Combinatorial interpretations of the Kreweras triangle in terms of subset tuples

Ange Bigeni
Faculty of Mathematics
National Research University Higher School of Economics
Moscow, Russian Federation
ange.bigeni@gmail.com

Submitted: Dec 7, 2017; Accepted: Jul 18, 2018; Published: Nov 30, 2018
© The author. Released under the CC BY license (International 4.0).

Abstract

We show how the combinatorial interpretation of the normalized median Genocchi numbers in terms of multiset tuples, defined by Hetyei in his study of the alternation acyclic tournaments, is bijectively equivalent to previous models like the normalized Dumont permutations or the Dellac configurations, and we extend the interpretation to the Kreweras triangle.

Keywords: Genocchi numbers; Kreweras triangle; Dumont permutations; Dellac configurations

Mathematics Subject Classifications: 05A05, 05A15, 05A19

1 Introduction

For all pair of integers \( n < m \), the set \( \{n, n+1, \ldots, m\} \) is denoted by \([n, m]\), and the set \([1, n]\) by \([n]\). The set of the permutations of \([n]\) is denoted by \(\mathfrak{S}_n\).

1.1 Genocchi numbers, Kreweras triangle, Dumont permutations

The Genocchi numbers \((G_{2n})_{n \geq 1} = (1, 1, 3, 17, 155, 2073, \ldots)\) [13] and median Genocchi numbers \((H_{2n+1})_{n \geq 0} = (1, 2, 8, 56, 608, \ldots)\) [14] can be defined as the positive integers \(G_{2n} = g_{2n-1,n}\) and \(H_{2n+1} = g_{2n+2,1}\) [7] where \((g_{i,j})_{1 \leq j \leq i}\) is the Seidel triangle, defined by \(g_{1,1} = 1\) and

\[

g_{2p,j} = g_{2p-1,j} + g_{2p,j+1},
\]

\[

g_{2p+1,j} = g_{2p+1,j-1} + g_{2p,j}
\]
for all \( p \geq 1 \), where \( g_{2p,p+1} = g_{2p+1,0} = 0 \). It is well known that \( H_{2n+1} \) is divisible by \( 2^n \) for all \( n \geq 0 \) [1]. The normalized median Genocchi numbers \((h_n)_{n \geq 0} = (1, 1, 2, 7, 38, 295, \ldots)\) [15] are the positive integers defined by

\[
h_n = H_{2n+1}/2^n.\]

Among the first combinatorial models of the (median) Genocchi numbers [5, 7, 1, 6], there is the set \( PD_2^n \) of the Dumont permutations of the second kind, that is, the permutations \( \sigma \in S_{2n+2} \) such that \( \sigma(2i-1) > 2i-1 \) and \( \sigma(2i) < 2i \) for all \( i \in [n+1] \), whose cardinality \( \#PD_2^n \) equals \( H_{2n+1} \) for all \( n \geq 0 \). In [11], Kreweras introduced the subset \( PD_2N_n \subset PD_2^n \) of the normalized permutations, i.e., the permutations \( \sigma \in PD_2^n \) such that \( \sigma^{-1}(2i) < \sigma^{-1}(2i+1) \) for all \( i \in [n] \), whose number is \( \#PD_2N_n = h_n \).

**Remark 1.** For all \((k,l) \in [n]^2\), let \( PD_2N_{n,k} \) (respectively \( PD_2N'_{n,l} \)) be the subset of the permutations \( \sigma \in PD_2N_n \) such that \( \sigma(1) = 2k \) (respectively \( \sigma(2n+2) = 2l+1 \)). It is easy to see that \( \{PD_2N_{n,k} : k \in [n]\} \) and \( \{PD_2N'_{n,l} : l \in [n]\} \) are partitions of \( PD_2N_n \).

