On certain combinatorial expansions of the Legendre-Stirling numbers

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Abstract

The Legendre-Stirling numbers of the second kind were introduced by Everitt et al. in the spectral theory of powers of the Legendre differential expressions. As a continuation of the work of Andrews and Littlejohn (Proc. Amer. Math. Soc., 137 (2009), 2581–2590), we provide a combinatorial code for Legendre-Stirling set partitions. As an application, we obtain expansions of the Legendre-Stirling numbers of both kinds in terms of binomial coefficients.

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1 Introduction

The study on Legendre-Stirling numbers and Jacobi-Stirling numbers has become an active area of research in the past decade. In particular, these numbers are closely related to set partitions [3], symmetric functions [12], special functions [11] and so on.

Let $\ell[y](t) = -(1-t^2)y''(t) + 2ty'(t)$ be the Legendre differential operator. Then the Legendre polynomial $y(t) = P_n(t)$ is an eigenvector for the differential operator ℓ

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corresponding to $n(n+1)$, i.e., $\ell[y](t) = n(n+1)y(t)$. Following Everitt et al. [6], for $n \in \mathbb{N}$, the *Legendre-Stirling numbers* $\mathrm{LS}\,(n,k)$ *of the second kind* appeared originally as the coefficients in the expansion of the n-th composite power of ℓ, i.e.,

$$\ell^n[y](t) = \sum_{k=0}^{n} (-1)^k \mathrm{LS}\,(n,k)((1-t^2)^k y^{(k)}(t))^{(k)}.$$

For each $k \in \mathbb{N}$, Everitt et al. [6, Theorem 4.1)] obtained that

$$\prod_{r=0}^{k} \frac{1}{1-r(r+1)x} = \sum_{n=0}^{\infty} \mathrm{LS}\,(n,k) x^{n-k}, \quad \left(|x| < \frac{1}{k(k+1)} \right). \tag{1}$$

According to [2, Theorem 5.4], the numbers $\mathrm{LS}\,(n,k)$ have the following horizontal generating function

$$x^n = \sum_{k=0}^{n} \mathrm{LS}\,(n,k) \prod_{i=0}^{k-1} (x - i(1+i)). \tag{2}$$

It follows from (2) that the numbers $\mathrm{LS}\,(n,k)$ satisfy the recurrence relation

$$\mathrm{LS}\,(n,k) = \mathrm{LS}\,(n-1,k-1) + k(k+1)\mathrm{LS}\,(n-1,k).$$

with the initial conditions $\mathrm{LS}\,(n,0) = \delta_{n,0}$ and $\mathrm{LS}\,(0,k) = \delta_{0,k}$, where $\delta_{i,j}$ is the Kronecker's symbol. By using (1), Andrews et al. [2, Theorem 5.2] derived that the numbers $\mathrm{LS}\,(n,k)$ satisfy the vertical recurrence relation

$$\mathrm{LS}\,(n,j) = \sum_{k=j}^{n} \mathrm{LS}\,(k-1,j-1)(j(j+1))^{n-k}.$$

Following [7, Theorem 4.1], the *Jacobi-Stirling number* $\mathrm{JS}\,_n^k(z)$ *of the second kind* may be defined by

$$x^n = \sum_{k=0}^{n} \mathrm{JS}\,_n^k(z) \prod_{i=0}^{k-1} (x - i(z+i)). \tag{3}$$

It follows from (3) that the numbers $\mathrm{JS}\,_n^k(z)$ satisfy the recurrence relation

$$\mathrm{JS}\,_n^k(z) = \mathrm{JS}\,_{n-1}^{k-1}(z) + k(k+z)\mathrm{JS}\,_{n-1}^k(z),$$

with the initial conditions $\mathrm{JS}\,_n^0(z) = \delta_{n,0}$ and $\mathrm{JS}\,_0^k(z) = \delta_{0,k}$ (see [11] for instance). It is clear that $\mathrm{JS}\,_n^k(1) = \mathrm{LS}\,(n,k)$. In [9], Gessel, Lin and Zeng studied generating function of the coefficients of $\mathrm{JS}\,_{n+k}^n(z)$. Note that $\mathrm{JS}\,_{n+k}^n(1) = \mathrm{LS}\,(n+k,n)$. This paper is devoted to the following problem.

