Equidistributed statistics on Fishburn matrices and permutations

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Submitted: Jun 12, 2018; Accepted: Dec 19, 2018; Published: Jan 25, 2019
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Abstract

Recently, Jelínek conjectured that there exists a bijection between certain restricted permutations and Fishburn matrices such that the bijection verifies the equidistribution of several statistics. The main objective of this paper is to establish such a bijection.

Mathematics Subject Classifications: 05A15, 05A17, 06A07

Keywords: ascent sequence, pattern avoiding permutation, Fishburn matrix.

1 Introduction

Given a sequence of integers \( x = x_1x_2\cdots x_n \), we say that the sequence \( x \) has an ascent at position \( i \) if \( x_i < x_{i+1} \). Let \( \text{ASC}(x) \) denote the set of the ascent positions of \( x \) and let \( \text{asc}(x) \) denote the number of ascents of \( x \). A sequence \( x = x_1x_2\cdots x_n \) is said to be an ascent sequence of length \( n \) if it satisfies \( x_1 = 0 \) and \( 0 \leq x_i \leq \text{asc}(x_1x_2\cdots x_{i-1}) + 1 \) for all \( 2 \leq i \leq n \). Let \( \mathcal{A}_n \) be the set of ascent sequences of length \( n \). For example,

\[ \mathcal{A}_3 = \{000, 001, 010, 011, 012\}. \]

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Ascent sequences were introduced by Bousquet-Mélou et al. [1] to unify three other combinatorial structures: (2 + 2)-free posets, a family of permutations avoiding a certain pattern and a class of involutions introduced by Stoimenow [12]. To be specific, Bousquet-Mélou et al. [1] constructed a bijection between ascent sequences and pattern avoiding permutations, a bijection between ascent sequences and (2 + 2)-free posets and a bijection between (2 + 2)-free posets and Stoimenow’s involutions. Dukes and Parviainen [3] completed the results of [1] by constructing a bijection between ascent sequences and Fishburn matrices. Hence, all these combinatorial objects are enumerated by the Fishburn number \( F_n \) (sequence A022493 in OEIS [10]) for memory of Fishburn’s pioneering work on the interval orders [4, 5, 6]. More examples of Fishburn objects are constantly being discovered. Levande [7] introduced the notion of Fishburn diagrams and proved that Fishburn diagrams are counted by Fishburn numbers, confirming a conjecture posed by Claesson and Linusson [2]. Jelínek [8] showed that some Fishburn triples are enumerated by Fishburn numbers.

Zagier [14] and Bousquet-Mélou et al. [1] obtained the generating function of \( F_n \), that is

\[
\sum_{n \geq 0} F_n x^n = \sum_{n \geq 0} \Pi_{k=1}^{n} (1 - (1 - x)^k).
\]


Let us recall the notions of pattern avoiding permutations and Fishburn matrices before we state our main results. Let \( S_n \) be the symmetric group on \( n \) elements and \( \pi = \pi_1 \pi_2 \cdots \pi_n \) be a permutation of \( S_n \). We say that \( \pi \) contains the pattern \( \pattern{13} \) if there is a subsequence \( \pi_i \pi_{i+1} \pi_j \) of \( \pi \) satisfying that \( \pi_i + 1 = \pi_j < \pi_{i+1} \), otherwise we say that \( \pi \) avoids the pattern \( \pattern{13} \). For example, the permutation 42513 contains the pattern \( \pattern{13} \) while the permutation 52314 avoids it.

The pattern \( \pattern{24} \) can be defined similarly. Let \( S_n(\pattern{24}) \) be the set of (\( \pattern{24} \))-avoiding permutations of \( [n] \) and \( S_n(\pattern{13}) \) be the set of (\( \pattern{13} \))-avoiding permutations of \( [n] \), respectively. These two sets are both enumerated by Fishburn numbers [1, 11]. In a permutation \( \pi \), we say \( \pi_i \) is a left-to-right maximum (or LR-maximum) if \( \pi_i \) is larger than any element among \( \pi_1, \pi_2, \ldots, \pi_{i-1} \). Let \( LRMAX(\pi) \) denote the set of LR-maxima of \( \pi \) and let \( LRmax(\pi) \) denote the number of LR-maxima of \( \pi \). Analogously, we can define LR-minima, RL-maxima, RL-minima of a permutation \( \pi \). Denote by \( LRMIN(\pi) \), \( RLMAX(\pi) \) and \( RLMIN(\pi) \) the set of LR-minima, RL-maxima and RL-minima of \( \pi \), their cardinalities being denoted by \( LRmin(\pi) \), \( RLmax(\pi) \) and \( RLMin(\pi) \), respectively.

Fishburn matrices were introduced by Fishburn [6] to represent interval orders. A Fishburn matrix is an upper triangular matrix with nonnegative integers whose every row
and every column contain at least one non-zero entry. The weight of a matrix is the sum of its entries. Similarly, the weight of a row (or a column) of a matrix is the sum of the entries in this row (or column). Denote by $\mathcal{M}_n$ the set of Fishburn matrices of weight $n$. For example,

$$\mathcal{M}_3 = \{(3), (2\ 0\ 0\ 1\ 1\ 0\ 0\ 0\ 1), (1\ 1\ 0\ 1\ 0\ 2), (1\ 0\ 0\ 0\ 1\ 1)\}.$$

Given a matrix $A$, we use the term cell $(i, j)$ of $A$ to refer to the the entry in the $i$-th row and $j$-th column of $A$, and we let $A_{i,j}$ denote its value. We assume that the rows of a matrix are numbered from top to bottom and the columns of a matrix are numbered from left to right in which the topmost row is numbered by 1 and the leftmost column is numbered by 1. A cell $(i, j)$ of a matrix $A$ is said to be zero if $A_{i,j} = 0$. Otherwise, it is said to be nonzero. A row (or column) is said be zero if it contains no nonzero cells. Otherwise, it is said to be nonzero row (or column).

