Enumerations, Forbidden Subgraph Characterizations, and the Split-Decomposition

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Abstract

Forbidden characterizations may sometimes be the most natural way to describe families of graphs, and yet these characterizations are usually very hard to exploit for enumerative purposes.

By building on the work of Gioan and Paul (2012) and Chauve et al. (2014), we show a methodology by which we constrain a split-decomposition tree to avoid certain patterns, thereby avoiding the corresponding induced subgraphs in the original graph.

We thus provide the grammars and full enumeration for a wide set of graph classes: ptolemaic, block, and variants of cactus graphs (2,3-cacti, 3-cacti and 4-cacti). In certain cases, no enumeration was known (ptolemaic, 4-cacti); in other cases, although the enumerations were known, an abundant potential is unlocked by the grammars we provide (in terms of asymptotic analysis, random generation, and parameter analyses, etc.).

We believe this methodology here shows its potential; the natural next step to develop its reach would be to study split-decomposition trees which contain certain prime nodes. This will be the object of future work.

Mathematics Subject Classifications: 05C30, 05A15, 05C05, 05A16.

Introduction

Many important families of graphs can be defined (sometimes exclusively) through a *forbidden graph characterization.* These characterizations exist in several flavors:

1. *Forbidden minors*, in which we try to avoid certain subgraphs from appearing after arbitrary edge contractions and vertex deletion.

2. *Forbidden subgraphs*, in which we try to avoid certain subgraphs from appearing as subsets of the vertices and edges of a graph.

3. *Forbidden induced subgraphs*, in which we try to avoid certain induced subgraphs from appearing (that is we pick a subset of vertices, and use all edges with both endpoints in that subset).

As far as we know, while these notions are part and parcel of the work of graph theorists, they are usually not exploited by analytic combinatorists. For forbidden minors, there is the penetrating article of Bousquet-Mélou and Weller [5]. For forbidden subgraphs or forbidden induced subgraphs, we know of few papers, except because of the simple nature of graphs [33], or because some other, alternate property is used instead [6], or only asymptotics are determined [34].

We are concerned, in this paper, with *forbidden induced subgraphs*.

Split-decomposition and forbidden induced subgraphs.

Chauve *et al.* [8, 9] observed that a relatively well-known graph decomposition, called the *split-decomposition*, could be a fruitful means to enumerate a class called *distance-hereditary graphs*, of which the enumeration had until then not been known (at the time, the best known result was the bound from Nakano *et al.* [27], which stated that there are at most $2^{\lfloor 3.59n \rfloor}$ unlabeled distance-hereditary graphs on n vertices).

In addition, the reformulated version of this split-decomposition introduced by Paul and Gioan, with internal graph-labels, considerably improved the legibility of the split-decomposition tree.

We have discovered, and we try to showcase in this paper, that the split-decomposition is a very convenient tool by which to find induced subpatterns: although various connected portions of the graphs may be broken down into far apart blocks in the split-decomposition tree, the property that there is an *alternated path* between any two vertices that are connected in the original graph is very powerful, and as we show in Section 2 of this paper, allows to deduce constraints following the appearance of an induced pattern or subgraph.

Outline of paper.

In Section 1, we introduce all the definitions and preliminary notions that we need for this paper to be relatively self-contained (although it is based heavily on work introduced by Chauve *et al.* [8]).

In Section 2, we introduced a collection of bijective lemmas, which translate several forbidden patterns (a cycle with 4 vertices, a diamond, cliques, a pendant vertex and a bridge, all illustrated in Figure 4) into constraints on the split-decomposition tree of a graph. In each of the subsequent sections, we show how these constraints can be used to express a formal symbolic grammar that describes the constrained tree—and by so doing, we obtain a grammar for the associated class of graphs.

We start by studying block graphs in Section 3, because their structure is sufficiently constrained as to yield a relatively simple grammar. We then study ptolemaic graphs in Section 4 (which allows us to showcase how to use the symbolic grammar to save "state" information, since we have to remember the provenance of the hierarchy of each node to determine whether it has a center as a starting point). And we finally look at some varieties of cactus graphs in Section 5.

Finally, in Section 6, we conclude and introduce possible future directions in which to continue this work.

1 Definitions and Preliminaries

In this rather large section, we introduce standard definitions from graph theory (1.1 to 1.3) and analytic combinatorics (1.4), and then present a summary of the work of Chauve *et al.* [8] (1.5), as well as a summary of how they used the dissymmetry theorem, introduced by Bergeron *et al.* [2] (1.6).

1.1 Graph definitions

For a graph G, we denote by $V(G)$ its vertex set and $E(G)$ its edge set. Moreover, for a vertex x of a graph G, we denote by $N(x)$ the (open) neighborhood of x, that is the set of vertices $y \neq x$ such that $\{x, y\} \in E(G)$; this notion extends naturally to vertex sets: if $V_1 \subseteq V(G)$, then $N(V_1)$ is the set of vertices defined by the (non-disjoint) union of the neighborhoods of the vertices in V_1, excluding V_1 itself. Finally, the subgraph of G induced by a subset V_1 of vertices is denoted by $G[V_1]$.

Given a graph G and two vertices $u, v \in V(G)$ in the same connected component of G, the distance between u and v denoted by $d_G(u, v)$ is defined as the length of the shortest path between u and v.

A graph on n vertices is *labeled* if its vertices are identified with the set $\{1, \ldots, n\}$, with no two vertices having the same label. A graph is *unlabeled* if its vertices are indistinguishable.

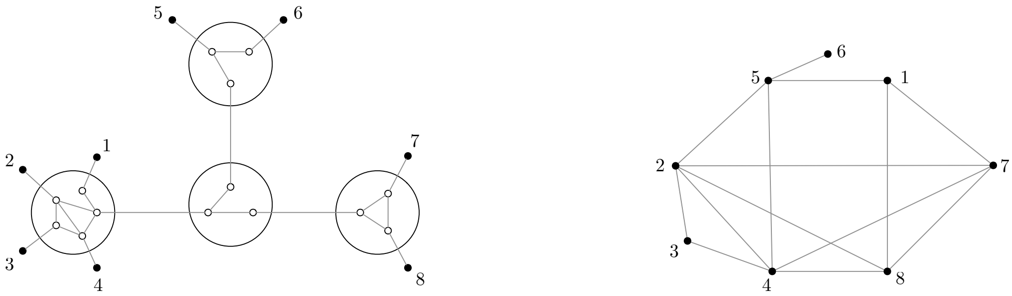
A clique on k vertices, denoted K_k is the complete graph on k vertices (*i.e.*, there exists an edge between every pair of vertices). A star on k vertices, denoted S_k, is the graph with one vertex of degree $k-1$ (the *center* of the star) and $k-1$ vertices of degree 1 (the *extremities* of the star).

1.2 Special graph classes

The following two graph classes are important because they are supersets of the classes we study in this paper.

Definition 1. A connected graph G is *distance-hereditary* if for every induced subgraph H and every $u, v \in V(H)$, $d_G(u, v) = d_H(u, v)$.

Definition 2. A connected graph is *chordal*, or *triangulated*, or $C_{\geqslant 4}$-free, if every cycle of length at least 4 has a chord.



(a) A graph-labeled tree with four graph-labels (one for each of the four internal nodes); the leaves also have labels, but these are of a separate kind.

(b) Original graph for (or *accessibility graph* of) the graph-labeled tree in Fig. 1a.

Figure 1. Two leaves of the split-decomposition graph-labeled tree (left) correspond to adjacent vertices in the original graph that was decomposed (right) if there exists an *alternated path*: a path between those leaves, which uses at most one interior edge of any given graph-label. For example, vertex 5 is adjacent to vertex 4 in the original graph, because there is an alternated path between the two corresponding leaves in the split-decomposition tree; vertex 5 is not adjacent to vertex 3 however, because that would require the path to take two interior edges of the (*prime*) leftmost graph-label.

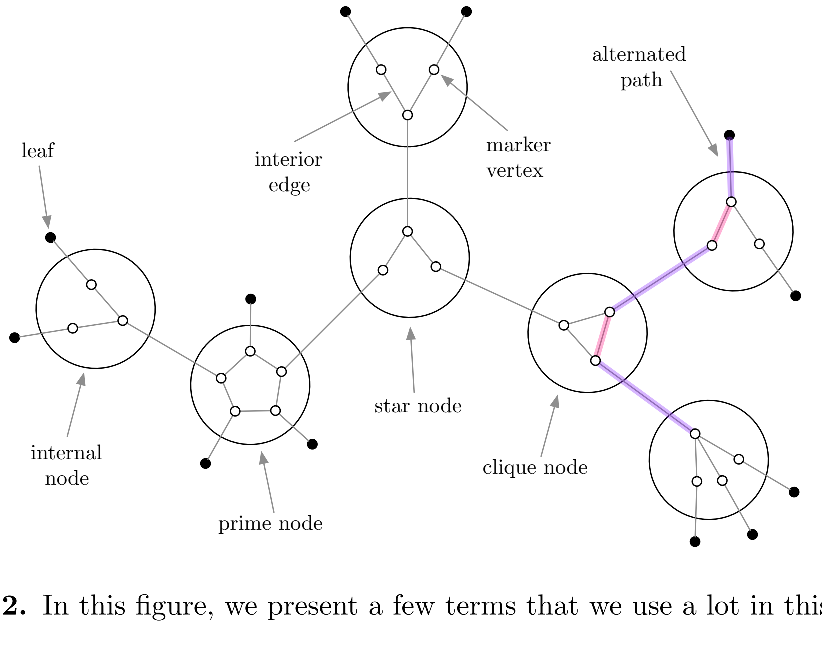
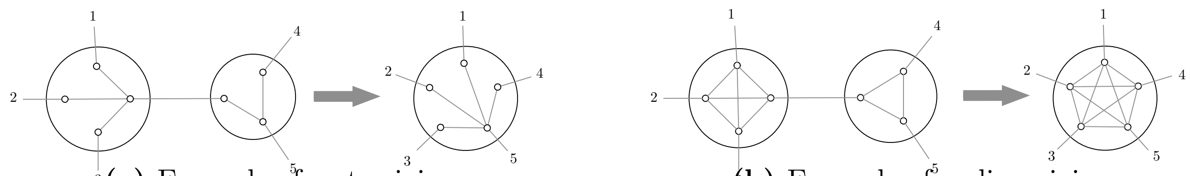


Figure 2. In this figure, we present a few terms that we use a lot in this article.



(a) Example of a star-join. **(b)** Example of a clique-join.

Figure 3. The star-join and clique-join operations result in the merging of two internal nodes of a split-decomposition tree. A split-decomposition tree in which neither one of these operations may be applied (and in which all non-clique and non-star nodes are prime nodes) is said to be *reduced*.

1.3 Split-decomposition

We first introduce the notion of *graph-labeled tree*, due to Gioan and Paul [19], then define the split-decomposition and finally give the characterization of a *reduced* split-decomposition tree, described as a graph-labeled tree.

Definition 3. A graph-labeled tree $(T, \mathcal{F})$ is a tree T in which every internal node v of degree k is labeled by a graph $G_v \in \mathcal{F}$ on k vertices, called *marker vertices*, such that there is a bijection ρ_v from the edges of T incident to v to the vertices of G_v.

For convenience, we refer to the graph G_v as the *graph-label* of vertex v. It is important to distinguish these graph-labels, from traditional labels.

For example, in Figure 1 the internal nodes of the tree T are represented as large circles, the marker vertices are represented with small hollow circles, the leaves of T are represented with small solid circles, and the bijection ρ_v is denoted by each edge that crosses the boundary of an internal node and ends at a marker vertex.

Importantly, the graph-labels of these internal nodes are a visual tool for convenience alone—indeed the split-decomposition tree itself does not have these graph-labels. However, as we will see in this paper, these graph-labeled trees are a powerful tool by which to look at the structure of the original graph they describe. Some elements of terminology have been summarized in Figure 2, as these are frequently referenced in the proofs of Section 2.

Definition 4. Let $(T, \mathcal{F})$ be a graph-labeled tree and let $\ell, \ell' \in V(T)$ be leaves of T. We say that there is an *alternated path* between ℓ and ℓ', if there exists a path from ℓ to ℓ' in T such that for any adjacent edges $e = (u, v)$ and $e' = (v, w)$ on the path, $(\rho_v(e), \rho_v(e')) \in E(G_v)$.

Definition 5. The *original graph*, also called *accessibility graph*, of a graph-labeled tree $(T, \mathcal{F})$ is the graph $G = G(T, \mathcal{F})$ where $V(G)$ is the leaf set of T and, for $x, y \in V(G)$, $(x, y) \in E(G)$ iff x and y are accessible[1] in $(T, \mathcal{F})$.

[1]That is, there is a path from the leaf representing x to the leaf representing y in the graph-labeled tree $(T, \mathcal{F})$.

Figures 1 and 2 illustrate the concept of alternated path: it is, more informally, a path that only ever uses at most one interior edge of each graph-label.

Definition 6. A *split* [10] of a graph G with vertex set V is a bipartition (V_1, V_2) of V (*i.e.*, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$) such that

(a) $|V_1| \geqslant 2$ and $|V_2| \geqslant 2$;

(b) every vertex of $N(V_1)$ is adjacent to every of $N(V_2)$.

A graph without any split is called a *prime* graph. A graph is *degenerate* if any partition of its vertices without a singleton part is a split: cliques and stars are the only such graphs.

Informally, the split-decomposition of a graph G consists in finding a split (V_1, V_2) in G, followed by decomposing G into two graphs $G_1 = G[V_1 \cup \{x_1\}]$ where $x_1 \in N(V_1)$ and $G_2 = G[V_2 \cup \{x_2\}]$ where $x_2 \in N(V_2)$ and then recursively decomposing G_1 and G_2. This decomposition naturally defines an unrooted tree structure of which the internal vertices are labeled by degenerate or prime graphs and whose leaves are in bijection with the vertices of G, called a *split-decomposition tree*. A split-decomposition tree $(T, \mathcal{F})$ with $\mathcal{F}$ containing only cliques with at least three vertices and stars with at least three vertices is called a *clique-star tree*[2].

It can be shown that the split-decomposition tree of a graph might not be unique (*i.e.*, several sequences of decompositions of a given graph can lead to different split-decomposition trees), but following Cunningham [10], we obtain the following uniqueness result, reformulated in terms of graph-labeled trees by Gioan and Paul [19].

