Boxicity, poset dimension, and excluded minors

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Abstract

In this short note, we relate the boxicity of graphs (and the dimension of posets) with their generalized coloring parameters. In particular, together with known estimates, our results imply that any graph with no K_t-minor can be represented as the intersection of $O(t^2 \log t)$ interval graphs (improving the previous bound of $O(t^4)$), and as the intersection of $\frac{15}{2}t^2$ circular-arc graphs.

Mathematics Subject Classifications: 05C15, 05C83, 06A07

1 Introduction

The *intersection* $G_1 \cap \cdots \cap G_k$ of k graphs $G_1, \ldots, G_k$ defined on the same vertex set V, is the graph $(V, E_1 \cap \ldots \cap E_k)$, where E_i $(1 \leqslant i \leqslant k)$ denotes the edge set of G_i. The *boxicity* $\mathrm{box}(G)$ of a graph G, introduced by Roberts [19], is defined as the smallest k such that G is the intersection of k interval graphs.

Scheinerman proved that outerplanar graphs have boxicity at most two [20] and Thomassen proved that planar graphs have boxicity at most three [24]. Outerplanar graphs have no K_4-minor and planar graphs have no K_5-minor, so a natural question is how these two results extend to graphs with no K_t-minor for $t \geqslant 6$.

It was proved in [6] that if a graph has acyclic chromatic number at most k, then its boxicity is at most $k(k-1)$. Using the fact that K_t-minor-free graphs have acyclic chromatic number at most $O(t^2)$ [10], it implies that graphs with no K_t-minor have boxicity $O(t^4)$. On the other hand, it was noted in [5] that a result of Adiga, Bhowmick and Chandran [1] (deduced from a result of Erdős, Kierstead and Trotter [4]) implies the existence of graphs with no K_t-minor and with boxicity $\Omega(t\sqrt{\log t})$.

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In this note, we relate the boxicity of graphs with their generalized coloring numbers (see the next section for a precise definition). Using this connection together with earlier results, we prove the following result.

Theorem 1. *There is a constant $C > 0$ such that every K_t-minor-free graph has boxicity at most $Ct^2 \log t$.*

Our technique can be slightly refined (and the bound can be slightly improved) if instead of considering boxicity we consider a variant, in which we seek to represent graphs as the intersection of circular-arc graphs (instead of interval graphs as in the definition of boxicity).

Theorem 2. *If G has no K_t-minor, then G can be represented as the intersection of at most $\frac{15}{2}t^2$ circular-arc graphs.*

The *dimension* of a poset $\mathcal{P}$, denoted by $\dim(\mathcal{P})$, is the minimum number of linear orders whose intersection is exactly $\mathcal{P}$. Adiga, Bhowmick and Chandran [1] discovered a nice connection between the boxicity of graphs and the dimension of posets, which has the following consequence: for any poset $\mathcal{P}$ with comparability graph $G_\mathcal{P}$, $\dim(\mathcal{P}) \leqslant 2\text{box}(G_\mathcal{P})$. In particular, our main result implies the following.

Theorem 3. *There is a constant $C > 0$ such that if $\mathcal{P}$ is a poset whose comparability graph $G_\mathcal{P}$ has no K_t-minor, then $\dim(\mathcal{P}) \leqslant Ct^2 \log t$.*

It should be noted that while Theorem 1 directly implies Theorem 3 (using the result of [1] mentioned above), deducing Theorem 1 from Theorem 3 does not look as straightforward. Note also that a direct proof of Theorem 3 can be obtained along the same lines as that of Theorem 1 (see [25]).

Our result is based on a connection between the boxicity of graphs and their weak 2-coloring number (defined in the next section). Thus our result can also be seen as a connection between the dimension of posets and the weak 2-coloring number of their comparability graphs. Interestingly, similar connections between the dimension of posets and weak colorings of their *cover graphs* have recently been discovered. The cover graph of a poset $\mathcal{P}$ can be seen as a minimal spanning subgraph of the comparability graph $G_\mathcal{P}$ of $\mathcal{P}$, from which $G_\mathcal{P}$ can be recovered via transitivity. In particular, the cover graph of $\mathcal{P}$ can be much sparser than the comparability graph of $\mathcal{P}$ (for a chain, the first is a path while the second is a complete graph). However, for posets of height two, the comparability graph and the cover graph coincide.