In [12], by introducing the model of the alternating diagrams and connecting them bijectively to the normalized Dumont permutations, Kreweras and Barraud proved that

\[
\#PD_2N_{n,k} = \#PD_2N'_{n,k} = h_{n,k}
\]

where the Kreweras triangle \((h_{n,k})_{n \geq 1, k \in [n]}\) [11] (see Figure 1.1) is defined by \( h_{1,1} = 1 \) and, for all \( n \geq 2 \) and \( k \in [3, n] \),

\[
\begin{align*}
\begin{cases} h_{n,1} = h_{n-1,1} + h_{n-1,2} + \ldots + h_{n-1,n-1}, \\
h_{n,2} = 2h_{n,1} - h_{n-1,1}, \\
h_{n,k} = 2h_{n,k-1} - h_{n,k-2} - h_{n-1,k-1} - h_{n-1,k-2}. 
\end{cases}
\end{align*}
\]

\[
\begin{array}{cccccc}
1 & & & & & \\
& 1 & 1 & & & \\
& & 2 & 3 & 2 & \\
& & 7 & 12 & 12 & 7 \\
& & 38 & 552 & 702 & 702 & 552 & 295 \\
\vdots & & & & & & & \\
\end{array}
\]

Figure 1: The Kreweras triangle.

For example, we depict in Figure 1.1 how are partitioned the \( h_3 = 2 + 3 + 2 \) elements of \( PD_2N_3 \).

For all \( n \geq 1 \) and \( k \in [n] \), the Kreweras triangle has the visible two properties

\[
\begin{align*}
h_{n,n} &= h_{n-1}, \\
h_{n,k} &= h_{n,n-k+1},
\end{align*}
\]

\[\text{THE ELECTRONIC JOURNAL OF COMBINATORICS 25(4) (2018), #P4.44}\]
of which \([12]\) implies interpretations in terms of \(PD2N_n\). Formula (2) follows from the bijection \(\sigma \in PD2N_{n,n} \mapsto \sigma \downarrow [2n] \in PD2N_{n-1}\). Afterwards, let \(\sigma \in PD2N_n\) and \((k, l) \in [n]^2\) such that \(\sigma(1) = 2k\) and \(\sigma(2n + 2) = 2l + 1\), we define two permutations \(\sigma^t\) and \(\sigma^r\) as follows.

- If \(k = l\), we define \(\sigma^t\) as \(\sigma\), otherwise it is defined as the following composition of \(\sigma\) with a 4-cycle:
  \[(2k \ 2l \ 2l + 1 \ 2k + 1) \circ \sigma.\]
- We define \(\sigma^r\) by \(\sigma^r(i) = 2n + 3 - \sigma(2n + 3 - i)\) for all \(i \in [2n + 2]\).

The maps \(\sigma \mapsto \sigma^t\) and \(\sigma \mapsto \sigma^r\) are involutions of \(PD2N_n\) which induce bijections

\[
\begin{align*}
PD2N_{n,k} \cap PD2N'_{n,l} & \longleftrightarrow PD2N_{n,l} \cap PD2N'_{n,k}, \\
PD2N_{n,l} \cap PD2N'_{n,k} & \longleftrightarrow PD2N_{n,n-k+1} \cap PD2N'_{n,n+1-l},
\end{align*}
\]

from which follows Formula (3), which can also be obtained by induction from System (1) through the following easy equality (see also \([12]\))

\[
h_{n,k} - h_{n,k-1} = \sum_{i=k}^{n-1} h_{n-1,i} - \sum_{i=1}^{k-2} h_{n-1,i}
\]

for all \(n \geq 1\) and \(k \in [n]\) (where \(h_{n,0}\) is defined as 0).

There are several other bijectively equivalent models of the Kreweras triangle \([4, 12, 9, 8, 2]\). The Kreweras triangle also appeared recently in the theory of finite type Vassiliev knot invariants \([3]\), more precisely through a polynomial generalization.

### 1.2 The Dellac configurations

The Dellac configurations \([4]\) form the earliest combinatorial model of the Kreweras triangle and provide a geometrical analogous of the previous results. Recall that a Dellac configuration of size \(n\) is a tableau \(D\), made of \(n\) columns and \(2n\) rows, that contains \(2n\) dots such that:

| \(PD2N_{3,1}\) | 21637485 | 21436587 |
| \(PD2N_{3,2}\) | 41627583 | 41627385 | 41526387 |
| \(PD2N_{3,3}\) | 61427583 | 61427385 |
| \(PD2N'_{3,1}\) | \(PD2N'_{3,2}\) | \(PD2N'_{3,3}\) |

Figure 2: The partition of \(PD2N_3\).
every row contains exactly one dot;

- every column contains exactly two dots;

- if there is a dot in the box \((j, i)\) of \(D\) (i.e., in the intersection of its \(j\)-th column from left to right and its \(i\)-th row from bottom to top), then \(j \leq i \leq j + n\).

