A Cornucopia of Quasi-Yamanouchi Tableaux

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Abstract

Quasi-Yamanouchi tableaux are a subset of semistandard Young tableaux and refine standard Young tableaux. They are closely tied to the descent set of standard Young tableaux and were introduced by Assaf and Searles to tighten Gessel's fundamental quasisymmetric expansion of Schur functions. The descent set and descent statistic of standard Young tableaux repeatedly prove themselves useful to consider, and as a result, quasi-Yamanouchi tableaux make appearances in many ways outside of their original purpose. Some examples, which we present in this paper, include the Schur expansion of Jack polynomials, the decomposition of Foulkes characters, and the bigraded Frobenius image of the coinvariant algebra. While it would be nice to have a product formula enumeration of quasi-Yamanouchi tableaux in the way that semistandard and standard Young tableaux do, it has previously been shown by the author that there is little hope on that front. The goal of this paper is to address a handful of the numerous alternative enumerative approaches. In particular, we present enumerations of quasi-Yamanouchi tableaux using q-hit numbers, semistandard Young tableaux, weighted lattice paths, and symmetric polynomials, as well as the fundamental quasisymmetric and monomial quasisymmetric expansions of their Schur generating function.

Mathematics Subject Classifications: 05A19, 05A15, 05E05

Contents

1 Introduction

Schur polynomials have an elegant combinatorial description as a sum of monomials indexed by semistandard Young tableaux. However, as the number of variables increases, the number of semistandard Young tableaux rises dramatically, quickly making a computation using this description intractable. Ira Gessel [7] defined the fundamental basis for quasisymmetric polynomials and proved that Schur polynomials can instead be expressed as a sum of fundamental quasisymmetric polynomials indexed by standard Young tableaux. Since the number of standard Young tableaux of a given shape is fixed, this provides a significant computational improvement over the unbounded number of semistandard Young tableaux. Still, when the number of variables is low enough, it turns out that some of the standard Young tableaux contribute nothing to the expansion, indicating that some further improvement can be made. Sami Assaf and Dominic Searles [2] introduced the concept of quasi-Yamanouchi tableaux and proved that we can tighten Gessel's expansion by summing over these tableaux instead, which give exactly the nonzero terms.

Quasi-Yamanouchi tableaux have a natural bijection with standard Young tableaux of the same shape. Quasi-Yamanouchi tableaux of a fixed shape can be partitioned into sets by the largest valued labels that appear in the tableaux, and the corresponding partitioning of standard Young tableaux groups standard Young tableaux by number of descents. In this sense, quasi-Yamanouchi tableaux refine standard Young tableaux by their descent statistic through this natural bijection. In general, descent statistics have formed a rich and fruitful area, for example in the past with Solomon's descent algebra [14] and subsequent study of it [5] or, more recently, work on shuffle-compatible permutation statistics by Gessel and Zhuang [8]. Descents of standard Young tableaux do not appear to be an exception. Quasi-Yamanouchi tableaux inherit this value, and consequently, applications are readily found. In §7, we discuss some of these applications, which involve

the Schur expansion of Jack polynomials, the decomposition of Foulkes characters into irreducible characters, and the bigraded Frobenius image of the coinvariant algebra.

Semistandard Young tableaux and standard Young tableaux are each enumerated by celebrated product formulas. Since quasi-Yamanouchi tableaux of a fixed shape are in bijection with standard Young tableaux of the same shape, we can enumerate the total number of quasi-Yamanouchi tableaux of a given shape using the same product formula. However, this is a somewhat trivial result, and it would be much more interesting to find a product formula enumeration for the parts of the aforementioned partitioning of quasi-Yamanouchi tableaux into sets by the largest value that appears in the tableaux. Unfortunately, in [18], the author demonstrates that there is likely only such a product formula for certain special cases due to large primes appearing for general shapes. The work in this paper takes a number of alternative approaches and was inspired by the surprising appearance of quasi-Yamanouchi tableaux in the coefficients of Jack polynomials under a binomial coefficient basis [1] and a hit number interpretation of the same coefficients. Comparing the two equivalent interpretations of the coefficients gives an enumeration of quasi-Yamanouchi tableaux in terms of hit numbers of certain Ferrers boards. We reproduce this enumeration in §3 and prove two q-analogues involving the major index and charge statistics. In §4, we give a summation formula in terms of semistandard Young tableaux, and in §5 we first give an enumeration first in terms of weighted lattice paths and then using symmetric polynomials. Finally, we consider the Schur generating function in §6 and give its fundamental quasisymmetric expansion and monomial symmetric expansion.

2 Preliminaries

We first present the more general concepts that will be used throughout the paper; the more specific concepts will be located at the start of their relevant section. A *partition* $\lambda = (\lambda_1 \geqslant \ldots \geqslant \lambda_k > 0)$ is a weakly decreasing sequence of positive integers. The *size* of λ is denoted $|\lambda|$ and is the sum of the integers of the sequence. The *length* of λ, $\ell(\lambda)$, is the number of integers in the partition. We also use the notation $d(\lambda) = \sum_{i=1}^{k}(i-1)\lambda_i$. We say that a partition λ dominates a partition μ if for all $i \geqslant 1$, $\lambda_1 + \cdots + \lambda_i \geqslant \mu_1 + \cdots + \mu_i$.

We identify a partition with its *diagram*, where rows are counted from bottom to top, the number of boxes in the ith row equals λ_i, and boxes are left justified. The conjugate of λ is written λ' and is obtained by reflecting the diagram across the diagonal. We identify $u = (i, j)$ with the box in the ith column and jth row. The *content* of a square is $c(u) = i - j$, and the *hook-length* of a box is $h(u)$, which counts the number of boxes (a, b) such that either $a > i, b = j$ or $a = i, b > j$ or $a = i, b = j$.

We write permutations $\pi \in S_n$ in one line notation, $\pi = \pi_1 \cdots \pi_n$, where $\pi_i = \pi(i)$. The *descent set* of π is $\mathrm{Des}(\pi) = \{i \in [n-1] \mid \pi_i > \pi_{i+1}\}$, and we denote the size $|Des(\pi)|$ by $des(\pi)$. The major index for a permutation is $\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i$.

Bases of the ring of symmetric functions are indexed by partitions, and in particular, e_λ, m_λ, and s_λ are the elementary symmetric, monomial symmetric, and Schur functions respectively. We also write $F_\sigma(x)$ for the fundamental quasisymmetric function, where
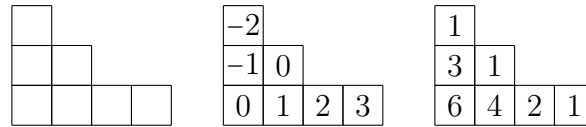
Figure 1: From left to right, the diagram of $(4, 2, 1)$, the cell contents, and the cell hook-lengths.