It was proved by Joret, Micek, Ossona de Mendez and Wiechert [12] that if $\mathcal{P}$ is a poset of height at most h, and the cover graph of $\mathcal{P}$ has weak $(3h-3)$-coloring number at most k, then $\dim(P) \leqslant 4^k$. For posets $\mathcal{P}$ of height $h = 2$, this implies that the dimension is at most 4^k, where k is the weak 3-coloring number of the comparability graph of $\mathcal{P}$. This will be significantly improved in Section 2 (see Theorem 4).

The *adjacency poset* of a graph $G = (V, E)$, introduced by Felsner and Trotter [9], is the poset $(W, \leqslant)$ with $W = V \cup V'$, where V' is a disjoint copy of V, and such that $u \leqslant v$

if and only if $u = v$, or $u \in V$ and $v \in V'$ and u, v correspond to adjacent vertices of G. It was proved in [6] that for any graph G, the dimension of the adjacency poset of G is at most $2\,\mathrm{box}(G) + \chi(G) + 4$, where $\chi(G)$ is the chromatic number of G. Since graphs with no K_t minor have chromatic number $O(t\sqrt{\log t})$ [15, 23], this implies that the dimension of the adjacency poset of any graph with no K_t-minor is $O(t^2 \log t)$.

2 Weak coloring

Let G be a graph and let $\Pi(G)$ denote the set of linear orders on $V(G)$. Fix some linear order $\pi \in \Pi(G)$ for the moment. We write $x <_\pi y$ if x is smaller than y in π, and we write $x \leqslant_\pi y$ if $x = y$ or $x <_\pi y$. For a set S of vertices, $x \leqslant_\pi S$ means that $x \leqslant_\pi y$ for every vertex $y \in S$. When π is clear from the context, we omit the subscript π and write $<$ and $\leqslant$ instead of $<_\pi$ and $\leqslant_\pi$.

For an integer $r \geqslant 0$, we say that a vertex u is weakly r-reachable from v in G if there is a path P of length (number of edges) at most r between u and v, such that $u \leqslant_\pi P$. In particular, u is weakly 2-reachable from v if $u \leqslant_\pi v$, and either $u = v$, or u and v are adjacent, or u and v have a common neighbor w with $u <_\pi w$.

The *weak r-coloring number* of a graph G, denoted by $\mathrm{wcol}_r(G)$, is the minimum (over all linear orders $\pi \in \Pi(G)$) of the maximum (over all vertices v of G) of the number of vertices that are weakly r-reachable from v with respect to π. For more background on weak coloring numbers, the reader is referred to [18].

In this section we will consider the following slight variant of weak coloring: let $\mathrm{wcol}^*_r(G)$ be the minimum k such that for some linear order $\pi \in \Pi(G)$, there exists a coloring of the vertices of G such that for any vertex v of G, all the vertices distinct from v that are weakly r-reachable from v have a color that is distinct from that of v. Note that the greedy algorithm trivially shows that for any graph G and integer $r \geqslant 0$, $\mathrm{wcol}^*_r(G) \leqslant \mathrm{wcol}_r(G)$.

There is an interesting connection between $\mathrm{wcol}^*_2(G)$ and the *star-chromatic number* $\chi_s(G)$ of G, which is defined as the minimum number of colors in a proper coloring of the vertices of G, such that any 4-vertex path contains at least 3 distinct colors. It was observed in [17] that $\chi_s(G)$ can be equivalently defined as the minimum number of colors in a coloring of some orientation of G, such that any two vertices are required to have distinct colors if they are connected by an edge, a directed 2-edge path, or a 2-edge-path where the two edges are directed toward the ends of the paths. An anonymous referee observed that if we add the constraint that the orientation of G is acyclic, the corresponding graph parameter is precisely $\mathrm{wcol}^*_2(G)$.

It is known that there exist graphs G of unbounded boxicity with $\mathrm{wcol}_1(G) \leqslant 2$ [1]. We now prove that the boxicity is bounded by a linear function of the weak 2-coloring number.

