Combinatorial interpretations of the Kreweras triangle in terms of subset tuples

Ange Bigeni
Faculty of Mathematics
National Research University Higher School of Economics
Moscow, Russian Federation

ange.bigeni@gmail.com

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Abstract

We show how the combinatorial interpretation of the normalized median Genocchi numbers in terms of multiset tuples, defined by Hetyei in his study of the alternation acyclic tournaments, is bijectively equivalent to previous models like the normalized Dumont permutations or the Dellac configurations, and we extend the interpretation to the Kreweras triangle.

Keywords: Genocchi numbers; Kreweras triangle; Dumont permutations; Dellac configurations

Mathematics Subject Classifications: 05A05, 05A15, 05A19

1 Introduction

For all pair of integers $n < m$, the set $\{n, n+1, \ldots, m\}$ is denoted by $[n, m]$, and the set $[1, n]$ by $[n]$. The set of the permutations of $[n]$ is denoted by $\mathfrak{S}_n$.

1.1 Genocchi numbers, Kreweras triangle, Dumont permutations

The Genocchi numbers $(G_{2n})_{n \geqslant 1} = (1, 1, 3, 17, 155, 2073, \ldots)$ [13] and median Genocchi numbers $(H_{2n+1})_{n \geqslant 0} = (1, 2, 8, 56, 608, \ldots)$ [14] can be defined as the positive integers $G_{2n} = g_{2n-1,n}$ and $H_{2n+1} = g_{2n+2,1}$ [7] where $(g_{i,j})_{1 \leqslant j \leqslant i}$ is the Seidel triangle, defined by $g_{1,1} = 1$ and

$$
\begin{aligned}
g_{2p,j} &= g_{2p-1,j} + g_{2p,j+1}, \\
g_{2p+1,j} &= g_{2p+1,j-1} + g_{2p,j}
\end{aligned}
$$

for all $p \geqslant 1$, where $g_{2p,p+1} = g_{2p+1,0} = 0$. It is well known that H_{2n+1} is divisible by 2^n for all $n \geqslant 0$ [1]. The normalized median Genocchi numbers $(h_n)_{n\geqslant 0} = (1, 1, 2, 7, 38, 295, \ldots)$ [15] are the positive integers defined by

$$h_n = H_{2n+1}/2^n.$$

Among the first combinatorial models of the (median) Genocchi numbers [5, 7, 1, 6], there is the set $PD2_n$ of the Dumont permutations of the second kind, that is, the permutations $\sigma \in \mathfrak{S}_{2n+2}$ such that $\sigma(2i-1) > 2i-1$ and $\sigma(2i) < 2i$ for all $i \in [n+1]$, whose cardinality $\#PD2_n$ equals H_{2n+1} for all $n \geqslant 0$. In [11], Kreweras introduced the subset $PD2N_n \subset PD2_n$ of the normalized permutations, *i.e.*, the permutations $\sigma \in PD2_n$ such that $\sigma^{-1}(2i) < \sigma^{-1}(2i+1)$ for all $i \in [n]$, whose number is $\#PD2N_n = h_n$.

Remark 1. For all $(k,l) \in [n]^2$, let $PD2N_{n,k}$ (respectively $PD2N'_{n,l}$) be the subset of the permutations $\sigma \in PD2N_n$ such that $\sigma(1) = 2k$ (respectively $\sigma(2n+2) = 2l+1$). It is easy to see that $\{PD2N_{n,k} : k \in [n]\}$ and $\{PD2N'_{n,l} : l \in [n]\}$ are partitions of $PD2N_n$.

In [12], by introducing the model of the alternating diagrams and connecting them bijectively to the normalized Dumont permutations, Kreweras and Barraud proved that

$$\#PD2N_{n,k} = \#PD2N'_{n,k} = h_{n,k}$$

where the Kreweras triangle $(h_{n,k})_{n\geqslant 1, k\in[n]}$ [11] (see Figure 1.1) is defined by $h_{1,1} = 1$ and, for all $n \geqslant 2$ and $k \in [3, n]$,

