Hypergraph coverings and their zeta functions[*]

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Abstract

Let H be a finite hypergraph. The concept of hypergraph coverings over H is introduced, and we can generate all hypergraph coverings by permutation voltage assignments of the edge-colored graph or the incidence graph of H. Consequently, we show two explicit decomposition formulae for the zeta function of any hypergraph covering $\overline{H}$ over H which indicates that the zeta function of H divides the zeta function of $\overline{H}$.

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1 Introduction

Let $G = (V_G, E_G)$ be a finite graph which multiple edges and loops are allowed. $\overleftrightarrow{G}$ is the symmetric digraph corresponding to G which is obtained from G by changing every undirected edge of G to a pair of oppositely directed edges (i.e., arcs). We denote the arc from the vertex u to the vertex v as ordered pair (u, v). For $e = (u, v) \in E_{\overleftrightarrow{G}}$, set $u = o(e)$ and $v = t(e)$, and $e^{-1} = (v, u)$ means the reverse arc to e.

A *covering projection* from a graph $\overline{G}$ to G is a surjection $\pi : V_{\overline{G}} \to V_G$ such that $\pi|_{N(\overline{v})} : N(\overline{v}) \to N(v)$ is a bijection for all vertices $v \in V_G$ and $\overline{v} \in \pi^{-1}(v)$, where $N(v)$, the neighborhood of the vertex v, is the set of vertices adjacent to v. The graph $\overline{G}$ is called a *graph covering* over G, and $\overline{G}$ is a *k-fold covering* if π is k-to-one. As an important class of graphs, covering graphs have been studied in many literatures, such as [12, 13, 15].

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Let S_k be the symmetric group on $[k] = \{1, 2, \ldots, k\}$. The function $\phi : E_{\overleftrightarrow{G}} \to S_k$ is called a *permutation voltage assignment* on G if $\phi(e)^{-1} = \phi(e^{-1})$ for each $e \in E_{\overleftrightarrow{G}}$. The permutation voltage assignment ϕ is *trivial* if $\phi(e)^{-1} = \phi(e^{-1}) = 1$ for each $e \in E_{\overleftrightarrow{G}}$, where 1 is the identity of S_k. Otherwise, ϕ is *nontrivial*. The pair (G, ϕ) is called a *permutation voltage graph*. A *derived graph* G^ϕ of (G, ϕ) is a graph with the vertex set $V_{G^\phi} = V_G \times [k] = \{(v, i) | v \in V_G, i \in [k]\}$, and (v, i) and (u, j) are adjacent if and only if $e = (v, u) \in E_{\overleftrightarrow{G}}$ and $i = \phi(e)j$. Gross and Tucker found the following lemma:

Lemma 1. *[2] There is a one-to-one correspondence between each k-fold covering $\overline{G}$ over G and some derived graph G^ϕ of (G, ϕ).*

The *zeta function* [3, 12] of a finite graph G is defined to be the function of a complex number u with $|u|$ sufficiently small, given by

$$\varsigma_G(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G. We refer the reader to [16] for an in-depth treatment of the zeta functions of graphs.

The degree of the vertex v_i in the graph G is denoted by $d_G(v_i)$. Let $A(G)$ and $D(G)$ denote the adjacency matrix and the degree-diagonal matrix of the graph G, respectively. Bass [1] proved that the zeta function of a graph G with minimum degree 2 satisfies the following determinant formula

$$\varsigma_G(u)^{-1} = (1 - u^2)^{|E_G| - |V_G|} \det(I_n - uA(G) + u^2 Q(G)), \tag{1}$$

where I_n is the identity matrix with order n and $Q(G) = D(G) - I_n$.

By Eq. (1), Terras [16] showed the zeta function of a finite graph divides the zeta function of any covering over this graph. Moreover, Mizuno and Sato in [7] gave an explicit composition formula for the zeta function of any covering over a finite graph by applying Eq. (1) and the representation theory of symmetric groups. More results on the zeta functions on graph coverings can see [5, 7, 8].

Subsequently, Storm [14] introduced the definition of the *zeta function* of a hypergraph being thought of the generalization of the zeta function of a graph. Let $H = (V_H, E_H)$ be a hypergraph on the vertex set V_H, and E_H is a set of non-empty subsets of V_H called *hyperedges*. A *hyperpath* P in H is a sequence $P = (x = v_1, e_1, v_2, e_2, v_3, \ldots, e_t, v_{t+1} = y)$ of $t + 1$ vertices and t hyperedges such that $v_j \in V_H, e_j \in E_H, v_1 \in e_1, v_{t+1} \in e_t$ and $v_i \in e_i, e_{i-1}$ for each $i = 2, \ldots, t$. The length of P is $|P| = t$. The hyperpath P has a *hyperedge backtrack* if there is a subsequence of P of the form (e, v, e). If $x = v_1 = v_{t+1} = y$ the hyperpath P is called a *closed hyperpath* or *hypercycle*. Let C^m denote the hypercycle obtained by going m times around the hypercycle C. A closed hyperpath C is *tail-less* if C^2 does not have a hyperedge backtrack, and a closed hyperpath P is *reduced* if it has no hyperedge backtrack or tail. A hypercycle C is said to be *prime* if $C \neq B^m$ for any hypercycle B and integer $m \geqslant 2$. Finally, two hypercycles $C_1 = (v_1, e_1, v_2, e_2, v_3, \ldots, e_t, v_1)$ and $C_2 = (w_1, f_1, w_2, f_2, w_3, \ldots, f_t, w_1)$ are called *equivalent* if $w_j = v_{j+k}$ and $f_j = e_{j+k}$ for

all j, i.e., C_1 is a cyclic permutation of C_2. The equivalence class containing C is denoted by $[C]$.