Theorem 4. *For any graph G,* $\mathrm{box}(G) \leqslant 2\,\mathrm{wcol}^*_2(G)$.

Proof. Let G be a graph on n vertices and let $c := \mathrm{wcol}^*_2(G)$. By definition, there exist a linear order π on $V(G)$ and a vertex coloring ϕ with colors from the set $\{1, \ldots, c\}$, such that whenever a vertex u is weakly 2-reachable from another vertex v with respect to π, then $\phi(u) \neq \phi(v)$.

We aim to show that G is the intersection of $2c$ interval graphs $I_1, \ldots, I_{2c}$. We associate to each color $i \in [c]$ the two interval graphs I_i and I_{i+c}. Fix color i for the moment. We explicitly define the intervals representing the vertices of $V(G)$ in I_i and I_{i+c}, respectively. Consider the vertices $v_1, \ldots, v_\ell$ that received color i by ϕ. By relabelling the vertices if needed, we may assume that $v_1 < \cdots < v_\ell$ holds in π.

We start with I_i. Here, we map v_j ($1 \leqslant j \leqslant \ell$) to the point $\{j\}$; and for every vertex u that is not colored with i, we consider two cases: if u has no neighbor colored i we map u to the point $\{n\}$, and otherwise we consider the minimal k ($1 \leqslant k \leqslant \ell$) such that u is adjacent to v_k, and then we map u to the interval $[k, n]$. Notice that I_i is a supergraph of G.

We now proceed with I_{i+c}. Here, we reverse the order of the vertices with color i, that is, we map v_j ($1 \leqslant j \leqslant \ell$) to the point $\{\ell - j + 1\}$; and for every vertex u not colored with i, we again map u to the point $\{n\}$ if u has no neighbor colored i, and otherwise we consider the maximal k' ($1 \leqslant k' \leqslant \ell$) such that u is adjacent to $v_{k'}$, and then we map u to the interval $[\ell - k' + 1, n]$. Notice that I_{i+c} is also a supergraph of G. In Figure 1 the two interval graphs I_i and I_{i+c} are illustrated by their induced box representation in dimensions i and $i + c$.

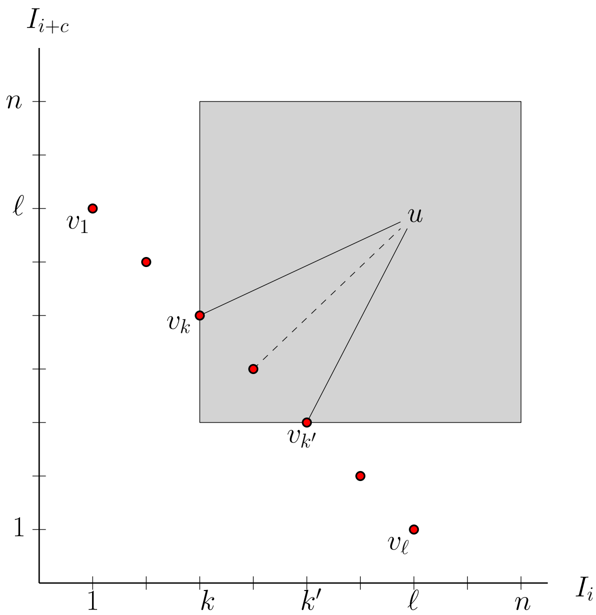

Figure 1: Illustration of I_i and I_{i+c} as the corresponding box representation. Vertices with color i are mapped to the red points. Projections onto the two axis yield the intervals representing the vertices.