$$\begin{aligned} h_{n,1} &= h_{n-1,1} + h_{n-1,2} + \ldots + h_{n-1,n-1}, \\ h_{n,2} &= 2h_{n,1} - h_{n-1,1}, \\ h_{n,k} &= 2h_{n,k-1} - h_{n,k-2} - h_{n-1,k-1} - h_{n-1,k-2}. \end{aligned} \tag{1}$$

$$\begin{array}{ccccccccccccc} & & & & & & 1 & & & & & & \\ & & & & & 1 & & 1 & & & & & \\ & & & & 2 & & 3 & & 2 & & & & \\ & & & 7 & & 12 & & 12 & & 7 & & & \\ & & 38 & & 69 & & 81 & & 69 & & 38 & & \\ & 295 & & 552 & & 702 & & 702 & & 552 & & 295 & \\ \iddots & & & & & & & & & & & & \ddots \end{array}$$

Figure 1: The Kreweras triangle.

For example, we depict in Figure 1.1 how are partitioned the $h_3 = 2 + 3 + 2$ elements of $PD2N_3$.

For all $n \geqslant 1$ and $k \in [n]$, the Kreweras triangle has the visible two properties

$$h_{n,n} = h_{n-1}, \tag{2}$$

$$h_{n,k} = h_{n,n-k+1}, \tag{3}$$

$PD2N_{3,1}$		21637485	21436587
$PD2N_{3,2}$	41627583	41627385	41526387
$PD2N_{3,3}$	61427583	61427385	
	$PD2N'_{3,1}$	$PD2N'_{3,2}$	$PD2N'_{3,3}$

Figure 2: The partition of $PD2N_3$.

of which [12] implies interpretations in terms of $PD2N_n$. Formula (2) follows from the bijection $\sigma \in PD2N'_{n,n} \mapsto \sigma_{|[2n]} \in PD2N_{n-1}$. Afterwards, let $\sigma \in PD2N_n$ and $(k,l) \in [n]^2$ such that $\sigma(1) = 2k$ and $\sigma(2n+2) = 2l+1$, we define two permutations σ^t and σ^r as follows.

– If $k = l$, we define σ^t as σ, otherwise it is defined as the following composition of σ with a 4-cycle :
$$\begin{pmatrix} 2k & 2l & 2l+1 & 2k+1 \end{pmatrix} \circ \sigma.$$

– We define σ^r by $\sigma^r(i) = 2n+3-\sigma(2n+3-i)$ for all $i \in [2n+2]$.

The maps $\sigma \mapsto \sigma^t$ and $\sigma \mapsto \sigma^r$ are involutions of $PD2N_n$ which induce bijections
$$\begin{aligned} PD2N_{n,k} \cap PD2N'_{n,l} &\longleftrightarrow PD2N_{n,l} \cap PD2N'_{n,k}, \\ PD2N_{n,l} \cap PD2N'_{n,k} &\longleftrightarrow PD2N_{n,n-k+1} \cap PD2N'_{n,n+1-l}, \end{aligned}$$

from which follows Formula (3), which can also be obtained by induction from System (1) through the following easy equality (see also [12])
$$h_{n,k} - h_{n,k-1} = \sum_{i=k}^{n-1} h_{n-1,i} - \sum_{i=1}^{k-2} h_{n-1,i}$$

for all $n \geqslant 1$ and $k \in [n]$ (where $h_{n,0}$ is defined as 0).

There are several other bijectively equivalent models of the Kreweras triangle [4, 12, 9, 8, 2]. The Kreweras triangle also appeared recently in the theory of finite type Vassiliev knot invariants [3], more precisely through a polynomial generalization.

1.2 The Dellac configurations

The Dellac configurations [4] form the earliest combinatorial model of the Kreweras triangle and provide a geometrical analogous of the previous results. Recall that a Dellac configuration of size n is a tableau D, made of n columns and $2n$ rows, that contains $2n$ dots such that :

– every row contains exactly one dot;

– every column contains exactly two dots;

– if there is a dot in the box (j, i) of D (*i.e.*, in the intersection of its j-th column from left to right and its i-th row from bottom to top), then $j \leqslant i \leqslant j + n$.