$\sigma \subseteq \{1, \ldots, n-1\}$ and $F_\sigma(x) = \sum_{\substack{i_1 \leqslant \cdots \leqslant i_n \\ j \in \sigma \Rightarrow i_j < i_{j+1}}} x_{i_1} \cdots x_{i_n}$.

A *semistandard Young tableau* (SSYT) *of shape* λ is a filling of a partition λ using positive integers that weakly increase along rows from left to right and strictly increase up columns, and the set of such fillings is denoted $\mathrm{SSYT}(\lambda)$. When the entries are unbounded, there are infinitely many semistandard Young tableaux of a given shape, so it is sometimes useful to consider instead $\mathrm{SSYT}_m(\lambda)$, the set of SSYT of shape λ with entries at most m. We can enumerate $\#\mathrm{SSYT}_m(\lambda)$ using Stanley's hook-content formula, which we reproduce below.

Theorem 1 (Hook-content formula [17]). *Given a partition* λ,

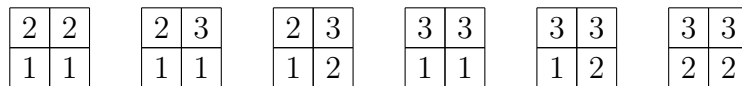$$\#\mathrm{SSYT}_m(\lambda) = \prod_{u \in \lambda} \frac{m + c(u)}{h(u)}. \tag{1}$$

Figure 2: All 6 elements of $\mathrm{SSYT}_3(2, 2)$.

The weight of a semistandard Young tableau T is $\mathbf{wt}(T) = (t_1, t_2, \ldots)$, where t_i is the number of times that i appears, and given partitions λ, μ, the *Kostka* numbers $K_{\lambda\mu}$ count the number of SSYT of shape λ and weight μ. A *standard Young tableau* (SYT) *of shape* λ is a semistandard Young tableau of shape λ with weight (1^n), where $n = |\lambda|$, and the set of such fillings is denoted $\mathrm{SYT}(\lambda)$. Frame, Robinson, and Thrall counted standard fillings using the hook-length formula.

Theorem 2 (Hook-length formula [4]). *Given a partition* λ,

$$\#\mathrm{SYT}(\lambda) = \frac{n!}{\prod_{u \in \lambda} h(u)}. \tag{2}$$

The *descent set* of $T \in \mathrm{SYT}(\lambda)$ is $\mathrm{Des}(T) = \{i \in [n-1] \mid i+1 \text{ is above } i\}$. If we write the descent set as $\{d_1, d_2, \ldots, d_{k-1}\}$ in increasing order, then the first *run* of the tableau

Figure 3: All 5 elements of SYT$(3, 2)$.

is the set of boxes that contain all the entries from 1 to d_1. For $1 < i < k$, the ith run is the set of boxes containing entries from $d_{i-1} + 1$ to d_i, and the kth run starts at $d_k + 1$ and ends at n. In other words, the runs of a standard Young tableau T partition the set $[n]$ into the maximal increasing subsequences of the *reading word* of T, which is obtained by reading entries of T along rows from left to right, then top to bottom.

Figure 4: This tableau has descent set $\{3, 6, 8, 11\}$ and has the fourth run bolded.

An SSYT is a *quasi-Yamanouchi tableau* (QYT) if when i appears in the tableau, some instance of i is in a higher row than some instance of $i - 1$ for all i. We write QYT(λ) to denote the set of QYT of shape λ, QYT$_{\leqslant m}(\lambda)$ to denote those with largest entry at most m, and QYT$_{=m}(\lambda)$ to denote those with largest entry exactly m. The definition implies that any quasi-Yamanouchi tableau that has largest value m must have at least one i for all $1 \leqslant i \leqslant m$. However, the reverse is not true in general.

Figure 5: The left is a quasi-Yamanouchi filling, while the right is not.

It turns out that there is a nice bijection between QYT(λ) and SYT(λ) via the following destandardization map [2]. Given a tableau $T \in$ SYT(λ), its *destandardization* is the semistandard Young tableau dst(T) constructed by changing the value of all entries in the ith run of T to i. From the definition of runs, we see that the resulting SSYT is quasi-Yamanouchi. A standard Young tableau is uniquely determined by the location of its runs, so this map has a well defined inverse, which we call standardization.

Proposition 3 ([2]). *For λ a partition, we have*
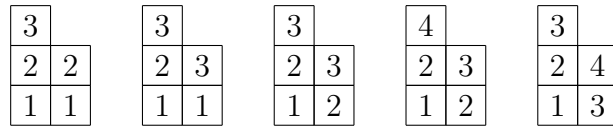
$$\text{QYT}(\lambda) \cong \text{SYT}(\lambda) \tag{3}$$

Figure 6: All 5 elements of $\mathrm{QYT}(2,2,1)$, showing that $\mathrm{QYT}_{=3}(2,2,1) = 3$ and $\mathrm{QYT}_{=4}(2,2,1) = 2$.

via the destandardization map dst.

By construction, it is clear that the map sends SYT with k runs (or $k-1$ descents) to QYT with maximum value k of the same shape. In this sense, the set $\{\mathrm{QYT}_{=m}(\lambda)\}$ refines $\mathrm{SYT}(\lambda)$ by number of descents, and we will often identify a QYT with its standardization. For further discussion on properties of QYT, see [18].

As with permutations, we define the *major index* of a tableau $T \in \mathrm{SYT}(\lambda)$ to be $\mathrm{maj}(T) = \sum_{i \in \mathrm{Des}(T)} i$. We also have the *charge* statistic for standard Young tableaux: each entry i in T has a charge defined recursively, where $ch(1) = 0$, $ch(i+1) = ch(i)$ if $i \notin \mathrm{Des}(T)$, $ch(i+1) = ch(i) + 1$ if $i \in \mathrm{Des}(T)$, and $ch(T) = \sum_{i=1}^{|\lambda|} ch(i)$. For a quasi-Yamanouchi tableau, we define its descent set, major index, and charge statistic to be those of its standardization.

There is a nice relationship between the charge of a standard Young tableau and its destandardization. The definition of the charge statistic implies that every entry in the ith run contributes $i-1$ to the total charge. The destandardization changes every entry in the ith run to i, so the charge of a standard Young tableau and its destandardization is the sum of all entries in the destandardization minus the number of entries.
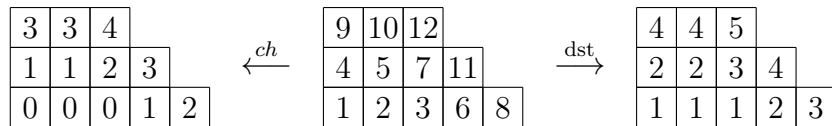

Figure 7: We have a standard Young tableau in the middle. On the left, each entry has been replaced with its charge, and on the right, the filling has been destandardized.

