ for Π_M.

(c) Suppose $S = [i, n+1]$ with $i \leqslant a$. Then S contains every interval of M^+ except $[1, b_1], \dots, [i-1, b_{i-1}]$, and so $\#S - \#S(M^+) = \#[i, n+1] - (\#M^+ - (i-1)) = k$: the rank condition (2) is vacuous.

□

Let M^{+r} denote the result of applying the $^+$ operation r times starting with M; when $f = f_M$, we also write f^{+r} and $\bar{f}^{+r}$ to mean $f_{M^{+r}}$ and $\bar{f}_{M^{+r}}$. Write S_∞ for the union $\bigcup_{n=0}^{\infty} S_n$, identifying S_n with the subgroup of S_{n+1} fixing $n+1$.

Lemma 25. *Let M be a rank set for $\mathrm{Gr}_k(n)$. There exists an integer R such that*

- *$f_M^{+r}\tau^{-k} \in S_{n+r}$ for $r \geqslant R$, and*
- *the permutations $f_M^{+r}\tau^{-k}$ for $r \geqslant R$ are all the same as members of S_∞.*

Proof. Suppose first $\bar{f}_M$ has domain $[1, n-k]$, so $\bar{f}_M = b_1 \cdots b_{n-k} \_ \cdots \_$. Then $f_M\tau^{-k} = d_1 \cdots d_k b_1 \cdots b_{n-k}$ is in S_n. In general, $\bar{f}^+$ is the partial permutation of $[n+1]$ agreeing with $\bar{f}$ on $\mathrm{dom}(\bar{f})$, and sending the minimal member of $[n+1] \setminus \mathrm{dom}(\bar{f})$ to $n+1$. Thus, $f_M^+\tau^{-k} = d_1 \cdots d_k b_1 \cdots b_{n-k}(n+1)$, which is equal to $f_M\tau^{-k}$ as a member of S_∞.

For an arbitrary $\bar{f}_M$, it suffices by the previous paragraph to find R such that $\bar{f}_M^{+R}$ has domain $[1, n+R-k]$. Any R such that $\mathrm{dom}(\bar{f}_M) \subseteq [1, R + \#\,\mathrm{dom}(\bar{f}_M)]$ does the job. □

Theorem 26. *For any interval positroid variety Π_M, there is an ordinary permutation w such that the Stanley symmetric function F_w is a stable representative for the class $[\Pi_M]$.*

Proof. Since the reduced words of a permutation w only depend on w as an element of S_∞, the same is true of F_w. Lemma 25 therefore shows that the sequence $\tilde{F}_{f_M^{+r}}$ for $r \geqslant 0$ is eventually constant and equal to some F_w. These symmetric functions represent the classes $[\Pi_M^{+r}]$ by Lemma 24, so F_w stably represents the class $[\Pi_M]$. □

Although $\phi(\tilde{F}_f)$ must be Schubert-positive, and it is known that F_w is Schur-positive [4], the symmetric functions $\tilde{F}_f$ are *not* always Schur-positive. For instance, if $M = \{[2,2],[4,4]\}$ with $\Sigma_M \subseteq \mathrm{Gr}_2(4)$, then $f_M = 5274$, and $\tilde{F}_{5274} = s_{22} + s_{211} - s_{1111}$. On the other hand, $M^{++} = \{[2,2],[4,4],[1,5],[3,6]\}$, $f_M^+ = 526479$, and $\tilde{F}_{526479} = F_{135264} = s_{22} + s_{211}$. Thus, Theorem 26 provides a canonical way to represent interval positroid classes by Schur-positive symmetric functions.