The set of the Dellac configurations of size \(n\) is denoted by \(DC_n\). It can be partitioned into \(\{DC_{n,k} : k \in [n]\}\) or \(\{DC'_{n,l} : l \in [n]\}\) where \(DC_{n,k}\) (respectively \(DC'_{n,l}\)) is the subset of the tableaux \(D \in DC_n\) whose box \((k, n+1)\) (respectively \((l, n)\)) contains a dot, for all \((k, l) \in [n]^2\). In [8, Proposition 3.3], Feigin constructs a bijection \(f_1 : PD2N_n \rightarrow DC_n\) such that \(f_1(PD2N_{n,k}) = DC_{n,k}\), hence \(h_{n,k} = \#DC_{n,k}\), for all \(k \in [n]\). One can also check that \(f_1(PD2N'_{n,k}) = DC'_{n,k}\), so \(h_{n,k} = \#DC'_{n,k}\). For example, the \(h_3 = 2 + 3 + 2\) elements of \(DC_3\) are partitionned as depicted in Figure 3.

\[
\begin{array}{ccc}
DC_{3,1} & DC_{3,2} & DC_{3,3} \\
\hline
DC'_{3,1} & DC'_{3,2} & DC'_{3,3}
\end{array}
\]

Figure 3: The partition of \(DC_3\).

The combinatorial interpretations of Formulas (2) and (3) in terms of Dellac configurations are simple. Every element of \(DC_{n-1}\) can be obtained by deleting the \(n\)-th column (from left to right) and the \((n + 1)\)-th and \(2n\)-th rows (from bottom to top) of a unique element of \(DC'_{n,n}\), which gives Formula (2). Afterwards, for all \(D \in DC_{n,k} \cap DC'_{n,l}\),

- let \(D' \in DC_{n,l} \cap DC'_{n,k}\) be obtained by deleting the dots of the boxes \((k, n + 1)\) and \((l, n)\) of \(D\) and placing dots in the boxes \((l, n+1)\) and \((k, n)\),

- let \(D' \in DC_{n,n+1-l} \cap DC'_{n,n-k+1}\) be obtained by rotating \(D\) through \(180^\circ\),
the maps \( D \mapsto D' \) and \( D \mapsto D^r \) are involutions of \( DC_n \) that induce bijections

\[
\begin{align*}
DC_{n,k} \cap DC_{n,l} & \leftrightarrow DC_{n,l} \cap DC_{n,k}', \\
DC_{n,l} \cap DC_{n,k}' & \leftrightarrow DC_{n,n-k+1} \cap DC_{n,n+1-l}'.
\end{align*}
\]

from which follows Formula (3).

### 1.3 Heteyi’s model

In his study of the alternation acyclic tournaments [10], Heteyi proved that the median Genocchi number \( H_{2n+1} \) is the number of pairs

\[ ((a_1, \ldots, a_n), (b_1, \ldots, b_n)) \in \mathbb{Z}^n \times \mathbb{Z}^n \]

such that \( (a_i, b_i) \in [0, n] \times [n] \) for all \( i \in [n] \), and the set \([n] \) is contained in the multiset \( \{a_1, b_1, \ldots, a_n, b_n\} \). He then defined a free group action of \( (\mathbb{Z} \setminus 2\mathbb{Z})^n \) on the set of these pairs, whose orbits are indexed by the \( n \)-tuples \( \{(u_i, v_i)\}_{i \in [n]} \) such that \( (u_i, v_i) \in [n]^2 \) for all \( i \in [n] \) and the multiset \( \{u_1, v_1, \ldots, u_n, v_n\} \) contains \([n] \), which raises a new proof of \( H_{2n+1} \) being a multiple of \( 2^n \), and a new combinatorial model of \( h_n \) through the set \( \mathcal{M}_n \) of these tuples \( \{(u_i, v_i)\}_{i \in [n]} \). For example, the \( h_3 = 7 \) elements of \( \mathcal{M}_3 \) are