Problem 1. Let k be a given nonnegative integer. Could the numbers $\mathrm{LS}\,(n+k,n)$ be expanded in the binomial basis?

A particular value of $\mathrm{LS}(n,k)$ is provided at the end of [3]:

$$\mathrm{LS}(n+1,n) = 2\binom{n+2}{3}. \tag{4}$$

In [5, Eq. (19)], Egge obtained that

$$\mathrm{LS}(n+2,n) = 40\binom{n+2}{6} + 72\binom{n+2}{5} + 36\binom{n+2}{4} + 4\binom{n+2}{3}.$$

Using the triangular recurrence relation $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$, we get

$$\mathrm{LS}(n+2,n) = 40\binom{n+3}{6} + 32\binom{n+3}{5} + 4\binom{n+3}{4}. \tag{5}$$

Egge [5, Theorem 3.1] showed that for $k \geqslant 0$, the quantity $\mathrm{LS}(n+k,n)$ is a polynomial of degree $3k$ in n with leading coefficient $\frac{1}{3^k k!}$.

As a continuation of [3] and [5], in Section 2, we give a solution of Problem 1. Moreover, we get an expansion of the Legendre-Stirling numbers of the first kind in terms of binomial coefficients.

2 Main results

The combinatorial interpretation of the Legendre-Stirling numbers $\mathrm{LS}(n,k)$ of the second kind was first given by Andrews and Littlejohn [3]. For $n \geqslant 1$, let M_n denote the multiset $\{1, \overline{1}, 2, \overline{2}, \ldots, n, \overline{n}\}$, in which we have one unbarred copy and one barred copy of each integer i, where $1 \leqslant i \leqslant n$. Throughout this paper, we always assume that the elements of M_n are ordered by

$$\overline{1} = 1 < \overline{2} = 2 < \cdots < \overline{n} = n.$$

A *Legendre-Stirling set partition* of M_n is a set partition of M_n with $k+1$ blocks $B_0, B_1, \ldots, B_k$ and with the following rules:

(r_1) The 'zero box' B_0 is the only box that may be empty and it may not contain both copies of any number;

(r_2) The 'nonzero boxes' $B_1, B_2, \ldots, B_k$ are indistinguishable and each is non-empty. For any $i \in [k]$, the box B_i contains both copies of its smallest element and does not contain both copies of any other number.

Let $\mathcal{LS}(n,k)$ denote the set of Legendre-Stirling set partitions of M_n with one zero box and k nonzero boxes. The *standard form* of an element of $\mathcal{LS}(n,k)$ is written as

$$\sigma = B_1 B_2 \cdots B_k B_0,$$

where B_0 is the zero box and the minima of B_i is less than that of B_j when $1 \leqslant i < j \leqslant k$. Clearly, the minima of B_1 are 1 and $\overline{1}$. Throughout this paper we always write $\sigma \in \mathcal{LS}(n,k)$ in the standard form. As usual, we let angle bracket symbol $< i, j, \ldots >$ and curly bracket symbol $\{k, \overline{k}, \ldots\}$ denote the zero box and nonzero box, respectively. In particular, let $<>$ denote the empty zero box. For example, $\{1, \overline{1}, 3\}\{2, \overline{2}\} < \overline{3} > \in \mathcal{LS}(3,2)$. A classical result of Andrews and Littlejohn [3, Theorem 2] says that

$$\mathrm{LS}\,(n,k) = \#\mathcal{LS}(n,k).$$

We now provide a combinatorial code for Legendre-Stirling partitions (CLS -sequence for short).

Definition 2. We call $Y_n = (y_1, y_2, \ldots, y_n)$ a CLS -*sequence* of length n if $y_1 = X$ and

$$y_{k+1} \in \{X, A_{i,j}, B_s, \overline{B}_s, 1 \leqslant i, j, s \leqslant n_x(Y_k), i \neq j\} \quad \text{for } k = 1, 2, \ldots, n-1,$$

where $n_x(Y_k)$ is the number of the symbol X in $Y_k = (y_1, y_2, \ldots, y_k)$.