For $u \in \mathbb{C}$ with $|u|$ sufficiently small, the (*generalized Ihara-Selberg*) *zeta function* of a finite hypergraph H is defined by

$$\varsigma_H(u) = \prod_{[C]} (1 - u^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime and reduced hypercycles of H. The zeta function of a hypergraph in the above form is generally an infinite product.

There are several methods in which a hypergraph can be transformed into a graph representation. For example, a hypergraph can be transformed into the edge-colored graph, the oriented line graph and the incidence graph of it. The *incidence graph* B_H of H is a bipartite graph with the vertex set $V_{B_H} = V_H \cup E_H$, and $v \in V_H$ and $e \in E_H$ are adjacent if and only if $v \in e$. Besides, we will construct two edges between v and e in B_H when e is a loop at the vertex v in H. The hyperedge set of a hypergraph H is labelled as $E_H = \{e_1, \ldots, e_m\}$ and fix m colors $\{c_1, \ldots, c_m\}$. The *edge-colored graph* GH_c of H is constructed as follows: The vertex set is the vertex set of H, and for each hyperedge $e_i \in E_H$, a $|e_i|$-clique (i.e., the complete graph with $|e_i|$ vertices) on the vertices in e_i is constructed such that all edges of the $|e_i|$-clique are colored by c_i. In addition, a loop is added at the vertex v in GH_c if the hyperedge of H is a loop at v. Here, it should be noted that the definitions of the incidence graph and the edge-colored graph of a hypergraph are a little bit different from those in [14] where loops in a hypergraph are not taken into account. For example, the hypergraph H and its edge-colored graph GH_c is showed in Figure 1. Let $\overleftrightarrow{GH_c}$ be the symmetric digraph corresponding to GH_c and
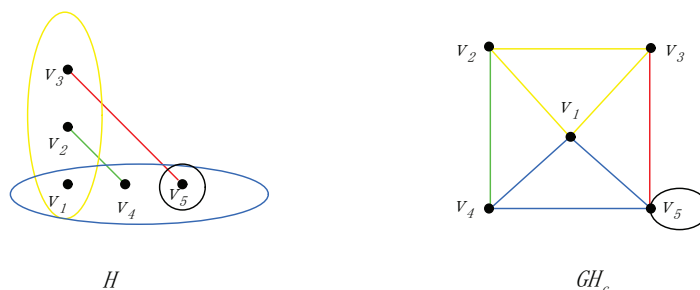

Figure 1: The hypergraph H, the edge-colored graph GH_c of H

color the arcs in $\overleftrightarrow{GH_c}$ keeping consistent to undirected edge of GH_c they are induced from. Let H_L be the *oriented line graph* of H defined by $V_{H_L} = E_{\overleftrightarrow{GH_c}}$ and $E_{H_L} = \{(e_i, e_j) \in E_{\overleftrightarrow{GH_c}} \times E_{\overleftrightarrow{GH_c}} : c(e_i) \neq c(e_j), t(e_i) = o(e_j)\}$, where $c(e_i)$ is the colored assigned to the arc $e_i \in E_{\overleftrightarrow{GH_c}}$. $A(H_L) = (a_{ij})$ is the adjacency matrix of H_L, i.e., $a_{ij} = 1$ if there is an arc from the vertex e_i to the vertex e_j, and $a_{ij} = 0$ otherwise. In terms of the oriented line graph and the incidence graph of a hypergraph, Storm [14] gave two explicit determinant expressions for the Ihara-Selberg zeta function of a hypergraph utilizing the results of Kotani and Sunada [3] and Bass [1].

Theorem 2. *[14] Let H be a finite connected hypergraph without loops, and each vertex of H is in at least two hyperedges. Then we have*

$$
\begin{aligned}
&\varsigma_H(u)^{-1} \\
&= \det(I - uA(H_L)) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (2) \\
&= \varsigma_{B_H}(\sqrt{u})^{-1} = (1-u)^{|E_{B_H}| - |V_{B_H}|} \det(I_{|V_{B_H}| + |E_{B_H}|} - \sqrt{u}A(B_H) + uQ(B_H)). \quad (3)
\end{aligned}
$$

Another determinant expression of the zeta function of a hypergraph is characterized in [9]. In addition, Sato [10] introduced the edge zeta function of a hypergraph and presented its determinant expression, and a decomposition formula for the edge zeta function of a group covering of a hypergraph was given.

In view of the above, in Section 2 we generalize the definition of graph coverings over a graph to hypergraphs. By permutation voltage assignments of the edge-colored graph or the incidence graph of a finite hypergraph, we can generate all hypergraph coverings. In Section 3, based on the representation theory of symmetric groups we find two explicit decomposition formulae for the zeta function of any hypergraph covering which indicates the zeta function of any hypergraph covering can be expressed by the zeta function of this hypergraph.

2 Hypergraph coverings over a finite hypergraph

Let $H = (V_H, E_H)$ be a finite hypergraph. We allow hyperedges to repeat and allow loops. E_v is the hyperedge set containing the vertex v. The *neighborhood* $N(v)$ of a vertex v in H is the set of vertices u such that u and v are adjacent, i.e., $N(v) = \{u \in V_H : u \in e \ and \ e \in E_v\}$.

A *homomorphism* from a finite hypergraph $\overline{H}$ to H is a map which sends vertices to vertices, hyperedges to hyperedges, and preserves the adjacency. A homomorphism map $\pi : \overline{H} \to H$ is a *hypergraph covering projection* if it is surjection such that $\pi|_{N(v)} : N(\widetilde{v}) \to N(v)$ is a bijection for all vertices $v \in V_H$ and $\widetilde{v} \in \pi^{-1}(v)$. The hypergraph $\overline{H}$ is called a *hypergraph covering* over H, or a *covering* for short. If π is k-to-one, we call $\overline{H}$ a *k-fold* hypergraph covering. It should be pointed out that in the special case where each of hyperedges contains only one vertex or two vertices, a hypergraph covering reduces to a graph covering. From the definition of k-fold hypergraph coverings, each vertex v and each hyperedge e of H have k vertices and k hyperedges in their respectively preimages $\pi^{-1}(v)$ and $\pi^{-1}(e)$. Also, the hypergraph covering over H is not unique. As an illustration, we give two 2-fold hypergraph coverings over the hypergraph H. See Figure 2.