Theorem 7 (Cunningham [10])**.** *For every connected graph G, there exists a unique split-decomposition tree such that:*

(a) every non-leaf node has degree at least three;

(b) no tree edge links two vertices with clique labels;

(c) no tree edge links the center of a star-node to the extremity of another star-node.

Such a tree is called *reduced*, and this theorem establishes a one-to-one correspondence between graphs and their reduced split-decomposition trees. So enumerating the split-decomposition trees of a graph class provides an enumeration for the corresponding graph class, and we rely on this property in the following sections.

Figure 3 demonstrates the *star-join* and *clique-join* operations which respectively allow trees that do not satisfy conditions (b) and (c) to be further reduced—in terms of number of internal nodes.

Definition 8. A graph is said to be *totally decomposable [with respect to the split-decomposition]*, if and only if its split-decomposition tree is a clique-star tree, which by definition contains no prime nodes.

[2] In this paper, we only consider split-decomposition trees which are clique-star trees. As such the family $\mathcal{F}$, to which our graph-labels belong, is understood to only contain cliques and stars. We thus omit $\mathcal{F}$, and simply refer to clique-star trees as T.

Lemma 9 (Split-decomposition tree characterization of distance-hereditary graphs [10, 19])**.** *A graph is* distance-hereditary *if and only if its split-decomposition tree is a clique-star tree.*

For this reason, distance-hereditary graphs are known as the largest graph class that is totally decomposable with respect to the split-decomposition.

1.4 Decomposable structures

In order to enumerate classes of split-decomposition trees, we use the framework of decomposable structures, described by Flajolet and Sedgewick [15]. We refer the reader to this book for details and outline below the basic idea.

We denote by $\mathcal{Z}$ the combinatorial family composed of a single object of size 1, usually called *atom* (in our case, these refer to a leaf of a split-decomposition tree, *i.e.*, a vertex of the corresponding graph).

Given two disjoint families $\mathcal{A}$ and $\mathcal{B}$ of combinatorial objects, we denote by $\mathcal{A}+\mathcal{B}$ the *disjoint union* of the two families and by $\mathcal{A}\times\mathcal{B}$ the *Cartesian product* of the two families.

Finally, we denote by $\textsc{Set}(\mathcal{A})$ (resp. $\textsc{Set}_{\geqslant k}(\mathcal{A})$, $\textsc{Set}_k(\mathcal{A})$) the family defined as all sets (resp. sets of size at least k, sets of size exactly k) of objects from $\mathcal{A}$, and by $\textsc{Seq}_{\geqslant k}(\mathcal{A})$, the family defined as all sequences of at least k objects from $\mathcal{A}$.

1.5 Split-decomposition trees expressed symbolically

While approaching graph enumeration from the perspective of tree decomposition is not a new idea (the recursively decomposable nature of trees makes them well suited to enumeration), Chauve *et al.* [8] brought specific focus to Cunningham's split-decomposition.

Their description of constrained split-decomposition trees using decomposable grammars is the starting point of this paper, so we briefly outline their method here.

Example. Let us consider the split-decomposition tree of Figure 1a, and illustrate how this tree[3] can be expressed recursively as a rooted tree.

Suppose the tree is rooted at vertex 5. Assigning a root immediately defines a direction for all tree edges, which can be thought of as oriented away from the root. Starting from the root, we can set out to traverse the tree in the direction of the edges, one internal node at a time.

We start at the root, vertex 5. The first internal node we encounter is a star-node, and since we are entering it from the star's center, we have to describe what is on each of its two remaining extremities. On one of the extremities there is a leaf, 6; on the other, there is another split-decomposition subtree, of which the first internal node we encounter happens to be another star-node.

This time, we enter the star-node through one of its extremity. So we must describe what is connected to its center and its remaining extremities (of which there is only one).

[3]Figure 1a is not a clique-star tree because it contains a prime node—the leftmost internal node that does not have any splits. We illustrate the method for this more general split-decomposition tree, noting that the process would be identical in the case of a clique-star tree.

Both of these are connected to smaller split-decomposition trees: the extremity is connected itself to a clique-node, which we enter through one of its undistinguished edges (leaving the two other to go to leaves, 7 and 8); the center of the star-node is connected to a prime node, and so on.

Grammar description. Now, to describe this tree symbolically, let's consider the rule for star-nodes (assuming we are, unlike in the tree of Figure 1a, in a clique-star tree that has no prime internal nodes). First assume like at the beginning of our example, that we enter a star-node through its center: we have to describe what the extremities can be connected to.

According to Cunningham's Theorem: we know that there are at least two extremities (since every non-leaf node has degree at least three); and we know that the star-node's extremities *cannot* be connected to the center of another star-node. We call $\mathcal{S}_C$ a split-decomposition tree that is traversed starting at a star-node entered through its center. We have

$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}\left(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X\right)$$

because indeed, we have at least two extremities, which are not ordered—so $\textsc{Set}_{\geqslant 2}(\ldots)$—and each of these extremities can either lead to a leaf, $\mathcal{Z}$, a clique-node entered through any edge, $\mathcal{K}$, and a star-node *entered through one of its extremities*, $\mathcal{S}_X$.

For a star-node entered through its extremity, we have a similar definition, with a twist,

$$\mathcal{S}_X = \left(\mathcal{Z} + \mathcal{K} + \mathcal{S}_C\right) \times \textsc{Set}_{\geqslant 1}\left(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X\right)$$

because the center—which can lead to a leaf, $\mathcal{Z}$, a clique-node, $\mathcal{K}$, or a star-node entered through its center, $\mathcal{S}_C$—is distinct from the extremities (which, from the perspective of the star-node itself, are undistinguishable). We thus express the subtree connected through the center as separate from those connected through the extremities: this is the reason for the Cartesian product (rather than strictly using non-ordered constructions such as $\textsc{Set}$).

Conventions. As explained above, we use rather similar notations to describe the combinatorial classes that arise from decomposing split-decomposition trees. These notations are summarized in Table 2, and the most frequently used are:

- $\mathcal{K}$ is a clique-node entered through one of its edges;
- $\mathcal{S}_C$ is a star-node entered through its center;
- $\mathcal{S}_X$ is a star-node entered through one of its extremities.

Furthermore because we provide grammars for tree classes that are both rooted and unrooted, we use some notation for clarity. In particular, we use $\mathcal{Z}_\bullet$ to denote the *rooted vertex*, although this object does not differ in any way from any other atom $\mathcal{Z}$.

Terminology. In the rest of this paper, we describe the combinatorial class $\mathcal{S}_X$ as representing "*a star-node entered through an extremity*", but others may have alternate descriptions, such as "*a star-node linked to its parent by an extremity*"; or such as Iriza [25], "*a star-node with the subtree incident to one of its extremities having been removed*"—all these descriptions are equivalent (but follow different viewpoints).

1.6 The dissymmetry theorem

All the grammars produced by this methodology are *rooted* grammars: the trees are described as starting at a root, and branching out to leaves—yet the split-decomposition trees are not rooted, since they decompose graphs which are themselves not rooted.

If we were limiting ourselves to *labeled* objects[4], it would be simple to move from a rooted object to an unrooted one, because there are exactly n ways to root a tree with n labeled leaves. But because we allow the graphs (and associated split-decomposition trees) to be *unlabeled*, some symmetries make the transition to unrooted objects less straightforward.

While this problem has received considerable attention since Pólya [30, 31], Otter [28] and others [23], we choose to follow the lead of Chauve *et al.* [8], and appeal to a more recent result, the *dissymmetry theorem*. This theorem was introduced by Bergeron *et al.* [2] in terms of ordered and unordered pairs of trees, and was eventually reformulated in a more elegant manner, for instance by Flajolet and Sedgewick [15, VII.26 p. 481] or Chapuy *et al.* [7, §3]. It states

$$\mathcal{A} + \mathcal{A}_{\circ\to\circ} \simeq \mathcal{A}_{\circ} + \mathcal{A}_{\circ-\circ} \tag{1}$$

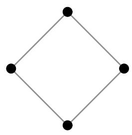
where $\mathcal{A}$ is the unrooted class of trees, and $\mathcal{A}_{\circ}$, $\mathcal{A}_{\circ-\circ}$, $\mathcal{A}_{\circ\to\circ}$ are the rooted classes of trees respectively where a root-vertex is distinguished, a root-edge is distinguished, and a directed, outgoing root-edge is distinguished. The proof is straightforward, see Drmota [11, §4.3.3, p. 293], and involves the notion of *center* of a tree.

For more details on the dissymmetry theorem, see Chauve *et al.* [8, §2.2 and §3]. We will content ourselves with some summary remarks:

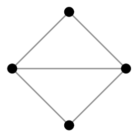
- The process of applying the dissymmetry theorem involves *rerooting* the trees described by a grammar in every possible way. Indeed, the trees obtained from our methodology will initially be rooted at their *leaves*. For the dissymmetry theorem, we re-express the grammar of the tree in all possible ways it can be rooted.

- In terms of notation, we systematically refer to $\mathcal{T}_\omega$ as trees re-rooted at a node (or edge) of type ω. Often these rerooted trees present the distinct characteristic that, unlike the trees described in the rooted grammars, they are not "missing a subtree." Thus the combinatorial class $\mathcal{T}_S$ refers to a split-decomposition tree (of some graph family) which has been rerooted at a star-node; in this context, we must account both for the center, and at least two extremities. See an unrooted grammar, *e.g.*, Theorem 26, for an example of how this notation is used: $\mathcal{T}_K$ refers to trees re-rooted at a clique-node K, while $\mathcal{T}_{S-S}$ refers to trees re-rooted at a star-star edge, $S-S$.

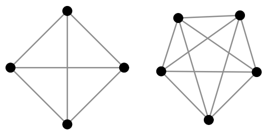
[4]Labeled objects are composed of atoms (think of atoms as being vertices in a graph, or leaves in a tree) that are each uniquely distinguished by an integer between 1 and n, the size of the object; each of these integer is called a *label*.



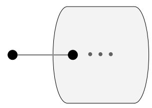
(a) A C_4 (cycle with 4 vertices).



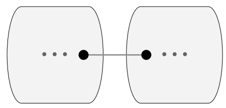
(b) A diamond.



(c) Cliques (C_4, C_5,...).



(d) A pendant vertex.



(e) A bridge.

Figure 4. These are the induced forbidden subgraphs that we investigate in this paper. In Section 2, we introduce a series of lemmas that characterize the split-decomposition tree of a (totally decomposable) graph which avoids one or some of these induced subgraphs.

- A particularity of the dissymmetry theorem is that we can ignore any term $\mathcal{T}_\omega$, in which ω involves a leaf (*e.g.*, we don't need to consider re-rooting the trees at edges which connect a star-node to a leaf). These terms algebraically cancel each-other out [8, Lemma 1] in the subtraction of Eq. (1):

  Lemma 10 (Dissymmetry theorem leaf-invariance [8]). *In the dissymmetry theorem for trees, when rerooting at the nodes (or atoms) of a combinatorial tree-like class $\mathcal{A}$, leaves do not need to be considered.*

- This is a relatively simple theorem to apply; the downside is that it only yields an equality of the coefficient, but it loses the symbolic meaning of a grammar. This is a problem when using the tools of analytic combinatorics [15], in particular those having to do with random generation [16, 12, 14].

- An alternate tool to unroot combinatorial classes, *cycle-pointing* [3], does not have this issue: it is a combinatorial operation (rather than algebraic one), and it allows for the creation of random samplers for a class. However it is more complex to use, though Iriza [25] has already applied it to the distance-hereditary and 3-leaf power grammars of Chauve *et al.* [8].

2 Characterization & Forbidden Subgraphs

In this section, we provide a set of bijective lemmas that characterize the split-decomposition tree of a graph that avoids any of the forbidden induced subgraphs of Figure 4.

2.1 Elementary lemmas

We first provide three simple lemmas, which essentially have to do with the fact that the split-decomposition tree is a *tree*. Their proofs are provided in Appendix B, and notably are still valid in the presence of prime nodes (*i.e.*, these elementary lemmas would still apply to a split-decomposition tree that while reduced, is not purely a clique-star tree—even though those are the only trees that we work with in the context of this paper).

Recall also that the notion of *totally decomposable* graph is introduced in Definition 8.

Lemma 11. *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T, any maximal*[5] *alternated path starting from any node in $V(T)$ ends in a leaf.*

Lemma 12. *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T and let $u \in V(T)$ be an internal node. Any two maximal alternated paths P and Q that start at distinct marker vertices of u but contain no interior edges from G_u end at distinct leaves.*

Lemma 13. *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T. If T has a clique-node of degree n, then G has a corresponding induced clique on (at least) n vertices.*

2.2 Forbidden subgraphs lemmas

Definition 14. Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T. A *center-center path* in T is an alternated path P, such that the endpoints of P are centers of star-nodes $(u, v) \in V(T)^2$ and P does not contain any interior edge of either star-node.

Lemma 15 (Split-decomposition tree characterization of $\boldsymbol{C_4}$-free graphs)**.** *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T. G does not have any induced C_4 if and only if T does not have any center-center paths.*

Proof. [$\Rightarrow$] Let T be a clique-star tree with a center-center path P between the centers of two star-nodes $u, v \in V(T)$; we will show that the accessibility graph $G(T)$ has an induced C_4.

Let $c_u \in G_u$ and $c_v \in G_v$ be the endpoints of P. Since T is assumed to be a reduced split-decomposition tree (Theorem 7), u and v have degree at least three and thus G_u and G_v have at least two extremities. Therefore, there are at least two maximal alternated paths out of u (resp. v), each beginning at an extremity of G_u (resp. G_v) and not using any interior edges of G_u (resp. G_v). By Lemma 12, these paths end at distinct leaves $a, b \in V(T)$ (resp. $c, d \in V(T)$), as shown in Figure 5.

[5] A maximal alternated path is one that cannot be extended to include more edges while remaining alternated.

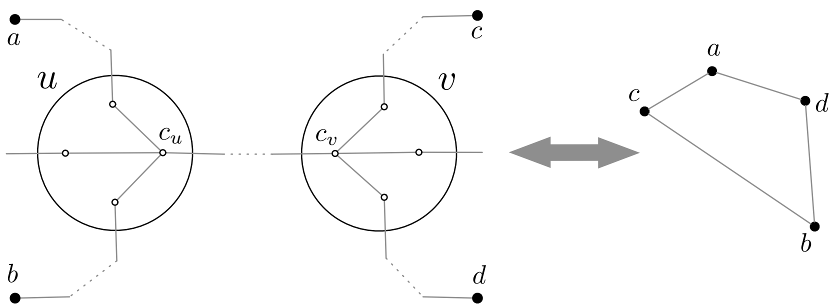


Figure 5. A center-center path in a split-decomposition tree translates to an induced C_4 in the corresponding accessibility graph, see Lemma 15.