Next, we show that G is the intersection of $I_1, \ldots, I_{2c}$. Since all involved interval graphs are supergraphs of G, we only need to show that for each pair of distinct non-adjacent vertices $u, v \in V(G)$ there is an interval graph I_j ($1 \leqslant j \leqslant 2c$) in which the two vertices are mapped to disjoint intervals. We may assume without loss of generality that $u < v$ in π. If u and v have the same color i, then their intervals are distinct points in I_i (and also in I_{i+c}) and thus disjoint. So suppose that u and v have distinct colors i and j, respectively. We assume for a contradiction that the intervals of u and v intersect in every interval graph $I_1, \ldots, I_{2c}$. This holds in particular in I_i and I_{i+c} (where u is mapped to a point and v to an interval containing point $\{n\}$); and from this we deduce that there are distinct vertices x and y with color i such that v is adjacent to both of them and $x < u < y$ in π. However, since we assumed that $u < v$ in π, this implies that $x < u < y < v$ or $x < u < v < y$. It follows that x is weakly 2-reachable from y with respect to π, as is witnessed by the path x, v, y. This is a contradiction to the properties of the coloring ϕ.

We conclude that G is indeed the intersection of $I_1, \ldots, I_{2c}$, and thus $\text{box}(G) \leqslant 2c$. $\square$

It is known that planar graphs have weak 2-coloring number (and thus wcol_2^*) at most 30 [10], so this implies that their boxicity is at most 60 (this is significantly worse than the result of Thomassen [24], who proved that planar graphs have boxicity at most 3). Given two integers s and t, let $K_{s,t}^*$ denote the complete join of K_s and $\overline{K_t}$. Using recent bounds on weak 2-coloring numbers by Van den Heuvel and Wood (Proposition 28 in [11]), Theorem 4 directly implies the following.

Theorem 5. *If G does not contain $K_{s,t}^*$ as a minor, then* $\text{box}(G) \leqslant 5s^3(t-1)$.

In particular, when s is a constant, the boxicity is linear in t. The result also directly implies that the boxicity of K_t-minor-free graphs is $O(t^3)$. This is also the order of magnitude of the best known bound on the weak 2-coloring number of K_t-minor-free graphs. To improve the bound on the boxicity of K_t-minor-free graphs, we will now use wcol_2^* as an alternative to wcol_2. We believe that considering wcol_r^* instead of wcol_r might yield to significant improvements in other problems as well.

It is proven in [17] that if every minor of a graph G has average degree at most d, then the star-chromatic number $\chi_s(G)$ is $O(d^2)$. A closer look at the proof contained in the paper reveals that this bound also holds for $\text{wcol}_2^*(G)$. We will indeed prove that a slightly stronger statement holds. Given a graph H, a *subdivision* of H is a graph obtained from H by subdividing some of the edges of H (i.e. replacing them by paths). The subdivision is said to be an $(\leqslant \ell)$-*subdivision* if each edge is subdivided at most ℓ times (i.e. replaced by a path on at most $\ell + 1$ edges). Given a half-integer $r \geqslant 0$, we denote by $\widetilde{\nabla}_r(G)$ the maximum average degree of a graph H such that G contains an $(\leqslant 2r)$-subdivision of H as a subgraph. Given an integer $r \geqslant 0$, let $\nabla_r(G)$ be the maximum average degree of a graph that can be obtained from G by contracting disjoint balls of radius at most r. For more on these notions and their connections with generalized coloring parameters, the reader is referred to the monograph [18].

We now prove the following (the first part of the result is a simple rewriting of the original argument of [17], while the second part was suggested to us by Sebastian Siebertz).

Theorem 6. *For any graph G,*

$$\text{wcol}_2^*(G) \leqslant 3\nabla_0(G)^2 + 1 + \min(\nabla_0(G)\nabla_1(G), \nabla_0(G)^2\widetilde{\nabla}_{1/2}(G)).$$

Proof. Any subgraph of G has average degree at most $k = \nabla_0(G)$ and in particular $\mathrm{wcol}_1(G) \leqslant k$ (i.e. G is k-degenerate). Let $\pi \in \Pi(G)$ be an order such that for any vertex u, at most k neighbors v of u are such that $v <_\pi u$ (in the remainder of the proof, we write $<$ instead of $<_\pi$ whenever there is no risk of confusion). Let H be obtained from G by adding an edge between u and w, for each $u < v < w$ such that uv and vw are edges of G, and by adding an edge between x and y for each $x < y < z$ such that xz and yz are edges of G. Observe that $\mathrm{wcol}_2^*(G) \leqslant \chi(H)$, where $\chi(H)$ denotes the chromatic number of H. Thus, it is sufficient to prove that H is c-colorable, with $c = 3k^2 + 1 + \min(k\nabla_1(G), k^2\widetilde{\nabla}_{1/2}(G))$. In order to do so, we will indeed prove that any subgraph of H has average degree at most $c-1$, which implies that H is $(c-1)$-degenerate and thus c-colorable. Consider a subset A of vertices of G. Each edge uv of $H[A]$, with $u < v$, corresponds to (at least) one of these cases:

- uv is an edge of G (there are at most $k|A|/2$ such edges, since all subgraphs of G have average degree at most k).
- there is a vertex x in G (not necessarily in A) with $u < x < v$, such that ux and xv are edges of G (there are at most $k^2|A|$ such edges, by definition of π).
- there is a vertex $w \in A$ with $u < v < w$ such that uw and vw are edges of G (there are at most $|A|k(k-1)/2$ such edges, since for each w in A there are at most $k(k-1)/2$ pairs of neighbors of w in G preceding it in π).
- there is a vertex $w \notin A$ with $u < v < w$ such that uw and vw are edges of G (in this case let us say that the edge uv of $H[A]$ is *special*).

It follows from the observations above that there are at most $3k^2|A|/2$ non-special edges in $H[A]$. We now bound the number of special edges uv of $H[A]$ in two different ways. For each vertex $x \notin A$, consider the (at most k) edges yx of G with $y \in A$ and $y < x$, and label them with distinct integers from the set $\{1, \dots, k\}$. For each $1 \leqslant i \leqslant k$, observe that the edges labelled i form disjoint unions of stars, centered in vertices of A. For $1 \leqslant i \leqslant k$, let G_i be the graph obtained from G by contracting each of these stars labelled i into a single vertex. Note that each special edge of $H[A]$ corresponds to an edge in at least one of the graphs $G_i[A]$. Since each $G_i[A]$ was obtained from G by contracting disjoint balls of radius at most 1, each $G_i[A]$ contains at most $\nabla_1(G)|A|/2$ edges, and thus $H[A]$ has at most $k\nabla_1(G)|A|/2$ special edges. It follows that H has average degree at most $3k^2 + k\nabla_1(G)$, as desired.

For each pair i, j with $1 \leqslant i < j \leqslant k$, consider the graph G_{ij} with vertex set A, such that any two vertices $u, v \in A$ are connected by an edge if in G, u and v are connected by a path on 2 edges, one labelled i and the other labelled j. Observe that each special edge of $H[A]$ is an edge of some G_{ij}, and G contains a 1-subdivision of each G_{ij}. It follows that $H[A]$ has at most $k^2\widetilde{\nabla}_{1/2}(G)|A|/2$ special edges, and thus average degree at most $3k^2 + k^2\widetilde{\nabla}_{1/2}(G)$, as desired. □

Since graphs with no K_t-minor have average degree $\mathcal{O}(t\sqrt{\log t})$ [15, 23], it follows that these graphs have $\mathrm{wcol}_2^*(G) = O(t^2 \log t)$. We thus obtain Theorem 1 as a direct consequence of Theorem 4.

A classic result [3, 14] states that graphs with no subdivision of K_t have average degree $O(t^2)$. So, for these graphs ∇_0 and $\widetilde{\nabla}_{1/2}$ are of order $O(t^2)$. An immediate consequence is the following.

Corollary 7. *There is a constant $C > 0$ such that if G has no subdivision of K_t, then $\mathrm{box}(G) \leqslant Ct^6$.*

3 Strong coloring and circular-arc graphs

The purpose of this section is to prove that if we consider a slightly larger class of graphs (circular-arc graphs instead of interval graphs), we can gain a multiplicative factor of $\log t$ in Theorem 1.

A *circular interval* is an interval of the unit circle, and a *circular-arc graph* is the intersection graph of a family of circular intervals. Equivalently, we can define a *circular interval* of $\mathbb{R}$ as being either an interval of $\mathbb{R}$, or the (closed) complement of an interval of $\mathbb{R}$. Note that this defines the same intersection graphs, and we will use whatever formulation is the most convenient, depending on the situation.