A cell $(i, j)$ of a matrix $A$ is a weakly north-east cell (or wNE-cell) if it is a nonzero cell and any other cell weakly north-east from it is a zero cell. More precisely, a nonzero cell $(i, j)$ of a matrix $A$ is a wNE-cell if $A_{s,t} = 0$ holds for all $s \leq i$ and $t \geq j$ and $(s,t) \neq (i,j)$.

Jelínek [8] posed the following conjecture.

**Conjecture 1.** (See [8], Conjecture 4.1) For every $n$, there is a bijection $\alpha$ between $S_n(\mathbb{F}_2)$ and $\mathcal{M}_n$ satisfying that:

- LRmax($\pi$) is the weight of the first row of $\alpha(\pi)$,
- RLmin($\pi$) is the weight of the last column of $\alpha(\pi)$,
- RLmax($\pi$) is the number of wNE-cells of $\alpha(\pi)$,
- LRmin($\pi$) is the number of nonzero cells of $\alpha(\pi)$ belonging to the main diagonal, and
- $\alpha(\pi^{-1})$ is obtained from $\alpha(\pi)$ by transposing along the North-East diagonal.

By using generating functions, Jelínek [8] proved the following symmetric joint distribution on $\mathcal{M}_n$.

**Theorem 2.** (See [8], Theorem 3.7) For any $n$, the number of wNE-cells and the weight of the first row have symmetric joint distribution on $\mathcal{M}_n$.

Jelínek [8] also posed the following weaker conjecture which follows directly from Theorem 2 and Conjecture 1.

**Conjecture 3.** (See [8], Conjecture 4.2) For any $n$, LRmax and RLmax have symmetric joint distribution on $S_n(\mathbb{F}_2)$.

The main objective of this paper is to establish a bijection between $S_n(\mathbb{F}_2)$ and $\mathcal{M}_n$ which satisfies the former four items of Conjecture 1, thereby confirming Conjecture 3.
2 Bijection between permutations and ascent sequences

In this section, we shall construct a bijection \( \theta \) between \( S_n(\mathcal{E}) \) and \( A_n \), and show that the map \( \theta \) proves the equidistribution of two 4-tuples of statistics.

Let \( \pi \) be a permutation in \( S_n(\mathcal{E}) \) and let \( \tau \) be the permutation obtained by deleting \( n \) from \( \pi \). Then we have that \( \tau \) is also a permutation in \( S_n(\mathcal{E}) \). If not, we assume that \( \pi_i \pi_{i+1} \pi_j \) is a \( \mathcal{E} \) pattern in \( \tau \). Since \( \pi \) is \( \mathcal{E} \)-avoiding, we have \( \pi_{i+1} = n \). Then \( \pi_i \pi_{i+1} \pi_{j+1} \) forms a \( \mathcal{E} \) pattern in \( \pi \), a contradiction. This property allows us to construct the permutation of \( S_n(\mathcal{E}) \) inductively, starting from the empty permutation and adding a new maximal value at each step.

Let \( \tau \) be a permutation in \( S_{n-1}(\mathcal{E}) \). The positions where we can insert the element \( n \) into \( \tau \) to obtain a \( \mathcal{E} \)-avoiding permutation are called active sites. The site after the maximal entry \( n \) in \( \pi \) is always an active site. We label the active sites in \( \pi \) from right to left with 0, 1, 2 and so on.

The bijection \( \theta \) between \( S_n(\mathcal{E}) \) and \( A_n \) can be defined recursively. Set \( \theta(1) = 0 \). Suppose that \( \pi \) is a permutation in \( S_n(\mathcal{E}) \) which is obtained from \( \tau \) by inserting the element \( n \) into the \( x_n \)-th active site of \( \tau \). Then we set \( \theta(\pi) = x_1 x_2 \cdots x_{n-1} x_n \), where \( x_1 x_2 \cdots x_{n-1} = \theta(\tau) \).

Example 4. The permutation 85231647 corresponds to the sequence 01102103 since it is obtained by the following insertions, where the subscripts indicate the labels of the active sites.

\[
\begin{align*}
11_0 & \xrightarrow{x_2=1} 211_0 \\
 & \xrightarrow{x_3=1} 231_0 \\
 & \xrightarrow{x_4=0} 231_40 \\
 & \xrightarrow{x_5=2} 35231_40 \\
 & \xrightarrow{x_6=1} 35231_640 \\
 & \xrightarrow{x_7=0} 35231_6470 \\
 & \xrightarrow{x_8=3} 4835231_6470.
\end{align*}
\]

Lemma 5. Let \( \pi = \pi_1 \pi_2 \cdots \pi_n \) be a permutation in \( S_n(\mathcal{E}) \) and \( \theta(\pi) = x = x_1 x_2 \cdots x_n \). Then we have that

\[
s(\pi) = 2 + \text{asc}(x) \quad \text{and} \quad a(\pi) = x_n,
\]

where \( s(\pi) \) denotes the number of active sites of \( \pi \) and \( a(\pi) \) denotes the label of the site located just after the entry \( n \) of \( \pi \).

Proof. Suppose that \( \pi \) is obtained from \( \tau \) by inserting the element \( n \) into the \( x_n \)-th active site of \( \tau \). Then we have \( \theta(\tau) = x' \), where \( x' = x_1 x_2 \cdots x_{n-1} \). For any entry \( i \) which is to the right of \( n \), \( i \) is followed by an active site in \( \pi \) if and only if \( i \) is followed by an active site in \( \tau \). Since the site after \( n \) in \( \pi \) is always active, we obtain \( a(\pi) = x_n \).
Now let us focus on the equation \( s(\pi) = 2 + \text{asc}(x) \). We will prove it by induction on \( n \). It obviously holds for \( n = 1 \). Assume that it holds for \( n - 1 \). For any entry \( i < n - 1 \), \( i \) is followed by an active site in \( \pi \) if and only if \( i \) is followed by an active site in \( \tau \). The site after \( n \) in \( \pi \) is always an active site. Thus, to determine \( s(\pi) \), the only question is whether the site after \( n - 1 \) is active. We need consider two cases.