Let $\overline{H}$ be a k-fold hypergraph covering over H. Let $G\overline{H}_c$ and $B_{\overline{H}}$ be the edge-colored graph and the incidence graph of $\overline{H}$, respectively. By applying the definition of hypergraph coverings, it is easy to find that $G\overline{H}_c$ and $B_{\overline{H}}$ must be the k-fold graph coverings over GH_c and B_H, respectively.

Now, we shall generate all hypergraph coverings using two distinct methods.

Firstly, we establish a one-to-one correspondence on any k-fold hypergraph covering over the hypergraph H with the help of the edge-colored graph of H. The hyperedge
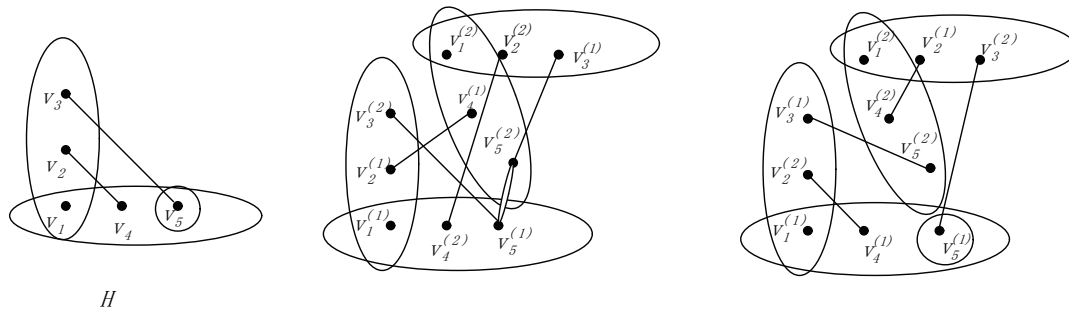
Figure 2: The hypergraph H and two 2-fold hypergraph coverings over H

set of H is labelled as $E_H = \{e_1, \ldots, e_m\}$ and fix m colors $\{c_1, \ldots, c_m\}$. An additional condition on k-fold coverings over H is assumed: For any hypergraph covering projection from the hypergraph $\overline{H}$ to H, any hyperedge of $\overline{H}$ has the same color of its image. With the same method, we color all edges of any k-fold covering over GH_c.

Lemma 3. *Let K_n be a complete graph with $n(n \geqslant 3)$ vertices. Then there exists a non-trivial permutation voltage assignment $\phi : E_{\overleftrightarrow{K_n}} \to S_k(k > 1)$ on K_n such that $\phi((v_i, v_j)) \phi((v_j, v_t)) \phi((v_t, v_i)) = 1$ for every three distinct vertices v_i, v_j, v_t of K_n.*

Proof. Let T_n be a spanning star of K_n with $n-1$ edges $v_1 v_2, v_1 v_3, \ldots, v_1 v_n$. Furthermore, we give a permutation voltage assignment $\phi((v_1, v_i))$ on the arc (v_1, v_i) arbitrarily for $i = 2, \ldots, n$. Then we decide $(n-1)(n-2)/2$ values $\phi((v_i, v_j))$ on the arc (v_i, v_j) ($2 \leqslant i < j \leqslant n$) as follows:

$$\phi((v_i, v_j)) = \phi((v_i, v_1)) \phi((v_1, v_j)).$$

Note that $\phi(e^{-1}) = \phi^{-1}(e)$ for any arc $e \in E_{\overleftrightarrow{K_n}}$. Thus, for every triangle with vertices v_i, v_j and v_t ($2 \leqslant i < j < t \leqslant n$) of K_n we have

$$
\begin{aligned}
& \phi((v_i, v_j)) \phi((v_j, v_t)) \phi((v_t, v_i)) \\
= \ & \phi((v_i, v_1)) \phi((v_1, v_j)) \phi((v_j, v_1)) \phi((v_1, v_t)) \phi((v_t, v_1)) \phi((v_1, v_i)) \\
= \ & 1.
\end{aligned}
$$

The proof is completed. $\qquad\square$

The assignment $\phi : E_{\overleftrightarrow{GH_c}} \to S_k$ of permutation voltage on GH_c such that $\phi((v_i, v_j)) \phi((v_j, v_t)) \phi((v_t, v_i)) = 1$ for any three distinct arcs with the same color, denoted by $\widetilde{\phi}$, is called the *strong permutation voltage assignment* on GH_c. Obviously, $\widetilde{\phi}$ is feasible from Lemma 3. Notice that $\widetilde{\phi}$ is a general permutation voltage assignment ϕ if edges in GH_c with the same color are only an edge or a loop. Let $C(\widetilde{\phi}; k)$ denote the set of all strong permutation voltage assignments $\widetilde{\phi} : E_{\overleftrightarrow{GH_c}} \to S_k$. From the construction of $\widetilde{\phi}$ and the definition of the covering projection, it is easy to check that the preimage of any clique (in GH_c) with identical color is k cliques. Moreover, we assign the same color to these k
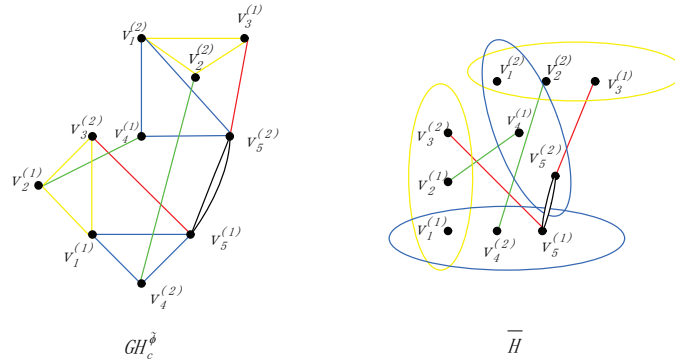
$GH_c^{\widetilde{\phi}}$
$\overline{H}$