3   Hit number formulas

Fix $n \in \mathbb{N}$. A *board* B is a subset of $[n] \times [n]$, which we view as an n by n array of squares. In particular, a *Ferrers board* is a board composed of adjacent columns whose heights are weakly increasing from left to right. If λ is a partition of size n, then we can construct a Ferrers board B_λ as follows. Take the contents $c_1, \ldots, c_n$ of the diagram λ arranged in weakly decreasing order, then let the heights of the columns of B_λ be $(c_i + i - 1)$. Let $B_\lambda \times 1$ be the board obtained by incrementing the height of every column by one. We

note that it is always possible to do this once for any λ, as the construction never creates a column with height n. From the definitions, we get a relation between B_λ and $B_{\lambda'}$.

Proposition 4. *Given a partition λ, the complement of $B_\lambda \times 1$ is $B_{\lambda'}$ up to rotation.*
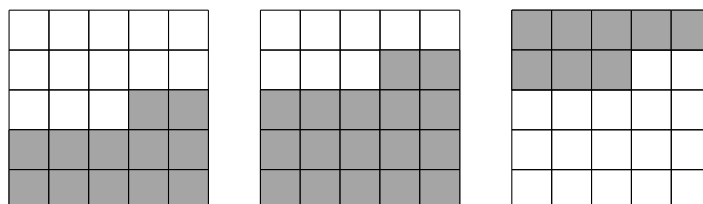

Figure 8: $B_{(3,2)}$, $B_{(3,2)} \times 1$, and $B_{(2,2,1)}$ rotated.

For $\pi \in S_n$, we let $\Gamma(\pi) = \{(i, \pi_i) \mid 1 \leqslant i \leqslant n\}$, and we define the number of *hits* of π on a given board B to be $|B \cap \Gamma(\pi)|$. We define the kth hit number $h_k(B)$ to be the number of permutations in S_n which have exactly k hits on B. Dworkin [3] gave a combinatorial interpretation of a q-analogue of hit numbers for boards B such that B is a Ferrers board up to column permutation, which we use as the definition. For $\pi \in S_n$, place a cross at each square in $\Gamma(\pi)$, and for any square to the right of a cross, put a bullet. Then from each cross, draw circles going up and wrapping around the top edge of the $[n] \times [n]$ array, skipping over bullets, and stopping after hitting the top border of the given board B. The q weight of π is the number of circles at the end of this process. The kth q-hit number $T_k(B)$ is the q weighted sum over all permutations that hit the board exactly k times.
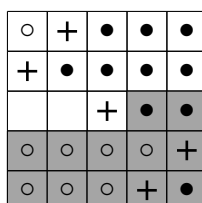

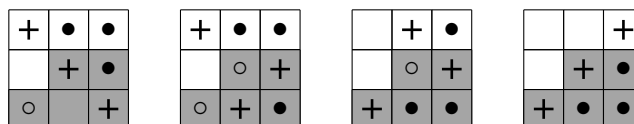Figure 9: The q weight of 45312 on $B_{3,2}$ is 8.

Figure 10: In this example, we can compute $T_2(B) = 1 + 2q + q^2$.

In [1], two separate interpretations of the coefficients of the Schur expansion of the one row case of Jack polynomials are given. One is in terms of quasi-Yamanouchi tableaux, and the other is in terms of hit numbers. By comparing these interpretations, we get the following hit number formula for quasi-Yamanouchi tableaux.

Theorem 5. *Given a partition λ of n and $0 \leqslant k \leqslant n - 1$,*

$$\#\mathrm{QYT}_{=k+1}(\lambda) = \frac{h_k(B_{\lambda'})}{\prod_{u \in \lambda} h(u)}. \tag{4}$$

In this section, we prove two q-analogues of this theorem, where the first weights each tableaux by its major index and the second by its charge. To do this, we use the theory of posets and (P, ω)-partitions, which were introduced by Stanley [16].

For a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, let P_λ be the subposet of $\mathbb{N} \times \mathbb{N}$ such that $(i, j) \in P_\lambda$ if $1 \leqslant j \leqslant k$, $1 \leqslant i \leqslant \lambda_j$. Given a poset P with n elements, a *labeling* ω is a bijective map $\omega : P \to [n]$. A labeling of some P_λ is *column-strict* if it is order reversing on columns and order preserving on rows.
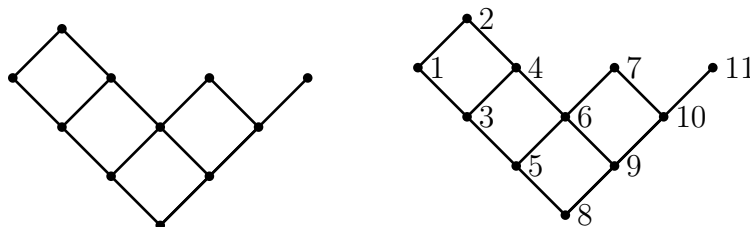

Figure 11: $P_{4,3,2,2}$ and a column-strict labeling of $P_{4,3,2,2}$.

For a fixed labeling ω, a (P, ω)-*partition* of size p is a map $\sigma : P \to \mathbb{N}_{\geqslant 0}$ satisfying
  1) $x \leqslant y \in P \implies \sigma(x) \geqslant \sigma(y)$, meaning σ is order reversing.
  2) $x < y \in P$ and $\omega(x) > \omega(y) \implies \sigma(x) > \sigma(y)$.
  3) $|\sigma| = \sum_{x \in P} \sigma(x) = p$.
The values $\sigma(x)$ are called the *parts* of σ, and a $(P, \omega; m)$-partition is a (P, ω)-partition with largest part at most m. $\mathcal{A}(P, \omega)$ denotes the set of (P, ω)-partitions, and $\mathcal{A}(P, \omega; m)$ denotes the set of $(P, \omega; m)$-partitions, which have generating function

$$U_m(P, \omega; m) = \sum_{\sigma \in \mathcal{A}(P, \omega; m)} q^{|\sigma|}. \tag{5}$$

The ω-*separator* $\mathcal{L}(P, \omega)$ is the set of permutations in S_n of the form $\omega(x_{i_1}) \cdots \omega(x_{i_n})$ where $x_{i_1} < \cdots < x_{i_n}$ forms a linear extension of P. For each $0 \leqslant k \leqslant n - 1$, define

$$W_k(P, \omega) = W_k(P, \omega; q) = \sum_{\pi} q^{\mathrm{maj}(\pi)}, \tag{6}$$

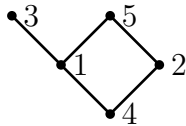where the sum is over all $\pi \in \mathcal{L}(P, \omega)$ with $des(\pi) = k$.