Case 1: If \( 0 \leq x_n \leq a(\tau) = x_{n-1} \), then the entry \( n \) in \( \pi \) is to the right of \( n - 1 \). It follows that the site after \( n - 1 \) is not an active site in \( \pi \). Since the site after \( n - 1 \) is an active site in \( \tau \), we have that \( s(\pi) = s(\tau) \). By the induction hypothesis, \( s(\tau) = 2 + \text{asc}(x') = 2 + \text{asc}(x) \). Hence we deduce that \( s(\pi) = 2 + \text{asc}(x) \).

Case 2: If \( x_n > a(\tau) = x_{n-1} \), then the entry \( n \) in \( \pi \) is to the left of \( n - 1 \). It yields that the site after \( n - 1 \) is also an active site in \( \pi \). Hence \( s(\pi) = s(\tau) + 1 \). Since \( x_n > x_{n-1} \), we have that \( \text{asc}(x) = \text{asc}(x') + 1 \). By the induction hypothesis, \( s(\tau) = 2 + \text{asc}(x') \). Thus we have \( s(\pi) = 2 + \text{asc}(x) \). This completes the proof.

**Theorem 6.** The map \( \theta \) is a bijection between \( S_n(\mathbb{Z}^2) \) and \( A_n \).

**Proof.** We prove this conclusion by induction on \( n \). It obviously holds for \( n = 1 \). Assume that \( \theta \) is a bijection between \( S_{n-1}(\mathbb{Z}^2) \) and \( A_{n-1} \).

We first show that \( \theta \) is a map from \( S_n(\mathbb{Z}^2) \) to \( A_n \). Let \( \pi = \pi_1 \pi_2 \cdots \pi_n \) be a permutation in \( S_n(\mathbb{Z}^2) \) which is obtained from \( \tau \) by inserting a maximal entry \( n \) in the active site labeled by \( x_n \) in \( \tau \). Then \( \theta(\pi) = x = x_1 x_2 \cdots x_n \), where \( \theta(\tau) = x' = x_1 x_2 \cdots x_{n-1} \). To prove that \( x \in A_n \), it suffices to show that \( x_n \leq \text{asc}(x') + 1 \). Recall that the rightmost active site is labeled 0. Hence the leftmost active site in \( \tau \) is labeled \( s(\tau) - 1 \). By the recursive description of the map \( \theta \), we have that \( x_n \leq s(\tau) - 1 \). From Lemma 5 we see that \( s(\tau) = 2 + \text{asc}(x') \). Thus we have \( x_n \leq \text{asc}(x') + 1 \). Since \( x \) encodes the construction of \( \pi \), \( \theta \) is an injective map from \( S_n(\mathbb{Z}^2) \) to \( A_n \).

It remains to show that \( \theta \) is surjection. Let \( y = y_1 y_2 \cdots y_n \) be an ascent and \( p = p_1 p_2 \cdots p_{n-1} = \theta^{-1}(y') \), where \( y' = y_1 y_2 \cdots y_{n-1} \). From the definition of ascent sequence and Lemma 5, we have that \( y_n \leq \text{asc}(y') + 1 = s(p) - 1 \). Let \( q \) be the permutation obtained from \( p \) by inserting the maximal entry \( n \) into the active site labeled \( y_n \) in \( p \). By the construction of the map \( \theta \), it can be easily seen that \( \theta(q) = y \). This concludes the proof.

Let \( x = x_1 x_2 \cdots x_n \) be an ascent sequence in \( A_n \). The modified ascent sequence of \( x \), denoted by \( \hat{x} \), is defined by the following procedure:

for \( i \in \text{ASC}(x) \)
for \( j = 1, 2, \ldots, i - 1 \)
if \( x_j \geq x_{i+1} \) then \( x_j := x_j + 1 \).

For example, for \( x = 01012213 \), we have \( \text{ASC}(x) = \{1, 3, 4, 7\} \) and \( \hat{x} = 04012213 \). Modified ascent sequences were introduced by Bousquet-Mérou et al., see more details in [1].

For a permutation \( \pi = \pi_1 \pi_2 \cdots \pi_n \in S_n(\mathbb{Z}^2) \), let \( l(\pi_i) \) be the largest label of the active site to the right of \( \pi_i \) and let \( LMAX(\pi) \) be the multiset of \( l(\pi_i) \) when \( \pi_i \) ranges over all LR-maxima of \( \pi \). That is

\[
LMAX(\pi) = \{l(\pi_i) \mid \pi_i \in LMAX(\pi)\}.
\]
Similarly, let 

\[ \text{RMAXL}(\pi) = \{ l(\pi_i) \mid \pi_i \in \text{RLMAX}(\pi) \}. \]

For example, for \( \pi = 42178536 \), its active sites are labeled as 42178536. Then we have \( \text{RMAXL}(\pi) = \{0, 2\} \) and \( \text{LMAXL}(\pi) = \{2, 2, 3\} \).

For an ascent sequence \( x = x_1x_2 \cdots x_n \), let 

\[ \text{ZERO}(x) = \{ i \mid x_i = 0 \}, \]

and 

\[ \text{MAX}(x) = \{ i \mid x_i = \text{asc}(x_1x_2 \cdots x_{i-1}) + 1 \}, \]

with their cardinalities being denoted by \( \text{zero}(x) \) and \( \text{max}(x) \) respectively.