Figure 3: The derived graph $GH_c^{\widetilde{\phi}}$, and the corresponding 2-fold hypergraph coverings $\overline{H}$ over H

cliques as their image. Subsequently, the hypergraph $H^{\widetilde{\phi}}$ associated with $\widetilde{\phi}$ is established as follows: The vertex set $H^{\widetilde{\phi}}$ is the vertex set of the derived graph $GH_c^{\widetilde{\phi}}$ of $(GH_c, \widetilde{\phi})$. For each clique with identical color in $GH_c^{\widetilde{\phi}}$ a hyperedge is formed on the vertices of this clique, and a loop is formed on the vertex where there exists a loop in $GH_c^{\widetilde{\phi}}$. $H^{\widetilde{\phi}}$ is called the *derived hypergraph* associated with the strong permutation voltage assignment $\widetilde{\phi}$. From the construction of $H^{\widetilde{\phi}}$, it is apparent that $H^{\widetilde{\phi}}$ must be a k-fold hypergraph covering over H and $GH_c^{\widetilde{\phi}}$ can be considered as the edge-colored graph of $H^{\widetilde{\phi}}$.

Theorem 4. *Let $\overline{H}$ be any k-fold hypergraph covering over the hypergraph H. Then we can establish a one-to-one correspondence between $\overline{H}$ and some derived hypergraph $H^{\widetilde{\phi}}$ associate with $\widetilde{\phi} \in C(\widetilde{\phi}; k)$.*

Proof. Since the edge-colored graph $G\overline{H}_c$ of $\overline{H}$ is a k-fold covering over GH_c, then by Lemmas 1 and 3, there exists a strong permutation voltage assignment $\widetilde{\phi}$ on GH_c such that the derived graph $GH_c^{\widetilde{\phi}}$ associated with $\widetilde{\phi}$ is corresponding to $G\overline{H}_c$. The proof is completed by noting that every hypergraph is one-to-one corresponding to its edge-colored graph. □

Example 5. In Figure 1, we assign the strong permutation voltage $\widetilde{\phi}$ on GH_c as follows: $\widetilde{\phi}((v_2, v_3)) = \widetilde{\phi}((v_4, v_5)) = \widetilde{\phi}((v_1, v_3)) = \widetilde{\phi}((v_3, v_5)) = \widetilde{\phi}((v_1, v_4)) = \widetilde{\phi}((v_5, v_5)) = (12)$, $\widetilde{\phi}((v_1, v_2)) = \widetilde{\phi}((v_1, v_5)) = \widetilde{\phi}((v_2, v_4)) = 1$. Then the derived graph $GH_c^{\widetilde{\phi}}$ and the corresponding 2-fold hypergraph covering $\overline{H}$ over H are obtained, see Figure 3.

Next, we find another method to establish a representation to all k-fold hypergraph coverings over a finite hypergraph. For a finite hypergraph H with the vertex set $V_H = \{v_1, v_2, \ldots, v_n\}$ and the hyperedge set $E_H = \{e_1, e_2, \ldots, e_m\}$. Suppose that $\phi: E_{\overleftrightarrow{B_H}} \to S_k$ is a permutation voltage assignment on the incidence graph B_H of H. It is known that the derived graph B_H^{ϕ} of (B_H, ϕ) brings into a correspondence with some k-fold covering over B_H. The *derived hypergraph* $H^{B_H^{\phi}}$ associated with B_H^{ϕ} is constructed as follows: The vertex set $V_{H^{B_H^{\phi}}} = \{(v_i, t) | v_i \in V_H, t \in [k]\}$, and the vertices (v_i, t) and (v_j, s) are adjacent

if and only if there is a vertex (e_l, h) in B_H^ϕ such that $\phi((v_i, e_l))t = h$ and $\phi((e_l, v_j))h = s$. That is to say all vertices are adjacent to (e_l, h) in B_H^ϕ form a hyperedge of $H^{B_H^\phi}$. It is worth nothing that a loop is formed at the vertex (v_i, t) in $H^{B_H^\phi}$ if $(v_i, t) = (v_j, s)$. Obviously, $H^{B_H^\phi}$ is a k-fold hypergraph covering over H by the construction of $H^{B_H^\phi}$. Thus, for each permutation voltage assignment ϕ on B_H, there must exist a k-fold hypergraph covering $\overline{H}$ over H corresponding to $H^{B_H^\phi}$. On the other hand, the graph B_H^ϕ is considered as the incidence graph of $H^{B_H^\phi}$ if the hyperedge containing the vertices (v_i, t) and (v_j, s) is labelled to (e_l, h). However, different permutation voltage assignments on B_H may lead to the same derived hypergraph $H^{B_H^\phi}$ because the vertex (e_l, h) in B_H^ϕ transforms into a hyperedge when B_H^ϕ acts as a graph representation of $H^{B_H^\phi}$. Therefore, each hypergraph covering over H can be generated by at least one permutation voltage assignment ϕ on B_H.