The *circular dimension* of a graph G, denoted by $\dim^\circ(G)$, is the minimum integer k such that G can be represented as the intersection of k circular-arc graphs. This parameter was introduced by Feinberg [7]. Since every interval graph is a circular-arc graph, $\dim^\circ(G) \leqslant \mathrm{box}(G)$ for any graph G.

Let G be a graph, let $\pi \in \Pi(G)$, and let $r \geqslant 0$ be an integer. Following [13], we say that a vertex u is *strongly r-reachable* from v if there is a path P of length at most r between u and v, such that $u \leqslant_\pi P$ and $v \leqslant_\pi P - u$. In particular, u is strongly 2-reachable from v if $u \leqslant_\pi v$, and either $u = v$, or u and v are adjacent, or u and v have a common neighbor w with $u <_\pi v <_\pi w$.

The *strong r-coloring number* of a graph G, introduced in [13] and denoted by $\mathrm{col}_r(G)$, is the minimum (over all linear orders $\pi \in \Pi(G)$) of the maximum (over all vertices v of G) of the number of vertices that are strongly r-reachable from v.

Theorem 8. *For any graph G, $\dim^\circ(G) \leqslant 3\mathrm{col}_2(G)$.*

Proof. The proof proceeds similarly as the proof of Theorem 4. Let n be the number of vertices in G. We consider a total order $\pi \in \Pi(G)$ on the vertices of G such that for any v, at most $c = \mathrm{col}_2(G)$ vertices are strongly 2-reachable from v. Again, any notion of order between the vertices of G in this proof will implicitly refer to π. As before, we start by greedily coloring G, with at most c colors, such that for any v and any vertex $u \neq v$ that is strongly 2-reachable from v, the colors of u and v are distinct. For each color class $1 \leqslant i \leqslant c$, we consider the two interval graphs I_i and I_{i+c} of the proof of Theorem 4, and a circular-arc graph I_{i+2c} defined as follows. Let $v_1 < \ldots < v_\ell$ be the vertices colored i in G. Again, each vertex v_j ($1 \leqslant j \leqslant \ell$) is mapped to the point $\{j\}$. Each vertex v not colored i is mapped (1) to the point $\{n\}$ if v has no neighbor colored i, (2) to the interval $[j, n]$ if v_j is the unique neighbor of v colored i, and otherwise (3) to the complement of the open interval (j, k), where v_j and v_k are the smallest and second smallest neighbors of v colored i (with respect to π). An example of construction of I_{i+2c} is illustrated in Figure 2.

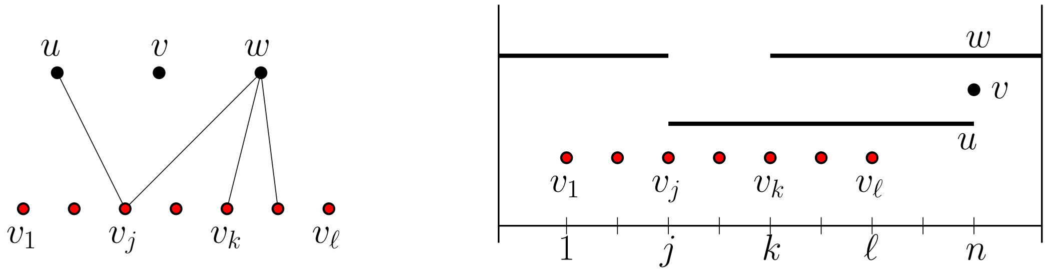

Figure 2: Left: A graph, with the vertices colored i depicted in red; Right: The corresponding circular-arc graph I_{i+2c}.

We now prove that G is precisely the intersection of the graphs I_i for $1 \leqslant i \leqslant 3c$, which will show that G is the intersection of at most $3c = 3\mathrm{col}_2(G)$ circular-arc graphs. We already proved in the previous section that for each $1 \leqslant i \leqslant 2c$, the graphs I_i are supergraphs of G. We prove that it is also the case for the graphs I_{i+2c} with $1 \leqslant i \leqslant c$. Observe that in the graph I_{i+2c}, any vertex v not colored i is adjacent to all the vertices not colored i, and to all its neighbors in G that are colored i. Since there is no egde between two vertices colored i in G, it follows that every edge uv in G is also an edge of I_{i+2c}.