The set of the Dellac configurations of size n is denoted by DC_n. It can be partitionned into $\{DC_{n,k} : k \in [n]\}$ or $\{DC'_{n,l} : l \in [n]\}$ where $DC_{n,k}$ (respectively $DC'_{n,l}$) is the subset of the tableaux $D \in DC_n$ whose box $(k, n+1)$ (respectively (l, n)) contains a dot, for all $(k, l) \in [n]^2$. In [8, Proposition 3.3], Feigin constructs a bijection $f_1 : PD2N_n \to DC_n$ such that $f_1(PD2N_{n,k}) = DC_{n,k}$, hence $h_{n,k} = \#DC_{n,k}$, for all $k \in [n]$. One can also check that $f_1(PD2N'_{n,k}) = DC'_{n,k}$, so $h_{n,k} = \#DC'_{n,k}$. For example, the $h_3 = 2 + 3 + 2$ elements of DC_3 are partitionned as depicted in Figure 3.

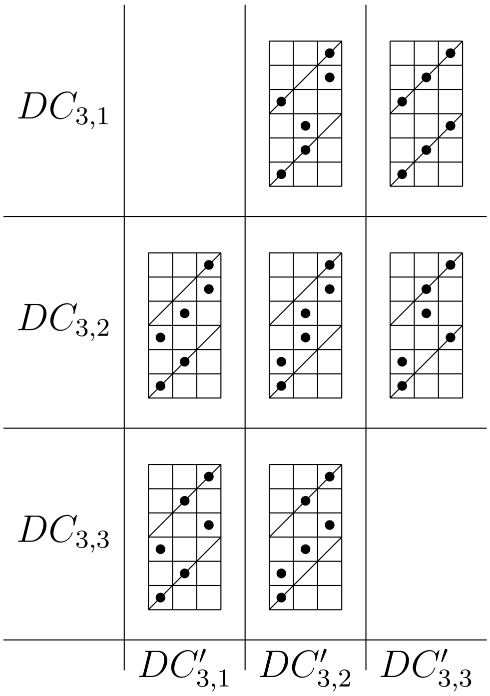

Figure 3: The partition of DC_3.

The combinatorial interpretations of Formulas (2) and (3) in terms of Dellac configurations are simple. Every element of DC_{n-1} can be obtained by deleting the n-th colomn (from left to right) and the $(n+1)$-th and $2n$-th rows (from bottom to top) of a unique element of $DC'_{n,n}$, which gives Formula (2). Afterwards, for all $D \in DC_{n,k} \cap DC'_{n,l}$,

– let $D^t \in DC_{n,l} \cap DC'_{n,k}$ be obtained by deleting the dots of the boxes $(k, n+1)$ and (l, n) of D and placing dots in the boxes $(l, n+1)$ and (k, n),

– let $D^r \in DC_{n,n+1-l} \cap DC'_{n,n-k+1}$ be obtained by rotating D through 180°, the maps $D \mapsto D^t$ and $D \mapsto D^r$ are involutions of DC_n that induce bijections

$$\begin{aligned} DC_{n,k} \cap DC'_{n,l} &\longleftrightarrow DC_{n,l} \cap DC'_{n,k}, \\ DC_{n,l} \cap DC'_{n,k} &\longleftrightarrow DC_{n,n-k+1} \cap DC'_{n,n+1-l}, \end{aligned}$$

from which follows Formula (3).

1.3 Hetyei's model

In his study of the alternation acyclic tournaments [10], Hetyei proved that the median Genocchi number H_{2n+1} is the number of pairs

$$((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \in \mathbb{Z}^n \times \mathbb{Z}^n$$

such that $(a_i, b_i) \in [0, n] \times [n]$ for all $i \in [n]$, and the set $[n]$ is contained in the multiset $\{a_1, b_1, \ldots, a_n, b_n\}$. He then defined a free group action of $(\mathbb{Z}\backslash 2\mathbb{Z})^n$ on the set of these pairs, whose orbits are indexed by the n-tuples $(\{u_i, v_i\})_{i \in [n]}$ such that $(u_i, v_i) \in [i]^2$ for all $i \in [n]$ and the multiset $\{u_1, v_1, \ldots, u_n, v_n\}$ contains $[n]$, which raises a new proof of H_{2n+1} being a multiple of 2^n, and a new combinatorial model of h_n through the set $\mathcal{M}_n$ of these tuples $(\{u_i, v_i\})_{i \in [n]}$. For example, the $h_3 = 7$ elements of $\mathcal{M}_3$ are

$$\begin{gathered} \{1,1\}, \{2,2\}, \{3,3\} \\ \{1,1\}, \{1,2\}, \{3,3\} \\ \{1,1\}, \{2,2\}, \{2,3\} \\ \{1,1\}, \{1,2\}, \{2,3\} \\ \{1,1\}, \{1,1\}, \{2,3\} \\ \{1,1\}, \{2,2\}, \{1,3\} \\ \{1,1\}, \{1,2\}, \{1,3\}. \end{gathered}$$