Now consider the accessibility graph G of T. First, we observe that the pairs[6] (a, c), (a, d), (b, c), (b, d) all belong to the edge set of G. We will show this for the edge (a, c) by extending P into an alternated path in T from a to c. The argument extends symmetrically to the other three edges.

Let P_a be the alternated path between a and an extremity of G_u, and let P_c be the alternated path between c and an extremity of G_v. To show $(a, c) \in E(G)$, we extend P into the following alternated path:

$$P_a, c_u, P, c_v, P_c$$

We next observe that (a, b) and symmetrically (c, d) cannot belong to the edge set of G. Since T is a tree, there is a unique path in T between a and b, which passes through u. This unique path must use two interior edges within G_u and therefore cannot be alternated. Consequently, $(a, b) \notin E(G)$. It can be shown by a similar argument that $(c, d) \notin E(G)$. Therefore, the induced subgraph of G consisting of a, b, c, d is a C_4 illustrated in Figure 5.

[$\Leftarrow$] Let G be a totally decomposable graph with an induced C_4 with its vertices arbitrarily labeled $(a, c, b, d) \in V(G)$ as in Figure 5. We will show that the reduced split-decomposition tree T of G has a center-center path.

First, we will show that there is a star-node $v \in V(T)$ that has alternated paths out of its extremities ending in c and d. Since $(a, c), (a, d) \in E(G)$, there must exist alternated paths $P_{a,c}$ and $P_{a,d}$, which begin at the leaf a and end at the leaf c or d respectively. Let $v \in V(T)$ be the internal node that both $P_{a,c}$ and $P_{a,d}$ enter via the same edge ρ_{c_v} but exit via different edges ρ_{x_c} and ρ_{x_d} respectively. We claim that v must be a star-node, such that c_v is its center and x_c and x_d are two of its extremities. It is sufficient to show $(x_c, x_d) \notin E(G_v)$, which is indeed true because otherwise, we could use that edge and the disjoint parts of $P_{a,c}$ and $P_{a,d}$ to construct the alternated path between c and d, contradicting the fact that $(c, d) \notin E(G)$.

Next, we will show that there is a star-node $u \in V(T)$ that has alternated paths out of its extremities ending in a and b and forms a center-center path with v. Consider this

[6]Out of what is, perhaps, notational abuse, we refer to both vertices of the accessibility graph, and leaves of the split-decomposition tree as the same objects.

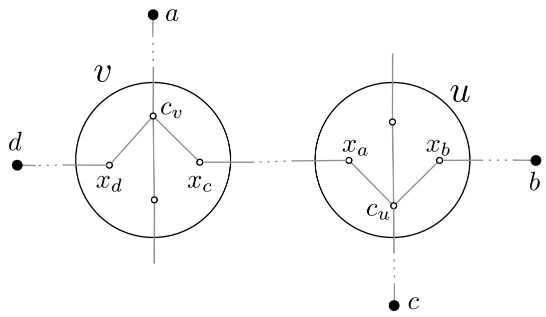


Figure 6. A graph that has an induced C_4 subgraph on vertices a, c, b, d must have a center-center path. In this figure, the star u is where $P_{a,c}$ and $P_{b,c}$ branch apart, and the star-node v is where $P_{a,c}$ and $P_{a,d}$ branch apart. If u and v do not form a center-center path, b and d cannot be adjacent in the accessibility graph G.

time the alternated path $P_{b,c}$ between leaves b and c, as well as $P_{a,c}$ defined above. Similar to the argument above, let $u \in V(T)$ be the internal node that both $P_{a,c}$ and $P_{b,c}$ enter via the same edge ρ_{c_u} but exit via different edges ρ_{x_a} and ρ_{x_b} respectively. With the same argument outlined above, u must be a star-node, such that c_u is its center and x_a and x_b are two of its extremities. It remains to show that u and v form a center-center path.

Suppose u and v do not form a center-center path. Then u must be on the common part of $P_{a,c}$ and $P_{b,c}$ between x_c and c. However, in this case, both u and v lie on the unique path $P_{b,d}$ in T between b and d, in such a way that $P_{b,d}$ must use two interior edges of both G_u and G_v, which is a contradiction since $(b, d) \in E(G)$ (see Figure 6). □

Remark 16. Importantly, a *center-center path* is defined as being an alternated path between the centers of two star-nodes, as reflected in Figure 6. In this manner, the definition excludes the possibility that, somewhere on the path between the c_u and c_v marker vertices, there is a star (or for that matter a prime node) which *breaks* the alternating path—in the sense that it requires taking at least two interior edges.

But while the definition excludes it, it is a very real possibility to keep in mind when decomposing the grammar of the tree. As we will see in Section 4 on ptolemaic graphs, specifically for the case of the clique-node $\mathcal{K}$, we may need to engineer the grammar in such a way that it keeps track of whether a path between two nodes is alternated (or not).

Definition 17. Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T. A *clique-center path* in T is an alternated path P, such that the endpoints of P are the center of a star-node $u \in V(T)$ and a marker vertex of a clique-node $v \in V(T)$ and P does not contain any interior edge of the clique-node or the star-node.

Lemma 18 (Split-decomposition tree characterization of diamond-free graphs)**.** *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T. G does not have any induced diamonds if and only if T does not have any induced clique-center paths.*

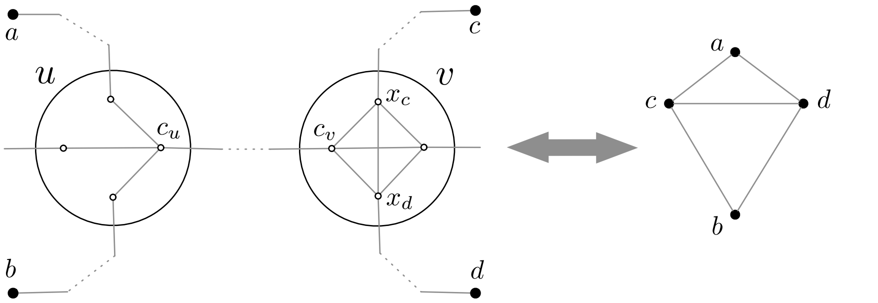


Figure 7. A clique-center path in a split-decomposition tree translates to an induced diamond in the accessibility graph.

Proof. [⇒] Let T be a clique-star tree containing a clique-center path P between the center of a star-node $u \in V(T)$ and a marker vertex of a clique-node $v \in V(T)$. We will show that the accessibility graph $G(T)$ has an induced diamond.

Let $c_u \in G_u$ and $c_v \in G_v$ be the endpoints of P. By an argument similar to the one in the proof of Lemma 15, it follows from Lemma 12 that there must be at least two disjoint maximal alternated paths out of u, each beginning at an extremity of G_u and ending at leaves $a, b \in V(T)$. Similarly, there must be at least two disjoint maximal alternated paths out of the clique-node v ending at leaves $c, d \in V(T)$ (Figure 7).

We can now show that this clique-center path translates to an induced diamond in the accessibility graph G of T. Given this established labeling of the leaves a, b, c, d and internal nodes u, v, the exact same argument outlined in the proof of Lemma 15 directly applies here, showing that $(a, c), (a, d), (b, c), (b, d) \in E(G)$. Similarly, it can be shown that $(a, b) \notin E(G)$.

Where this proof diverges from the proof of Lemma 15 is in the existence of the edge $(c, d) \in E(G)$. This is easy to show: Let $x_c, x_d \in V(G_v)$ be the marker vertices of the clique-node v that mark the end points of the paths out of v to the leaves c and d respectively[7]. We have $(c, d) \in E(G)$ by tracing the following alternated path: c, x_c, x_d, d (see Figure 7).

[⇐] Let G be a totally decomposable graph with an induced diamond on vertices $(a, c, b, d) \in V(G)$ labeled as illustrated in Figure 7. We need to show that the reduced split-decomposition tree T of G has a clique-center path.

It can be shown by a similar argument to the proof of Lemma 15 that there must exist a star node $u \in V(T)$ that has alternated paths $P_{a,c}$ and $P_{b,c}$ out of its extremities ending in a and b respectively. Let $c_u \in G_u$ be the center of this star-node.

Similarly, we can show that there is a clique-node out of which maximal alternated paths lead to c and d. Let $P_{a,d}$ be the unique path in T between leaves a and d, and

[7]We chose here to use the same notation as the proof of Lemma 15 in referring to marker vertices of the clique-node v by names that might be reminiscent of the center and extremities of a star-node. This notation is not meant to imply that v is a star-node, but rather aims to highlight the parallelism between the two proofs, hinting at the ease by which our methods can be generalized to derive split-decomposition tree characterizations for different classes of graphs defined in terms of forbidden subgraphs.

consider the node $v \in V(T)$ where $P_{a,c}$ and $P_{a,d}$ branch apart. Let $c_v \in V(G_v)$ denote the marker vertex in common between the two paths, and let $x_c, x_d \in V(G_v)$ be the marker vertices out of which $P_{a,c}$ and $P_{a,d}$ exit v respectively. Since $(c,d) \in E(G)$, there must be an alternated path in T between c and d that uses at most one interior edge from G_v, so we must have $(x_c, x_d) \in E(G_v)$. Therefore, G_v has an induced K_3 on the marker vertices x_c, x_d, and c_v. Since T is a clique-star tree and v cannot be a star-node, it has to be a clique node.

Finally, we need to show that u and v form a clique-center path. This is indeed the case since, if the path P between u and v connected to either extremity of G_u, one of the following cases would occur:

- P connects to the extremity of u ending in a, which implies $(b,c), (b,d) \notin E(G)$;
- P connects to the extremity of u ending in b, which implies $(a,c), (a,d) \notin E(G)$;
- P connects to another extremity of u (if one exists), which implies (a,c), (a,d), (b,c), $(b,d) \notin E(G)$.

Since all the above cases contradict the fact that $\{a,b,c,d\}$ induces a diamond in G, P must be a clique-center path between u and v. □

Lemma 19 (Split-decomposition tree characterization of graphs without induced cliques on 4 (or more) vertices)**.** *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T. G does not contain any induced $K_{\geqslant 4}$ subgraphs if and only if T does not have:*

- *any clique-nodes of degree 4 or more;*
- *any alternated paths between different clique-nodes.*

Proof. [⇒] We will show that for any clique-star tree T breaking either of the conditions of this lemma, the accessibility graph $G(T)$ must have an induced clique on at least 4 vertices as a subgraph.

First, suppose T has a clique-node of degree 4 or more. It follows from Lemma 13 that $G(T)$ must have an induced $K_{\geqslant 4}$ subgraph.

Second, suppose there are two clique-nodes $u, v \in V(T)$ connected via an alternated path P. Each of G_u and G_v must have at least three marker vertices, one of which belongs to P. Therefore, u and v each have at least two marker vertices with outgoing maximal alternated paths that end in two distinct leaves by Lemma 12. The four leaves at the end of these alternated paths are pairwise adjacent in G, thus inducing a K_4.

[⇐] Let G be a totally decomposable graph with an induced clique subgraph on 4 or more vertices, including $a, b, c, d \in V(G)$. We will show that the split-decomposition tree T of G breaks at least one the conditions listed in this lemma, i.e. either T has a clique-node of degree 4 or more, or it has two clique-nodes (of degree 3) connected via an alternated path.

Consider the alternated paths $P_{a,b}$, $P_{a,c}$, and $P_{a,d}$ between the pairs of leaves $\{a,b\}$, $\{a,c\}$, and $\{a,d\}$ respectively. Let $u_{b,c} \in P_{a,b} \cap P_{a,c}$ be the closest internal node to a in common between $P_{a,b}$ and $P_{a,c}$.

We observe that $u_{b,c}$ must be a clique-node. This is the case because if $u_{b,c}$ were a star-node, at least two of the alternated paths would have to enter $u_{b,c}$ at two extremities

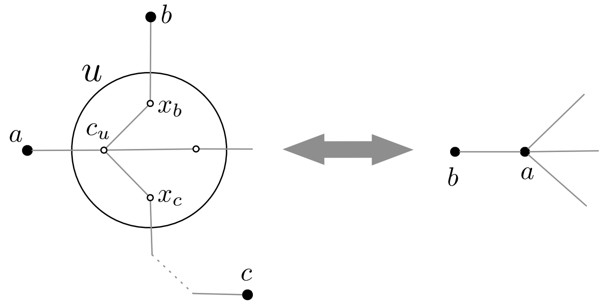


Figure 8. In a clique-star tree, a star-node with both its center and one of its extremities adjacent to leaves translates to a pendant edge in the accessibility graph.

and use two interior edges of the graph-label $g_{u_{b,c}}$. In this case, the leaves at the end of those two paths could not be adjacent in G.

By a symmetric argument, it can be shown that $u_{b,d}$, the closest internal node to a in common between $P_{a,b}$ and $P_{a,c}$, must also be a clique-node.

Depending on whether or not $u_{b,c}$ and $u_{b,d}$ are distinct nodes, one of the conditions of the lemma is contradicted:

- if $u_{b,c}$ and $u_{b,d}$ are the same clique node, there are four disjoint outgoing alternated paths out it, implying that it must have a degree of at least four, contradiction the first condition of the lemma;
- if $u_{b,c}$ and $u_{b,d}$ are distinct clique nodes, they are connected by an alternated path that is a part of $P_{a,b}$ between them, contradicting the second condition of the lemma.

□

Lemma 20 (Split-decomposition tree characterization of graphs without pendant edges)**.** *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T. G does not have any pendant edges if and only if T does not have any star-node with its center and an extremity adjacent to leaves.*

Proof. [$\Rightarrow$] Let T be a clique-star tree, and let $u \in V(T)$ be a star-node, such that its center c_u is adjacent to a leaf $a \in V(T)$ and one of its extremities x_b is adjacent to a leaf $b \in V(T)$. We will show that b does not have any neighbors beside a in the accessibility graph $G(T)$ and thus, the edge (a, b) is a pendant edge of $G(T)$.

Suppose, on the contrary, that b has a neighbor $c \in V(G(T))$, $c \neq a$. Then there must be an alternated path P in T that connects b and c. Note that P must go through u, entering it at an extremity $x_c \in V(G_u)$. The path P must thus use two interior edges (x_b, c_u) and (c_u, x_c) and cannot be alternated (Figure 8).