4 Degenerations of dual interval positroid varieties

Given a subset $E \subseteq [k] \times [n]$, define

$$\Sigma_E^\circ = \{\mathrm{rowspan}\, A : A \in M_{k,n} \text{ such that } A_{pq} = 0 \text{ whenever } (p,q) \notin E\} \subseteq \mathrm{Gr}_k(n)$$

and $\Sigma_E = \overline{\Sigma_E^\circ}$. For a generic $V = \mathrm{rowspan}\, A \in \Sigma_E^\circ$, the matroid of V is the *transversal matroid* associated to the columns of E: the matroid on $[n]$ whose bases are the sets $\{j_1, \ldots, j_k\}$ for which $(1, j_1), \ldots, (k, j_k) \in E$. Thus, Σ_E is the closure of a matroid stratum.

We identify a rank set (or any collection of intervals) $M = \{S_1, \ldots, S_k\}$ in $[n]$ with the subset

$$\{(i,j) : i \in [k], j \in S_i\} \subseteq [k] \times [n],$$

and define Σ_M accordingly. For instance, if $M = \{[1,3],[3,6],[4,5]\}$ and $n = 6$, then Σ_M° is the set of rowspans of full rank matrices of the form

$$\begin{bmatrix} * & * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & 0 \end{bmatrix}.$$

The varieties Σ_M are the "rank varieties" defined in [1], where it is shown that they are exactly the projections of Schubert varieties in partial flag varieties $\mathrm{Fl}(k_1,\dots,k_p;\mathbb{C}^n)$ with $k_p = k$ to $\mathrm{Gr}_k(n)$ (see also [3]).

Lemma 27. $\dim \Sigma_M = \sum_{S\in M}(\#S - \#S(M))$ *for a rank set* M.

Proof. Write $M = \{[a_1,b_1],\dots,[a_k,b_k]\}$ where $a_1 < \cdots < a_k$. Let V be the set of $k\times(n-k)$ matrices A such that

- $A_{i,a_i} = 1$ for each i;
- If $j \notin [a_i,b_i]$, then $A_{ij} = 0$;
- If $[a_\ell,b_\ell] \subseteq [a_i,b_i]$ with $\ell \neq i$, then $A_{i,a_\ell} = 0$;
- If A_{ij} has not been defined already, it is nonzero.

For example, if $M = \{[1,4],[2,6],[4,5]\}$, then

$$V = \left\{ \begin{bmatrix} 1 & * & * & * & 0 & 0 \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & 0 \end{bmatrix} : \text{all } * \text{ nonzero} \right\}$$

Note that $\dim V = \sum_{S\in M}(\#S - \#S(M))$. The map $A \mapsto \mathrm{rowspan}(A)$ takes V onto a dense subset of Σ_M°, so to prove the lemma it suffices to show that this map is injective, i.e. that if $A, gA \in V$ for some $g \in \mathrm{GL}_k(\mathbb{C})$, then $g = 1$.

Use the Bruhat decomposition of GL_k to write $g = u_1 t u_2$, where t is diagonal and u_1, u_2 are respectively upper and lower triangular with 1's on the diagonal. If $gA \in V$, then $u_2 = 1$, for otherwise gA would have a nonzero entry below some position (i,a_i). Next, $t = 1$, for otherwise gA would have an entry other than 1 in some position (i,a_i). Finally, $u_1 = 1$, for otherwise if u_1 added a multiple of some row ℓ to a row $i < \ell$, then gA would have a nonzero entry in position A_{i,a_ℓ} (if $b_\ell \leqslant b_i$) or position A_{i,a_i+1} (if $b_\ell > b_i$). □

We will not need this fact, but it is worth noting that the proof of Lemma 27 only requires that all left endpoints of intervals in M are distinct, or that all right endpoints are, but not both (as required by the definition of a rank set).

Lemma 28. *The Grassmann dual to an interval positroid variety* $\Pi_M \subseteq \mathrm{Gr}^{n-k}(n)$ *is* $\Sigma_M \subseteq \mathrm{Gr}_k(n)$.