For a sequence \( x = x_1x_2 \cdots x_n \), let 

\[ \text{RMIN}(x) = \{ x_i \mid x_i < x_j \text{ for all } j > i \}, \]

\[ \text{RMAX}(x) = \{ x_i \mid x_i \geq x_j \text{ for all } j > i \}. \]

It should be noted that the set \( \text{RMAX}(x) \) is a multiset. Denote by \( \text{Rmin}(x) \) and \( \text{Rmax}(x) \) the cardinalities of the sets \( \text{RMIN}(x) \) and \( \text{RMAX}(x) \), respectively. For example, let \( x = 01012201 \). We have \( \text{RMIN} = \{0, 1\} \), \( \text{RMAX} = \{1, 2, 2\} \), \( \text{Rmin}(x) = 2 \) and \( \text{Rmax}(x) = 3 \).

**Theorem 7.** For any \( \pi = \pi_1\pi_2 \cdots \pi_n \in S_n(\mathbb{P}_3) \) and \( x = x_1x_2 \cdots x_n \in \mathcal{A}_n \) with \( \theta(\pi) = x \), we have

1. \( \text{RLMIN}(\pi) = \text{ZERO}(x) \);
2. \( \text{LRMIN}(\pi) = \text{MAX}(x) \);
3. \( \text{RMAXL}(\pi) = \text{RMIN}(x) \);
4. \( \text{LMAXL}(\pi) = \text{RMAX}(\hat{x}) \).

**Proof.** We will prove points (1)-(4) by induction on \( n \). It is easily checked that the statement holds for \( n = 1 \). Assume that it also holds for some \( n - 1 \) with \( n \geq 2 \). Let \( \tau \) be the permutation which is obtained from \( \pi \) by deleting the largest entry \( n \) in \( \pi \). Then we have that \( x' = x_1x_2 \cdots x_{n-1} = \theta(\tau) \). From the construction of the bijection \( \theta \) and the induction hypothesis, one can easily verify that

\[
\text{RLMIN}(\pi) = \begin{cases} 
\text{RLMIN}(\tau) \cup \{n\} = \text{ZERO}(x') \cup \{n\} = \text{ZERO}(x) & \text{if } x_n = 0, \\
\text{RLMIN}(\tau) = \text{ZERO}(x') = \text{ZERO}(x) & \text{otherwise}, 
\end{cases}
\]

\[
\text{LRMIN}(\pi) = \begin{cases} 
\text{LRMIN}(\tau) = \text{MAX}(x') = \text{MAX}(x) & \text{if } x_n \leq \text{asc}(x'), \\
\text{LRMIN}(\tau) \cup \{n\} = \text{MAX}(x') \cup \{n\} = \text{MAX}(x) & \text{if } x_n = \text{asc}(x') + 1,
\end{cases}
\]
and

\[
RMAXL(\pi) = \{i \mid i \in RMAXL(\tau), i < x_n\} \cup \{x_n\} \\
= \{i \mid i \in RMIN(x'), i < x_n\} \cup \{x_n\} \\
= RMIN(x).
\]

For point (4), we consider two cases. If \(x_n \leq x_{n-1}\), then \(n\) is to the right of \(n-1\) in \(\pi\). Notice that all the LR-maxima in \(\tau\) are also LR-maxima in \(\pi\). One can easily check that \(LMAXL(\pi) = LMAXL(\tau) \cup \{x_n\}\) and \(RMAX(\hat{x}) = RMAX(\hat{x}') \cup \{x_n\}\). By the induction hypothesis, we have

\[
LMAXL(\pi) = LMAXL(\tau) \cup \{x_n\} = RMIN(x') \cup \{x_n\} = RMAX(\hat{x}).
\]

If \(x_n > x_{n-1}\), then \(n\) is to the left of \(n-1\) in \(\pi\). In this case, \(\tau_t\) is a LR-maximum in \(\pi\) if and only if \(\tau_t\) is a LR-maximum in \(\tau\) and \(l(\tau_t) \geq x_n\). After the inserting \(n\) into \(\tau\), \(l(\tau_t)\) is increased by 1 if \(\tau_t\) is also a LR-maximum in \(\pi\). Hence we have that

\[
LMAXL(\pi) = \{i + 1 \mid i \in LMAXL(\tau), i \geq x_n\} \cup \{x_n\}.
\]

From the definition of the modified ascent sequence, it follows that

\[
RMAX(\hat{x}) = \{i + 1 \mid i \in RMAX(\hat{x}'), i \geq x_n\} \cup \{x_n\}.
\]

By the induction hypothesis, we immediately deduce that \(LMAXL(\pi) = RMAX(\hat{x})\) as desired. This completes the proof.

Combining Theorems 6 and 7, we are led to the following result.

**Theorem 8.** The map \(\theta\) is a bijection between \(S_n(\frac{[n]}{[n]}\) and \(A_n\). Moreover, for any \(\pi \in S_n(\frac{[n]}{[n]}\) and \(x \in A_n\) with \(\theta(\pi) = x\), we have

\[
(RLmin, LRmin, RLmax)\pi = (zero, max, Rmin)x
\]

and \(LRmax(\pi) = Rmax(\hat{x})\).

### 3 Bijection between ascent sequences and Fishburn matrices

The main objective of this section is to establish a bijection \(\phi\) between \(A_n\) and \(M_n\). To this end, we will define a removal operation and an addition operation on the matrices of \(M_n\).

Given a matrix \(A\) in \(M_n\), let \(dim(A)\) denote the number of rows of the matrix \(A\) and let \(index(A)\) denote the smallest value of \(i\) such that \(A_{i,dim(A)} > 0\). Denote by \(rsum_i(A)\) and \(csum_i(A)\) the sum of the entries in row \(i\) and column \(i\) of \(A\), respectively. We define a removal operation \(f\) on a given matrix \(A \in M_n\) as follows.