Example 6. In Figure 4, B_H is the incidence graph of the hypergraph H. The permutation voltage assignments ϕ_1 and ϕ_2 on B_H can generate two distinct derived graphs $B_H^{\phi_1}$ and $B_H^{\phi_2}$ depicted in Figure 5, but they induce to the same 2-fold hypergraph covering $\overline{H}$ over H, where $\phi_1((v_1, e_1)) = \phi_1((e_1, v_3)) = \phi_1((v_1, e_3)) = \phi_1((v_3, e_2)) = \phi_1((e_2, v_4)) = (12)$, $\phi_1((e_1, v_2)) = \phi_1((v_2, e_2)) = \phi_1((e_3, v_4)) = 1$, and $\phi_2((v_1, e_1)) = \phi_2((e_1, v_3)) = \phi_2((v_1, e_3)) = \phi_2((v_2, e_2)) = 1$, $\phi_2((v_3, e_2)) = \phi_2((e_2, v_4)) = \phi_2((e_1, v_2)) = \phi_2((e_3, v_4)) = (12)$.
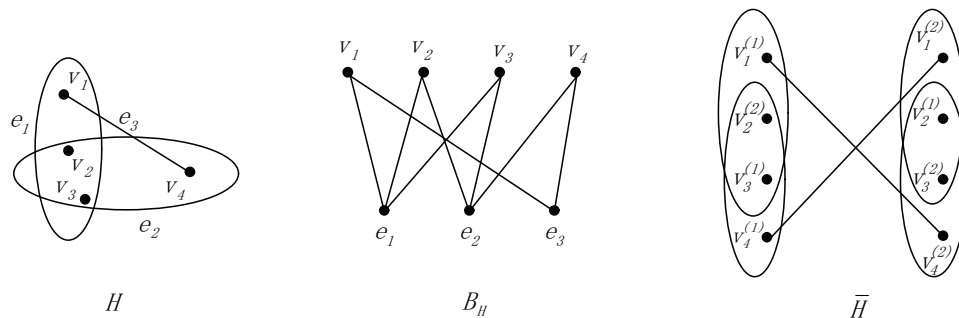
Figure 4: The hypergraph H, the incidence graph B_H of H, and a 2-fold hypergraph covering $\overline{H}$ over H

Theorem 7. *Let $\overline{H}$ be any k-fold hypergraph covering over the hypergraph H. Then there is a permutation voltage assignment $\phi : E_{\overleftrightarrow{B_H}} \to S_k$ on the incidence graph B_H of H such that the derived hypergraph $H^{B_H^\phi}$ associated with B_H^ϕ is isomorphic to $\overline{H}$.*

Proof. A hypergraph is one-to-one corresponding to its incidence graph if giving the vertices and the hyperedges with a fixed labelling. Thus, H is one-to-one corresponding to its incidence graph B_H, and do $\overline{H}$ and its incidence graph $B_{\overline{H}}$. Note that $B_{\overline{H}}$ is a k-fold
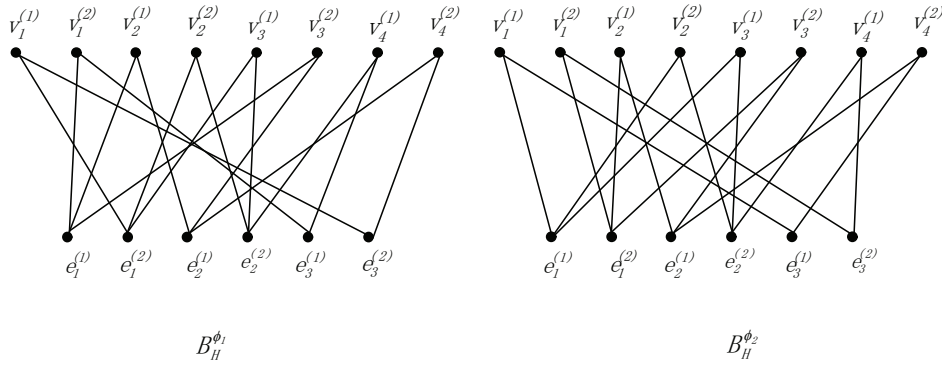
Figure 5: The derived graphs $B_H^{\phi_1}$ and $B_H^{\phi_2}$ with the permutation voltage assignments ϕ_1 and ϕ_2 on B_H given in Figure 4.

covering over B_H. Then by Lemma 1, there exists a permutation voltage assignment $\phi : E_{\overleftrightarrow{B_H}} \to S_k$ on B_H such that B_H^ϕ is correspondence with $B_{\overline{H}}$. The theorem is obtained by the fact that B_H^ϕ being the incidence graph of $H^{B_H^\phi}$. $\qquad\qquad\square$

3 The zeta function of a hypergraph covering

The *complex general linear group* of degree k, denoted by $GL(k, \mathbb{C})$, is the group of all $k \times k$ invertible matrices over the complex field $\mathbb{C}$ with respect to multiplication. Let $\mathcal{H}$ be a finite group. A *representation* ρ of the group $\mathcal{H}$ over $\mathbb{C}$ is a homomorphism from $\mathcal{H}$ to $GL(k, \mathbb{C})$, meanwhile k is called the *degree* of the representation ρ. If the representation $\rho : \mathcal{H} \to GL(k, \mathbb{C})$ sends each $g \in \mathcal{H}$ to a permutation matrix, then ρ is called the *permutation representation* of $\mathcal{H}$. The permutation representation $\mathbf{P}$ of the symmetric group S_k sends each $g \in S_k$ to the $k \times k$ permutation matrix $\mathbf{P}_g = (p_{ij}^{(g)})$, where

$$p_{ij}^{(g)} = \begin{cases} 1 & if \quad i = g(j), \\ 0 & otherwise. \end{cases} \tag{4}$$

The *Kronecker product* $A \otimes B$ of matrices A and B is considered as the matrix A having the element a_{ij} replaced by the matrix $a_{ij}B$. Let

$$A_1 \oplus \cdots \oplus A_n = \oplus_{i=1}^n A_i = diag(A_1, A_2, \cdots, A_n)$$

for square matrices $A_1, \ldots, A_n$, and it is written as $n \circ A$ for short when $A_1 = \cdots = A_n = A$.