[$\Leftarrow$] Let G be a totally decomposable graph with a pendant edge $(a, b) \in E(G)$ such that b has degree 1 (Figure 8). We will show that the corresponding leaves a and b in the reduced clique-star tree T of G are attached to a star-node u, with its center adjacent to a and one of its extremities adjacent to b. Let $u \in V(T)$ be the internal node to which b is attached, and let $x_b \in V(G_u)$ be the marker vertex adjacent to b.

First, we will show that u is indeed a star-node and x_b is one of its extremities. To do so, it suffices to show that x_b has degree 1 in G_u. Suppose, on the contrary, that there are two marker vertices $y, z \in V(G_u)$ that are adjacent to x_b, and consider two maximal alternated paths out of y and z. By Lemma 12, these paths end at two distinct leaves of T, both of which much be adjacent to b in G, contradicting the assumption that b has degree 1.

Next, we will show that a must be attached to the center c_u of G_u. Otherwise, one of the following cases will occur:

- *c_u is adjacent to a leaf $c \in V(T)$.* In this case, we have the alternated path b, x_b, c_u, c, implying $(b, c) \in E(G)$, contradicting the assumption that b has degree 1.
- *c_u is adjacent to a clique-node $v \in V(T)$.* With an argument similar to the previous case, it can be shown that in this case, there must exist at least two alternated paths out of v that lead to leaves, all of which must be adjacent to b.
- *c_u is adjacent to the center of a star-node $v \in V(T)$.* Similar to the previous case, there must exist at least two alternated paths out extremities of v that lead to leaves, all of which must be adjacent to b.
- *c_u is adjacent to an extremity of a star-node $v \in V(T)$.* This case never happens, since T is assumed to be a *reduced* clique-star tree.

Therefore, a must be attached to the center of G_u, so u is a star-node with a adjacent to its center and b adjacent to one of its extremities. □

Lemma 21 (Split-decomposition tree characterization of graphs without bridges)**.** *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T. G does not have any bridges if and only if T does not have:*

- *any star-node with its center and an extremity adjacent to leaves;*
- *any two star-nodes adjacent via their extremities, with their centers adjacent to leaves.*

Proof. We distinguish between two kinds of bridges: pendant edges and other bridges, which we will call *internal* bridges. Lemma 20 states that a star-node with its center and an extremity adjacent to leaves in T corresponds to a pendant edge in G. Therefore, it suffices to show G has no internal bridges if and only if the second condition holds in T.

[⇒] Let T be a clique-star tree, and let $u, v \in V(T)$ be two star-nodes, with the center $c_u \in G_u$ adjacent to a leaf $a \in V(T)$, the center $c_v \in G_v$ adjacent to a leaf $b \in V(T)$, and two of their extremities $x_u \in G_u$ and $x_v \in G_v$ adjacent to each other. We will show that (a, b) is an internal bridge in $G(T)$.

First, let us define the following partition of the leaves of T into two sets: Since every edge in a tree is a bridge, removing (u, v) from T breaks T into two connected components. Let $V_u, V_v \in V(T)$ be the leaves of these components respectively, and note that $a \in V_u$ and $b \in V_v$ (Figure 9).

Next, note that $(a, b) \in E(G(T))$ by tracing the alternated path a, c_u, x_u, x_v, c_v, b. To show that (a, b) must be an internal bridge, we will show that the edge (a, b) is a bridge in $G(T)$ by showing it does not belong to any cycles. We will then confirm that (a, b) must be an *internal* bridge.

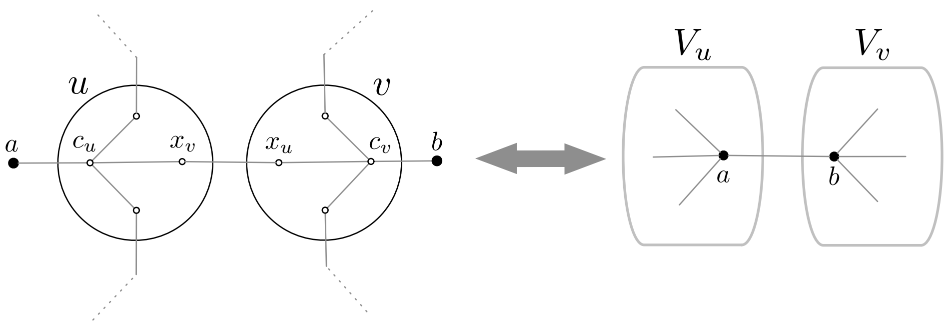


Figure 9. In a clique-star tree, a structure consisting of two star-nodes adjacent via their extremities, with their centers adjacent to leaves, translates to an *internal* bridge in the accessibility graph.

Suppose, on the contrary, that $G(T)$ has a cycle C of vertices $(x_1 = a,\ x_2,\ \ldots,\ x_{k-1},\ x_k = b) \in V(G(T))^k$ for some $k \geqslant 3$. Clearly, $x_1 = a \in V_u$. Additionally, for every edge $(x_i, x_{i+1}) \in E(G(T))$, $i = 1 \ldots k-1$, there must be an alternated path P_i in T between leaves x_i and x_{i+1}. Furthermore, if $x_i \in V_u$, we must also have $x_{i+1} \in V_u$, since otherwise, P_i must use the only edge crossing the cut V_u, V_v; this requires P_i to enter and exit u via two extremities of G_u, which requires using two interior edges from G_u. Applying a similar argument for every edge (x_i, x_{i+1}) of C up to b implies that $b \in V_u$. Therefore, we must have $b \in V_u \cap V_v$, contradicting the fact that V_u and V_v are disjoint.

Finally, we can show via Lemma 12 that (a, b) must be an *internal* bridge, by showing that a and b must have neighbors besides each other in $G(T)$). We will confirm this for a, and the argument applies symmetrically to b. Since T is reduced, u has degree at least three, so there is at least one alternating path out of an extremity of G_u other than x_u ending in a leaf of T other than b, implying that a must be adjacent to that leaf in $G(T)$. Similarly, b must have a neighbor in $G(T)$ other than a. Therefore, (a, b) cannot be a pendant edge and must be an internal bridge.

[$\Leftarrow$] Let G be a totally decomposable graph with an internal bridge $(a, b) \in E(G)$ (Figure 9). We show that the corresponding leaves a and b in the reduced clique-star tree T of G are respectively attached to centers of two star-node u and v adjacent via their extremities.

Let $u \in V(T)$ be the internal node to which a is attached, and let $c_u \in V(G_u)$ be the marker vertex adjacent to a. Similarly, let $v \in V(T)$ be the internal node to which b is attached, and let $c_v \in V(G_v)$ be the marker vertex adjacent to b.

Now, we show that u and v are star-nodes. Suppose, on the contrary, that u is a clique-node, and note that G_u must have at least two other marker vertices besides c_u. Consider the two maximal alternated paths P_x and P_y out of these two marker vertices, respectively ending in leaves $x, y \in V(T)$ by Lemma 12. We first observe that $(a, x) \in E(G)$ by the union of the alternated path P_x and the interior edge of G_u between c_u and the marker vertex at the end of P_x. Similarly, we have $(a, y) \in E(G)$. Furthermore, $(x, y) \in E(G)$ by the union of the two alternated paths P_x and P_y and the interior edge of G_u between the ends of these paths.The trio of vertices $a, x, y \in V(G)$ thus induces a C_3 in G, contradicting

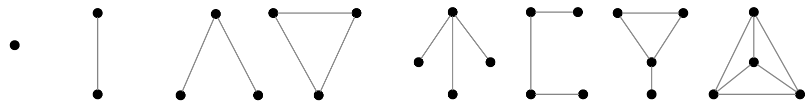


Figure 10. All block graphs on four vertices or less.

(a) A diamond.

(b) A C_4.

Figure 11. The class of *weakly geodetic* graphs can be characterized in terms of forbidden subgraphs as (C_4, diamond)-free. This is the case because, while in a weakly geodetic graph, a pair of vertices of distance 2 has a unique common neighbor, in an induced C_4 or diamond subgraph, the highlighted vertices a and b are of distance 2 while having at least two common neighbors c and d.

the assumption that (a, b) is a bridge.

Next, we will show that u and v are adjacent to each other via their extremities x_u and x_v. Otherwise, since no star centers are adjacent to extremities of other star-nodes in a reduced split-decomposition tree, u and v would have to be adjacent via their centers. This would constitute a center-center path, which would, by Lemma 15, imply that (a, b) belongs to a C_4 and cannot be a bridge.

Finally, we confirm that c_u and c_v, the marker vertices to which a and b are attached, are the centers of G_u and G_v respectively. It suffices to show this claim for a, as the argument symmetrically applies to b as well. If, on the contrary, a were attached to an extremity of G_u, the only path in T between a and b would have to use two interior edges of G_u, one from c_u to the center of G_u and one from the center to x_u. This would imply $(a, b) \notin E(G)$, a contradiction. □

3 Block graphs

In this section, we analyze a class of graphs called block graphs. After providing a general definition of this class, we present its well-known forbidden induced subgraph characterization, and using a lemma we proved in Section 2, we deduce a characterization of the split-decomposition tree of graphs in this class.

Block graphs are the (weakly geodetic, as defined a few paragraphs below) subset of ptolemaic graphs—themselves the (chordal) subset of distance-hereditary graphs. Thus, their split-decomposition tree is a more constrained version of that of ptolemaic graphs. As such, we use block graphs as a case study to prepare for ptolemaic graphs, for which the grammar is a bit more complicated.

3.1 Characterization

For any graph G, a vertex v is a *cut vertex* if the number of connected components is increased after removing v, and a *block* is a maximal connected subgraph without any cut vertex.

A graph is then called a *block graph* [21] if and only if its blocks are complete graphs (or cliques) and the intersection of two blocks is either empty or a cut vertex. Block graphs are the intersection of ptolemaic graphs and weakly geodetic graphs, as was shown by Kay and Chartrand [26].

Definition 22 (Kay and Chartrand [26, §2])**.** A graph is *weakly geodetic* if for every pair of vertices of distance 2 there is a unique common neighbor of them.

It is relatively intuitive to figure out from this definition, that weakly geodetic graphs are exactly (C_4, diamond)-free graphs, but surprisingly we were only able to find this result mentioned relatively recently [13].

Lemma 23. *A graph is weakly geodetic if and only if it contains no induced C_4 or diamond subgraphs.*

Proof. [$\Rightarrow$] We show a weakly geodetic graph is (C_4, diamond)-free by arguing that graphs with induced C_4 subgraphs and diamonds as induced subgraphs are not weakly geodetic. This is illustrated in Figure 11, in which the highlighted pairs of vertices in a C_4 and a diamond are of distance 2 and have more than one neighbor in common.
[$\Leftarrow$] Let G be a (C_4, diamond)-free, and let $a, b \in V(G)$ be vertices of distance 2 with two neighbors $c, d \in V(G)$ in common. Since a, b have distance 2, $(a, b) \notin E(G)$. Depending on whether or not (c, d) belongs to the edge set of G, we have $\{a, b, c, d\}$ inducing a diamond (Figure 11a) or a C_4 (Figure 11b) respectively. □

Since we have established that block graphs are the subset of totally decomposable (distance-hereditary) graphs which are also (C_4, diamond)-free, we can now characterize their split-decomposition tree by applying our two lemmas from Section 2 and deducing the overall constraint on the split-decomposition trees that these imply.

Theorem 24 (Split-decomposition tree characterization of block graphs)**.** *A graph G with the reduced split-decomposition tree $(T, \mathcal{F})$ is a block graph if and only if*

(a) T is a clique-star tree;

(b) the centers of all star-nodes are attached to leaves.

Proof. We have introduced block graphs as being the intersection class of ptolemaic graphs and weakly geodetic graphs. As we will see again in Section 4, Howorka [24, §2] has shown that ptolemaic graphs are the intersection class of distance-hereditary graphs and chordal (triangulated) graphs.

A chordal graph is a graph in which any cycle of size larger than 3 has a chord; because distance-hereditary graphs are themselves $C_{\geqslant 5}$-free, chordal distance-hereditary (ptolemaic) graphs are the C_4-free distance-hereditary graphs. The additional constraint that comes with being weakly geodetic, implies that block graphs are the (C_4, diamond)-free distance-hereditary graphs[8].

The first condition in this theorem is due to the total decomposability of block graphs as a subset of distance-hereditary graphs. The second condition forbids having any center-center or clique-center paths, which, by Lemma 15 and Lemma 18 respectively, ensures that G does not have any induced C_4 or diamond. □

3.2 Rooted grammar

Using the split-decomposition tree characterization derived above, we can provide a symbolic grammar that can be used to enumerate labeled and unlabeled block graphs. This provides a new comprehensive approach for this class, previously enumerated by Harary [22, §3.4.14].

Theorem 25. *The class $\mathcal{BG}_\bullet$ of block graphs rooted at a vertex is specified by*

$$\mathcal{BG}_\bullet = \mathcal{Z}_\bullet \times (\mathcal{S}_C + \mathcal{S}_X + \mathcal{K}) \tag{2}$$

$$\mathcal{K} = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) \tag{3}$$

$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{4}$$

$$\mathcal{S}_X = \mathcal{Z} \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X)\,. \tag{5}$$

This grammar is similar to that of distance-hereditary graphs [8]. The constraint that the centers of all star-nodes are attached to leaves means essentially that the rule $\mathcal{S}_C$ can only be reached as a starting point when we are describing what the root vertex might be connected to (from the initial rule, $\mathcal{BG}_\bullet$).

For the sake of comprehensiveness, we give this proof in full detail. However since the following proofs are fairly similar, we will tend to abbreviate them.

Proof. We begin with the class of split-decomposition trees for block graphs, rooted at a star-node entered by its center,

$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X)\,.$$

This equation specifies that a subtree rooted at a star-node, linked to its parent by its center, has at least 2 unordered children attached to the extremities of the star-node: each extremity can either lead to a leaf, a regular clique-node, or another star-node entered through an extremity (but not another star-node entered through its center since the tree is reduced). The lower bound of 2 children is due to the fact that in a reduced

[8]Alternatively block graphs can be characterized as the class of ($C_{\geqslant 4}$, diamond)-free graphs. Since block graphs are also distance-hereditary, and since distance-hereditary graphs do not have any induced $C_{\geqslant 5}$, we conclude again that block graphs can be thought of as (C_4, diamond)-free distance-hereditary graphs.

split-decomposition tree, every internal node has degree at least 3, one of which is the star-node's center.