Figure 12: We have $\mathcal{L}(P, \omega) = \{\mathbf{4}1235, \mathbf{42}135, \mathbf{41}3\mathbf{2}5, \mathbf{41}2\mathbf{5}3, \mathbf{42}1\mathbf{5}3\}$ with descents in bold. Computing the major index of elements with two descents gives $W_2(P, \omega; q) = q^3 + q^4 + q^5$.

3.1 Major index formula

We first prove the following major index q-analogue of Theorem 5.

Theorem 6. *Given a partition λ and $0 \leqslant k \leqslant n - 1$,*

$$\sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} q^{maj(T)} = \frac{q^{d(\lambda)}}{\prod_{u \in \lambda} [h(u)]} T_{n-k}(B_\lambda \times 1). \tag{7}$$

Proof. Fix a partition λ, and let ω be a column-strict labeling on P_λ. For each $\pi \in \mathcal{L}(P_\lambda, \omega)$, we have some corresponding linear extension $x_{k_1} < \cdots < x_{k_n}$ of P_λ. We can identify each $(i, j) \in P_\lambda$ with the cell (i, j) in the diagram of λ and then label the cell of λ corresponding to x_{k_m} with the entry m. This gives a bijection between $\mathcal{L}(P_\lambda, \omega)$ and $\mathrm{SYT}(\lambda)$. Since the labeling ω is order reversing on columns and order preserving on rows, π has a descent in position i if and only if i is a descent in the tableau corresponding to π.
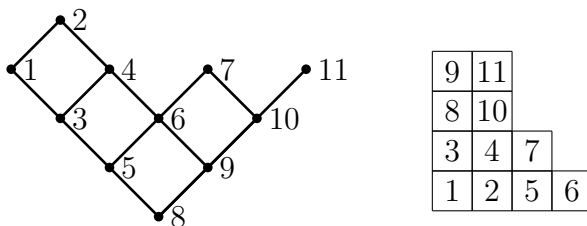
Figure 13: On the left, we have a column-strict labeling ω of $P_{4,3,2,2}$. On the right, we have a filling of $(4, 3, 2, 2)$ corresponding to the permutation $8\ 9\ 5\ 6\ 10\ 11\ 7\ 3\ 1\ 4\ 2 \in \mathcal{L}(P_{4,3,2,2}, \omega)$.

Therefore, we have

$$W_k(P_\lambda, \omega) = \sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} q^{\mathrm{maj}(T)}. \tag{8}$$

By Proposition 21.3 of [16],

$$U_m(P_\lambda, \omega; m) = q^{d(\lambda)} \prod_{u \in \lambda} \frac{[m + c(u) + 1]}{[h(u)]}. \tag{9}$$

Proposition 8.2 of [16] gives

$$U_m(P_\lambda, \omega) = \sum_{k=0}^{n-1} \begin{bmatrix} m+n-k \\ n \end{bmatrix} W_k(P_\lambda, \omega). \tag{10}$$

Then since there is no restriction on m, it follows that

$$\sum_{k=0}^{n-1} \begin{bmatrix} x+n-k \\ n \end{bmatrix} W_k(P_\lambda, \omega) = q^{d(\lambda)} \prod_{u \in \lambda} \frac{[x+c(u)+1]}{[h(u)]} \tag{11}$$

as polynomials over x and q. On the right hand side, we apply the following q-analogue [10] of the Goldman, Joichi, Write identity [9]

$$\prod_{i=1}^{n} [x+b_i-i+1] = \sum_{k=0}^{n} \begin{bmatrix} x+k \\ n \end{bmatrix} T_k(B), \tag{12}$$

where B is a Ferrers board with column heights b_i. Comparing coefficients of $\begin{bmatrix} x+k \\ n \end{bmatrix}$ completes the proof. $\qquad\square$

Setting $q = 1$ and applying Proposition 4 to Theorem 6 recovers Theorem 5. We note that since $T_k(B)$ is Mahonian [3] for a Ferrers board, summing over k gives a nice (known) q-analogue of the hook-length formula,

$$\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} = \frac{q^{d(\lambda)}[n]!}{\prod_{u \in \lambda}[h(u)]}. \tag{13}$$

We briefly attempted to prove Theorem 6 bijectively but were unsuccessful. It would be nice to know what such a bijective algorithm might look like, and such an algorithm could be an interesting project to revisit in the future.

3.2 Charge formula

In this section, we prove a charge statistic q-analogue of Theorem 5. Before we begin, we first introduce some notation and prove a technical lemma.

Fix a permutation λ of size n, and let ω be a column-strict labeling on P_λ. We write P_λ^* for the dual of P_λ and write ω^* for the labeling defined by $\omega^*(x_i) = n + 1 - \omega(x_i)$ for all $x_i \in P_\lambda$.

The algorithm for computing the q weight of a permutation π for a given board B produces a particular arrangement of crosses, bullets, and circles on the $[n] \times [n]$ grid of squares. We will call this arrangement the *regular arrangement* of π on B. We define the *dual arrangement* of π on B as follows.

Starting with the regular arrangement, simultaneously turn every empty square into a circle and every circle into an empty square by drawing circles downwards from crosses instead of upwards, wrapping around the bottom edge of the $[n] \times [n]$ array, skipping over
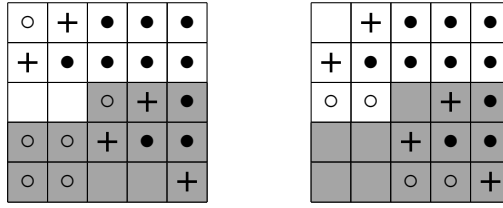
Figure 14: On the left, we have the regular arrangement of 45231 on $B_{2,2,1} \times 1$, and on the right, we have the dual arrangement.

bullets, and stopping after hitting the top border of the Ferrers board. This gives the dual arrangement of π on B.

Finally, we define the polynomial $T_k^*(B)$ on a given Ferrers board or column permutation of a Ferrers board B as follows. For a permutation π that hits B exactly k times, assign a q weight by taking the regular arrangement and giving circles a $1/q$ weight instead of a q weight as you normally would, and then take the weighted sum over all such permutations. Equivalently, if $T_k(B) = \sum a_j q^{i_j}$, then $T_k^*(B) = \sum a_j q^{-i_j}$.

Lemma 7. *For a permutation λ and $1 \leqslant k \leqslant n$, we have*

$$T_k^*(B_\lambda \times 1) = \frac{T_{n-k}(B_{\lambda'})}{q^{\binom{n}{2}}}. \tag{14}$$

Proof. Observe that the number of squares containing either a cross or a bullet in the regular arrangement of a permutation on a board is always fixed. As a result, the number of squares that contain either a circle or nothing is also fixed, and in particular there are always $\binom{n}{2}$ many such squares.