The following lemma is needed for the main result:

Lemma 8. *[11] Let $\rho_1 = I_1, \rho_2, \ldots, \rho_s$ form a complete set of inequivalent irreducible representations of S_k. Furthermore, let f_i be the degree of ρ_i and m_i the multiplicity of ρ_i in the permutation representation $\mathbf{P}$ which is defined by Eq. (4) for each $i \in [s]$, where $f_1 = 1$. Then there exists a nonsingular matrix M such that for all $g \in S_k$*

$$M^{-1}\mathbf{P}_g M = m_1 \circ I_1 \oplus m_2 \circ \rho_2(g) \oplus \cdots \oplus m_s \circ \rho_s(g).$$

Theorem 9. *Let H be a finite hypergraph with n vertices and no loops, and GH_c be the edge-colored graph of H. Suppose that $H^{\widetilde{\phi}}$ is a k-fold hypergraph covering over H associated with a strong permutation voltage assignment $\widetilde{\phi}$ on GH_c. Let $\Gamma = \langle \widetilde{\phi}(e) | e \in E_{\overleftrightarrow{GH_c}} \rangle$ be the subgroup of S_k generated by $\{\widetilde{\phi}(e) | e \in E_{\overleftrightarrow{GH_c}}\}$, whose elements act as permutations on the set $[k]$. Furthermore, let $\mathbf{P}$ be a permutation representation of Γ associated with the set $[k]$ which is defined as Eq. (4). Moreover, let $\rho_1 = I_1, \rho_2, \cdots, \rho_s$ be all inequivalent irreducible representations of Γ, and the multiplicity of ρ_i in $\mathbf{P}$ and the degree of ρ_i be denoted by m_i and f_i for each $i \in [s]$, respectively. Then the zeta function of $H^{\widetilde{\phi}}$ can be decomposed as follows:*

$$\varsigma_{H^{\widetilde{\phi}}}(u)^{-1} = \varsigma_H(u)^{-m_1} \prod_{i=2}^{s} \det(I_{2lf_i} - u\Sigma_{g\in\Gamma}(\rho_i(g) \otimes \mathbf{A}_g))^{m_i},$$

where $\mathbf{A}_g$ is defined by Eq. (5), and l is the size of GH_c.

Proof. Let $E_{\overleftrightarrow{GH_c}} = \{e_1, \ldots, e_l, e_{1+1}, \ldots, e_{2l}\}$, where $e_i = e_{i-l}^{-1}$ for $i = l+1, \ldots, 2l$. $GH_c^{\widetilde{\phi}}$ is a k-fold covering associated with $\widetilde{\phi}$ over GH_c. Arrange arcs of $GH_c^{\widetilde{\phi}}$ into k blocks:

$$e_1^{(1)}, \ldots, e_l^{(1)}, e_{1+1}^{(1)}, \ldots, e_{2l}^{(1)}; \ldots; e_1^{(k)}, \ldots, e_l^{(k)}, e_{1+1}^{(k)}, \ldots, e_{2l}^{(k)},$$

where $e_i^{(j)} = (e_i, j)$.

Since $GH_c^{\widetilde{\phi}}$ is the edge-colored graph of $H^{\widetilde{\phi}}$, the vertices of the oriented line graph $H_L^{\widetilde{\phi}}$ of $H^{\widetilde{\phi}}$ is the arcs of $\overleftrightarrow{GH_c^{\widetilde{\phi}}}$. We consider the adjacency matrix $A(H_L^{\widetilde{\phi}})$ under this order. For any $g \in \Gamma$, the $2l \times 2l$ matrix $\mathbf{A}_g = (a_{ij}^{(g)})$ is defined by

$$a_{ij}^{(g)} = \begin{cases} 1 & \text{if } (e_i, e_j) \in E_{H_L} \text{ and } \widetilde{\phi}(e_i) = g, \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Thus, we have

$$A(H_L) = \sum_{g\in\Gamma} \mathbf{A}_g, \tag{6}$$

and from [8] it follows that

$$A(H_L^{\widetilde{\phi}}) = \sum_{g\in\Gamma}(\mathbf{P}_g \otimes \mathbf{A}_g). \tag{7}$$

By Lemma 8, there exists a nonsingular matrix M such that $M^{-1}\mathbf{P}_g M = m_1 \circ \rho_1(g) \oplus m_2 \circ \rho_2(g) \oplus \cdots \oplus m_s \circ \rho_s(g)$ for all $g \in \Gamma$, where $\rho_1(g) = I_1$. Set $T = M \otimes I_{2l}$, then by

Eqs. (7) and (6) we have

$$
\begin{aligned}
T^{-1}A(H_L^{\widetilde{\phi}})T &= (M \otimes I_{2l})^{-1}[\sum_{g \in \Gamma}(\mathbf{P}_g \otimes A_g)](M \otimes I_{2l}) \\
&= \sum_{g \in \Gamma}(M^{-1}\mathbf{P}_g M \otimes A_g) \\
&= \sum_{g \in \Gamma}[(\oplus_{i=1}^s m_i \circ \rho_i(g)) \otimes A_g] \\
&= \bigoplus_{i=1}^s \{\Sigma_{g \in \Gamma}[(m_i \circ \rho_i(g)) \otimes A_g]\} \\
&= [\Sigma_{g \in \Gamma}(I_{m_1} \otimes A_g)] \bigoplus \bigoplus_{i=2}^s \{\Sigma_{g \in \Gamma}[(m_i \circ \rho_i(g)) \otimes \mathbf{A}_g]\} \\
&= (m_1 \circ A(H_L)) \bigoplus \bigoplus_{i=2}^s \{m_i \circ [\Sigma_{g \in \Gamma}(\rho_i(g) \otimes \mathbf{A}_g)]\}. \qquad (8)
\end{aligned}
$$