For example, $(X, X, A_{1,2})$ is a CLS -sequence, while $(X, X, A_{1,2}, B_3)$ is not since $y_4 = B_3$ and $3 > n_x(Y_3) = 2$. Let $\mathcal{CLS}_n$ denote the set of CLS -sequences of length n.

The following lemma is a fundamental result.

Lemma 3. *For $n \geqslant 1$, we have* $\mathrm{LS}\,(n,k) = \#\{Y_n \in \mathcal{CLS}_n \mid n_x(Y_n) = k\}$.

Proof. Let $\mathcal{CLS}(n,k) = \{Y_n \in \mathcal{CLS}_n \mid n_x(Y_n) = k\}$. Now we start to construct a bijection, denoted by Φ, between $\mathcal{LS}(n,k)$ and $\mathcal{CLS}(n,k)$. When $n = 1$, we have $y_1 = X$. Set $\Phi(Y_1) = \{1, \overline{1}\} <>$. This gives a bijection from $\mathcal{CLS}(1,1)$ to $\mathcal{LS}(1,1)$. Let $n = m$. Suppose Φ is a bijection from $\mathcal{CLS}(n,k)$ to $\mathcal{CLS}(n,k)$ for all k. Consider the case $n = m+1$. Let

$$Y_{m+1} = (y_1, y_2, \ldots, y_m, y_{m+1}) \in \mathcal{CLS}_{m+1}.$$

Then $Y_m = (y_1, y_2, \ldots, y_m) \in \mathcal{CLS}(m,k)$ for some k. Assume $\Phi(Y_m) \in \mathcal{LS}(m,k)$. Consider the following three cases:

- (i) If $y_{m+1} = X$, then let $\Phi(Y_{m+1})$ be obtained from $\Phi(Y_m)$ by putting the box $\{m+1, \overline{m+1}\}$ just before the zero box. In this case, $\Phi(Y_{m+1}) \in \mathcal{LS}(m+1, k+1)$.
- (ii) If $y_{m+1} = A_{i,j}$, then let $\Phi(Y_{m+1})$ be obtained from $\Phi(Y_m)$ by inserting the entry $m+1$ to the ith nonzero box and inserting the entry $\overline{m+1}$ to the jth nonzero box. In this case, $\Phi(Y_{m+1}) \in \mathcal{LS}(m+1, k)$.
- (iii) If $y_{m+1} = B_s$ (resp. $y_{m+1} = \overline{B}_s$), then let $\Phi(Y_{m+1})$ be obtained from $\Phi(Y_m)$ by inserting the entry $m+1$ (resp. $\overline{m+1}$) to the sth nonzero box and inserting the entry $\overline{m+1}$ (resp. $m+1$) to the zero box. In this case, $\Phi(Y_{m+1}) \in \mathcal{LS}(m+1, k)$.

After the above step, it is clear that the obtained $\Phi(Y_{m+1})$ is in standard form. By induction, we see that Φ is the desired bijection from $\mathcal{CLS}(n,k)$ to $\mathcal{CLS}(n,k)$, which also gives a constructive proof of Lemma 3. □

Example 4. Let $Y_5 = (X, X, A_{2,1}, B_2, \overline{B}_1)$. The correspondence between Y_5 and $\Phi(Y_5)$ is built up as follows:

$$\begin{aligned} X &\Leftrightarrow \{1, \overline{1}\} <>; \\ X &\Leftrightarrow \{1, \overline{1}\}\{2, \overline{2}\} <>; \\ A_{2,1} &\Leftrightarrow \{1, \overline{1}, \overline{3}\}\{2, \overline{2}, 3\} <>; \\ B_2 &\Leftrightarrow \{1, \overline{1}, \overline{3}\}\{2, \overline{2}, 3, 4\} < \overline{4} >; \\ \overline{B}_1 &\Leftrightarrow \{1, \overline{1}, \overline{3}, \overline{5}\}\{2, \overline{2}, 3, 4\} < \overline{4}, 5 > . \end{aligned}$$

As an application of Lemma 3, we present the following lemma.