We compute $T_k^*(B_\lambda \times 1)$ using the regular arrangement of every π for which $|B_\lambda \times 1 \cap \Gamma(\pi)| = k$ and giving each circle a $1/q$ weight and each empty square a q^0 weight to get the weight of π. We can therefore compute $q^{\binom{n}{2}} T_k^*(B_\lambda \times 1)$ by using the regular arrangement for the same set of π and giving each circle a q^0 weight and each empty square a q^1 weight. However, this is equivalent to using the dual arrangement for each π and giving each circle a q^1 weight and each empty square a q^0 weight.

By Proposition 4, taking the complement of $B_\lambda \times 1$ and reflecting vertically gives $B_{\lambda'}$ with columns in reverse order, which we will call $B_{\lambda'}^{\text{rev}}$. If we start with the dual arrangement of some π on $B_\lambda \times 1$ and take the complement of $B_\lambda \times 1$ and reflect squares and their contents vertically on columns, then we get a regular arrangement of π' on $B_{\lambda'}^{\text{rev}}$, where π' is obtained from π by replacing π_i with $n + 1 - \pi_i$.

The correspondence between π and π' forms a bijection between permutations that hit $B_\lambda \times 1$ exactly k times and permutations that hit $B_{\lambda'}^{\text{rev}}$ exactly $n-k$ times. Furthermore, if π is given a q weight by assigning circles a q^1 weight in the dual arrangement of π on $B_\lambda \times 1$ and π' is given a q weight by assigning circles a q^1 weight in the regular arrangement of π' on $B_{\lambda'}^{\text{rev}}$, then this bijection preserves q weight from π on $B_\lambda \times 1$ to π' on $B_{\lambda'}^{\text{rev}}$. This gives $q^{\binom{n}{2}} T_k^*(B_\lambda \times 1) = T_{n-k}(B_{\lambda'}^{\text{rev}})$. By Theorem 7.13 of [3], $T_k(B)$ is invariant under column permutations for Ferrers boards, so $T_k(B_{\lambda'}^{\text{rev}}) = T_k(B_{\lambda'})$ and the claim follows. $\square$
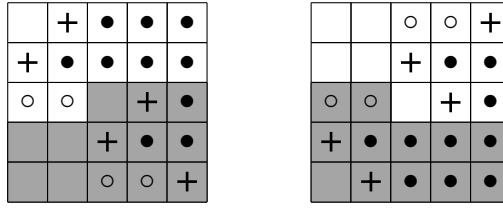
Figure 15: On the left, we have the dual arrangement of 45231 on $B_{2,2,1} \times 1$, and on the right, we have the regular arrangement of $(45231)' = 21435$ on $B_{3,2}^{\mathrm{rev}}$.

Theorem 8. *Given a partition λ and $0 \leqslant k \leqslant n-1$,*

$$\sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} q^{ch(T)} = \frac{q^{nk+d(\lambda')-\binom{n}{2}}}{\prod_{u \in \lambda}[h(u)]} T_k(B_{\lambda'}). \tag{15}$$

Proof. Proposition 12.1 of [16] details what effect dualization on P_λ and ω has on W_k, which is that

$$W_k(P_\lambda^*, \omega^*; q) = q^{nk} W_k(P_\lambda, \omega; \frac{1}{q}). \tag{16}$$

We note that since there are k descents,

$$q^{nk} W_k(P_\lambda, \omega; \frac{1}{q}) = \sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} q^{nk-\mathrm{maj}(T)} = \sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} \sum_{i \in \mathrm{Des}(T)} q^{n-i}. \tag{17}$$

Then since a descent at position i increments the charge value of the $n-i$ remaining entries by one, we get

$$W_k(P_\lambda^*, \omega^*; q) = \sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} q^{ch(T)}. \tag{18}$$

Using the facts that $[k] \mapsto [k]\frac{1}{q^{k-1}}$ when substituting $1/q$ and that $\sum_{u \in \lambda} h(u) = n + d(\lambda) + d(\lambda')$, we get

$$W_k(P_\lambda, \omega; \frac{1}{q}) = \frac{q^{d(\lambda')}}{\prod_{u \in \lambda}[h(u)]} T_{n-k}^*(B_\lambda \times 1), \tag{19}$$

and combining equations (16), (18), and (19) with Lemma 7 gives proves the result. $\square$

As with Theorem 6, we can substitute $q = 1$ to reduce Theorem 8 to Theorem 5. Summing over k in this case also gives some sort of q-analogue of the hook-length formula, although it does not appear to immediately give a nice form. Recalling the relationship between charge and the destandardization map at the end of Section 2, we can also interpret equation (15) in terms of the sum of entries of each QYT.

Corollary 9. *Given a partition λ and $0 \leqslant k \leqslant n-1$,*

$$\sum_{T \in \mathrm{QYT}_{=k+1}(\lambda)} \prod_{u \in T} q^{\mathrm{entry}(u)} = \frac{q^{n(k+1)+d(\lambda')-\binom{n}{2}}}{\prod_{u \in \lambda}[h(u)]} T_k(B_{\lambda'}). \tag{20}$$

4 A summation formula

We prove the following theorem in two ways, first with a q-hit number identity and then using (P, ω)-partitions. This gives a relatively clean enumeration for quasi-Yamanouchi tableaux compared to the product formula of [18], the downside being that it is not a positive summation.

Theorem 10. *Given a partition λ and $0 \leqslant k \leqslant n - 1$,*

$$\#\mathrm{QYT}_{=k+1}(\lambda) = \sum_{m=0}^{k} \binom{n+1}{k-m} (-1)^{k-m} \#\mathrm{SSYT}_{m+1}(\lambda). \tag{21}$$

First proof. We begin with equation (24) in [10], where we set $t = n$ and where for all cases that we consider, $d_i = 1$ and $D_i = i$. This simplifies (24) in [10] to

$$T_{n-k}(B) = \sum_{m=0}^{k} \begin{bmatrix} n+1 \\ k-m \end{bmatrix} (-1)^{k-m} q^{\binom{k-m}{2}} \prod_{i=1}^{n} [m + H_i - i + 1], \tag{22}$$

where H_i is the height of the ith column of B. By the way B_λ is constructed, the sequence $H_i - i$ for $B_\lambda \times 1$ becomes exactly the cell contents of λ, so setting $q = 1$ gives

$$h_{n-k}(B_\lambda \times 1) = \sum_{m=0}^{k} \binom{n+1}{k-m} (-1)^{k-m} \prod_{i=1}^{n} (m + c_i + 1). \tag{23}$$