Proof. Let Π_M^* denote the Grassmann dual of Π_M. Recall that $V \in \Pi_M$ if and only if $\dim \operatorname{Prj}_{S^*} V \leqslant \#S - \#S(M)$ for all $S \in M$. As in the proof of Lemma 16,

$$\dim \operatorname{Prj}_{S^*} V = \#S - k + \dim \operatorname{Prj}_{\bar{S}} \operatorname{ann}(V) = \#S - \dim(\operatorname{ann}(V) \cap \langle S \rangle).$$

Thus, $W = \operatorname{ann}(V) \in \Pi_M^*$ if and only if $\dim(W \cap \langle S \rangle) \geqslant \#S(M)$ for $S \in M$. These rank conditions hold when $W \in \Sigma_M^\circ$, so $\Sigma_M \subseteq \Pi_M^*$. Since Π_M^* is irreducible and has the same dimension as Σ_M by Lemmas 22 and 27, we are done. $\square$

Let $\phi_{t,i\to j}$ be the linear transformation sending e_i to $te_i + (1-t)e_j$. For $t \neq 0$, the varieties $\phi_{t,i\to j}\Sigma_M$ are all isomorphic, so they form a flat family [5, Proposition III-56]. The flat limit $\lim_{t\to 0} \phi_{t,i\to j}\Sigma_M$ then exists as a scheme [7, Proposition 9.8]. The key fact for us is that Σ_M and $\lim_{t\to 0} \phi_{t,i\to j}\Sigma_M$ have the same Chow ring class, hence the same cohomology class. Other authors have used these degenerations to calculate cohomology classes or K-theory classes of subvarieties of Grassmannians, including Coskun [3] and Vakil [22]. Our goal in this section is to exhibit a degeneration of Σ_M, for an appropriate M, which contains the diagram variety $X_{D(w)}$ as an irreducible component.

For a closed subscheme $X \subseteq \operatorname{Gr}_k(n)$, let $C_{i\to j}X = \lim_{t\to 0} \phi_{t,i\to j}X$. For $E \subseteq [k] \times [n]$, let $C_{i\to j}E$ be the subset of $[k]\times[n]$ obtained from E by replacing columns i and j by their intersection and union, respectively. That is, $(p,q) \in C_{i\to j}E$ if and only if

- $q \notin \{i,j\}$ and $(p,q) \in E$, or
- $q = i$ and $(p,i), (p,j) \in E$, or
- $q = j$ and $(p,i) \in E$ or $(p,j) \in E$.

Lemma 29 ([14], Proposition 5.3.3)**.** *For any $E \subseteq [k]\times[n]$ we have $\Sigma_{C_{i\to j}E} \subseteq C_{i\to j}\Sigma_E$.*

Given a permutation $w \in S_n$, define a rank set $M(w) = \{[w(i), i+n] : 1 \leqslant i \leqslant n\}$, so $\Sigma_{M(w)} \subseteq \operatorname{Gr}_n(2n)$. Then

$$\tau^{2n} f_{M(w)}^{-1} = (n+1)\cdots(2n)(w(1)+2n)\cdots(w(n)+2n) = (w \times 12\cdots n)\tau^{-n}.$$

Here, for $w \in S_n$ and $v \in S_m$, $w \times v$ is the permutation in S_{n+m} sending i to $w(i)$ if $i \leqslant n$ and to $v(i-n)+n$ otherwise. By Lemmas 28 and 16, $\Sigma_{M(w)} = \Pi_{\tau^{2n} f_{M(w)}^{-1}}$. It is clear from Definition 14 that $F_{w\times 12\cdots n} = F_w$, so Theorem 15 gives

$$[\Sigma_{M(w)}] = [\Pi_{\tau^{2n} f_{M(w)}^{-1}}] = \phi(F_{w\times 12\cdots n}) = \phi(F_w).$$

In fact, $\Sigma_{M(w)}$ is a *graph Schubert variety* as defined in [10, §6], where it is also shown that $[\Sigma_{M(w)}] = \phi(F_w)$.