\[
\begin{align*}
\{1,1\}, \{2,2\}, \{3,3\} \\
\{1,1\}, \{1,2\}, \{3,3\} \\
\{1,1\}, \{2,2\}, \{2,3\} \\
\{1,1\}, \{1,2\}, \{2,3\} \\
\{1,1\}, \{1,1\}, \{2,3\} \\
\{1,1\}, \{2,2\}, \{1,3\} \\
\{1,1\}, \{1,2\}, \{1,3\}.
\end{align*}
\]

As for the Dumont permutations and the Dellac configurations, we now intend to define two partitions of \( \mathcal{M}_n \).

**Definition 2.** Let \( M = \{(u_i, v_i)\}_{i \in [n]} \in \mathcal{M}_n \), we define a tuple

\[ n = i_1 > i_2 > \ldots > i_m \geq 1 \]

as follows : if \( u_{i_p} = v_{i_p} = i_p \), then \( m \) is defined as \( p \), otherwise \( i_{p+1} \) is defined as \( \min\{u_{i_p}, v_{i_p}\} < i_p \). This tuple is well-defined because \( u_1 = v_1 = 1 \) in general.

Afterwards, for all integer \( i \in [i_m, n] \), let \( p \in [m] \) such that \( i \in [i_p, i_p - 1] \) (where \( i_0 \) is defined as \( n + 1 \)), we say that \( i \) is \( M \)-redundant if \( i_p \in \{u_i, v_i\} \). Note that the set of such integers is not empty because it contains \( i_m \).

We are ready to define two partitions \( \{\mathcal{M}_{n,k} : k \in [n]\} \) and \( \{\mathcal{M}_{n,l} : l \in [n]\} \) of \( \mathcal{M}_n \).
Definition 3. For all \( n \geq 1 \) and \((k, l) \in [n]^2\), we define \( \mathcal{M}_{n,k} \) (respectively \( \mathcal{M}'_{n,l} \)) as the set of the tuples \( M \in \mathcal{M}_n \) such that
\[
\max\{i \in [n] : i \text{ is } M\text{-redundant}\} = n - k + 1
\]
(respectively
\[
\max\{i \in [n] : 1 \in \{u_i, v_i\}\} = n - l + 1.
\]

For example, consider the tuple \( M_0 = (\{1, 1\}, \{1, 2\}, \{2, 2\}, \{3, 4\}, \{3, 5\}) \in \mathcal{M}_5 \). We can see in Picture 4 (in which the multisets \( \{u_i, v_i\} \) are encircled and the multisets \( \{u_i, v_i\} \) where \( i \) is \( M_0 \)-redundant are underlined) that \( M_0 \in \mathcal{M}_{5,2} \cap \mathcal{M}'_{5,4} \).

\[
\begin{align*}
\{1, 1\} & \quad \{1, 2\} & \quad \{2, 2\} & \quad \{3, 4\} & \quad \{3, 5\} \\
\text{i_1 = 1} & \quad \text{i_2 = 2} & \quad \text{i_3 = 3} & \quad \text{i_4 = 5}
\end{align*}
\]

Figure 4: Tuple \( M_0 \in \mathcal{M}_{5,2} \cap \mathcal{M}'_{5,4} \).

The \( h_3 = 7 \) elements of \( \mathcal{M}_3 \) are partitioned as depicted in Figure 5.

\[
\begin{array}{c|c|c|c|c|c}
\mathcal{M}_{3,1} & \{1, 1\}, \{1, 2\}, \{3, 3\} & \{1, 1\}, \{2, 2\}, \{3, 3\} & \{1, 1\}, \{2, 2\}, \{3, 3\} \\
\hline
\mathcal{M}_{3,2} & \{1, 1\}, \{1, 2\}, \{1, 3\} & \{1, 1\}, \{1, 2\}, \{2, 3\} & \{1, 1\}, \{2, 2\}, \{2, 3\} \\
\hline
\mathcal{M}_{3,3} & \{1, 1\}, \{2, 2\}, \{1, 3\} & \{1, 1\}, \{1, 1\}, \{2, 3\} & \{1, 1\}, \{2, 2\}, \{2, 3\} \\
\hline
\mathcal{M}'_{3,1} & \mathcal{M}'_{3,2} & \mathcal{M}'_{3,3}
\end{array}
\]

Figure 5: The partition of \( \mathcal{M}_3 \).