Substituting this into Theorem 5 after applying Proposition 4 and comparing with the hook-content formula proves Theorem 10. $\qquad \square$

Second proof. Let ω be a column-strict labeling on P_λ. The set $\mathcal{A}(P_\lambda, \omega; m)$ can be naturally identified with the set of fillings of the diagram λ with entries at most $m + 1$ and with entries weakly decreasing along rows from left to right and strictly increasing up columns. By replacing each entry i with $m + 2 - i$, we can in turn identify such fillings with semistandard Young tableaux of shape λ with entries at most $m + 1$. Therefore, setting $q = 1$ in $U_m(P_\lambda, \omega)$ gives $\#\mathrm{SSYT}_{m+1}(\lambda)$. Proposition 8.4 in [16] says that

$$W_k(P, \omega) = \sum_{m=0}^{k} (-1)^m q^{\binom{n}{2}} \begin{bmatrix} n+1 \\ m \end{bmatrix} U_{k-m}(P, \omega). \tag{24}$$

Then setting $q = 1$, and reversing the order of summation proves Theorem 10. $\qquad \square$

5 The polynomials $P_{n,k}$

For any partition λ with $|\lambda| = n$, we can actually express $\#\mathrm{QYT}_{=k+1}(\lambda)$ using certain symmetric functions $P_{n,k}$ as follows. We begin with Lemma 4 of [10], where we set $q = 1$, $t = n$, $d_i = 1$, $e_i \in \{0, 1\}$, E_i the partial sums of the e_i, and $D_i = i$. We also recall

as before that for $B = B_\lambda \times 1$, we have $H_i - D_i = c_i$, the cell contents of λ in weakly decreasing order. After all of that, we get

$$h_{n-k}(B_\lambda \times 1) = \sum_{e_1 + \cdots + e_n = k} \prod_{i=1}^{n} \binom{c_i + E_i - e_i + 1}{1 - e_i} \binom{i - 1 - c_i - E_i + e_i}{e_i}. \quad (25)$$

Since exactly one of $1 - e_i$ or e_i are 1 and the other is 0, we can rewrite equation (25) as

$$h_{n-k}(B_\lambda \times 1) = \sum_{e_1 + \cdots + e_n = k} \prod_{i=1}^{n} (c_i + E_i + 1)^{1-e_i} (i - c_i - E_i)^{e_i}. \quad (26)$$

This is the same as summing over weighted lattice paths with n steps from $(0, 0)$ to $(k, n - k)$. Let E_i count the cumulative east steps and $N_i = i - E_i$ count the cumulative north steps. Then for each path, weight the ith step by $x_i + E_i + 1$ if it is a north step and $N_i - x_i$ if it is an east step, and let the weight of a path be the product of the weights of its steps.
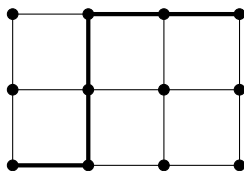

Figure 16: A path with weight $(-x_1)(x_2 + 2)(x_3 + 2)(2 - x_4)(2 - x_5)$.

Let $P_{n,k}(x_1, \ldots, x_n)$ denote the sum of the weights of all such paths. This gives the following weighted lattice path interpretation for QYT enumeration.

Theorem 11. *Given a partition λ of n with contents $c_1, \ldots, c_n$ and $0 \leqslant k \leqslant n$,*

$$\#\mathrm{QYT}_{=k+1}(\lambda) = \frac{P_{n,k}(c_1, \ldots, c_n)}{\prod_{u \in \lambda} h(u)}. \quad (27)$$

The set of such lattice paths can be partitioned into ones that end on an east step and ones that end on a north step, giving the following recursion.

Proposition 12. *The polynomials $P_{n,k}$ satisfy the relation*

$$P_{n,k}(x_1, \ldots, x_n) = (x_n + k + 1)P_{n-1,k}(x_1, \ldots, x_{n-1}) + (n - k - x_n)P_{n-1,k-1}(x_1, \ldots, x_{n-1}). \quad (28)$$

We can use this to get a more concrete idea of what these polynomials look like. By their construction, it is not obvious that these polynomials are symmetric, but computing

small cases seems to indicate they are.

$$
\begin{aligned}
P_{1,0}(x_1) &= e_1(x_1) + 1 \\
P_{1,1}(x_1) &= e_1(x_1) \\
P_{2,0}(x_1, x_2) &= e_2(x_1, x_2) + e_1(x_1, x_2) + 1 \\
P_{2,1}(x_1, x_2) &= -2e_2(x_1, x_2) - e_1(x_1, x_2) + 1 \\
P_{2,2}(x_1, x_2) &= e_2(x_1, x_2) \\
P_{3,0}(x_1, x_2, x_3) &= e_3(x_1, x_2, x_3) + e_2(x_1, x_2, x_3) + e_1(x_1, x_2, x_3) + 1 \\
P_{3,1}(x_1, x_2, x_3) &= -3e_3(x_1, x_2, x_3) - 2e_2(x_1, x_2, x_3) + 4 \\
P_{3,2}(x_1, x_2, x_3) &= 3e_3(x_1, x_2, x_3) + e_2(x_1, x_2, x_3) - e_1(x_1, x_2, x_3) + 1 \\
P_{3,3}(x_1, x_2, x_3) &= e_3(x_1, x_2, x_3)
\end{aligned}
\tag{29}
$$

Let $a(n, k, m)$ denote the coefficient of e_m in $P_{n,k}$, and assume that $P_{i,j}$ is symmetric for $j \leqslant i < n$. Using the recursion, the coefficient of the degree m monomials containing x_n in $P_{n,k}$ is $a(n-1, k, m-1) - a(n-1, k-1, m-1)$ and the coefficient of the degree m monomials not containing x_n is $(k+1)a(n-1, k, m) + (n-k)a(n-1, k-1, m)$. Then to show that $P_{n,k}$ is symmetric, it is sufficient to show that $a(n-1, k, m-1) - a(n-1, k-1, m-1) = (k+1)a(n-1, k, m) + (n-k)a(n-1, k-1, m)$, which can be done with a straightforward induction argument.

Theorem 13. *Given a partition λ of n with contents $c_1, \ldots, c_n$ and $0 \leqslant k \leqslant n$,*

$$
\#\mathrm{QYT}_{=k+1}(\lambda) = \frac{\sum_{m=0}^{n} a(n, k, m) e_m(c_1, \ldots, c_n)}{\prod_{u \in \lambda} h(u)}.
\tag{30}
$$

The recursion and initial conditions imply that $a(n, k, 0)$ is the Eulerian number $A(n, k)$, and using the recursion, we can generate the other coefficients using the relation $a(n, k, m) = a(n-1, k, m-1) - a(n-1, k-1, m-1)$ for $1 < m \leqslant n$. We also note that for a fixed value of $n - m$ with varying n and k, this gives something close to a Pascal's triangle for the coefficients.