**Rem1** If \(rsum_{\text{index}(A)}(A) > 1\), let \(f(A)\) be the matrix \(A\) with the entry \(A_{\text{index}(A),dim(A)}\) reduced by 1.
(Rem2) If \( rsum_{\text{index}(A)}(A) = 1 \) and \( \text{index}(A) = \text{dim}(A) \), then let \( f(A) \) be the matrix \( A \) with row \( \text{dim}(A) \) and column \( \text{dim}(A) \) removed.

(Rem3) If \( rsum_{\text{index}(A)}(A) = 1 \) and \( \text{index}(A) < \text{dim}(A) \), then we construct \( f(A) \) in the following way. Let \( S \) be the set of indices \( j \) such that \( j \geq \text{index}(A) \) and column \( j \) contains at least one nonzero entry above row \( \text{index}(A) \). Suppose that \( S = \{c_1, c_2, \ldots, c_\ell\} \) with \( c_1 < c_2 \ldots < c_\ell \). Clearly we have \( c_1 = \text{index}(A) \). Let \( c_{\ell+1} = \text{dim}(A) \). For all \( 1 \leq i < \text{index}(A) \) and \( 1 \leq j \leq \ell \), move all the entries in the cell \((i, c_j)\) to the cell \((i, c_{j+1})\). Simultaneously delete row \( \text{index}(A) \) and column \( \text{index}(A) \).

Example 9. Let \( A, B, C \) be the following three Fishburn matrices:

\[
A = \begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 2 & 1 \\
0 & 0 & 0 & 2
\end{pmatrix}; \quad B = \begin{pmatrix}
1 & 0 & 2 & 0 \\
0 & 3 & 0 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}; \quad C = \begin{pmatrix}
2 & 4 & 1 & 3 & 0 \\
0 & 5 & 2 & 2 & 0 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 2
\end{pmatrix}.
\]

For Matrix \( A \), rule (Rem1) is applied since \( rsum_{\text{index}(A)}(A) = 3 \) and

\[
f(A) = \begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 2 & 1 & 0 \\
0 & 0 & 2 & 0 \\
0 & 0 & 0 & 2
\end{pmatrix}.
\]

For Matrix \( B \), since \( rsum_{\text{index}(B)}(B) = 1 \) and \( \text{index}(B) = \text{dim}(B) \), rule (Rem2) is applied and

\[
f(B) = \begin{pmatrix}
1 & 0 & 2 \\
0 & 3 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

For matrix \( C \), since \( rsum_{\text{index}(C)}(C) = 1 \) and \( \text{index}(C) < \text{dim}(C) \), rule (Rem3) is applied. It is easy to check that \( S = \{3, 4\} \), and thus we have

\[
f(C) = \begin{pmatrix}
2 & 4 & 1 & 3 \\
0 & 5 & 2 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 2
\end{pmatrix}.
\]

The following lemma shows that the removal operation on a Fishburn matrix of \( \mathcal{M}_n \) will yield a Fishburn matrix in \( \mathcal{M}_{n-1} \).

Lemma 10. Let \( n \geq 2 \) be an integer and \( A \in \mathcal{M}_n \), then we have that \( f(A) \in \mathcal{M}_{n-1} \).

Proof. It is easily seen that for any removal operation applied on the matrix \( A \), the weight of \( f(A) \) is one less than the weight of \( A \). It is trivial to check that there exists no zero columns or rows in \( f(A) \). Moreover, the removal operation also preserves the property of being upper-triangular. Thus, \( f(A) \in \mathcal{M}_{n-1} \). This completes the proof. \( \square \)
Lemma 10 tells us that for any \( A \in \mathcal{M}_n \), after \( n \) applications of the removal operation \( f \) to \( A \), we will get a sequence of Fishburn matrices, say \( A^{(1)}, A^{(2)}, \ldots, A^{(n)} \), where \( A^{(k)} = f(A^{(k)}) \) for all \( 1 < k \leq n \) and \( A^{(n)} = A \). Define \( \psi(A) = x_1x_2\ldots x_n \) where \( x_k = \text{index}(A^{(k)}) - 1 \).

We now define an addition operation \( g \) on a Fishburn matrix which is shown to be the inverse of the removal operation later. Given a matrix \( A \in \mathcal{M}_n \) and \( i \in [0, \text{dim}(A)] \), we construct a matrix \( g(A, i) \) in the following manner.

1. **Add1** If \( 0 \leq i \leq \text{index}(A) - 1 \), then let \( g(A, i) \) be the matrix obtained from \( A \) by increasing the entry in the cell \((i + 1, \text{dim}(A))\) by 1.

2. **Add2** If \( i = \text{dim}(A) \), then let \( g(A, i) \) be the matrix \(
\begin{pmatrix}
A & 0 \\
0 & 1
\end{pmatrix}
\).

3. **Add3** If \( \text{index}(A) \leq i < \text{dim}(A) \), then we construct \( g(A, i) \) in the following way. In \( A \), insert a new (empty) row between rows \( i \) and \( i + 1 \), and insert a new (empty) column between columns \( i \) and \( i + 1 \). Let the new row be filled with all zeros except for the rightmost cell which is filled with a 1. Denote by \( A' \) the resulting matrix. Let \( T \) be the set of indices \( j \) such that \( j \geq i + 1 \) and column \( j \) contains at least one nonzero cell above row \( i + 1 \). Suppose that \( T = \{c_1, c_2, \ldots, c_{\ell}\} \). Clearly we have \( c_\ell = \text{dim}(A') \). Let \( c_0 = i + 1 \). For all \( 1 \leq a \leq i \) and \( 1 \leq b \leq \ell \), move all the entries in the cell \((a, c_b)\) to the cell \((a, c_{b-1})\), and fill all the cells which are in column \( \text{dim}(A') \) and above row \( i + 1 \) with zeros.