One of the results of this paper is to show that the properties of Hetyei’s model extend to the Kreweras triangle, \textit{i.e.}, that \( \#\mathcal{M}_{n,k} = \#\mathcal{M}'_{n,k} = h_{n,k} \) for all \( k \in [n] \). To do so, we connect \( \mathcal{M}_n \) bijectively to the previous models of \( h_n \). In Section 2, we describe a model introduced by Feigin is his study of the degenerate flag varieties \([8]\), and whose construction fits \( \mathcal{M}_n \) in the best way. Incidentally, we define a slight adjustment of this model in a way that describes its inner construction. In Section 3, we construct a bijection between Feigin’s and Hetyei’s model, which provides the wanted combinatorial interpretation of the Kreweras triangle in terms of \( \mathcal{M}_n \).
2 Feigin’s model

In order to label the torus fixed points of the degenerate flag variety $F_n^a$, Feigin [8] introduced the set $I_n$ of the tuples $(I_0, \ldots, I_n)$ where $I_i \subset [n]$ has the conditions

$$\#I_i = i, \quad I_{i-1}\{i\} \subset I_i.$$  \hfill (4)

In [8, Proposition 3.1], Feigin constructs a bijection $f_2 : I_n \rightarrow DC_n$, thus $\#I_n = h_n$. The set $I_n$ can be partitioned into $\{I_{n,k} : k \in [n]\}$ or $\{I_{n,l} : l \in [n]\}$ where $I_{n,k}$ (respectively $I_{n,l}$) is the subset of elements $(I_0, \ldots, I_n) \in I_n$ such that $k = \min\{i : 1 \in I_i\}$ (respectively $l = \min\{i : n \in I_i\}$). One can check that $f_2(I_{n,k}) = DC_{n,k}$ and $f_2(I_{n,l}) = DC'_{n,l}$; so $\#I_{n,k} = \#I'_{n,k} = h_{n,k}$. For example, the $h_3 = 2 + 3 + 2$ elements of $I_3$ are partitioned as depicted in Figure 6.

| $I_{3,1}$ | $\emptyset, \{1\}, \{1,3\}, [3]$ | $\emptyset, \{1\}, \{1,2\}, [3]$ |
| $I_{3,2}$ | $\emptyset, \{3\}, \{1,3\}, [3]$ | $\emptyset, \{2\}, \{1,3\}, [3]$ | $\emptyset, \{2\}, \{1,2\}, [3]$ |
| $I_{3,3}$ | $\emptyset, \{3\}, \{2,3\}, [3]$ | $\emptyset, \{2\}, \{2,3\}, [3]$ |
| $I'_{3,1}$ | $I'_{3,2}$ | $I'_{3,3}$ |

Figure 6: The partition of $I_3$.

In the following, we define a tweaking of this model.

Notation. For all $n$-tuple $(S_1, \ldots, S_n)$ of subsets of $[n]$ and for all $i \in [n]$, the set $\{j \in [n] : i \in S_j\}$ is denoted by $S_i^{-1}$.

Definition 4. For all $n \geq 1$, let $S_n$ be the set of the tuples $(S_1, \ldots, S_n)$ of subsets of $[n]$ with the conditions

- $\#S_1 = \#S_1^{-1} = 1$ or 2,
- if $\#S_i = 2$, then $S_i^{-1} = \{i_1, i_2\}$ for some $i_1 < i < i_2$.

Remark 5. We can partition $S_n$ into $\{S_{n,k} : k \in [n]\}$ and $\{S'_{n,l} : l \in [n]\}$ where $S_{n,k}$ (respectively $S'_{n,l}$) is the set of the $(S_1, \ldots, S_n)$ such that $S_1^{-1} = \{k\}$ (respectively $S_n^{-1} = \{l\}$).