$n = 3$				1		4		1		
$n = 4$			1		3		-3		-1		
$n = 5$		1		2		-6		2		1	
$n = 6$	1		1		-8		8		-1		-1

Figure 17: $a(n, k, m)$ for fixed $n - m = 3$ and $0 \leqslant k \leqslant n - 1$ increasing along rows.

Each term contributes its positive absolute value and its negative absolute value to the next line of the triangle, so summing over a line gives 0 except when $m = 0$. Therefore, summing over $P_{n,k}$ for all $0 \leqslant k \leqslant n$ leaves only the constant terms, and the hook-length formula is easily recovered.

6 Generating functions

We define the Schur basis generating function for quasi-Yamanouchi tableaux to be

$$\sum_{|\lambda|=n} \sum_{k=1}^{n} \#\mathrm{QYT}_{=k}(\lambda) t^{k-1} s_\lambda, \tag{31}$$

which has a natural q-analogue

$$\sum_{|\lambda|=n} \sum_{T \in \mathrm{QYT}(\lambda)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)} s_\lambda. \tag{32}$$

In this section, we present the fundamental quasisymmetric and monomial expansions of this q-analogue of the generating function. We note that the fundamental quasisymmetric expansion is an extension of Theorem 3.8 in [1].

Theorem 14. *For $n \in \mathbb{N}$,*

$$\sum_{\pi \in S_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} F_{\mathrm{Des}(\pi^{-1})}(x) = \sum_{|\lambda|=n} \sum_{T \in \mathrm{QYT}(\lambda)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)} s_\lambda. \tag{33}$$

Proof. Connect all $\pi \in S_n$ by colored edges corresponding to dual Knuth relations to get a graph G and identify each permutation π with its image $(P(\pi), Q(\pi))$ through RSK. Dual Knuth relations do not change the descent set of a permutation, and the descent set of a permutation corresponds to the descent set of its recording tableau $Q(\pi)$. Therefore, all permutations in a connected component of G have the same descent and major index statistics and map to the same recording tableau.

On the other hand, RSK respects dual Knuth relations between permutations and their insertion tableaux, so the equivalence classes formed by dual Knuth relations guarantee that the insertion tableaux on a connected component range over exactly all $T \in \mathrm{SYT}(\lambda)$ for some λ. The descent set of an insertion tableau $P(\pi)$ is the same as the descent set of π^{-1}.

We can give each vertex of a connected component the weight $q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} F_{\mathrm{Des}(\pi^{-1})}(x)$ and apply Gessel's fundamental quasisymmetric expansion to show that each connected component has summed weight $q^{\mathrm{maj}(Q)} t^{\mathrm{des}(Q)} s_{sh(Q)}$, where Q is the recording tableau shared by the connected component. RSK forms a bijection between $\pi \in S_n$ and pairs of standard Young tableaux (P, Q) of the same shape, so summing over all connected components of G, applying a counting argument, and using the correspondence between SYT and QYT completes the proof. $\square$

For the monomial symmetric function expansion, we use multiset permutations. We can define descents and major index for multiset permutations in the same way as for permutations in S_n, and we write $S(1^{\lambda_1}, 2^{\lambda_2}, \ldots)$ for the set of multiset permutations of $\{1^{\lambda_1}, 2^{\lambda_2}, \ldots\}$.

Lemma 15. *Given a partition λ of n,*

$$\sum_{\pi \in S(1^{\lambda_1}, 2^{\lambda_2}, \ldots)} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} = \sum_{\nu \geqslant \lambda} K_{\nu\lambda} \sum_{T \in \mathrm{QYT}(\nu)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)}. \tag{34}$$

Proof. RSK gives a bijection between multiset permutations $\pi \in S(1^{\lambda_1}, 2^{\lambda_2}, \ldots)$ and pairs of tableaux (P, Q) of the same shape $\nu \geqslant \lambda$. In particular, P is an SYT with descents in the same positions as π and Q has weight λ. Then since the descent set and major index are preserved, using the correspondence between SYT and QYT proves the claim. $\square$

Theorem 16. *For $n \in \mathbb{N}$,*

$$\sum_{|\lambda|=n} \sum_{\pi \in S(1^{\lambda_1}, 2^{\lambda_2}, \ldots)} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} m_\lambda = \sum_{|\nu|=n} \sum_{T \in \mathrm{QYT}(\nu)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)} s_\lambda. \tag{35}$$

Proof. We proceed by induction on the poset of partitions of n under dominance order. The inductive claim is that the coefficient of s_λ on the right hand side is the desired coefficient, and the inductive assumption is that the claim is true for all $\nu > \lambda$. As a base case, this clearly holds for $\lambda = (n)$ by computation. By the triangularity of the expansion of Schur functions into monomials, the coefficients of m_λ on each side forces

$$\sum_{\pi \in S(1^{\lambda_1}, 2^{\lambda_2}, \ldots)} q^{\mathrm{maj}(\pi)} t^{\mathrm{des}(\pi)} = C_\lambda + \sum_{\nu > \lambda} K_{\nu\lambda} \sum_{T \in \mathrm{QYT}(\nu)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)}, \tag{36}$$

where C_λ is the coefficient of s_λ on the right hand side, and the second term comes from the expansion of each s_ν, $\nu > \lambda$. Applying Lemma 15 immediately shows that $C_\lambda = \sum_{T \in \mathrm{QYT}(\lambda)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)}$. Continuing this induction downwards on the poset eventually proves the claim for all partitions of n. $\square$

7 Applications

We have already mentioned two instances of quasi-Yamanouchi tableaux proving to be a useful concept. The first was in Gessel's fundamental quasisymmetric expansion of Schur polynomials, due to Assaf and Searles [2]. To make the statements in the introduction more precise, when the number of variables $x_1, x_2, \ldots$ is k, the standard Young tableaux that index the nonzero terms of the expansion are exactly those corresponding to quasi-Yamanouchi tableaux with maximum value at most k.

Theorem 17 (Theorem 2.7, [2]). *The Schur polynomial $s_\lambda(x_1, \ldots, x_k)$ is given by*

$$s_\lambda(x_1, \ldots, x_k) = \sum_{T \in \mathrm{QYT}_{\leqslant k}(\lambda)} F_{\mathbf{wt}(T)}(x_1, \ldots, x_k), \tag{37}$$

where all terms on the right hand side are nonzero.