As for the Dumont permutations and the Dellac configurations, we now intend to define two partitions of $\mathcal{M}_n$.

Definition 2. Let $M = (\{u_i, v_i\})_{i \in [n]} \in \mathcal{M}_n$, we define a tuple

$$n = i_1 > i_2 > \ldots > i_m \geqslant 1$$

as follows : if $u_{i_p} = v_{i_p} = i_p$, then m is defined as p, otherwise i_{p+1} is defined as $\min\{u_{i_p}, v_{i_p}\} < i_p$. This tuple is well-defined because $u_1 = v_1 = 1$ in general.

Afterwards, for all integer $i \in [i_m, n]$, let $p \in [m]$ such that $i \in [i_p, i_{p-1} - 1]$ (where i_0 is defined as $n+1$), we say that i is *M-redundant* if $i_p \in \{u_i, v_i\}$. Note that the set of such integers is not empty because it contains i_m.

We are ready to define two partitions $\{\mathcal{M}_{n,k} : k \in [n]\}$ and $\{\mathcal{M}'_{n,l} : l \in [n]\}$ of $\mathcal{M}_n$.

Definition 3. For all $n \geqslant 1$ and $(k,l) \in [n]^2$, we define $\mathcal{M}_{n,k}$ (respectively $\mathcal{M}'_{n,l}$) as the set of the tuples $M \in \mathcal{M}_n$ such that

$$\max\{i \in [n] : \ i \text{ is } M\text{-redundant}\} = n-k+1$$

(respectively

$$\max\{i \in [n] : 1 \in \{u_i, v_i\}\} = n-l+1).$$

For example, consider the tuple $M_0 = (\{1,1\},\{1,2\},\{2,2\},\{3,4\},\{3,5\}) \in \mathcal{M}_5$. We can see in Picture 4 (in which the multisets $\{u_{i_p}, v_{i_p}\}$ are encircled and the multisets $\{u_i, v_i\}$ where i is M_0-redundant are underlined) that $M_0 \in \mathcal{M}_{5,2} \cap \mathcal{M}'_{5,4}$.

$$\underset{i_4=1}{\{\underline{1,1}\}},\ \underset{i_3=2}{\{\underline{1,2}\}},\ \underset{i_2=3}{\{2,2\}},\ \{\underline{3,4}\},\ \underset{i_1=5}{\{3,5\}}$$

Figure 4: Tuple $M_0 \in \mathcal{M}_{5,2} \cap \mathcal{M}'_{5,4}$.

The $h_3 = 7$ elements of $\mathcal{M}_3$ are partitionned as depicted in Figure 5.

$\mathcal{M}_{3,1}$		$\{1,1\},\{1,2\},\{3,3\}$	$\{1,1\},\{2,2\},\{3,3\}$
$\mathcal{M}_{3,2}$	$\{1,1\},\{1,2\},\{1,3\}$	$\{1,1\},\{1,2\},\{2,3\}$	$\{1,1\},\{2,2\},\{2,3\}$
$\mathcal{M}_{3,3}$	$\{1,1\},\{2,2\},\{1,3\}$	$\{1,1\},\{1,1\},\{2,3\}$	
	$\mathcal{M}'_{3,1}$	$\mathcal{M}'_{3,2}$	$\mathcal{M}'_{3,3}$

Figure 5: The partition of $\mathcal{M}_3$.