Lemma 5. *Let k be a given positive integer. Then for $n \geqslant 1$, we have*

$$\mathrm{LS}\,(n+k, n) = 2^k \sum_{t_k=1}^{n} \binom{t_k+1}{n} \sum_{t_{k-1}=1}^{t_k} \binom{t_{k-1}+1}{2} \cdots \sum_{t_2=1}^{t_3} \binom{t_2+1}{2} \sum_{t_1=1}^{t_2} \binom{t_1+1}{2}. \tag{6}$$

Proof. It follows from Lemma 3 that

$$\mathrm{LS}\,(n+k, n) = \#\{Y_{n+k} \in \mathcal{CLS}_{n+k} \mid n_x(Y_{n+k}) = n\}.$$

Let $Y_{n+k} = (y_1, y_2, \ldots, y_{n+k})$ be a given element in $\mathcal{CLS}_{n+k}$. Since $n_x(Y_{n+k}) = n$, it is natural to assume that $y_i = X$ except $i = t_1+1, t_2+2, \cdots, t_k+k$. Let σ be the corresponding Legendre-Stirling partition of Y_{n+k}. For $1 \leqslant \ell \leqslant k$, consider the value of $y_{t_\ell+\ell}$. Note that the number of the symbols X before $y_{t_\ell+\ell}$ is t_ℓ. Let $\widehat{\sigma}$ be the corresponding Legendre-Stirling set partition of $(y_1, y_2, \ldots, y_{t_\ell+\ell-1})$. Now we insert $y_{t_\ell+\ell}$. We distinguish two cases:

- (i) If $y_{t_\ell+\ell} = A_{i,j}$, then we should insert the entry $t_\ell+\ell$ to the ith nonzero box of $\widehat{\sigma}$ and insert $\overline{t_\ell+\ell}$ to the jth nonzero box. This gives $2\binom{t_\ell}{2}$ possibilities, since $1 \leqslant i, j \leqslant t_\ell$ and $i \neq j$.
- (ii) If $y_{t_\ell+\ell} = B_s$ (resp. $y_{t_\ell+\ell} = \overline{B}_s$), then we should insert the entry $t_\ell+\ell$ (resp. $\overline{t_\ell+\ell}$) to the sth nonzero box of $\widehat{\sigma}$ and insert $\overline{t_\ell+\ell}$ (resp. $t_\ell+\ell$) to the zero box. This gives $2\binom{t_\ell}{1}$ possibilities, since $1 \leqslant s \leqslant t_\ell$.

Therefore, there are exactly $2\binom{t_\ell}{2} + 2\binom{t_\ell}{1} = 2\binom{t_\ell+1}{2}$ Legendre-Stirling set partitions of $\mathrm{M}_{\,t_\ell+\ell}$ can be generated from $\widehat{\sigma}$ by inserting the entry $y_{t_\ell+\ell}$. Note that $1 \leqslant t_{j-1} \leqslant t_j \leqslant n$ for $2 \leqslant j \leqslant k$. Applying the product rule for counting, we immediately get (6). ☐

The following simple result will be used in our discussion.

Lemma 6. *Let a and b be two given integers. Then*

$$\binom{x-b}{2}\binom{x}{a} = \binom{a+2}{2}\binom{x}{a+2} + (a+1)(a-b)\binom{x}{a+1} + \binom{a-b}{2}\binom{x}{a}.$$

In particular,

$$\binom{x-1}{2}\binom{x}{a} = \binom{a+2}{2}\binom{x}{a+2} + (a^2-1)\binom{x}{a+1} + \binom{a-1}{2}\binom{x}{a}.$$

Proof. Note that

$$\binom{a+2}{2}\frac{(x-a)(x-a-1)}{(a+2)(a+1)}+(a+1)(a-b)\frac{x-a}{a+1}+\binom{a-b}{2}=\binom{x-b}{2}.$$

This yields the desired result. □

We can now conclude the main result of this paper from the discussion above.