Proposition 6. The map $(I_i)_{i \in [0,n]} \mapsto (I_i \setminus I_{i-1})_{i \in [n]}$ is a bijection between $I_n$ and $S_n$, which sends $I_{n,k}$ and $I'_{n,l}$ to $S_{n,k}$ and $S'_{n,l}$ respectively. In particular $h_{n,k} = \#S_{n,k} = \#S'_{n,k}$.

Proof. For all $i \in [n]$, let $S_i = I_i \setminus I_{i-1}$. There are two situations.

1. If $\emptyset \subset I_i \ni i \not\subset I_{i-1}$, then $I_i = I_{i-1} \cup \{j\}$ for some $j \not\subset [n]$, and $\#S_i = \#S_i^{-1} = 1$. 

THE ELECTRONIC JOURNAL OF COMBINATORICS 25(4) (2018), #P4.44
2. Else \( i \in I_{i-1} \) and \( i \not\in I_i \), in which case \( I_i = (I_{i-1} \setminus \{i\}) \cup \{j_1, j_2\} \) for some \((j_1, j_2) \in \{n\}^2\), and \( \#S_i = 2 \). Also, let

\[
i_1 = \min \{ j \in [n] : i \in I_j \} < i,
\]

\[
i_2 = \min \{ j \in [i, n] : i \in I_j \} > i,
\]

then \( S_i^{-1} = \{i_1, i_2\} \). So \((S_i)_{i \in [n]} \in S_n\). The inverse map is obtained as follows. Let \((S_i)_{i \in [n]} \in S_n\) and \( I_0 = \emptyset \). For all \( i \in [n] \), suppose that we have defined \( I_0, \ldots, I_i \) with Conditions (4) and (5), and the additional condition for all \( j \in [n] \):

\[
\min \{ k \in [i - 1] : j \in I_k \} = \min S_j^{-1}.
\]

(6)

If \( \#S_i = 1 \), then \( I_i \) is defined as \( I_{i-1} \cup S_i \). Otherwise \( S_i^{-1} = \{i_1, i_2\} \) with \( i_1 < i < i_2 \), so \( i \in I_i \) in view of Condition (6), hence \( i \in I_{i-1} \) by Condition (5), and \( I_i \) is defined as \((I_{i-1} \setminus \{i\}) \cup S_i\). In both cases \( I_0, \ldots, I_i \) have Conditions (4),(5) and (6), and \((I_i)_{i \in [0, n]} \in \mathcal{I}_n\). The rest of the lemma is straightforward.

Remark 7. For all \((S_i)_{i \in [n]} \in S_n\), the inverse image \((I_i)_{i \in [0, n]}\) is also given by \( I_i = \left( \bigcup_{j=1}^n S_j \right) \setminus \{j \in [i] : \min S_j^{-1} < i < \max S_j^{-1} \} \).

For example, the \( h_3 = 2 + 3 + 2 \) elements of \( S_3 \) are partitionned as depicted in Figure 7.

| \( S_{3,1} \) | \{1\}, \{3\}, \{2\} | \{1\}, \{2\}, \{3\} |
| \( S_{3,2} \) | \{3\}, \{1\}, \{2\} | \{2\}, \{1\}, \{3\} |
| \( S_{3,3} \) | \{3\}, \{2\}, \{1\} | \{2\}, \{3\}, \{1\} |

\( S'_{3,1} \) \hspace{2cm} \( S'_{3,2} \) \hspace{2cm} \( S'_{3,3} \)

Figure 7: The partition of \( S_3 \).

Remark 8. There is a natural injection \( \mathcal{G}_n \hookrightarrow S_n : \sigma \mapsto (\{\sigma(i)\})_{i \in [n]} \), which is the analogous of the elements \((I_i)_{i \in [0, n]}\) with the conditions

\[
\#I_i = i,
\]

\[
I_{i-1} \subset I_i
\]

forming a subset of \( \mathcal{I}_n \) and labelling the torus fixed points of the flag variety \( \mathcal{F}_n \) [8].