One of the results of this paper is to show that the properties of Hetyei's model extend to the Kreweras triangle, *i.e.*, that $\#\mathcal{M}_{n,k} = \#\mathcal{M}'_{n,k} = h_{n,k}$ for all $k \in [n]$. To do so, we connect $\mathcal{M}_n$ bijectively to the previous models of h_n. In Section 2, we describe a model introduced by Feigin is his study of the degenerate flag varieties [8], and whose construction fits $\mathcal{M}_n$ in the best way. Incidentally, we define a slight adjustment of this model in a way that describes its inner construction. In Section 3, we construct a bijection between Feigin's and Hetyei's model, which provides the wanted combinatorial interpretation of the Kreweras triangle in terms of $\mathcal{M}_n$.

2 Feigin's model

In order to label the torus fixed points of the degenerate flag variety $\mathcal{F}_n^a$, Feigin [8] introduced the set $\mathcal{I}_n$ of the tuples $(I_0, \ldots, I_n)$ where $I_i \subset [n]$ has the conditions

$$\#I_i = i, \tag{4}$$

$$I_{i-1} \backslash \{i\} \subset I_i. \tag{5}$$

In [8, Proposition 3.1], Feigin constructs a bijection $f_2 : \mathcal{I}_n \to DC_n$, thus $\#\mathcal{I}_n = h_n$. The set $\mathcal{I}_n$ can be partitionned into $\{\mathcal{I}_{n,k} : k \in [n]\}$ or $\{\mathcal{I}'_{n,l} : l \in [n]\}$ where $\mathcal{I}_{n,k}$ (respectively $\mathcal{I}'_{n,l}$) is the subset of the elements $(I_0, \ldots, I_n) \in \mathcal{I}_n$ such that $k = \min\{i : 1 \in I_i\}$ (respectively $l = \min\{i : n \in I_i\}$). One can check that $f_2(\mathcal{I}_{n,k}) = DC_{n,k}$ and $f_2(\mathcal{I}'_{n,l}) = DC'_{n,l}$, so $\#\mathcal{I}_{n,k} = \#\mathcal{I}'_{n,k} = h_{n,k}$. For example, the $h_3 = 2 + 3 + 2$ elements of $\mathcal{I}_3$ are partitionned as depicted in Figure 6.

$\mathcal{I}_{3,1}$		$\varnothing, \{1\}, \{1,3\}, [3]$	$\varnothing, \{1\}, \{1,2\}, [3]$
$\mathcal{I}_{3,2}$	$\varnothing, \{3\}, \{1,3\}, [3]$	$\varnothing, \{2\}, \{1,3\}, [3]$	$\varnothing, \{2\}, \{1,2\}, [3]$
$\mathcal{I}_{3,3}$	$\varnothing, \{3\}, \{2,3\}, [3]$	$\varnothing, \{2\}, \{2,3\}, [3]$	
	$\mathcal{I}'_{3,1}$	$\mathcal{I}'_{3,2}$	$\mathcal{I}'_{3,3}$

Figure 6: The partition of $\mathcal{I}_3$.

In the following, we define a tweaking of this model.

Notation. For all n-tuple $(S_1, \ldots, S_n)$ of subsets of $[n]$ and for all $i \in [n]$, the set $\{j \in [n] : i \in S_j\}$ is denoted by S_i^{-1}.

Definition 4. For all $n \geqslant 1$, let $\mathcal{S}_n$ be the set of the tuples $(S_1, \ldots, S_n)$ of subsets of $[n]$ with the conditions

- $\#S_i = \#S_i^{-1} = 1$ or 2,
- if $\#S_i = 2$, then $S_i^{-1} = \{i_1, i_2\}$ for some $i_1 < i < i_2$.

Remark 5. We can partition $\mathcal{S}_n$ into $\{S_{n,k} : k \in [n]\}$ and $\{S'_{n,l} : l \in [n]\}$ where $S_{n,k}$ (respectively $S'_{n,k}$) is the set of the $(S_1, \ldots, S_n)$ such that $S_1^{-1} = \{k\}$ (respectively $S_n^{-1} = \{l\}$).