Next, we consider:

$$\mathcal{S}_X = \mathcal{Z} \times \textsc{Set}_{\geqslant 1}\left(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X\right),$$

corresponding to a subtree rooted at a star-node, linked to its parent by an extremity. This star-node can be be exited either via its center and lead to a leaf $\mathcal{Z}$ (the only type of element the center of a star-node can be connected to, following Theorem 24), or via some extremity and lead to a leaf, a regular clique-node, or another star-node entered through an extremity (but not another star-node entered through its center, as that is forbidden in reduced trees).

Next, we have the equation corresponding to a clique-node,

$$\mathcal{K} = \textsc{Set}_{\geqslant 2}\left(\mathcal{Z} + \mathcal{S}_X\right).$$

A clique-node has a degree of at least three, so a clique-rooted subtree can be exited from a set of at least two children and reach a leaf or a star-node through its extremity. It cannot reach another clique-node since the tree is reduced, and it cannot enter a star-node through its center since, again according to Theorem 24, star centers are only adjacent to leaves.

Finally, this equation

$$\mathcal{BG}_\bullet = \mathcal{Z}_\bullet \times (\mathcal{S}_C + \mathcal{S}_X + \mathcal{K}),$$

combines the previously introduced terms into a specification for rooted split-decomposition trees of block graphs, which are combinatorially equivalent to the class of rooted block graphs. It states that a rooted split-decomposition tree of a block graph consists of a distinguished leaf $\mathcal{Z}_\bullet$, which is attached to an internal node. The internal node could be a clique-node, or a star-node entered through either its center or an extremity. □

With this symbolic specification, and a computer algebra system, we may extract an arbitrarily long enumeration (we have easily extracted 10 000 terms).

3.3 Unrooted grammar

Applying the dissymmetry theorem to the internal nodes and edges of split-decomposition trees for block graphs gives the following grammar.

Theorem 26. *The class* $\mathcal{BG}$ *of unrooted block graphs is specified by*

$$\mathcal{BG} = \mathcal{T}_K + \mathcal{T}_S + \mathcal{T}_{S-S} - \mathcal{T}_{S\to S} - \mathcal{T}_{S-K} \tag{6}$$

$$\mathcal{T}_K = \textsc{Set}_{\geqslant 3}\left(\mathcal{Z} + \mathcal{S}_X\right) \tag{7}$$

$$\mathcal{T}_S = \mathcal{Z} \times \mathcal{S}_C \tag{8}$$

$$\mathcal{T}_{S-S} = \textsc{Set}_{=2}\left(\mathcal{S}_X\right) \tag{9}$$

$$\mathcal{T}_{S\to S} = \mathcal{S}_X \times \mathcal{S}_X \tag{10}$$

$$\mathcal{T}_{S-K} = \mathcal{K} \times \mathcal{S}_X \tag{11}$$

$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{12}$$

$$\mathcal{S}_X = \mathcal{Z} \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{13}$$

$$\mathcal{K} = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) \tag{14}$$

As noted in Subsection 1.6, in the unrooted specification, the classes denoted by $\mathcal{T}_{\ldots}$ correspond to trees introduced by the dissymmetry theorem, whereas the specification of all other classes is identical to the rooted grammar for block graph split-decomposition trees given in Theorem 25.

Proof. From the dissymmetry theorem, we have the following bijection linking rooted and unrooted split-decomposition trees of block graphs,

$$\mathcal{BG} = \mathcal{BG}_{\circ} + \mathcal{BG}_{\circ - \circ} - \mathcal{BG}_{\circ \to \circ}.$$

Lemma 10 allows us to consider only internal nodes for the rooted terms. Since block graphs are totally decomposable into star-nodes and clique-nodes, we have the following symbolic equation for split-decomposition trees of block graphs rooted at an internal node,

$$\mathcal{BG}_{\circ} = \mathcal{T}_K + \mathcal{T}_S.$$

Additionally, when rooting split-decomposition trees of block graphs at an undirected edge between internal nodes, the edge could either connect two star-nodes or a star-node and a clique-node (recall that clique-nodes cannot be adjacent in reduced trees by Theorem 7), which yields the following symbolic equation for block graph split-decomposition trees rooted at an internal undirected edge,

$$\mathcal{BG}_{\circ - \circ} = \mathcal{T}_{S-S} + \mathcal{T}_{S-K}.$$

Finally, when rooting split-decomposition trees of block graphs at a directed edge between internal nodes, the edge could either go from a star-node to a clique-node, a clique-node to a star-node, or a star-node to another star-node (again, there are no adjacent clique-nodes by Theorem 7 of reduced trees), giving the following symbolic equation for block graph split-decomposition trees rooted at an internal directed edge,

$$\mathcal{BG}_{\circ \to \circ} = \mathcal{T}_{S \to K} + \mathcal{T}_{K \to S} + \mathcal{T}_{S \to S}.$$

Combining the above equations with the dissymmetry theorem, as stated in Equation 1, yields

$$\begin{aligned} \mathcal{BG} = \ & \mathcal{T}_K + \mathcal{T}_S \\ & + \mathcal{T}_{S-S} + \mathcal{T}_{S-K} \\ & - \mathcal{T}_{S \to K} - \mathcal{T}_{K \to S} - \mathcal{T}_{S \to S}. \end{aligned}$$

We next observe the following bijection between split-decomposition trees rooted at an edge between a clique-node and a star-node, $\mathcal{T}_{S \to K} \simeq \mathcal{T}_{K \to S} \simeq \mathcal{T}_{S-K}$. This is due to the

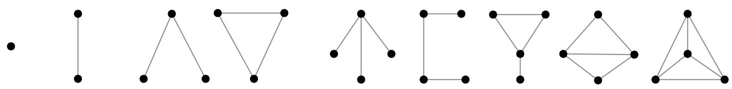


Figure 12. Small (unrooted, unlabeled) ptolemaic graphs.

fact that star- and clique-nodes are distinguishable, so an edge connecting a star-node and a clique-node bears an implicit direction. (One can, for example, define the direction to always be out of the clique-node into the star-node.) Simplifying accordingly, we arrive at Equation (6),

$$\mathcal{PG} = \mathcal{T}_K + \mathcal{T}_S + \mathcal{T}_{S-S} - \mathcal{T}_{S\to S} - \mathcal{T}_{S-K}$$

We will now discuss the symbolic equations for rooted split-decomposition trees of block graphs, starting with the following equation,

$$\mathcal{T}_K = \textsc{Set}_{\geqslant 3}\left(\mathcal{Z} + \mathcal{S}_X\right).$$

This equation states that the split-decomposition tree of a block graph rooted at a clique-node can be specified as a set of at least three subtrees (since internal nodes in reduced split-decomposition trees have degree $\geqslant 3$), each of which can lead to either a leaf or a star-node entered through its center; they cannot lead to clique-nodes as there are no adjacent clique-nodes in reduced split-decomposition trees, and they cannot lead to star-nodes through their centers, as centers of star-nodes in block graph split-decomposition trees only connect to leaves.

Next, we will consider the equation,

$$\mathcal{T}_S = \mathcal{Z} \times \mathcal{S}_C.$$

which specifies a block graph split-decomposition tree rooted at a star-node. The specification of the subtrees of the distinguished star-node depends on whether they are connected to the center or an extremity of the root. The center of the root can only be attached to a leaf, while the subtrees connected to the extremities of the distinguished star-node are exactly those specified by an $\mathcal{S}_C$.

The other three rooted tree equations follow from with the same logic. □

4 Ptolemaic graphs

Ptolemaic graphs were introduced by Kay and Chartrand [26] as the class of graph that satisfied the same properties as a ptolemaic space. Later, it was shown by Howorka [24] that these graphs are exactly the intersection of distance-hereditary graphs and chordal graphs; beyond that, relatively little is known about ptolemaic graphs [37], and in particular, their enumeration was hitherto unknown.

4.1 Characterization

Definition 27. A graph G is *ptolemaic* if any four vertices u, v, w, x in the same connected component satisfy the ptolemaic inequality [26]:

$$d_G(u,v) \cdot d_G(w,x) \leqslant d_G(u,w) \cdot d_G(v,x) + d_G(u,x) \cdot d_G(v,w).$$

Equivalently, ptolemaic graphs are graphs that are both chordal and distance-hereditary [24, §2].

This second characterization is the one that we will use: indeed, by a reasoning similar to that provided in the proof of Theorem 24, we have that distance-hereditary graphs do not contain any $C_{\geqslant 5}$, and chordal graphs do not contain any $C_{\geqslant 4}$; by virtue of being a distance-hereditary graph (described by a clique-star tree), we thus need only worry about the forbidden C_4 induced subgraphs. As it so happens, we already have a characterization of a split-decomposition tree which avoids such cycles.

Theorem 28 (Split-decomposition tree characterization of ptolemaic graphs[9])**.** *A graph G with the reduced split-decomposition tree $(T, \mathcal{F})$ is ptolemaic if and only if*

(a) T is a clique-star tree;

(b) there are no center-center paths in T.

Proof. Ptolemaic graphs are exactly the intersection of distance-hereditary graphs and chordal graphs. The first condition in this theorem addresses the fact that distance-hereditary graphs are exactly the class of totally decomposable with respect to the split-decomposition, and the second condition reflects the fact that, by Lemma 15, center-center paths correspond to induced C_4 subgraphs, the defining forbidden subgraphs for chordal graphs. □

4.2 Rooted grammar

Equipped with the characterization of a bijective split-decomposition tree representation of ptolemaic graphs, we are now ready to enumerate ptolemaic graphs. In this subsection, we begin by providing a grammar for rooted split-decomposition trees of ptolemaic graphs, which can be used to enumerate labeled ptolemaic graphs. Next, we derive the unlabeled enumeration.

Theorem 29. *The class $\mathcal{PG}_\bullet$ of ptolemaic graphs rooted at a vertex is specified by*

$$\mathcal{PG}_\bullet = \mathcal{Z}_\bullet \times (\mathcal{S}_C + \mathcal{S}_X + \mathcal{K}) \tag{15}$$

$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{16}$$

$$\mathcal{S}_X = (\mathcal{Z} + \overline{\mathcal{K}}) \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{17}$$

$$\mathcal{K} = \mathcal{S}_C \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{S}_X) + \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) \tag{18}$$

$$\overline{\mathcal{K}} = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) \tag{19}$$

[9] This characterization was given, but not proven, by Paul [29, p. 4] in an enlightening encyclopedia article related to split-decomposition.

Proof. The interesting part of this grammar is that, to impose the restriction on center-center paths (condition (b) of Theorem 28), we must distinguish between two classes of clique-nodes, depending on the path through which we have reached them in the rooted tree:

- $\overline{\mathcal{K}}$: these are clique-nodes for which the most recent star-node on their ancestorial path has been exited through its center; we call these clique-nodes *prohibitive* to indicate that they cannot be connected to the center of a star-node;
- $\mathcal{K}$: all other clique-nodes, which we by contrast call *regular*.

Recall that the split-decomposition tree of ptolemaic graphs must, overall, satisfy the following constraints:

(a) center-center paths are forbidden (Theorem 28);
(b) internal nodes must have degree at least 3 (Thm. 7);
(c) the center of a star-node cannot be incident to the extremity of another star-node (Theorem 7);
(d) two clique-nodes cannot be adjacent (Theorem 7).

We can now prove the correctness of the grammar. We begin with the following equation

$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X)$$

which specifies that a subtree rooted at a star-node, linked to its parent by its center, has at least 2 unordered children as the extremities of the star-node: each extremity can either lead to a leaf, a regular clique-node, or another star-node entered through an extremity. The children subtrees cannot be star-nodes entered through their center, since the tree is reduced. The lower bound of two children is due to the first condition of reduced split-decomposition trees (Theorem 7), which specifies that every internal node has degree at least 3.

We now consider the next equation

$$\mathcal{S}_X = (\mathcal{Z} + \overline{\mathcal{K}}) \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X)$$

The ordered pair in this equation indicates that a subtree rooted at a star-node, linked to its parent by an extremity, can be exited in two ways, either through the center, which yields the term $(\mathcal{Z} + \overline{\mathcal{K}})$, or through another extremity, which yields the non-empty Set.

If the star-node is exited through its center, it can either enter a leaf or a prohibitive clique-node. It cannot enter a $\mathcal{S}_X$ by the third condition of reduced split-decomposition trees (Theorem 7), and it cannot enter a $\mathcal{S}_C$ as that would be a center-center path. Furthermore, it has to enter a prohibitive clique-node $\overline{\mathcal{K}}$ rather than a regular clique-node $\mathcal{K}$ to keep track of the fact that a star-node has been exited from its center on the current path and ensure that no another star-node will not be entered through its center.

If the star-node entered from an extremity is exited through an extremity, it has a set of at least one other extremity to choose from. Each of those extremities can lead to a either a leaf, a clique-node, or another star-node entered through its center. It cannot lead to an $\mathcal{S}_C$, as that would be a center-center path.

We next discuss the equation

$$\mathcal{K} = \mathcal{S}_C \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{S}_X) + \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X)$$

The disjoint union specifies that a subtree rooted at a regular clique-node can have exactly zero or one $\mathcal{S}_C$ as a child. First, a regular-clique-rooted subtree is allowed to have a $\mathcal{S}_C$ as a child, since regular clique-nodes are by definition not on potential center-center paths. However, a regular-clique-rooted subtree cannot have more than one $\mathcal{S}_C$ child, since otherwise there would be a center-center path between the $\mathcal{S}_C$ children through the clique-node.

The first summand corresponds to the case where the regular clique-node has exactly one $\mathcal{S}_C$ as a child, which can be used to exit the tree. Additionally, the clique-node can be exited via any of the other children besides $\mathcal{S}_C$ and reach either a leaf or a star-node entered through an extremity. Note that the clique-node cannot be exited into another clique-node of any kind by the second condition of reduced split-decomposition trees (Theorem 7), which indicates that no two clique-nodes are adjacent in a reduced split-decomposition tree.

The second summand corresponds to the case where the regular clique-node has no $\mathcal{S}_C$ children. In this case, the regular clique-node can be exited via any of the remaining two or more subtrees that have not been used to enter it. After exiting the clique-node, one arrives at either a leaf or a star-node entered through its extremity. As explained above, is not possible to arrive at a clique-node, since there are no adjacent clique-nodes in reduced split-decomposition trees.

We now take a look at the equation specifying subtrees rooted at prohibitive clique-nodes

$$\overline{\mathcal{K}} = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X)$$

A subtree rooted at a prohibitive clique-node can be exited via any of its set of at least two children and either enter a leaf or enter a star-node through its extremity. Since a prohibitive clique-nodes lies on a path from the center of a star-node, it cannot enter a $\mathcal{S}_C$. Additionally, it cannot enter another clique-node of any kind since reduced clique-nodes cannot be adjacent.