Hence, in order to prove that G is precisely the intersection of the graphs I_i for $1 \leqslant i \leqslant 3c$, it is sufficient to prove that each non-edge uv of G is also a non-edge in a graph I_i for some $1 \leqslant i \leqslant 3c$. Consider two non-adjacent vertices $u < v$ in G. We can assume that u and v have distinct colors i and j, respectively (otherwise uv is a non-edge in the three graphs I_i, I_{i+c}, and I_{i+2c} corresponding to their common color class). If v has no neighbor colored i in G, then v has no neighbor colored i in each of the three graphs I_i, I_{i+c}, and I_{i+2c}, and thus u and v are non-adjacent in each of these graphs. If v has a unique neighbor colored i in G, call it w (it is different from u, since u and v are non-adjacent), then it follows from the construction of I_i and I_{i+c} that w is the unique neighbor of v colored i in $I_i \cap I_{i+c}$, and thus u and v are non-adjacent in $I_i \cap I_{i+c}$. So we can assume that v has at least two neighbors colored i. As in the proof of Theorem 4 (using the definition of I_i and I_{i+c}) we can assume that v has two neighbors x and y colored i, such that $x < u < y$. Take x and y minimal (with respect to π) with this property.

By the definition of the strong 2-coloring number, we can assume that at most one neighbor of v colored i precedes v in π (since otherwise the smaller neighbor would be strongly 2-reachable from the larger neighbor, via v, which would contradict the fact that the two neighbors have the same color). Hence, it follows that x and y are respectively the smallest and second smallest neighbors of v colored i. But since $x < u < y$ and u is colored i, it follows from the definition of I_{i+3c} that u and v are non-adjacent in I_{i+3c}, as desired. $\square$

The following result was recently proved by Van den Heuvel, Ossona de Mendez, Quiroz, Rabinovich and Siebertz [10].

Theorem 9. *If G has no K_t-minor, then* $\mathrm{col}_2(G) \leqslant \frac{5}{2}(t-1)(t-2)$.

Together with Theorem 8, this immediately implies Theorem 2.

The $\log t$ factor between Theorems 1 and 2 raises some interesting questions about the parameter $\dim^\circ$. It is known that every n-vertex graph has boxicity at most $n/2$, and equality holds only for the complete graph K_n (n even) minus a perfect matching. However this graph is a circular-arc graph (see Figure 3) and thus has circular dimension equal to 1.

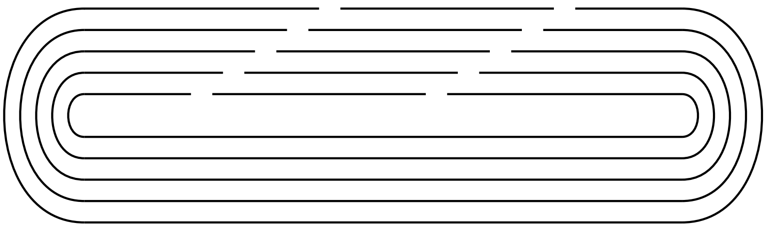

Figure 3: A circular-arc graph representation of the complete graph K_{10} minus a perfect matching.

Question 10. What is the maximum circular dimension of a graph on n vertices?

It was observed in [22] that there are $2^{\Theta(bn \log n)}$ n-vertex graphs of circular dimension at most b, and thus almost all n-vertex graphs have circular dimension $\Omega(n/\log n)$.

It is known that every graph of maximum degree Δ has boxicity $O(\Delta \log^{1+o(1)} \Delta)$ [21], while there are graphs of maximum degree Δ with boxicity $\Omega(\Delta \log \Delta)$ [1].

Question 11. What is the maximum circular dimension of a graph of maximum degree Δ?

Since there are $2^{\Theta(\Delta n \log n)}$ n-vertex graphs of maximum degree Δ, it follows that almost all graphs of maximum degree Δ have circular dimension $\Omega(\Delta)$. On the other hand, it was proved by Aravind and Subramanian [2] that every graph of maximum degree Δ has circular dimension $O(\Delta \log \Delta / \log \log \Delta)$.

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