Theorem 7. *Let k be a given nonnegative integer. For $n \geqslant 1$, the numbers* $\mathrm{LS}\,(n+k,n)$ *can be expanded in the binomial basis as*

$$\mathrm{LS}\,(n+k,n)=2^k\sum_{i=k+2}^{3k}\gamma(k,i)\binom{n+k+1}{i}, \tag{7}$$

where the coefficients $\gamma(k,i)$ are all positive integers for $k+2 \leqslant i \leqslant 3k$ and satisfy the recurrence relation

$$\gamma(k+1,i)=\binom{i-k-1}{2}\gamma(k,i-1)+(i-1)(i-k-2)\gamma(k,i-2)+\binom{i-1}{2}\gamma(k,i-3), \tag{8}$$

with the initial conditions $\gamma(0,0)=1$, $\gamma(0,i)=\gamma(i,0)=0$ for $i\neq 0$. Let $\gamma_k(x)=\sum_{i=k+2}^{3k}\gamma(k,i)x^i$. Then the polynomials $\gamma_k(x)$ satisfy the recurrence relation

$$\gamma_{k+1}(x)=\left(\frac{k(k+1)}{2}-kx+x^2\right)x\gamma_k(x)-(k+(k-2)x-2x^2)x^2\gamma_k'(x)+\frac{(1+x)^2x^3}{2}\gamma_k''(x), \tag{9}$$

with the initial conditions $\gamma_0(x)=1, \gamma_1(x)=x^3$ and $\gamma_2(x)=x^4+8x^5+10x^6$.

Proof. We prove (7) by induction on k. It is clear that $\mathrm{LS}\,(n,n)=1=\binom{n+1}{0}$. When $k=1$, by using the *Chu Shih-Chieh's identity*

$$\binom{n+1}{k+1}=\sum_{i=k}^{n}\binom{i}{k},$$

we obtain

$$\sum_{t_1=1}^{n}\binom{t_1+1}{2}=\binom{n+2}{3},$$

and so (4) is established. When $k=2$, it follows from Lemma 5 that

$$\begin{aligned}\mathrm{LS}\,(n+2,n)&=4\sum_{t_2=1}^{n}\binom{t_2+1}{2}\sum_{t_1=1}^{t_2}\binom{t_1+1}{2}\\&=4\sum_{t_2=1}^{n}\binom{t_2+1}{2}\binom{t_2+2}{3}.\end{aligned}$$

Setting $x = t_2 + 2$ and $a = 3$ in Lemma 6, we get

$$\begin{aligned}\mathrm{LS}\,(n+2,n) &= 4\sum_{t_2=1}^{n}\left(10\binom{t_2+2}{5}+8\binom{t_2+2}{4}+\binom{t_2+2}{3}\right)\\ &= 4\left(10\binom{n+3}{6}+8\binom{n+3}{5}+\binom{n+3}{4}\right),\end{aligned}$$

which yields (5). Along the same lines, it is not hard to verify that

$$\begin{aligned}\mathrm{LS}\,(n+3,n) &= 8\sum_{t_3=1}^{n}\binom{t_3+1}{2}\left(10\binom{t_3+3}{6}+8\binom{t_3+3}{5}+\binom{t_3+3}{4}\right)\\ &= 8\left(280\binom{n+4}{9}+448\binom{n+4}{8}+219\binom{n+4}{7}+34\binom{n+4}{6}+\binom{n+4}{5}\right).\end{aligned}$$

Hence the formula (7) holds for $k = 0, 1, 2, 3$, so we proceed to the inductive step. For $k \geqslant 3$, assume that

$$\mathrm{LS}\,(n+k,n) = 2^k \sum_{i=k+2}^{3k} \gamma(k,i)\binom{n+k+1}{i}.$$

It follows from Lemma 5 that

$$\mathrm{LS}\,(n+k+1,n) = 2^{k+1}\sum_{t_{k+1}=1}^{n}\binom{t_{k+1}+1}{2}\sum_{i=k+2}^{3k}\gamma(k,i)\binom{t_{k+1}+k+1}{i}$$