The bijection \( S'_{n,n} \rightarrow S_{n-1} \), from which arises Formula (2), is the plain map \((S_1, \ldots, S_n) \mapsto (S_1, \ldots, S_{n-1})\). The involution \((S_1, \ldots, S_n) \in S_n \mapsto (S'_1, \ldots, S'_n)\), defined by replacing every occurrence of 1 (respectively \( n \)) by \( n \) (respectively 1) in all \( S'_i \), induces the bijection \( S_{n,k} \cap S'_{n,l} \rightarrow S_{n,l} \cap S'_{n,k} \). The inversion \((S_1, \ldots, S_n) \mapsto (S'_1, \ldots, S'_n)\), defined by \( S'_i = \{n + 1 - j : j \in S_{n+1-i}\} \), induces the bijection \( S_{n,k} \cap S'_{n,l} \rightarrow S_{n,n+1-l} \cap S'_{n,n-k+1} \), from which follows Formula (3).
3 Bijective equivalence with Hetyei’s model

Definition 9 (map $\varphi: \mathcal{I}_n \to \mathcal{M}_n$). Let $I = (I_0, \ldots, I_n) \in \mathcal{I}_n$ and $L_0 = (n, \ldots, 1)$. Consider $k \in [n]$ and suppose that we have defined:

- a multiset $\{u_{n-k+2}, \ldots, u_n, v_n\}$, such that $(u_i, v_i) \in [i]^2$ for all $i \in [n-k+2, n]$, which contains the set $[n-k+2, n]$;
- a tuple $L_{k-1} = (j_1^{k-1}, \ldots, j_{n-k-1}^{k-1})$ such that
  $$\{j_1^{k-1}, \ldots, j_{n-k-1}^{k-1}\} = [n]\setminus I_{k-1}.$$  

We now define $(u_{n-k+1}, v_{n-k+1}) \in [n-k+1]^2$ and $L_k$ as follows.

1. If $I_{k-1} \subset I_k$, let $p \in [n-k+1]$ such that $I_k = I_{k-1} \uplus \{j_p^{k-1}\}$.
   a) If $k \in I_{k-1}$, we define $(u_{n-k+1}, v_{n-k+1})$ as $(p, p)$.
   b) Otherwise, we define $(u_{n-k+1}, v_{n-k+1})$ as $(p, n-k+1)$.

In either case, let

$$L_k = (j_1^{k-1}, \ldots, j_p^{k-1}, j_{n-k+1}^{k-1}, j_{p+1}^{k-1}, \ldots, j_{n-k}^{k-1}).$$

2. Otherwise $k \in I_{k-1}$ and $k \notin I_k$, hence $I_k = (I_{k-1}\setminus\{k\}) \uplus \{j_p^{k-1}, j_q^{k-1}\}$ for some $1 \leq p < q \leq n-k+1$. We define $(u_{n-k+1}, v_{n-k+1})$ as $(p, q)$, and

$$L_k = (j_1^{k-1}, \ldots, j_p^{k-1}, j_{n-k+1}^{k-1}, j_q^{k-1}, \ldots, j_{n-k}^{k-1}).$$

For the algorithm to move to $k+1$, we only need to show that the integer $n-k+1$ belongs to $\{u_{n-k+1}, v_{n-k+1}, \ldots, u_n, v_n\}$. It is obvious if $\{u_{n-k+1}, v_{n-k+1}\}$ is defined by Rule 1.b). Otherwise, by hypothesis, we have $k \in I_{k-1}$. Let

$$i_0 = \min\{i \in [n] : k \in I_i\} \in [k-1].$$

By construction of $L_1, \ldots, L_{k-1}$, it is easy to see that $j_{i_0}^{n-k+1} = k$, hence $n-k+1 \in \{u_{n+1-i_0}, v_{n+1-i_0}\}$ by either Rule 1.a) or Rule 2.

This algorithm provides a tuple $(\{u_i, v_i\})_{i \in [n]} \in \mathcal{M}_n$, that we denote by $\varphi(I)$.