Finally, the following equation

$$\mathcal{PG}_\bullet = \mathcal{Z}_\bullet \times (\mathcal{S}_X + \mathcal{S}_C + \mathcal{K})$$

combines all pieces into a symbolic specification for rooted ptolemaic graphs. It states that a rooted ptolemaic graph consists of a distinguished leaf $\mathcal{Z}_\bullet$, which is attached to an internal node. The internal node could be star-node entered through either its center or an extremity, or it could be a regular clique-node. □

Given this grammar for ptolemaic graphs, we can produce the exact enumeration for rooted labeled ptolemaic graphs using a computer algebraic system. Furthermore, we can derive the enumeration of *unrooted* labeled ptolemaic graphs by normalizing the counting sequence by the number of possible ways to distinguish a vertex as the root. This normalization is easy for labeled graphs, since the labels prevent the formation of symmetries. Therefore, since each vertex is equally likely to be chosen as the root, the number of *unrooted* labeled graphs of size n is simply the number of *rooted* labeled graphs divided by n.

4.3 Unrooted grammar

Theorem 30. *The class $\mathcal{PG}$ of unrooted ptolemaic graphs is specified by*

$$\mathcal{PG} = \mathcal{T}_K + \mathcal{T}_S + \mathcal{T}_{S-S} - \mathcal{T}_{S\to S} - \mathcal{T}_{S-K} \tag{20}$$
$$\mathcal{T}_K = \mathcal{S}_C \times \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) + \textsc{Set}_{\geqslant 3}(\mathcal{Z} + \mathcal{S}_X) \tag{21}$$
$$\mathcal{T}_S = \mathcal{S}_C \times (\mathcal{Z} + \overline{\mathcal{K}}) \tag{22}$$
$$\mathcal{T}_{S-S} = \textsc{Set}_{=2}(\mathcal{S}_X) \tag{23}$$
$$\mathcal{T}_{S\to S} = \mathcal{S}_X \times \mathcal{S}_X \tag{24}$$
$$\mathcal{T}_{S-K} = \mathcal{K} \times \mathcal{S}_X + \overline{\mathcal{K}} \times \mathcal{S}_C \tag{25}$$
$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{26}$$
$$\mathcal{S}_X = (\mathcal{Z} + \overline{\mathcal{K}}) \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{27}$$
$$\mathcal{K} = \mathcal{S}_C \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{S}_X) + \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) \tag{28}$$
$$\overline{\mathcal{K}} = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) \tag{29}$$

Proof. Applying the dissymmetry theorem in a similar manner to the proof of the unrooted grammar of block graphs, we obtain the following formal equation:

$$\mathcal{PG} = \mathcal{T}_K + \mathcal{T}_S + \mathcal{T}_{S-S} + \mathcal{T}_{S-K} \\ - \mathcal{T}_{S\to K} - \mathcal{T}_{K\to S} - \mathcal{T}_{S\to S}.$$

Notably, even though we distinguish between prohibitive $\overline{\mathcal{K}}$ and regular $\mathcal{K}$ clique-nodes in the rooted grammar, this distinction disappears when rerooting the trees for the dissymmetry theorem. This is because the prohibitive or regular nature of a clique-node depends on an *implicitly directed* path leading to it from the root; however when rerooting the tree, the clique-node in question becomes the new root, and all (implicitly directed) paths originate from it[10].

We can then simplify the dissymmetry theorem equation above in the same manner as for block graphs:

$$\mathcal{PG} = \mathcal{T}_K + \mathcal{T}_S + \mathcal{T}_{S-S} - \mathcal{T}_{S\to S} - \mathcal{T}_{S-K}.$$

We then discuss the rerooted terms of the grammar, starting with the following equation,

$$\mathcal{T}_K = \mathcal{S}_C \times \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) + \textsc{Set}_{\geqslant 3}(\mathcal{Z} + \mathcal{S}_X).$$

The disjoint union translates the fact that in a ptolemaic tree rerooted at a clique-node, the root can have either zero or one $\mathcal{S}_C$ as a subtree. (A clique-node having more than one $\mathcal{S}_C$ subtree would induce a center-center path, which cannot exist in ptolemaic split-decomposition trees by Theorem 28.)

[10]This notion is implicitly used in the unrooted grammar for block graphs—and previously by Chauve *et al.* [8], for distance-hereditary and 3-leaf power graphs—in which we only reroot at a star-node $\mathcal{S}$ without distinguishing whether it was entered by its center or an extremity, precisely because it is the new root, and therefore all paths lead away from it.

The first summand corresponds to the case where the clique-node at which the split-decomposition tree is rooted has one $\mathcal{S}_C$ as a subtree. The clique-node root can have a set of at least two other subtrees, each of which can lead to either a leaf or a star-node entered from an extremity.

The second summand corresponds to the case where the clique-node at which the split-decomposition tree is rooted has no $\mathcal{S}_C$ subtrees, in which case it can have a set of at least three other subtrees leading to leaves or $\mathcal{S}_X$ nodes, but not clique-nodes or $\mathcal{S}_C$ as explained.

Next, we will consider the equation,

$$\mathcal{T}_S = \mathcal{S}_C \times (\mathcal{Z} + \overline{\mathcal{K}})$$

which specifies a ptolemaic split-decomposition tree rooted at a star-node. The specification of the subtrees of the distinguished star-node depends on whether they are connected to the center or an extremity of the star-node. The subtrees connected to the extremities of the distinguished star-node are exactly those specified by an $\mathcal{S}_C$, and the subtree attached to the center of the distinguished star-node can lead to either a leaf or a prohibitive clique-node.

The other three rooted tree equations follow from with the same logic. □

The first few terms of the enumeration of unlabeled, unrooted ptolemaic graphs (among others) are available in the Table 3 at the end of this paper.

5 2,3-Cactus and 3-Cactus

The definition of *cactus graphs* is similar to that of block graphs. Yet whereas in block graphs (discussed in Section 3), each block[11] is a clique, in a cactus graph each block is an edge or a simple cycle. Thus, just as block graphs can be called clique trees, cacti can be seen as "*cycle trees*"[12]. An alternate definition:

Definition 31. A *cactus* is a connected graph in which every edge belongs to at most one simple cycle [22].

We can also conjure further variations on this definition, with cactus graphs having as blocks, cycles that have size constrained to a set of positive integers; thus given a set of integers Ω, an Ω-cactus graph[13] is the class of cactus graphs of which the cycles have

[11] Recall that a block, or *biconnected component*, is a maximal subgraph in which every edge belongs to a simple cycle.

[12] Although cactus graphs have been known by many different names, including *Husimi Trees* (a term that grew contentious because the graphs are not in fact trees [22, §3.4]—although this seems not to have been an issue for k-trees and related classes!), they have not generally been known by the name "*cycle trees*", except in a non-graph theoretical publication, which rediscovered the concept [17].

[13] Note that it makes no sense for 1 to be in Ω given this definition. We can however have $2 \in \Omega$, in which case we treat an edge as a cycle of size 2. For example, if $2 \notin \Omega$, every vertex must be part of a cycle.

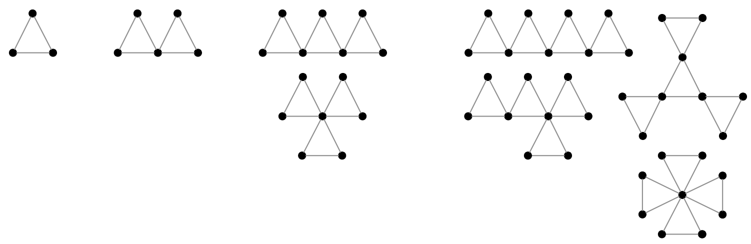


Figure 13. Small (unrooted, unlabeled) triangular cacti.

size $m \in \Omega$. In this section, we discuss cactus graphs for the sets: $\Omega = \{2, 3\}$, $\Omega = \{3\}$ and $\Omega = \{4\}$, following an article by Harary and Uhlenbeck [23], who use dissimilarity characteristics derived from Otter's theorem [28].

5.1 3-Cactus Graphs

In this section, we enumerate the family of cactus graph that is constrained to $\Omega = \{3\}$, which we refer to as the family as 3-cacti or *triangular* cacti.

Lemma 32 (Forbidden subgraph characterization of 3-cacti)**.** *A graph G is a triangular cactus if and only if G is a block graph with no bridges or induced $K_{\geqslant 4}$.*

Proof. [$\Rightarrow$] Given a triangular cactus G, we will show that G is a block graph and does not have any bridges or induced $K_{\geqslant 4}$.

We first note that G is a block graph by showing that it is ($C_{\geqslant 4}$, diamond)-free. There cannot be any induced $C_{\geqslant 4}$ in G, because every edge of a 3-cactus is in exactly one triangle and no other cycle. There cannot be any induced diamonds in a G because diamonds have an edge in common between two cycles[14].

We next observe that G cannot have bridges, as a bridge is by definition not part of any cycles, including triangles. Furthermore, G cannot have any induced cliques on 4 or more vertices, as such a clique would involve edges shared between triangles. Therefore, G must be a block graph and with no pendant edges or induced $K_{\geqslant 4}$.

[$\Leftarrow$] Given a block graph G without any bridges or induced $K_{\geqslant 4}$, we need to show that G is a 3-cactus. We do so by showing that every edge $(a, b) \in E(G)$ is in exactly one triangle and no other cycle.

First, since (a, b) cannot be a bridge, it must lie on some cycle C. Since G is a block graph and thus $C_{\geqslant 4}$-free, C must be a triangle.Furthermore, if (a, b) belonged to another cycle C', by the same argument, C' would also be a triangle. Let c be the third vertex of C other than a and b, and let c' the third vertex of C'. Depending on the adjacency of c and c', we have one of the following two cases:

[14] Here is another way to see why 3-cacti are a subset of block graphs. Block graphs can be thought of a set of cliques sharing at most one vertex pairwise, and cactus graphs can be thought of a set of cycles sharing at most one vertex pairwise. Since triangles are both cycles and cliques, a pairwise edge-disjoint collection of them is both a cactus graph and block graph.

- $(c,c') \in E(G)$, in which case $\{a,c,b,c'\}$ induces a K_4, which we assumed G does not include;
- $(c,c') \notin E(G)$, in which case $\{a,c,b,c'\}$ induces a diamond, which G, as a block graph, cannot contain.

Therefore, no such cycle C' can exist, implying that (a,b) belongs to one and exactly one triangle in G and no other cycle. Extending this argument to all edges of G ensures that G is a 3-cactus. □

Theorem 33 (Split-decomposition tree characterization of 3-cacti)**.** *A graph G with the reduced split-decomposition tree $(T, \mathcal{F})$ is a triangular cactus graph if and only if*

(a) T is a clique-star tree;
(b) the centers of all star-nodes are attached to leaves;
(c) the extremities of star-nodes are only attached to clique-nodes;
(d) every clique-node has degree 3.

Proof. By Lemma 32, we know that 3-cacti can be described exactly as the class of block graphs with no bridges or induced $K_{\geqslant}4$.

The first and second conditions of this theorem duplicate the split-decomposition tree characterization of block graphs outlined in Theorem 24.

The third condition uses Lemma 21 to forbid bridges. Since by the second condition, all star centers in T are adjacent to leaves, a star extremity adjacent to a leaf of T would correspond to a bridge in the form of a pendant edge in G, and a star extremity adjacent to another star extremity would correspond to a non-pendant bridge in G.

Finally, the last condition applies Lemma 19 to disallow $K_{\geqslant}4$, the last set of forbidden induced subgraphs for 3-cactus. □

The split-decomposition tree characterization of 3-cacti derived in the previous section naturally defines the following symbolic grammar for rooted block graphs.

Theorem 34. *The class $\mathcal{TCG}_\bullet$ of triangular cactus graphs rooted at a vertex is specified by*

$$\mathcal{TCG}_\bullet = \mathcal{Z}_\bullet \times (\mathcal{S}_C + \mathcal{K}) \tag{30}$$
$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{K}) \tag{31}$$
$$\mathcal{S}_X = \mathcal{Z} \times \textsc{Set}_{\geqslant 1}(\mathcal{K}) \tag{32}$$
$$\mathcal{K} = \textsc{Set}_{=2}(\mathcal{Z} + \mathcal{S}_X) \tag{33}$$

Theorem 35. *The class $\mathcal{TCG}$ of unrooted triangular cactus graphs is specified by*

$$\mathcal{TCG} = \mathcal{T}_K + \mathcal{T}_S - \mathcal{T}_{S-K} \tag{34}$$
$$\mathcal{T}_K = \textsc{Set}_{=3}(\mathcal{Z} + \mathcal{S}_X) \tag{35}$$
$$\mathcal{T}_S = \mathcal{Z} \times \mathcal{S}_C \tag{36}$$
$$\mathcal{T}_{S-K} = \mathcal{K} \times \mathcal{S}_X \tag{37}$$
$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{K}) \tag{38}$$
$$\mathcal{S}_X = \mathcal{Z} \times \textsc{Set}_{\geqslant 1}(\mathcal{K}) \tag{39}$$
$$\mathcal{K} = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{S}_X) \tag{40}$$

5.2 2,3-Cactus Graphs

In this section, we now enumerate the family of cactus graphs with $\Omega = \{2, 3\}$. The class of 2,3-cactus graphs is equivalent to the intersection of block graphs and of cactus graphs[15], not to be confused with the class of block-cactus graphs (which are the *union* of block graphs and cactus graphs [32]).

Theorem 36 (Split-decomposition tree characterization of 2,3-cactus graphs)**.** *A graph G with the reduced split-decomposition tree $(T, \mathcal{F})$ is a 2,3-cactus graph if and only if*

(a) T is a clique-star tree;

(b) every clique-node has degree 3;

(c) the centers of all star-nodes are attached to leaves;

Proof. This split-decomposition tree characterization is identical to the characterization for 3-cactus graphs, except the last condition in the characterization of 3-cacti (stating that leaves cannot be attached to extremities of star-nodes) is missing here. As we outlined in the proof of the characterization of 3-cacti, a leaf attached to an extremity of a star-node corresponds to a vertex of degree 1 in the original graph. Unlike with 3-cacti, which required that all vertices be in some cycle of size 3, having such a vertex of degree 1 here corresponds to a C_2 and is allowed. Therefore, the correctness of this characterization follows from the proof of the characterization for 3-cactus graphs. □

Remark 37. In the characterizations of 3–cacti (Theorem 33) and 2,3–cacti (Theorem 36), we do not restrict alternated paths between different clique–nodes, which was a required condition for forbidding induced cliques of size 4 or larger in Lemma 19. This is because forcing centers of star–nodes to be attached to leaves prohibits the existence of alternated paths between different clique–nodes. Indeed, since clique–nodes cannot be directly adjacent in a reduced split–decomposition tree, if two clique–nodes have an alternated path between them, there must be a star–node on that path. Since star–node centers are assumed to be connected to leaves, the alternated clique–clique path must at some point connect to two extremities of the same star–node, and thus cannot be alternated.