By using Lemma 6 and the Chu Shih-Chieh's identity, it is routine to verify that the coefficients $\gamma(k, i)$ satisfy the recurrence relation (8), and so (7) is established for general k. Multiplying both sides of (8) by x^i and summing for all i, we immediately get (9). □

In [2], Andrews et al. introduced the *(unsigned) Legendre-Stirling numbers* $\mathrm{Lc}\,(n, k)$ *of the first kind*, which may be defined by the recurrence relation

$$\mathrm{Lc}\,(n,k) = \mathrm{Lc}\,(n-1,k-1) + n(n-1)\mathrm{Lc}\,(n-1,k),$$

with the initial conditions $\mathrm{Lc}\,(n, 0) = \delta_{n,0}$ and $\mathrm{Lc}\,(0, n) = \delta_{0,n}$. Let $f_k(n) = \mathrm{LS}\,(n + k, n)$. According to Egge [5, Eq. (23)], we have

$$\mathrm{Lc}\,(n-1,n-k-1) = (-1)^k f_k(-n) \tag{10}$$

for $k \geqslant 0$. For $m, k \in \mathbb{N}$, we define

$$\binom{-m}{k} = \frac{(-m)(-m-1)\cdots(-m-k+1)}{k!}.$$

Combining (7) and (10), we immediately get the following result.

Corollary 8. *Let k be a given nonnegative integer. For $n \geqslant 1$, the numbers* $\mathrm{Lc}\,(n-1, n-k-1)$ *can be expanded in the binomial basis as*

$$\mathrm{Lc}\,(n-1, n-k-1) = (-1)^k 2^k \sum_{i=k+2}^{3k} \gamma(k,i) \binom{-n+k+1}{i}, \tag{11}$$

where the coefficients $\gamma(k,i)$ are defined by (8).

It follows from (9) that

$$\begin{aligned}
\gamma(k+1, k+3) &= \left(\frac{k(k+1)}{2} - k(k+2) + \frac{(k+2)(k+1)}{2}\right)\gamma(k, k+2),\\
\gamma(k+1, 3k+3) &= \left(1 + 6k + \frac{3k(3k-1)}{2}\right)\gamma(k, 3k),\\
\gamma_{k+1}(-1) &= -\left(\frac{k(k+1)}{2} + k + 1\right)\gamma_k(-1).
\end{aligned}$$

Since $\gamma(1,3) = 1$ and $\gamma_1(-1) = -1$, it is easy to verify that for $k \geqslant 1$, we have

$$\gamma(k, k+2) = 1,\ \gamma(k, 3k) = \frac{(3k)!}{k!(3!)^k},\ \gamma_k(-1) = (-1)^k \frac{(k+1)!k!}{2^k}.$$

It should be noted that the number $\gamma(k, 3k)$ is the number of partitions of $\{1, 2, \ldots, 3k\}$ into blocks of size 3, and the number $\frac{(k+1)!k!}{2^k}$ is the product of first k positive triangular numbers. Moreover, if the number $\mathrm{LS}\,(n+k, n)$ is viewed as a polynomial in n, then its degree is $3k$, which is implied by the quantity $\binom{n+k+1}{3k}$. Furthermore, the leading coefficient of $\mathrm{LS}\,(n+k, n)$ is given by $2^k\gamma(k, 3k)\frac{1}{(3k)!} = 2^k \frac{(3k)!}{k!(3!)^k}\frac{1}{(3k)!} = \frac{1}{k!3^k}$, which yields [5, Theorem 3.1].

3 Concluding remarks

In this paper, by introducing the CLS-sequence, we present a combinatorial expansion of $\mathrm{LS}\,(n+k, n)$. It should be noted that the CLS-sequence has several other variants.

For an alphabet A, let $\mathbb{Q}[[A]]$ be the rational commutative ring of formal Laurent series in monomials formed from letters in A. Following Chen [4], a context-free grammar over A is a function $G : A \to \mathbb{Q}[[A]]$ that replace a letter in A by a formal function over A. The formal derivative $D = D_G$: $\mathbb{Q}[[A]] \to \mathbb{Q}[[A]]$ is defined as follows: for $x \in A$, we have $D(x) = G(x)$; for a monomial u in $\mathbb{Q}[[A]]$, $D(u)$ is defined so that D is a derivation, and for a general element $q \in \mathbb{Q}[[A]]$, $D(q)$ is defined by linearity. The reader is referred to [10] for recent results on context-free grammars.