**Example 11.** Consider the matrix

\[
A = \begin{pmatrix}
2 & 4 & 0 & 3 \\
0 & 5 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 2
\end{pmatrix}.
\]

Obviously, we have \( \text{dim}(A) = 4 \) and \( \text{index}(A) = 1 \). For \( i = 0 \), since \( i \leq \text{index}(A) - 1 \), rule **Add1** applies and we get

\[
g(A, 0) = \begin{pmatrix}
2 & 4 & 0 & 4 \\
0 & 5 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 2
\end{pmatrix}.
\]

For \( i = 4 \), since \( i = \text{dim}(A) \), rule **Add2** applies and we get

\[
g(A, 4) = \begin{pmatrix}
2 & 4 & 0 & 3 & 0 \\
0 & 5 & 0 & 2 & 0 \\
0 & 0 & 1 & 3 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
For $i = 1$, since $\text{index}(A) \leq i < \text{dim}(A)$, rule (Add3) applies and we get

$$A' = \begin{pmatrix}
2 & 0 & 4 & 0 & 3 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 5 & 0 & 2 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix},$$

where the new inserted row and column are illustrated in bold. Then we have $T = \{3, 5\}$. Finally, we get

$$g(A, 1) = \begin{pmatrix}
2 & 4 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 5 & 0 & 2 \\
0 & 0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.$$

By similar arguments as in the proof of Lemma 10, one can easily verify that the addition operation will also yield a Fishburn matrix.

**Lemma 12.** For any matrix $A \in \mathcal{M}_{n-1}$ and $i \in [0, \text{dim}(A)]$, we have that $g(A, i) \in \mathcal{M}_n$.

We now define a map $\phi$ from $\mathcal{A}_n$ to $\mathcal{M}_n$ recursively as follows. Given an ascent sequence $x = x_1x_2\ldots,x_n$, we define $A^{(1)} = (1)$ and $A^{(k)} = g(A^{(k-1)}, x_k)$ for all $1 < k \leq n$. Set $\phi(x) = A^{(n)}$.

Next we aim to show that the map $\phi$ is well defined and has the following desired properties.

**Lemma 13.** For any $x = x_1x_2\ldots x_n \in \mathcal{A}_n$, we have $\phi(x) \in \mathcal{M}_n$ satisfying $\text{dim}(\phi(x)) = \text{asc}(x) + 1$ and $\text{index}(\phi(x)) = x_n + 1$.

**Proof.** We will prove by induction on $n$. It is trivial to check that the statement holds for $n = 1$. Assume that it also holds for $n - 1$, that is,

$$\phi(x') \in \mathcal{M}_{n-1}, \text{ dim}(\phi(x')) = \text{asc}(x') + 1 \text{ and index}(\phi(x')) = x_{n-1} + 1,$$

where $x' = x_1x_2\ldots x_{n-1}$. Since $0 \leq x_n \leq \text{asc}(x') + 1 = \text{dim}(\phi(x'))$, from Lemma 12 we see that $\phi(x) = g(\phi(x'), x_n) \in \mathcal{M}_n$. From the construction of the addition operation, one can easily verify that $\text{index}(\phi(x)) = x_n + 1$ and

$$\begin{align*}
\text{dim}(\phi(x)) &= \begin{cases} 
\text{dim}(\phi(x')) = \text{asc}(x') + 1 = \text{asc}(x) + 1 & \text{if } x_n \leq x_{n-1}, \\
\text{dim}(\phi(x')) + 1 = \text{asc}(x') + 2 = \text{asc}(x) + 1 & \text{if } x_n > x_{n-1}.
\end{cases}
\end{align*}$$

The result follows. \qed

For a matrix $A$, let $\text{NE}(A) = \{i - 1 | \text{row } i \text{ contains a wNE-cell} \}$ and let $n_e(A)$ denote the number of wNE-cells of $A$. Denote by $\text{diag}(A)$ the number of nonzero cells belonging to the main diagonal of $A$. Let $\text{LAST}(A)$ be the multiset of integers such that there are exactly $c$ occurrences of $i$ if and only if $A_{i+1, \text{dim}(A)} = c$ and $c > 0$. 

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Theorem 14. For any \(x = x_1 x_2 \cdots x_n \in A_n\) and \(A \in M_n\) with \(A = \phi(x)\), we have the following relations.

1. \(\text{zero}(x) = \text{rsum}_1(A)\);
2. \(\text{max}(x) = \text{diag}(A)\);
3. \(\text{RMIN}(x) = \text{NE}(A)\);
4. \(\text{RMAX}(\hat{x}) = \text{LAST}(A)\);
5. \(\text{Rmin}(x) = \text{ne}(A)\);
6. \(\text{Rmax}(\hat{x}) = \text{csum}_{\text{dim}(A)}(A)\).

Proof. Points (5) and (6) follow directly from points (3) and (4). Now we verify points (1)-(4) by induction on \(n\). Clearly, the statement holds for \(n = 1\). Assume that it also holds for any some \(n - 1\) with \(n \geq 2\). Let \(x' = x_1 x_2 \cdots x_{n-1}\) and \(B = \phi(x')\). Recall that \(A = g(B, x_n)\). From the definition of the addition operation \(g\) and the induction hypothesis, it is not difficult to verify that

\[
\text{rsum}_1(A) = \begin{cases} 
\text{rsum}_1(B) + 1 = \text{zero}(x') + 1 = \text{zero}(x) & \text{if } x_n = 0, \\
\text{rsum}_1(B) = \text{zero}(x') = \text{zero}(x) & \text{otherwise}.
\end{cases}
\]

and

\[
\text{diag}(A) = \begin{cases} 
\text{diag}(B) = \text{max}(x') = \text{max}(x) & \text{if } x_n \leq \text{asc}(x'), \\
\text{diag}(B) + 1 = \text{max}(x') + 1 = \text{max}(x) & \text{if } x_n = \text{asc}(x') + 1.
\end{cases}
\]