Note that $\sum_{i=1}^s m_i f_i = k$ and $f_1 = 1$. It follows that

$$
T^{-1}(I_{2lk} - uA(H_L^{\widetilde{\phi}}))T = m_1 \circ (I_{2l} - uA(H_L)) \bigoplus \bigoplus_{i=2}^s \{m_i \circ [I_{2lf_i} - u\Sigma_{g \in \Gamma}(\rho_i(g) \otimes \mathbf{A}_g)]\}.
$$

The theorem is obtained by Eq. (2). $\qquad \square$

Theorem 10. *Let H be a finite hypergraph with n vertices and m hyperedges such that each vertex is in at least two hyperedges and loops are not allowed. Furthermore, let B_H be the incidence graph of H. Suppose that $\phi : E_{\overleftrightarrow{B_H}} \to S_k$ is a permutation voltage assignment on B_H such that $\overline{H}$ is a k-fold hypergraph covering over H corresponding to ϕ and $\Gamma = \langle \phi(e) | e \in E_{\overleftrightarrow{B_H}} \rangle$. Let $\mathbf{P}$ be a permutation representation of Γ associated with the set $[k]$ which is defined as Eq. (4). Moreover, let $\rho_1 = I_1, \rho_2, \cdots, \rho_s$ be all inequivalent irreducible representations of Γ, and the multiplicity of ρ_i in $\mathbf{P}$ and the degree of ρ_i be denoted by m_i and f_i for each $i \in [s]$, respectively. Then the zeta function of $\overline{H}$ can be decomposed as follows:*

$$
\varsigma_{\overline{H}}(u)^{-1} = \varsigma_H(u)^{-m_1}(1 - u)^{(k-m_1)(|E_{B_H}|-m-n)} \prod_{i=2}^s M_i^{m_i}.
$$

where $M_i = \det(I_{f_i(m+n)} - \sqrt{u}\sum_{g \in \Gamma}(\rho_i(g) \otimes \mathbf{A}_g) + uf_i \circ Q(B_H))$, and $\mathbf{A}_g$ is defined by Eq. (9).

Proof. We utilize Eq. (3) to prove this theorem. The proof is an analogue of the method in [6] and [7].

Firstly, we have $(1-u)^{|E_{B_{\overline{H}}}|-|V_{B_{\overline{H}}}|} = (1-u)^{k(|E_{B_H}|-|V_{B_H}|)} = (1-u)^{k(|E_{B_H}|-m-n)}$ from the definition of hypergraph coverings.

Set $V_H = \{v_1, v_2, \ldots, v_n\}$ and $E_H = \{e_1, e_2, \ldots, e_m\}$. Arrange the vertices of B_H into 2 blocks:

$$v_1, v_2, \ldots, v_n; e_1, e_2, \ldots, e_m,$$

and we arrange the vertices of the incidence graph $B_{\overline{H}}$ into $2k$ blocks:

$$v_1^{(1)}, \ldots, v_n^{(1)}; e_1^{(1)}, \ldots, e_m^{(1)}; \ldots; v_1^{(k)}, \ldots, v_n^{(k)}; e_1^{(k)}, \ldots, e_m^{(k)},$$

where $v_i^{(j)} = (v_i, j)$. Then the adjacency matrix $A(B_{\overline{H}})$ and the degree matrix $D(B_{\overline{H}})$ of $B_{\overline{H}}$ are considered under this order.

Let $\overline{H}$ be a k-fold hypergraph covering over H. We can find a permutation voltage assignment $\phi : E_{\overleftrightarrow{B_H}} \to S_k$ on B_H such that $B_{\overline{H}}$ is corresponding to B_H^ϕ from Theorem 7. For any $g \in \Gamma$, the $(n+m) \times (n+m)$ matrix $\mathbf{A}_g = (a_{ij}^{(g)})$ is defined by

$$a_{ij}^{(g)} = \begin{cases} 1 & if \quad e = (v_i, e_j) \ or \ e = (e_i, v_j) \in E_{\overleftrightarrow{B_H}} \ \ and \ \ \phi(e) = g, \\ 0 & otherwise. \end{cases} \tag{9}$$

It is easily checked that

$$A(B_H) = \sum_{g \in \Gamma} \mathbf{A}_g. \tag{10}$$

Moreover, from the definition of graph coverings, the matrix $Q(B_{\overline{H}}) = D(B_{\overline{H}}) - I_{k(m+n)}$ can be expressed as:

$$Q(B_{\overline{H}}) = I_k \otimes Q(B_H). \tag{11}$$

As $B_{\overline{H}}$ is a k-fold covering over B_H corresponding to B_H^ϕ, Kwak and Lee [4] showed the adjacency matrix $A(B_{\overline{H}})$ of $B_{\overline{H}}$ is

$$A(B_{\overline{H}}) = \sum_{g \in \Gamma} (\mathbf{P}_g \otimes \mathbf{A}_g). \tag{12}$$

By Lemma 8, there exists a nonsingular matrix M such that $M^{-1}\mathbf{P}_g M = m_1 \circ \rho_g \oplus m_2 \circ \rho_2(g) \oplus \cdots \oplus m_s \circ \rho_s(g)$ for all $g \in \Gamma$, where $\rho_g = I_1$. Set $T = M \otimes I_{m+n}$. Similar to Eq. (8), by Eqs. (12) and (10) we have

$$T^{-1}A(B_{\overline{H}})T = (I_{m_1} \otimes A(B_H)) \bigoplus \bigoplus_{i=2}^{s} \{m_i \circ [\Sigma_{g \in \Gamma}(\rho_i(g) \otimes \mathbf{A}_g)]\}. \tag{13}$$

Besides, we have

$$T^{-1}Q(B_{\overline{H}})T = (M^{-1} \otimes I_{m+n})(I_k \otimes Q(B_H))(M \otimes I_{m+n}) = I_k \otimes Q(B_H). \tag{14}$$