Proposition 6. *The map $(I_i)_{i\in[0,n]} \mapsto (I_i \backslash I_{i-1})_{i\in[n]}$ is a bijection between $\mathcal{I}_n$ and $\mathcal{S}_n$, which sends $\mathcal{I}_{n,k}$ and $\mathcal{I}'_{n,l}$ to $\mathcal{S}_{n,k}$ and $\mathcal{S}'_{n,l}$ respectively. In particular $h_{n,k} = \#\mathcal{S}_{n,k} = \#\mathcal{S}'_{n,k}$.*

Proof. For all $i \in [n]$, let $S_i = I_i \backslash I_{i-1}$. There are two situations.

1. If $i \in I_{i-1} \cap I_i$ or $i \notin I_{i-1}$, then $I_i = I_{i-1} \sqcup \{j\}$ for some $j \notin [n]$, and $\#S_i = \#S_i^{-1} = 1$.

2. Else $i \in I_{i-1}$ and $i \notin I_i$, in which case $I_i = (I_{i-1} \backslash \{i\}) \sqcup \{j_1, j_2\}$ for some $(j_1, j_2) \in [n]^2$, and $\# S_i = 2$. Also, let
$$i_1 = \min\{j \in [n] : i \in I_j\} < i,$$
$$i_2 = \min\{j \in [i, n] : i \in I_j\} > i,$$
then $S_i^{-1} = \{i_1, i_2\}$.

So $(S_i)_{i \in [n]} \in \mathcal{S}_n$. The inverse map is obtained as follows. Let $(S_i)_{i \in [n]} \in \mathcal{S}_n$ and $I_0 = \varnothing$. For all $i \in [n]$, suppose that we have defined $I_0, \ldots, I_{i-1}$ with Conditions (4) and (5), and the additional condition for all $j \in [n]$:
$$\min\{k \in [i-1] : j \in I_k\} = \min S_j^{-1}. \tag{6}$$
If $\# S_i = 1$, then I_i is defined as $I_{i-1} \sqcup S_i$. Otherwise $S_i^{-1} = \{i_1, i_2\}$ with $i_1 < i < i_2$, so $i \in I_{i_1}$ in view of Condition (6), hence $i \in I_{i-1}$ by Condition (5), and I_i is defined as $(I_{i-1} \backslash \{i\}) \sqcup S_i$. In both cases $I_0, \ldots, I_i$ have Conditions (4),(5) and (6), and $(I_i)_{i \in [0,n]} \in \mathcal{I}_n$. The rest of the lemma is straightforward. □

Remark 7. For all $(S_i)_{i \in [n]} \in \mathcal{S}_n$, the inverse image $(I_i)_{i \in [0,n]}$ is also given by $I_i = \left(\bigcup_{j=1}^{i} S_j \right) \backslash \{j \in [i] : \min S_j^{-1} < i < \max S_j^{-1}\}$.

For example, the $h_3 = 2+3+2$ elements of $\mathcal{S}_3$ are partitionned as depicted in Figure 7.

$\mathcal{S}_{3,1}$		$\{1\}, \{3\}, \{2\}$	$\{1\}, \{2\}, \{3\}$
$\mathcal{S}_{3,2}$	$\{3\}, \{1\}, \{2\}$	$\{2\}, \{1,3\}, \{2\}$	$\{2\}, \{1\}, \{3\}$
$\mathcal{S}_{3,3}$	$\{3\}, \{2\}, \{1\}$	$\{2\}, \{3\}, \{1\}$	
	$\mathcal{S}'_{3,1}$	$\mathcal{S}'_{3,2}$	$\mathcal{S}'_{3,3}$

Figure 7: The partition of $\mathcal{S}_3$.

Remark 8. There is a natural injection $\mathfrak{S}_n \hookrightarrow \mathcal{S}_n : \sigma \mapsto (\{\sigma(i)\})_{i \in [n]}$, which is the analogous of the elements $(I_i)_{i \in [0,n]}$ with the conditions
$$\# I_i = i,$$
$$I_{i-1} \subset I_i$$
forming a subset of $\mathcal{I}_n$ and labelling the torus fixed points of the flag variety $\mathcal{F}_n$ [8].