For point (3), from the construction of the addition operation \(g\), we see that the cell \((x_n + 1, \text{dim}(A))\) is always a wNE-cell. Moreover, there is a wNE-cell in row \(i\) of \(A\) if and only if there is a wNE-cell in row \(i\) of \(B\) and \(i < x_n + 1\). This yields that

\[
\text{NE}(A) = \{i \mid i \in \text{NE}(B), i < x_n\} \cup \{x_n\} = \{i \mid i \in \text{RMIN}(x'), i < x_n\} \cup \{x_n\} = \text{RMIN}(x).
\]

For point (4), we have two cases.

If \(x_n \leq x_{n-1} = \text{index}(B) - 1\), then rule (Add1) applies. It is trivial to check that \(\text{RMAX}(\hat{x}) = \text{RMAX}(\hat{x'}) \cup \{x_n\}\) and \(\text{LAST}(A) = \text{LAST}(B) \cup \{x_n\}\).

If \(x_n > x_{n-1} = \text{index}(B) - 1\), then either rule (Add2) or rule (Add3) applies. It is not difficult to verify that

\[
\text{RMAX}(\hat{x}) = \{i + 1 \mid i \in \text{RMAX}(\hat{x'}), i \geq x_n\} \cup \{x_n\},
\]

and

\[
\text{LAST}(A) = \{i + 1 \mid i \in \text{LAST}(B), i \geq x_n\} \cup \{x_n\}.
\]

By the induction hypothesis, we have concluded that \(\text{RMAX}(\hat{x}) = \text{LAST}(A)\). This completes the proof. \(\square\)
Lemma 15. For any \( x = x_1x_2 \cdots x_n \in A_n \), we have \( \psi(\phi(x)) = x \).

Proof. Suppose that we get a sequence of matrices \( A^{(1)}, A^{(2)}, \ldots, A^{(n)} \) when we apply the map \( \phi \) to \( x \), where \( A^{(1)} = (1) \) and \( A^{(k)} = g(A^{(k-1)}, x_k) \) for all \( 1 < k \leq n \). Similarly, suppose that when we apply the map \( \psi \) to \( \phi(x) \), we get a sequence \( y = y_1y_2 \cdots y_n \) and a sequence of matrices \( B^{(1)}, B^{(2)}, \ldots, B^{(n)} \), where \( B^{(n)} = \phi(x) \), \( B^{(k)} = f(B^{(k+1)}) \) for all \( 1 \leq k < n \), and \( y_k = index(B^{(k)}) - 1 \). Lemma 13 ensures that \( \text{index}(A^{(k)}) = x_k + 1 \). In order to prove \( x = y \), it suffices to show that \( A^{(k)} = B^{(k)} \) for all \( 1 \leq k \leq n \). We proceed to prove this assertion by induction on \( n \). Clearly, we have \( B^{(n)} = \phi(x) = A^{(n)} \). Assume that we have \( A^{(j)} = B^{(j)} \) for all \( j \geq k + 1 \). In the following we aim to show that \( A^{(k)} = B^{(k)} \).

By the induction hypothesis, it suffices to show that \( f(A^{(k+1)}) = A^{(k)} \). We have three cases.

Let us assume that \( 0 \leq x_{i+1} < \text{index}(A^{(k)}) \). Then rule (Add1) applies and \( A^{(k+1)} \) is simply a copy of \( A^{(k)} \) with the entry in the cell \( (x_{i+1} + 1, \dim(A^{(k)})) \) increased by one. Clearly, we have \( \dim(A^{(k+1)}) = \dim(A^{(k+1)}) \), \( \text{index}(A^{(k+1)}) = x_{i+1} + 1 \) and \( \text{rsum}_{x_{i+1}+1}(A^{(k+1)}) > 1 \). So rule (Rem1) applies and \( f(A^{(k+1)}) \) is obtained from \( A^{(k+1)} \) by decreasing the entry in the cell \( (x_{i+1} + 1, \dim(A^{(k+1)})) \) by one. Thus we have \( f(A^{(k+1)}) = A^{(k)} \).

Next assume that \( x_{i+1} = \dim(A^{(k)}) \). Then rule (Add2) applies and \( A^{(k+1)} = \begin{pmatrix} A^{(k)} & 0 \\ 0 & 1 \end{pmatrix} \).

In this case, we have \( \text{index}(A^{(k+1)}) = x_{i+1} + 1 = \dim(A^{(k+1)}) \) and \( \text{rsum}_{x_{i+1}+1}(A^{(k+1)}) = 1 \). So rule (Rem2) applies and \( f(A^{(k+1)}) \) is obtained from \( A^{(k+1)} \) by removing column \( \dim(A^{(k+1)}) \) and row \( \dim(A^{(k+1)}) \). Thus we have \( f(A^{(k+1)}) = A^{(k)} \).