For example, let $I_0 = (\emptyset, \{3\}, \{1, 3\}, \{1, 3, 4\}, \{1, 2, 3, 5\}, [5]) \in \mathcal{I}_5$ and $L_0 = 54321$. We obtain $\varphi(I_0) = (\{u_i, v_i\})_{i \in [5]}$ where

- $\{u_5, v_5\} = \{3, 5\}, L_1 = 5412$ (rule 1.b)),
- $\{u_4, v_4\} = \{3, 4\}, L_2 = 542$ (rule 1.b)),$
- $\{u_3, v_3\} = \{2, 2\}, L_3 = 52$ (rule 1.a)),
- $\{u_2, v_2\} = \{1, 2\}, L_4 = 4$ (rule 2),
- $\{u_1, v_1\} = \{1, 1\}, L_5 = \emptyset$ (rule 1.a)).
Proposition 10. The map \( \varphi : \mathcal{I}_n \to \mathcal{M}_n \) is a bijection which sends \( \mathcal{I}_{n,k} \) and \( \mathcal{I}_{n,l}' \) to \( \mathcal{M}_{n,k} \) and \( \mathcal{M}_{n,l}' \) respectively for all \( (k,l) \in [n]^2 \). In particular \( h_{n,k} = \# \mathcal{M}_{n,k} = \# \mathcal{M}_{n,k}' \).

Proof. We construct the inverse map of \( \varphi \). Let \( M = (\{u_i, v_i\})_{i \in [n]} \in \mathcal{M}_n \), \( L_0 = (n, \ldots, 1) \) and \( I_0 = \emptyset \). Suppose that, for some \( k \in [n] \), we defined subsets \( I_0, \ldots, I_{k-1} \) of \([n]\) with Conditions (4) and (5), and a tuple \( L_{k-1} = (j_1^{k-1}, \ldots, j_{n-k+1}^{k-1}) \) with \( \{j_1^{k-1}, \ldots, j_{n-k+1}^{k-1}\} \in [n]\setminus I_{k-1} \). We define \( I_k \) and \( L_k \) as follows.

I. If \( u_{n-k+1} = v_{n-k+1} \) or \( n - k + 1 \not\in \{u_{n-k+2}, v_{n-k+2}, \ldots, u_n, v_n\} \), there exists \( p \in [n-k+1] \) such that \( \{u_{n-k+1}, v_{n-k+1}\} = \{p, p\} \) or \( \{p, n-k+1\} \). We define \( I_k \) as \( I_{k-1} \cup \{j_p^{k-1}\} \), and \( L_k \) as in Rule 1.

II. Otherwise \( \{u_{n-k}, v_{n-k}\} = \{p, q\} \) for some \( 1 \leq p < q \leq n-k+1 \). We define \( I_k \) as \( (I_{k-1}\setminus \{k\}) \cup \{j_p^{k-1}, j_q^{k-1}\} \), and \( L_k \) as in Rule 2.

For the algorithm to iterate, we only need to prove that \( \#I_k = k \) if it is defined by Rule II. In this context, let \( n - i_0 + 1 = \max\{l \in [n] : n - k + 1 \in \{u_l, v_l\}\} \), by hypothesis \( i_0 \in [k-1] \). By construction of \( L_1, \ldots, L_{k-1} \), we have \( j_{n-k+1}^0 = k \), hence \( k \in I_{i_0} \), which implies that \( k \in I_{k-1} \) in view of Condition (5).

So this algorithm provides an element \( (I_0, \ldots, I_n) \in \mathcal{I}_n \) that we denote by \( \phi(M) \), and it is straightforward that \( \varphi \) and \( \phi \) are inverse maps.

One can then thoroughly check that \( \varphi(\mathcal{I}_{n,k}) = \mathcal{M}_{n,k} \) and \( \varphi(\mathcal{I}_{n,l}') = \mathcal{M}_{n,l}' \) for all \( (k,l) \in [n]^2 \). \( \square \)

For example, the tuple \( M_0 = (\{1,1\}, \{1,2\}, \{2,2\}, \{3,4\}, \{3,5\}) \in \mathcal{M}_{5,2} \cap \mathcal{M}_{5,4} \) (see Figure 4), to which \( I_0 = (\emptyset, \{3\}, \{1,3\}, \{1,3,4\}, \{1,2,3,5\}, \{5\}) \in \mathcal{I}_{5,2} \cap \mathcal{I}_{5,4} \) is mapped by \( \varphi \) as seen earlier, is indeed sent back to \( I_0 \) by the map \( \phi \) defined in the proof of Proposition 10.

References