The bijection $\mathcal{S}'_{n,n} \to \mathcal{S}_{n-1}$, from which arises Formula (2), is the plain map $(S_1, \ldots, S_n) \mapsto (S_1, \ldots, S_{n-1})$. The involution $(S_1, \ldots, S_n) \in \mathcal{S}_n \mapsto (S_1^t, \ldots, S_n^t)$, defined by replacing every occurrence of 1 (respectively n) by n (respectively 1) in all S_i^t, induces the bijection $\mathcal{S}_{n,k} \cap \mathcal{S}'_{n,l} \to \mathcal{S}_{n,l} \cap \mathcal{S}'_{n,k}$. The involution $(S_1, \ldots, S_n) \to (S_1^r, \ldots, S_n^r)$, defined by $S_i^r = \{n+1-j : j \in S_{n+1-i}\}$, induces the bijection $\mathcal{S}_{n,k} \cap \mathcal{S}'_{n,l} \to \mathcal{S}_{n,n+1-l} \cap \mathcal{S}'_{n,n-k+1}$, from which follows Formula (3).

3 Bijective equivalence with Hetyei's model

Definition 9 (map $\boldsymbol{\varphi : \mathcal{I}_n \to \mathcal{M}_n}$). Let $I = (I_0, \dots, I_n) \in \mathcal{I}_n$ and $L_0 = (n, \dots, 1)$. Consider $k \in [n]$ and suppose that we have defined :

- a multiset $\{u_{n-k+2}, v_{n-k+2}, \dots, u_n, v_n\}$, such that $(u_i, v_i) \in [i]^2$ for all $i \in [n-k+2, n]$, which contains the set $[n-k+2, n]$;
- a tuple $L_{k-1} = (j_1^{k-1}, j_2^{k-1}, \dots, j_{n-k+1}^{k-1})$ such that

$$\{j_1^{k-1}, \dots, j_{n-k+1}^{k-1}\} = [n] \backslash I_{k-1}.$$

We now define $(u_{n-k+1}, v_{n-k+1}) \in [n-k+1]^2$ and L_k as follows.

1. If $I_{k-1} \subset I_k$, let $p \in [n-k+1]$ such that $I_k = I_{k-1} \sqcup \{j_p^{k-1}\}$.

   a) If $k \in I_{k-1}$, we define $\{u_{n-k+1}, v_{n-k+1}\}$ as $\{p, p\}$.
   b) Otherwise, we define $\{u_{n-k+1}, v_{n-k+1}\}$ as $\{p, n-k+1\}$.

   In either case, let

$$L_k = (j_1^{k-1}, \dots, j_{p-1}^{k-1}, j_{n-k+1}^{k-1}, j_{p+1}^{k-1}, \dots, j_{n-k}^{k-1}).$$

2. Otherwise $k \in I_{k-1}$ and $k \notin I_k$, hence $I_k = (I_{k-1} \backslash \{k\}) \sqcup \{j_p^{k-1}, j_q^{k-1}\}$ for some $1 \leqslant p < q \leqslant n-k+1$. We define $\{u_{n-k+1}, v_{n-k+1}\}$ as $\{p, q\}$, and

$$L_k = (j_1^{k-1}, \dots, j_{p-1}^{k-1}, j_{n-k+1}^{k-1}, j_{p+1}^{k-1}, \dots, j_{q-1}^{k-1}, k, j_{q+1}^{k-1}, \dots, j_{n-k}^{k-1}).$$

For the algorithm to move to $k+1$, we only need to show that the integer $n-k+1$ belongs to $\{u_{n-k+1}, v_{n-k+1}, \dots, u_n, v_n\}$. It is obvious if $\{u_{n-k+1}, v_{n-k+1}\}$ is defined by Rule 1.b). Otherwise, by hypothesis, we have $k \in I_{k-1}$. Let

$$i_0 = \min\{i \in [n] : k \in I_i\} \in [k-1].$$

By construction of $L_1, \dots, L_{k-1}$, it is easy to see that $j_{n-k+1}^{i_0-1} = k$, hence $n-k+1 \in \{u_{n+1-i_0}, v_{n+1-i_0}\}$ by either Rule 1.a) or Rule 2.

This algorithm provides a tuple $(\{u_i, v_i\})_{i \in [n]} \in \mathcal{M}_n$, that we denote by $\varphi(I)$.