The second instance was in the coefficients of Jack polynomials under a certain binomial coefficient basis [1]. More specifically, if we let $J_\mu^{(\alpha)}(x)$ denote the integral form, type A Jack polynomials and write $\tilde{J}_\mu^{(\alpha)}(x) = \alpha^n J_\mu^{(1/\alpha)}(x)$, then we have the following theorem.

Theorem 18 (Theorem 3.4, [1]). *Let λ be a partition of n and λ' be its conjugate. Then for the coefficient of s_λ in $\tilde{J}_{(n)}^{(\alpha)}(X)$*

$$\langle \tilde{J}_{(n)}^{(\alpha)}(x), s_\lambda \rangle = \sum_{k=0}^{n-1} a_k((n), \lambda) \binom{\alpha + k}{n}, \tag{38}$$

we have $a_k((n), \lambda) = n! \# \mathrm{QYT}_{=k+1}(\lambda')$.

On the representation theoretic side, QYT make another appearance with *Foulkes characters*. First, associate a skew partition with a permutation as follows. For $\pi \in S_n$, let the signature $\sigma(\pi)$ of π be a sequence of length $n-1$ of $+$s and $-$s so that $\sigma(\pi)_i = +$ if $i \notin \mathrm{Des}(\pi)$ and $\sigma(\pi)_i = -$ if $i \in \mathrm{Des}(\pi)$. We can extend this definition to standard Young tableaux as well, using their definition of descent set. To produce a ribbon shape given a signature σ, begin with a cell on a square grid, then at the ith step for $1 \leqslant i \leqslant n-1$, if $\sigma_i = -$, take a south step, and if $\sigma_i = +$, then take a west step, adding the cell at each step to the diagram. This gives a ribbon of length n, which we denote $R(\sigma)$.
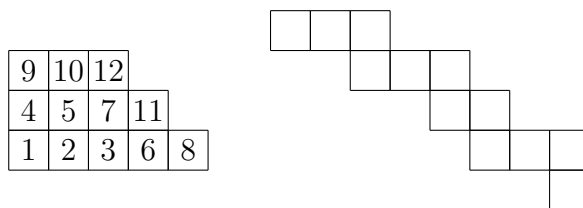

Figure 18: The tableau on the left has signature $\sigma = + + - + + - + - + + -$, which gives the ribbon $R(\sigma)$ on the right.

Kerber and Thürling [13] obtain the decomposition of the skew representation $[R(\sigma)]$ using a "cascade of diagrams" (see [13] for the full algorithm). They define the Foulkes character $\chi^{n,k}$ in to be

$$\chi^{n,k} = \sum_\sigma \chi^{R(\sigma)}, \tag{39}$$

where the sum is over signatures σ with exactly k many $+$s. If the order of nodes added in the cascade is recorded by placing an i at the position of the ith node, their rules for producing the cascade from a particular signature create exactly all standard Young tableaux with the corresponding signature. Since the signature is essentially just the descent set, the correspondence between QYT and SYT means that the Foulkes character can be expressed as

$$\chi^{n,k} = \sum_{|\lambda|=n} \mathrm{QYT}_{=n-k}(\lambda) \chi^\lambda. \tag{40}$$

Define the *Polya character* χ_n by

$$\chi_n(\pi) = m^{\# \text{ of cycles of } \pi}, \tag{41}$$

where $\pi \in S_n$. Kerber and Thürling showed that this has the decomposition

$$\chi_n = \sum_k \binom{m+k}{n} \chi^{n,k}, \tag{42}$$

which from the QYT perspective gives the following result.

Proposition 19. *The Polya character has the decomposition*

$$\chi_n = \sum_k \binom{m+k}{n} \sum_{|\lambda|=n} \mathrm{QYT}_{=n-k}(\lambda)\chi^\lambda. \tag{43}$$

Now we get a nice surprise: replacing χ^λ by s_λ, letting $m = \alpha$, and multiplying by $n!$ gives precisely $\tilde{J}_{(n)}^{(\alpha)}(x)$.

Our final example concerns the *coinvariant algebra* R_n, defined as

$$R_n = \frac{\mathbb{Q}[x_1, \ldots, x_n]}{\langle e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n) \rangle}. \tag{44}$$

The coinvariant algebra has been closely studied in both algebraic and geometric combinatorics. Recently for example, the Delta Conjecture of Haglund, Remmel, and Wilson [11] has received a great deal of attention. Haglund, Rhoades, and Shimozono [12] showed that a specialization of the combinatorial side of the Delta conjecture is the graded Frobenius image of a generalization of the coinvariant algebra.

The *Garsia-Stanton basis* is a common basis of R_n, where for $\pi \in S_n$, we have a monomial $gs_\pi(x_1, \ldots, x_n)$ defined as

$$gs_\pi(x_1, \ldots, x_n) = \prod_{d \in \mathrm{Des}(\pi)} x_{\pi_1} \cdots x_{\pi_d}, \tag{45}$$

and Lusztig and Stanley [15] gave the graded Frobenius image of R_n:

$$\mathrm{grFrob}(R_n; q) = \sum_{|\lambda|=n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} s_\lambda(x). \tag{46}$$

Comparing with equation (32), we can see that $\mathrm{grFrob}(R_n; q)$ is equation (32) at $t = 1$. In this case, the degree of the monomial gs_π captures the major index statistic. We can view the coinvariant algebra in another way that allows us to capture the descent statistic as well. Let $\mathcal{Y} = \{y_S \mid S \subseteq \{1, \ldots, n\}\}$ and

$$\theta_i = \sum_{\substack{S \subseteq \{1, \ldots, n\} \\ |S|=i}} y_S.$$

Garsia and Stanton [6] proved that R_n is isomorphic to

$$\mathcal{R}_n = \frac{\mathbb{Q}[\mathcal{Y}]}{\langle y_S \cdot y_T, \theta_1, \ldots, \theta_n \rangle},$$

where $y_S \cdot y_T$ is the product over S and T that are incomparable under inclusion ordering. They also showed that we get a basis $\{y_\pi \mid \pi \in S_n\}$ defined by

$$y_\pi = \prod_{d \in \mathrm{Des}(\pi)} y_{\{\pi_1, \ldots, \pi_d\}}.$$

Note that we can go between y_π and gs_π via the map that sends $y_{\{i_1, \ldots, i_k\}}$ to $x_{i_1} \cdots x_{i_k}$. If we assign to y_S a q-degree of $|S|$ and a t-degree of 1, then y_π has a q-degree of $\mathrm{maj}(\pi)$ and t-degree of $\mathrm{des}(\pi)$. This produces a bigraded Frobenius image which is precisely equation (32).

Proposition 20. *The bigraded Frobenius image of R_n is*

$$\mathrm{grFrob}(\mathcal{R}_n; q, t) = \sum_{|\lambda|=n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} t^{\mathrm{des}(T)} s_\lambda.$$

Acknowledgements

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