Theorem 38. *The class $\mathcal{TTCG}_\bullet$ of 2,3-cactus graphs rooted at a vertex is specified by*

$$\mathcal{TTCG}_\bullet = \mathcal{Z}_\bullet \times (\mathcal{S}_C + \mathcal{S}_X + \mathcal{K}) \tag{41}$$

$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{42}$$

$$\mathcal{S}_X = \mathcal{Z} \times \textsc{Set}_{\geqslant 1}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{43}$$

$$\mathcal{K} = \textsc{Set}_{=2}(\mathcal{Z} + \mathcal{S}_X) \tag{44}$$

[15]Block graphs can be thought of as a set of cliques sharing at most one vertex pairwise, and cactus graphs can be thought of as a set of cycles sharing at most one vertex pairwise. The intersection of cycles and cliques are those of sizes 1, 2, and 3; however, in the case of one vertex, adding a single vertex in this manner to a connected block or cactus graph does not change the size of the graph, contradicting the requirement that in a combinatorial class, there must be a finite number of objects of any fixed size. Therefore, the intersection of block graphs and cactus graphs is the family of 2,3-cactus graphs.

Theorem 39. *The class $\mathcal{TTCG}$ of unrooted 2,3-cactus graphs is specified by*

$$\mathcal{TTCG} = \mathcal{T}_K + \mathcal{T}_S + \mathcal{T}_{S-S} - \mathcal{T}_{S\to S} - \mathcal{T}_{S-K} \tag{45}$$
$$\mathcal{T}_K = \text{SET}_{=3}(\mathcal{Z} + \mathcal{S}_X) \tag{46}$$
$$\mathcal{T}_S = \mathcal{Z} \times \mathcal{S}_C \tag{47}$$
$$\mathcal{T}_{S-S} = \text{SET}_{=2}(\mathcal{S}_X) \tag{48}$$
$$\mathcal{T}_{S\to S} = \mathcal{S}_X \times \mathcal{S}_X \tag{49}$$
$$\mathcal{T}_{S-K} = \mathcal{K} \times \mathcal{S}_X \tag{50}$$
$$\mathcal{S}_C = \text{SET}_{\geqslant 2}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{51}$$
$$\mathcal{S}_X = \mathcal{Z} \times \text{SET}_{\geqslant 1}(\mathcal{Z} + \mathcal{K} + \mathcal{S}_X) \tag{52}$$
$$\mathcal{K} = \text{SET}_{=2}(\mathcal{Z} + \mathcal{S}_X) \tag{53}$$

6 Conclusion

In this paper, we follow the ideas of Gioan and Paul [19] and Chauve *et al.* [8], and provide full analyses of several important subclasses of distance-hereditary graph. Some of these analyses have lead us to uncover previously unknown enumerations (ptolemaic graphs, ...), while for other classes for which enumerations were already known (block graphs, 2,3-cactus and 3-cactus graphs), we have provided symbolic grammars which are a more powerful starting point for future work: such as parameter analyses, exhaustive and random generation and the empirical analyses that the latter enables. For instance, Iriza [25, §7] provided a nice tentative preview of the type of results unlocked by these grammars, when he empirically observed the linear growth of clique-nodes and star-nodes in the split-decomposition tree of a random distance-hereditary graph.

Our main idea is encapsulated in Section 2: we think that the split-decomposition, coupled with analytic combinatorics, is a powerful way to analyze classes of graphs specified by their forbidden induced subgraph. This is remarkably noteworthy, because forbidden characterizations are relatively common, and yet they generally are very difficult to translate to specifications. What we show is that this can be (at least for subclasses of distance-hereditary graphs which are totally decomposable by the split-decomposition) fairly automatic, in keeping with the spirit of analytic combinatorics:

(i) identify forbidden induced subgraphs;

(ii) translate each forbidden subgraph into constraints on the (clique-star) split-decomposition tree;

(iii) describe rooted grammar, apply unrooting, *etc.*.

This allows us to systematically derive the grammar of a number of well-studied classes of graphs, and to compute full enumerations, asymptotic estimates, and so on. In Figure 14, for instance, we have used the results from this paper to provide some intuition as to the relative "density" of these graph classes. A fairly attainable goal would be to use the asymptotic estimates which can be derived automatically from the grammars, to compute the asymptotic probability that a random block graph is also a 2,3-cactus graph.

Naturally, this raises a number of interesting questions, but possibly the most natural one to ask is: can we expand this methodology beyond distance-hereditary graphs, to classes for which the split-decomposition tree contains *prime nodes* (which are neither clique-nodes nor star-nodes).

Beyond distance-hereditary graphs, another perfect (pun intended) candidate is the class of *parity graphs*: these are the graphs whose split-decomposition tree has prime nodes that are bipartite graphs. But while bipartite graphs have been enumerated by Hanlon [20], and more recently Gainer-Dewar and Gessel [18], it is unclear whether this is sufficient to derive a grammar for parity graphs. Indeed, the advantage of the degenerate nodes (clique-nodes and star-nodes) is that their symmetries are fairly uncomplicated (all the vertices of a clique are undistinguished; all the vertices of a star, save the center, are undistinguished), as is in fact their enumeration (for each given size, there is only one clique or one star). An empirical study by Shi [35] showed that lower and upper bounds can be derived by plugging in the enumeration as an artificial generating function—either assuming all vertices of a bipartite prime node to be distinguished or undistinguished.

Other classes present a similar challenge, in that the subset of allowable prime nodes is itself too challenging.

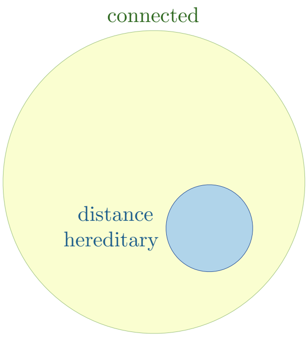
A likely more fruitful direction to pursue this work is to first start with classes of graphs which have small, predictable subsets of prime nodes. We discovered one such family of classes in a paper by Harary and Uhlenbeck [23]; in this paper, they discuss the enumeration of unlabeled and unrooted 3-cactus graphs and 4-cactus graphs (which we studied and enumerated using a radically different methodology in this paper), while suggesting that they would have liked to provide some general methodology to obtain the enumeration of m-cactus graphs, for generalized polygons on m sides[16].

There is some evidence to suggest that m-cactus graphs would yield split-decomposition trees with prime nodes that are undirected cycles of size m; likewise the split-decomposition tree of a general cactus graph (in which the blocks are cycles of any size larger than 3) would likely have prime nodes that are undirected cycles. In the same vein, block-cactus graphs (in which each block is a clique, a simple cycle, or an edge *i.e.*, the constraints on each block are the union of block graphs enumerated in Section 3, and of generalized cactus graphs) would likely also have the same type of prime nodes. All of these are more manageable subset, and it is likely that the various intersection classes with cactus graphs would be a more promising avenue by which to determine whether the split-decomposition can be reliably used for the enumeration of supersets of distance-hereditary graphs. Between the submission and publication of this article, this is a question we have begun to investigate [1].

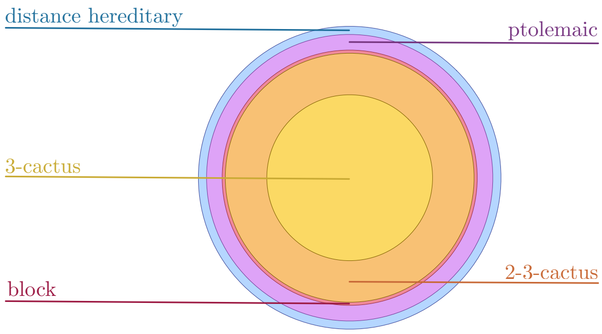
[16]It seems that Harary and Uhlenbeck have never published such a paper; and it appears that the closest there is in terms of a general enumeration of m-cactus graphs is by Bona *et al.* [4]—yet they enumerate graphs which are embedded in the plane, while we seek to enumerate the non-plane, unlabeled and unrooted m-cactus graphs.

Graph Class	Number
General [36, A000088]	2.86×10^{685}
General Connected [36, A001349]	2.86×10^{685}
Distance-Hereditary [8]	3.38×10^{56}
3-Leaf Power [8]	8.40×10^{37}
Ptolemaic	3.78×10^{50}
Block	1.44×10^{40}
2,3-Cacti	1.55×10^{38}
3-Cacti	9.13×10^{16}
4-Cacti	5.73×10^{14}

Table 1. Number of unlabeled graphs of size $n = 73$ for different graph classes.



(a) Ratio of the size of distance-hereditary graphs to general connected graphs on a logarithmic scale.



(b) Ratios of the size of various subsets of distance-hereditary graphs on a logarithmic scale.

Figure 14. Illustration of the ratios of sizes of various graph classes, for $n = 73$ as depicted in Table 1. The radii are the square roots of the ratios of the logarithms of the enumeration for a given class to the logarithm of the enumeration for the base class. In addition, only strict subsets are displayed on the right. For instance, 3-leaf power graphs are a subset of ptolemaic graphs, since their characterization [19, §3.3] does not allow for center-center paths; but they are not a subset of block graphs. Similarly, we cannot represent 4-cacti as they have no intersection with 3-cacti, and it is not physically possible to represent them here given our logarithmic scale.

Symbol	Explanation
$\mathcal{K}$	a clique-node entered from one of its vertices (and missing the corresponding subtree)
$\mathcal{S}_C$	a star-node entered through its *center* (and missing the corresponding subtree)
$\mathcal{S}_X$	a star-node entered through one of its (at least two) *extremities* (and missing the corresponding subtree)
$\mathcal{Z}$	a leaf of the split-decomposition tree (an atom with unit size)
$\mathcal{Z}_\bullet$	the *rooted* leaf of the split-decomposition tree (an atom with unit size)
$\mathcal{T}_K$	a split-decomposition tree rerooted at a clique-node (all subtrees are present)
$\mathcal{T}_S$	a split-decomposition tree rerooted at a star-node (all subtrees are present)
$\mathcal{T}_{K-S}$	a split-decomposition tree rerooted at an *edge* connecting a clique-node to a star-node (the edge can either connect the clique-node to the star-node's center or an extremity; the edge accounts for one subtree of the clique-node and one subtree of the star-node)
$\mathcal{T}_{S-S}$	a split-decomposition tree rerooted at an *edge* connecting two star-nodes; in the general case this can either be a center-center edge, or an extremity-extremity edge; some classes, such as ptolemaic graphs, may restrict this (and as before the edge accounts for a subtree of each of the nodes)
$\mathcal{T}_{S\to S}$	a split-decomposition tree rerooted at a *directed edge*; similar to $\mathcal{T}_{S-S}$, except there is a direction to the edge—and thus an order to the star-nodes
$\overline{\mathcal{K}}$	a *prohibitive* clique-node—used in the grammar for ptolemaic graphs—entered through an edge (and missing the corresponding subtree) that is on a path that is connected to the center of a star; this clique-node disallows outgoing connections to a star-node's center, to avoid the formation of a *center-center path*, as stated by Lemma 15
$\mathcal{Q}_C$	a *"quadrilateral" star-node*, as introduced in the grammars for 4-cactus graphs of Appendix A; this is one half of a group of two star-nodes, each with two extremities, and linked at their center as illustrated in Figure 5; here we are entering one such star-node from the center (or equivalently the center of star-node is the subtree that is missing), which means the parent node/missing subtree is the other part of the two star-node group
$\mathcal{Q}_X$	a "quadrilateral" star-node, entered from an extremity (or with a subtree rooted at an extremity missing), which means that we must now connect the center to a matching "quadralateral" star-node, and the remaining extremity to something else

Table 2. Main symbols used to define the split-decomposition tree of the classes of graphs analyzed in this paper. Refer to §1.5 for details on the terminology; and §1.6 for details on the dissymmetry theorem, from which all the rerooted trees, denoted $\mathcal{T}_\omega$, come from. (We omit rerooted trees from the treatment of 4-cactus graphs in Appendix A.)

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All of the figures in this article were created in OmniGraffle 6 Pro. Some were borrowed with permission from Iriza [25], and others were redone from original figures by Gioan and Paul [19] (notably Figures 1 and 3).

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A 4-Cactus Graphs

While investigating block graphs and the related class of 3-cactus graphs, we found an article by Harary and Uhlenbeck [23] in which they investigate both 3-cactus graphs and 4-cactus graphs. This prompted us to enumerate 4-cactus graphs.

This enumeration appears in appendix because it does not involve forbidden induced subgraphs, and so is somewhat of a non-sequitur as far as the point we would like to make in this paper.

The appeal of this enumeration is that it revisits a trick that is similar to that used to enumerate ptolemaic graphs. In Section 4, we introduced two symbols to express clique-nodes, $\mathcal{K}$ and $\overline{\mathcal{K}}$. These two symbols were used to keep track of whether, in the rooted decomposition of the split-decomposition tree, we were traveling down an alternated path starting at the center of a star-node or not. This "state" information was essential to prevent the formation of center-center paths that induce C_4.

In this grammar for 4-cacti, we use a similar idea. The quadrilaterals of these graphs translate to a very specific pattern in the split-decomposition tree: two star-nodes of size 3, connected at their center. We could well translate this in the grammar as a "meta" internal node that is just those star-nodes combined. Instead, we define the two symbols $\mathcal{Q}_C$ and $\mathcal{Q}_X$ to denote these special star-nodes; and because they always come in pairs connected at their centers, we know that if we encounter $\mathcal{Q}_C$ we are "inside" the pattern, and if we encounter $\mathcal{Q}_X$ we are entering this pattern from the outside.