As a variant of the CLS-sequence, we now introduce a marked scheme for Legendre-Stirling set partitions. Given a set partition $\sigma = B_1B_2\cdots B_kB_0 \in \mathcal{LS}(n,k)$, where B_0 is the zero box of σ. We mark the box vector $(B_1, B_2, \ldots, B_k)$ by the label a_k. We mark any box pair (B_i, B_j) by a label b and mark any box pair (B_s, B_0) by a label c, where $1 \leqslant i < j \leqslant k$ and $1 \leqslant s \leqslant k$. Let σ' denote the Legendre-Stirling set partition that generated from σ by inserting $n+1$ and $\overline{n+1}$. If $n+1$ and $\overline{n+1}$ are in the same box, then $\sigma' = B_1 B_2 \cdots B_k B_{k+1} B_0$, where $B_{k+1} = \{n+1, \overline{n+1}\}$. This case corresponds to the operator $a_k \to a_{k+1} b^k c$.

If $n+1$ and $\overline{n+1}$ are in different boxes, then we distinguish two cases:

(i) Given a box pair (B_i, B_j), where $1 \leqslant i < j \leqslant k$. We can put $n+1$ (resp. $\overline{n+1}$) into the box B_i and put $\overline{n+1}$ (resp. $n+1$) into the box B_j. This case corresponds to the operator $b \to 2b$.

(ii) Given a box pair (B_i, B_0), where $1 \leqslant i \leqslant k$. We can put $n+1$ (resp. $\overline{n+1}$) into the box B_i and put $\overline{n+1}$ (resp. $n+1$) into the zero box B_0. Moreover, we mark each barred entry in the zero box B_0 by a label z. This case corresponds to the operator $c \to (1+z)c$.

Let $A = \{a_0, a_1, a_2, a_3, \ldots, b, c\}$ be a set of alphabet. Following the above marked scheme, we consider the grammars

$$G_k = \{a_0 \to a_1 c, a_1 \to a_2 bc, \ldots, a_{k-1} \to a_k b^{k-1} c, b \to 2b, c \to (1+z)c\},$$

where $k \geqslant 1$. It is a routine check to verify that

$$D_n D_{n-1} \cdots D_1(a_0) = \sum_{k=1}^{n} \mathrm{JS}\,{}_n^k(z) a_k b^{\binom{k}{2}} c^k.$$

Therefore, it is clear that for $n \geqslant k$, the number $\mathrm{JS}\,{}_n^k(z)$ is a polynomial of degree $n-k$ in z, and the coefficient z^i of $\mathrm{JS}\,{}_n^k(z)$ is the number of Legendre-Stirling partitions in $\mathcal{LS}(n,k)$ with i barred entries in zero box, which gives a grammatical proof of [8, Theorem 2].

We end our paper by proposing the following.

Conjecture 9. For any $k \geqslant 1$, the polynomial $\gamma_k(x)$ has only real zeros. Set

$$\gamma_k(x) = \gamma(k, 3k) x^{k+2} \prod_{i=1}^{2k-2} (x - r_i),\ \gamma_{k+1}(x) = \gamma(k+1, 3k+3) x^{k+3} \prod_{i=1}^{2k} (x - s_i),$$

where $r_{2k-2} < r_{2k-3} < \cdots < r_2 < r_1$ and $s_{2k} < s_{2k-1} < s_{2k-2} < \cdots < s_2 < s_1$. Then

$$s_{2k} < r_{2k-2} < s_{2k-1} < r_{2k-3} < s_{2k-2} < \cdots < r_k < s_{k+1} < s_k < r_{k-1} < \cdots < s_2 < r_1 < s_1,$$

in which the zeros s_{k+1} and s_k of $\gamma_{k+1}(x)$ are continuous appearance, and the other zeros of $\gamma_{k+1}(x)$ separate the zeros of $\gamma_k(x)$.

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