For example, let $I_0 = (\varnothing, \{3\}, \{1,3\}, \{1,3,4\}, \{1,2,3,5\}, [5]) \in \mathcal{I}_5$ and $L_0 = 54321$. We obtain $\varphi(I_0) = (\{u_i, v_i\})_{i \in [5]}$ where

$$\begin{aligned}
&\{u_5, v_5\} = \{3,5\}, L_1 = 5412 \text{ (rule 1.b))},\\
&\{u_4, v_4\} = \{3,4\}, L_2 = 542 \text{ (rule 1.b))},\\
&\{u_3, v_3\} = \{2,2\}, L_3 = 52 \text{ (rule 1.a))},\\
&\{u_2, v_2\} = \{1,2\}, L_4 = 4 \text{ (rule 2.)},\\
&\{u_1, v_1\} = \{1,1\}, L_5 = \varnothing \text{ (rule 1.a))}.
\end{aligned}$$

Proposition 10. *The map $\varphi : \mathcal{I}_n \to \mathcal{M}_n$ is a bijection which sends $\mathcal{I}_{n,k}$ and $\mathcal{I}'_{n,l}$ to $\mathcal{M}_{n,k}$ and $\mathcal{M}'_{n,l}$ respectively for all $(k,l) \in [n]^2$. In particular $h_{n,k} = \#\mathcal{M}_{n,k} = \#\mathcal{M}'_{n,k}$.*

Proof. We construct the inverse map of φ. Let $M = (\{u_i, v_i\})_{i\in[n]} \in \mathcal{M}_n$, $L_0 = (n, \dots, 1)$ and $I_0 = \varnothing$. Suppose that, for some $k \in [n]$, we defined subsets $I_0, \dots, I_{k-1}$ of $[n]$ with Conditions (4) and (5), and a tuple $L_{k-1} = (j_1^{k-1}, \dots, j_{n-k+1}^{k-1})$ with $\{j_1^k, \dots, j_{n-k+1}^{k-1}\} = [n]\backslash I_{k-1}$. We define I_k and L_k as follows.

I. If $u_{n-k+1} = v_{n-k+1}$ or $n-k+1 \notin \{u_{n-k+2}, v_{n-k+2}, \dots, u_n, v_n\}$, there exists $p \in [n-k+1]$ such that $\{u_{n-k+1}, v_{n-k+1}\} = \{p,p\}$ or $\{p, n-k+1\}$. We define I_k as $I_{k-1} \sqcup \{j_p^{k-1}\}$, and L_k as in Rule 1.

II. Otherwise $\{u_{n-k}, v_{n-k}\} = \{p,q\}$ for some $1 \leqslant p < q \leqslant n-k+1$. We define I_k as $(I_{k-1}\backslash\{k\}) \sqcup \{j_p^{k-1}, j_q^{k-1}\}$, and L_k as in Rule 2.

For the algorithm to iterate, we only need to prove that $\#I_k = k$ if it is defined by Rule II. In this context, let $n - i_0 + 1 = \max\{l \in [n] : n-k+1 \in \{u_l, v_l\}\}$, by hypothesis $i_0 \in [k-1]$. By construction of $L_1, \dots, L_{k-1}$, we have $j_{n-1+k}^{i_0-1} = k$, hence $k \in I_{i_0}$, which implies that $k \in I_{k-1}$ in view of Condition (5).

So this algorithm provides an element $(I_0, \dots, I_n) \in \mathcal{I}_n$ that we denote by $\phi(M)$, and it is straightforward that φ and ϕ are inverse maps.

One can then thoroughly check that $\varphi(\mathcal{I}_{n,k}) = \mathcal{M}_{n,k}$ and $\varphi(\mathcal{I}'_{n,l}) = \mathcal{M}'_{n,l}$ for all $(k,l) \in [n]^2$. $\square$

For example, the tuple $M_0 = (\{1,1\},\{1,2\},\{2,2\},\{3,4\},\{3,5\}) \in \mathcal{M}_{5,2} \cap \mathcal{M}'_{5,4}$ (see Figure 4), to which $I_0 = (\varnothing, \{3\}, \{1,3\}, \{1,3,4\}, \{1,2,3,5\}, [5]) \in \mathcal{I}_{5,2} \cap \mathcal{I}'_{5,4}$ is mapped by φ as seen earlier, is indeed sent back to I_0 by the map ϕ defined in the proof of Proposition 10.

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