Theorem 40. *The class $\mathcal{FCG}_\bullet$ of 4-cactus graphs rooted at a vertex is specified by*

$$\mathcal{FCG}_\bullet = \mathcal{Z}_\bullet \times (\mathcal{Q}_X + \mathcal{S}_C) \tag{54}$$

$$\mathcal{Q}_C = \textsc{Set}_{=2}(\mathcal{Z} + \mathcal{S}_X) \tag{55}$$

$$\mathcal{Q}_X = \mathcal{Q}_C \times (\mathcal{Z} + \mathcal{S}_X) \tag{56}$$

$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Q}_X) \tag{57}$$

$$\mathcal{S}_X = \mathcal{Z} \times \textsc{Set}_{\geqslant 1}(\mathcal{Q}_X) \tag{58}$$

Sketch of proof. We first note that 4-cacti's property of not having any induced clique-nodes of 3 or more vertices, by Lemma 19, translates to the split-decomposition tree of 4-cacti having no clique-nodes. Therefore, the only internal nodes to consider are star-nodes.

Furthermore, by Lemma 15, every C_4 in a 4-cactus corresponds to a center-center path in the split-decomposition tree. Since we already ruled out the existence of clique-nodes, the only possible center-center paths in the split-decomposition tree of 4-cacti are two star-nodes adjacent via their centers, corresponding to an induced C_4 in the accessibility graph. Along this line, we distinguish between two types of star-nodes, with $\mathcal{Q}$ (for quadrilateral) representing star-nodes with their center adjacent to another star-node (another $\mathcal{Q}$) and $\mathcal{S}$ representing all other star-nodes, which we refer to as *regular* star-nodes.

We next observe that the centers of all regular star-nodes must be attached to leaves. They cannot be attached to extremities of other star nodes as that would allow for a star-join operation, and they cannot be attached to centers of other star-nodes since otherwise, they would be considered quadrilateral star-nodes instead of regular ones.

Additionally, we note that extremities of regular star-nodes must be attached to quadrilateral star-nodes. First, these extremities cannot be attached to centers of other star-nodes to avoid star-join operations. Furthermore, as we already established that the centers of these regular star-node are attached to leaves, having their extremities adjacent to leaves or extremities of other star-nodes would induce bridges in the accessibility graph by Lemma 21 (a pendant bridge in the former case, and an internal bridge in latter). However, every edge in a 4-cactus graph belongs to a C_4 and thus cannot be a bridge.

Finally, we show that every quadrilateral star-node must have exactly two extremities. This is because there are no edges in an accessibility graph between the leaves at the ends of maximal alternated paths out of any star-node, as an alternated path between two such leaves would require using two interior edges from that star-node. Therefore, if two adjacent quadrilateral star-nodes respectively have x_1 and x_2 extremities, then the corresponding leaves in their subtrees would induce a K_{x_1,x_2} in the accessibility graph. We have $x_1, x_2 \geqslant 2$ since the split-decomposition tree is assumed to be reduced. In a 4-cactus graph, the only allowed complete bipartite induced subgraph, where each side of the bipartition has size at least 2, is a $K_{2,2}$, i.e. C_4, implying that every quadrilateral node must have exactly two extremities. ☐

Theorem 41. *The class* $\mathcal{FCG}$ *of unrooted 4-cactus graphs is specified by*

$$\mathcal{FCG} = \mathcal{T}_Q + \mathcal{T}_S + \mathcal{T}_{Q-Q} - \mathcal{T}_{Q\to Q} - \mathcal{T}_{Q-S} \tag{59}$$

$$\mathcal{T}_Q = \mathcal{Q}_C \times \mathcal{Q}_C \tag{60}$$

$$\mathcal{T}_S = \mathcal{Z} \times \mathcal{S}_C \tag{61}$$

$$\mathcal{T}_{Q-Q} = \textsc{Set}_{=2}(\mathcal{Q}_C) \tag{62}$$

$$\mathcal{T}_{Q\to Q} = \mathcal{Q}_C \times \mathcal{Q}_C \tag{63}$$

$$\mathcal{T}_{Q-S} = \mathcal{Q}_X \times \mathcal{S}_X \tag{64}$$

$$\mathcal{Q}_C = \textsc{Set}_{=2}(\mathcal{Z} + \mathcal{S}_X) \tag{65}$$

$$\mathcal{Q}_X = \mathcal{Q}_C \times (\mathcal{Z} + \mathcal{S}_X) \tag{66}$$

$$\mathcal{S}_C = \textsc{Set}_{\geqslant 2}(\mathcal{Q}_X) \tag{67}$$

$$\mathcal{S}_X = \mathcal{Z} \times \textsc{Set}_{\geqslant 1}(\mathcal{Q}_X) \tag{68}$$

The first few terms of the enumeration of unlabeled, unrooted 4-cactus graphs are available in the Table 4 at the end of this paper.

B Proof of Lemmas 11, 12, and 13

We here restate and provide the full proof of two straight-forward lemmas, which

Lemma 11. *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T, any maximal*[17] *alternated path starting from any node in $V(T)$ ends in a leaf.*

Proof. Let P be a maximal alternated path of length ℓ (edges) originating from a node $u \in V(T)$, and suppose P does not end in a leaf. Let v be the internal node that P ends in, and let $x \in V(G_v)$ be the marker vertex attached to the edge in P that enters v. Note that since P is an alternated path, it can include at most one edge from $E(G_v)$, so the part of P going from u to x has length at least $\ell - 1$ edges. Depending on the structure of G_v, we have the following cases:

- *G_v is a clique-node.* Since T is reduced, v has degree at least 3. Therefore, x has at least two neighbors in $V(G_v)$. Let y be one of these neighbors, and let ρ_y connect y to z (z is either a leaf or a marker vertex in a different internal node than v). We can construct a new path P' of length $\ell + 1$ from P by cutting off the part of P that follows x and adding (x, y) and (y, z) to the end of P, contradicting the maximality of P.

[17] A maximal alternated path is one that cannot be extended to include more edges while remaining alternated.

- *G_v is a star-node and x is its center.* Similarly to the previous case, G_v has degree at least 3 and thus at least two extremities. Let $y \in V(G_v)$ be one of these extremities and repeat the same argument in the previous case.
- *G_v is a star-node and x is one of its extremities.* Let $y \in V(G_v)$ be the center of the star-node v, and the same argument given for the previous two cases applies.

□

Lemma 12. *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T and let $u \in V(T)$ be an internal node. Any two maximal alternated paths P and Q that start at distinct marker vertices of u but contain no interior edges from G_u end at distinct leaves.*

Proof. Let P and Q be two such maximal alternated paths, starting at marker vertices $p, q \in V(G_u)$ respectively. We will show that P and Q are disjoint, and the result follows.

Note that P and Q are indeed disjoint in G_u, since they begin at different marker vertices and leave u immediately from there. Suppose P and Q are not disjoint, and let $v \in V(T)$ be the first such common node encountered when tracing P and Q out of u. Since P and Q are disjoint before reaching v, the part of P and Q between u and v form a simple cycle in T, which cannot happen in a tree. Therefore, no such common node v can exist, implying $P \cap Q = \{\}$.

□

Lemma 13. *Let G be a totally decomposable graph with the reduced clique-star split-decomposition tree T. If T has a clique-node of degree n, then G has a corresponding induced clique on (at least) n vertices.*

Proof. Let $u \in V(T)$ be a clique-node of degree n. For every marker vertex $v_i \in V(G_u)$, $i = 1 \dots n$, fix some maximal alternated path P_{v_i} that starts at the marker vertex v_i, uses no interior edges of G_u, and ends at leaf a_i. Note that the a_i are all distinct, by Lemma 12, and pairwise adjacent, since every pair of leaves a_i, a_j are connected in T via the alternated path consisting of P_{v_i}, P_{v_j}, and the edge $(v_i, v_j) \in E(G_u)$, thus inducing a clique of size (at least[18]) n in G. □

[18]Note that this clique of size n might be part of a larger clique, as illustrated in Figure 15.

Graph Class	Rooted	Labeled	EIS	Enumeration
Block graphs	✓	✓	**A035051**	1, 2, 12, 116, 1555, 26682, 558215, 13781448, 392209380, 12641850510, 455198725025, 18109373455164, 788854833679549, . . .
Block graphs		✓	**A030019**	1, 1, 4, 29, 311, 4447, 79745, 1722681, 43578820, 1264185051, 41381702275, 1509114454597, 60681141052273,. . .
Block graphs	✓		**A007563**	1, 1, 3, 8, 25, 77, 258, 871, 3049, 10834, 39207, 143609, 532193, 1990163, 7503471, 28486071, 108809503, 417862340,. . .
Block graphs			**A035053**	1, 1, 2, 4, 9, 22, 59, 165, 496, 1540, 4960, 16390, 55408, 190572, 665699, 2354932, 8424025, 30424768, 110823984,. . .
Ptolemaic graphs	✓	✓	**A287885**†	1, 2, 12, 140, 2405, 54252, 1512539, 50168456, 1928240622, 84240029730, 4121792058791, 223248397559376, . . .
Ptolemaic graphs		✓	**A287886**†	1, 1, 4, 35, 481, 9042, 216077, 6271057, 214248958, 8424002973, 374708368981, 18604033129948, 1019915376831963, . . .
Ptolemaic graphs	✓		**A287887**†	1, 1, 3, 10, 40, 168, 764, 3589, 17460, 86858, 440507, 2267491, 11819232, 62250491, 330794053, 1771283115, 9547905381, . . .
Ptolemaic graphs			**A287888**†	1, 1, 2, 5, 14, 47, 170, 676, 2834, 12471, 56675, 264906,1264851, 6150187, 30357300, 151798497, 767573729, 3919462385, . . .

Table 3. The first few terms of the enumerations of ptolemaic and block graphs. The dagger † indicates sequences which are new additions to the EIS.

Graph Class	Rooted	Labeled	EIS	Enumeration
3-cactus graphs	✓	✓	**A034940**	0, 0, 3, 0, 75, 0, 5145, 0, 688905, 0, 152193195, 0, 50174679555, 0, 23089081640625, 0, …
3-cactus graphs		✓	**A034941**	0, 0, 1, 0, 15, 0, 735, 0, 76545, 0, 13835745, 0, 3859590735, 0, 1539272109375, 0, 831766748637825, 0, …
3-cactus graphs	✓		**A003080**	0, 0, 1, 0, 2, 0, 5, 0, 13, 0, 37, 0, 111, 0, 345, 0, 1105, 0, 3624, 0, 12099, 0, 41000, 0, 140647, 0, 487440, 0, 1704115, 0, …
3-cactus graphs			**A003081**	0, 0, 1, 0, 1, 0, 2, 0, 4, 0, 8, 0, 19, 0, 48, 0, 126, 0, 355, 0, 1037, 0, 3124, 0, 9676, 0, 30604, 0, 98473, 0, 321572, 0, 1063146, 0, …
4-cactus graphs	✓	✓	**A287889**†	0, 0, 0, 12, 0, 0, 4410, 0, 0, 7560000, 0, 0, 35626991400, 0, 0, 357082280755200, 0, 0, 6536573599765809600, 0, 0, …
4-cactus graphs		✓	**A287890**†	0, 0, 0, 3, 0, 0, 630, 0, 0, 756000, 0, 0, 2740537800, 0, 0, 22317642547200, 0, 0, 344030189461358400, 0, 0, …
4-cactus graphs	✓		**A287891**†	0, 0, 0, 1, 0, 0, 3, 0, 0, 11, 0, 0, 46, 0, 0, 208, 0, 0, 1002, 0, 0, 5012, 0, 0, 25863, 0, 0, 136519, 0, 0, 733902, 0, 0, …
4-cactus graphs			**A287892**†	0, 0, 0, 1, 0, 0, 1, 0, 0, 3, 0, 0, 7, 0, 0, 25, 0, 0, 88, 0, 0, 366, 0, 0, 1583, 0, 0, 7336, 0, 0, 34982, 0, 0, 172384, 0, 0, …

Table 4. The first few terms of the enumerations of some subclasses of cactus graphs studied in this paper. The zero terms in the enumeration of 3-cacti and 4-cacti are due to the fact that the number of vertices in a 3-cactus graph must be odd, while the number of vertices in a 4-cactus graph must have a remainder of 1 modulus 3. Note that the EIS sequence for 3-cacti lists the enumeration for graphs of odd size only, thus omitting the zero terms; similarly the 4-cacti lists the enumeration of graphs on $3n+1$ vertices, thus also omitting the zero terms. The dagger † indicates sequences which are new additions to the EIS.

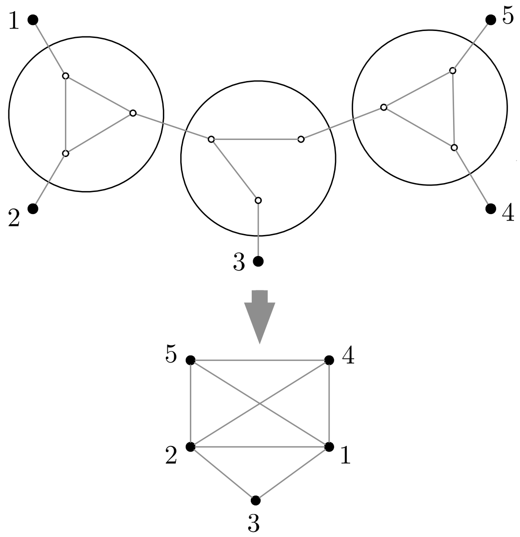

Figure 15. While the two clique-nodes of size 3 guarantee the presence of two corresponding induced cliques (one involving vertices 1 and 2, the other involving vertices 4 and 5), they do not allow us to rule out the existence of larger clique. This illustrates that, unlike many of our lemmas, the property presented in Lemma 13 is not bijective, and only works in one direction.

Graph Class	Rooted	Labeled	EIS	Enumeration
2,3-cactus graphs	✓	✓	**A091481**	1, 2, 12, 112, 1450, 23976, 482944, 11472896, 314061948, 9734500000, 336998573296,12888244482048, ...
2,3-cactus graphs		✓	**A091485**	1, 1, 4, 28, 290, 3996, 68992, 1434112, 34895772, 973450000, 30636233936, 1074020373504, 41510792057176, ...
2,3-cactus graphs	✓		**A091486**	1, 1, 3, 7, 21, 60, 190, 600, 1977, 6589, 22408, 77050, 268178, 941599, 3333585, 11882427, 42615480,153653039, ...
2,3-cactus graphs			**A091487**	1, 1, 2, 3, 7, 16, 41, 106, 304, 880, 2674, 8284, 26347, 85076, 279324, 928043, 3118915, 10580145, 36199094, 124774041, ...

Table 5. The first few terms of the enumerations of another subclass of cactus graphs studied in this paper.