On the other hand, it is known [17] that $s_{D(w)} = F_w$ where $D(w)$ is the *Rothe diagram* of w:

$$D(w) = \{(i, w(j)) \in [n]\times[n] : i < j, w(i) > w(j)\}.$$

For example,

$$D(3142) = \begin{matrix} \circ & \circ & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \circ & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{matrix}$$

Here we are using $\circ$ for points of $[n] \times [n]$ in $D(w)$ and $\cdot$ for points not in it. We also use matrix coordinates, meaning that $(1, 1)$ is at the upper left.

Let C_w be the composition of the (commuting) operators $C_{n+i\to w(i)}$ for $i \in [n]$, acting either on subsets of $[2n]$ or subschemes of $\mathrm{Gr}_n(2n)$ as before.

Theorem 30. *For $w \in S_n$, the diagram variety $X_{D(w)}$ is an irreducible component of $C_w\Sigma_{M(w)}$.*

Proof. Define

$$E(w) = ([n] \times [n] \setminus D(w)) \cup \{(i, n + i) : i \in [n]\},$$

so that $X_{D(w)} = \Sigma_{E(w)}$. Since $\operatorname{codim} \Sigma_{M(w)} = \ell(w) = \operatorname{codim} X_{D(w)}$, it suffices by Lemma 29 to show that $\Sigma_{C_w M(w)} = \Sigma_{E(w)}$.

Recall that we identify $M(w)$ with the set $\{(i, j) : i \in [n], w(i) \leqslant j \leqslant i + n\}$. First, if $j \leqslant n$ then $(i, j) \notin C_w M(w)$ if and only if $(i, j), (i, w^{-1}(j) + n) \notin M(w)$, if and only if $j < w(i)$ and $i < w^{-1}(j)$, if and only if $(i, j) \in D(w)$: thus $C_w M(w)$ and $E(w)$ agree on $[n] \times [n]$. For instance, $\Sigma_{M(3142)}$ contains

$$\left\{ \text{rowspan} \begin{bmatrix} 0 & 0 & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * & 0 \\ 0 & * & * & * & * & * & * & * \end{bmatrix} \right\}$$

as a dense subset, and $C_{3142}\Sigma_{M(3142)}$ accordingly contains

$$\left\{ \text{rowspan} \begin{bmatrix} 0 & 0 & * & * & * & 0 & 0 & 0 \\ * & * & * & * & * & * & 0 & 0 \\ * & 0 & * & * & 0 & 0 & * & 0 \\ * & * & * & * & * & 0 & * & * \end{bmatrix} \right\}.$$

As we see in this example, $C_w M(w)$ and $E(w)$ need *not* agree on $[n] \times [n + 1, 2n]$. However, note that $(i, j + n) \in C_w M(w)$ if and only if $i > j$ and $w(j) > w(i)$, and it is easy to check that this is equivalent to row j of $D(w)$ containing row i. Thus, if A is a matrix whose nonzero entries are exactly in positions $C_w M(w)$, then a row operation can be performed on rows i and j which replaces the $*$ in position $(i, j + n)$ by 0 without changing the pattern of $*$'s in $[n] \times [n]$. This shows that $\Sigma_{C_w M(w)} = \Sigma_{E(w)}$. $\square$

Since $[\lim_{t\to 0} \phi_{t,w}\Sigma_{M(w)}] = [\Sigma_{M(w)}]$, an immediate corollary is an upper bound on $[X_{D(w)}]$.

Theorem 31. *$\phi(F_w) - [X_{D(w)}]$ is a nonnegative combination of Schubert classes.*

However, this difference of classes can be nonzero. Indeed, the counterexample $D = \{(1,1),(2,2),(3,3),(4,4)\}$ to Conjecture 5 discussed in Section 2 provides an example. Take $w = 21436587$. Then $D(w) = \{(1,1),(3,3),(5,5),(7,7)\}$ can be obtained from D by permuting rows and columns, and viewing D in a larger rectangle. Neither of these operations on diagrams affects s_D or $[X_D]$, identifying the latter with its pullback along the embeddings of $\mathrm{Gr}_k(n)$ into $\mathrm{Gr}_k(n{+}1)$ or $\mathrm{Gr}_{k+1}(n{+}1)$.

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