Next, we consider $\det(I_{k(m+n)} - \sqrt{u}A(B_{\overline{H}}) + uQ(B_{\overline{H}}))$. Noting that $\sum_{i=1}^{s} m_i f_i = k$ and $f_1 = 1$, it follows that

$$
\begin{aligned}
&\det(I_{k(m+n)} - \sqrt{u}A(B_{\overline{H}}) + uQ(B_{\overline{H}})) \\
={} &\det(T^{-1}(I_k \otimes I_{m+n} - \sqrt{u}A(B_{\overline{H}}) + uQ(B_{\overline{H}}))T) \\
={} &\det\left(I_{\sum_{i=1}^{s} m_i f_i} \otimes I_{m+n} - \sqrt{u}I_{m_1} \otimes A(B_H)\bigoplus\bigoplus_{i=2}^{s}(-\sqrt{u}m_i) \circ (\Sigma_{g\in\Gamma}(\rho_i(g) \otimes \mathbf{A}_g))\right. \\
&\left. + uI_{\sum_{i=1}^{s} m_i f_i} \otimes Q(B_H)\right) \quad \text{(by Eqs. (13) and (14))} \\
={} &\det\left((I_{m_1} \otimes I_{m+n} - \sqrt{u}I_{m_1} \otimes A(B_H) + uI_{m_1} \otimes Q(B_H))\bigoplus\bigoplus_{i=2}^{s}(I_{m_i f_i} \otimes I_{m+n}\right. \\
&\left. -\sqrt{u}m_i \circ (\Sigma_{g\in\Gamma}(\rho_i(g) \otimes \mathbf{A}_g)) + uI_{m_i f_i} \otimes Q(B_H))\right) \\
={} &\det(I_{m+n} - \sqrt{u}A(B_H) + uQ(B_H))^{m_1}\prod_{i=2}^{s}\det(I_{f_i(m+n)} - \sqrt{u}\Sigma_{g\in\Gamma}(\rho_i(g) \otimes \mathbf{A}_g) \\
&+ uI_{f_i} \circ Q(B_H))^{m_i} \\
={} &\varsigma_H(u)^{-m_1}(1-u)^{-m_1(|E_{B_H}|-|V_{B_H}|)}\prod_{i=2}^{s}\det(I_{f_i(m+n)} - \sqrt{u}\Sigma_{g\in\Gamma}(\rho_i(g) \otimes \mathbf{A}_g) \\
&+ uf_i \circ Q(B_H))^{m_i} \quad \text{(by Eq. (3))}.
\end{aligned}
$$

The theorem is obtained by applying Eq. (3) again. $\qquad\square$

From the above expressions of the zeta function of a hypergraph covering, the following corollary can be obtained easily:

Corollary 11. *Let H be a hypergraph and $\overline{H}$ be a hypergraph covering over H, then the zeta function of H divides the zeta function of $\overline{H}$.*

4 Example

We demonstrate the use of Theorems 9 and 10 with an example. We consider the hypergraph given in Figure 4. The permutation representation $\mathbf{P}$ of S_2 is equivalent to $I_1 \oplus \rho_2$, where $\rho_2(1) = 1$ and $\rho_2((12)) = -1$.

The 2-fold hypergraph $\overline{H}$ is induced by the strong permutation voltage assignment $\widetilde{\phi}$ on GH_c, where $\widetilde{\phi}((v_1, v_4)) = \widetilde{\phi}((v_1, v_2)) = \widetilde{\phi}((v_2, v_4)) = (12)$, $\widetilde{\phi}((v_1, v_3)) = \widetilde{\phi}((v_3, v_4)) = 1$, and $\widetilde{\phi}((v_2, v_3)) = (12)$ for two arcs (v_2, v_3) with distinct color. By going through the

oriented line graph H_L of H, it follows that

$$
\mathbf{A}_1 =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

and

$$
\mathbf{A}_{(12)} =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

So, we have

$$
\begin{aligned}
\varsigma_H(u)^{-1} &= \det(I_{14} - uA(H_L)) \\
&= \det(I_{14} - u(\mathbf{A}_1 - \mathbf{A}_{(12)})) \\
&= -(u-1)^2(1+u)(1+2u+2u^2)(-1+u+2u^3).
\end{aligned}
$$

From Theorem 9, the zeta function of $\overline{H}$ is

$$
\begin{aligned}
\varsigma_{\overline{H}}(u)^{-1} &= \varsigma_H(u)^{-1} \det(I_{14} - u(\mathbf{A}_1 - \mathbf{A}_{(12)})) \\
&= (u-1)^3(1+u)^3(1+2u+2u^2)(1-2u+2u^2)(-1+u+2u^3)(1+u+2u^3).
\end{aligned}
$$

We also compute the zeta function of $\overline{H}$ by using Theorem 10. The permutation voltage assignment ϕ_1 on B_H is given in Example 6. We get

$$
Q(B_H) =
\begin{bmatrix}
1 & & & & & & \\
 & 1 & & & & & \\
 & & 1 & & & & \\
 & & & 1 & & & \\
 & & & & 2 & & \\
 & & & & & 2 & \\
 & & & & & & 1
\end{bmatrix}.
$$

The matrices $\mathbf{A}_g (g \in S_2)$ are listed as follows:

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad and \quad \mathbf{A}_{(12)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $m_1 = m_2 = f_1 = f_2 = 1$. By Theorem 10, it is calculated that

$$\begin{aligned} \varsigma_{\overline{H}}(u)^{-1} &= \varsigma_H(u)^{-1}(1-u)\det(I_7 - \sqrt{u}(\mathbf{A}_1 - \mathbf{A}_{(12)}) + uQ(B_H)) \\ &= (u-1)^3(1+u)^3(1+2u+2u^2)(1-2u+2u^2)(-1+u+2u^3)(1+u+2u^3). \end{aligned}$$

The same result is obtained by utilizing two different method.

Acknowledgements

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