If \( \text{index}(A^{(k)}) \leq x_{i+1} < \dim(A^{(k)}) \), then rule (Add3) applies and \( A^{(k+1)} \) is obtained from \( A^{(k)} \) in the following way. First we insert a new (empty) row between rows \( x_{i+1} \) and \( x_{i+1} + 1 \), and insert a new (empty) column between columns \( x_{i+1} \) and \( x_{i+1} + 1 \). Let the new row be filled with all zeros except for the rightmost cell which is filled with a 1. Denote by \( A' \) the resulting matrix. Let T be the set of indices \( j \) such that \( j \geq x_{i+1} + 1 \) and column \( j \) contains at least one nonzero cell above row \( x_{i+1} + 1 \). Suppose that \( T = \{c_1, c_2, \ldots, c_\ell\} \) with \( c_1 < c_2 < \ldots < c_\ell \). Let \( c_0 = x_{i+1} + 1 \). For all \( 1 \leq a \leq x_{i+1} + 1 \) and \( 1 \leq b \leq \ell \), move all the entries in the cell \( (a, c_b) \) to the cell \( (a, c_{b-1}) \), and fill all the cells in column \( \dim(A') \) and above row \( x_{i+1} + 1 \) with zeros. It is easy to check that \( \dim(A^{(k+1)}) = \dim(A^{(k)}) + 1 \), \( \text{index}(A^{(k+1)}) = x_{i+1} + 1 \) and \( \text{rsum}_{x_{i+1}+1}(A^{(k+1)}) = 1 \). So rule (Rem3) applies and \( f(A^{(k+1)}) \) is obtained from \( A^{(k+1)} \) by the following procedure. Let \( S \) be the set of indices \( j \) such that \( j \geq x_{i+1} + 1 \) and column \( j \) contains at least one nonzero entry above row \( x_{i+1} + 1 \). It is not difficult to check that \( S = \{c_0, c_1, c_2, \ldots, c_{\ell-1}\} \). Let \( c_\ell = \dim(A^{(k+1)}) \). For all \( 1 \leq a < x_{i+1} + 1 \) and \( 1 \leq b \leq \ell - 1 \), move all the entries in the cell \( (a, c_b) \) to the cell \( (a, c_{b+1}) \). Simultaneously delete row \( x_{i+1} + 1 \) and column \( x_{i+1} + 1 \). These operations simply reverse the construction of \( A^{(k+1)} \) from \( A^{(k)} \), and therefore \( f(A^{(k+1)}) = A^{(k)} \). This completes the proof.

\[ \square \]

Theorem 16. The map \( \phi \) is a bijection between \( A_n \) and \( M_n \). Moreover, for any \( x \in A_n \) and \( A \in M_n \) with \( \phi(x) = A \), we have
\[
(zero, max, Rmin)x = (\text{rsum}_1, \text{diag}, ne)A
\]
and $R_{\text{max}}(\hat{x}) = c\text{sum}_{\text{dim}}(A)(A)$.

**Proof.** By Theorem 14, it remains to show that $\phi$ is a bijection. Lemma 15 tells us that if $\phi(x) = \phi(y)$ then we have $x = y$ for any $x, y \in A_n$, and thus $\phi$ is injective. And, by cardinality reasons, it follows that $\phi$ is bijective. This completes the proof. \qed

**Remark 17.** Dukes and Parviainen [3] defined a bijection $\Gamma$ between $A_n$ and $M_n$, and showed that the bijection $\Gamma$ proves the equidistribution of two triples of statistics, that is, $(\text{zero}, \text{max})x = (r\text{sum}_1, \text{diag})\Gamma(x)$ and $R_{\text{max}}(\hat{x}) = c\text{sum}_{\text{dim}}(\Gamma(x))\Gamma(x)$. But unlike our bijection $\phi$, the bijection $\Gamma$ does not transform $R_{\text{min}}$ to $\text{ne}$. Our bijection $\phi$ is constructed in the sprit of $\Gamma$, and the two bijections are different from each other in the definition of rule (Add3) of the addition operation.

Combining Theorems 2 and 16, we are led to the following symmetric joint distribution on ascent sequences.

**Corollary 18.** For any $n$, the statistics zero and $R_{\text{min}}$ have symmetric joint distribution on $A_n$.

Given a matrix $A \in M_n$, the **flip** of $A$, denoted by $F(A)$, is the matrix obtained from $A$ by transposing along the North-East diagonal. It is not difficult to check that for any $A \in M_n$, we have $F(A) \in M_n$ satisfying that $(r\text{sum}_1, \text{diag}, \text{ne}, c\text{sum}_{\text{dim}}(A))A = (c\text{sum}_{\text{dim}}(F(A)), \text{diag}, \text{ne}, r\text{sum}_1)F(A)$.

In view of Theorems 8 and 16, we are led to the following result, confirming the former four items of Conjecture 1.

**Theorem 19.** The map $\alpha = F \cdot \phi \cdot \theta$ is a bijection between $S_n(\begin{bmatrix} 0 & \ldots & 0 \end{bmatrix})$ and $M_n$ satisfying that:

- $LR_{\text{max}}(\pi)$ is the weight of the first row of $\alpha(\pi)$,
- $RL_{\text{min}}(\pi)$ is the weight of the last column of $\alpha(\pi)$,
- $RL_{\text{max}}(\pi)$ is the number of wNE-cells of $\alpha(\pi)$,
- $LR_{\text{min}}(\pi)$ is the number of nonzero cells of $\alpha(\pi)$ belonging to the main diagonal.

**Remark 20.** It should be noted that our bijection $\alpha$ does not verify the last item of Conjecture 1. For example, let $\pi = 85231647$. Then we have $\pi^{-1} = 53472681$, $\theta(\pi) = x = 01102103$ and $\theta(\pi^{-1}) = y = 01223131$. It is easy to check that $\text{asc}(x) = 3$ and $\text{asc}(y) = 4$. By Lemma 13, we have $\text{dim}(\phi(x)) = 4$ and $\text{dim}(\phi(y)) = 5$. This implies that the resulting matrices $\alpha(\pi)$ and $\alpha(\pi^{-1})$ have different dimensions, and thus $\alpha(\pi^{-1}) \neq F(\alpha(\pi))$. 

Acknowledgements

We are grateful to the anonymous referee for his/her helpful comments and suggestions. This work was supported by the Zhejiang Provincial Natural Science Foundation of China (LQ17A010004) and the National Science Foundation of China (11671366 and 11626158).

References


