Equidistributed statistics on Fishburn matrices and permutations

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Abstract

Recently, Jelínek conjectured that there exists a bijection between certain restricted permutations and Fishburn matrices such that the bijection verifies the equidistribution of several statistics. The main objective of this paper is to establish such a bijection.

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1 Introduction

Given a sequence of integers $x = x_1x_2 \cdots x_n$, we say that the sequence x has an *ascent* at position i if $x_i < x_{i+1}$. Let $ASC(x)$ denote the set of the ascent positions of x and let $asc(x)$ denote the number of ascents of x. A sequence $x = x_1x_2 \cdots x_n$ is said to be an *ascent sequence of length* n if it satisfies $x_1 = 0$ and $0 \leqslant x_i \leqslant asc(x_1x_2 \cdots x_{i-1}) + 1$ for all $2 \leqslant i \leqslant n$. Let $\mathcal{A}_n$ be the set of ascent sequences of length n. For example,

$$\mathcal{A}_3 = \{000, 001, 010, 011, 012\}.$$

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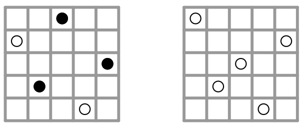
Ascent sequences were introduced by Bousquet-Mélou et al. [1] to unify three other combinatorial structures: $(2+2)$-free posets, a family of permutations avoiding a certain pattern and a class of involutions introduced by Stoimenow [12]. To be specific, Bousquet-Mélou et al. [1] constructed a bijection between ascent sequences and pattern avoiding permutations, a bijection between ascent sequences and $(2+2)$-free posets and a bijection between $(2+2)$-free posets and Stoimenow's involutions. Dukes and Parviainen [3] completed the results of [1] by constructing a bijection between ascent sequences and Fishburn matrices. Hence, all these combinatorial objects are enumerated by the Fishburn number F_n (sequence A022493 in OEIS [10]) for memory of Fishburn's pioneering work on the interval orders [4, 5, 6]. More examples of Fishburn objects are constantly being discovered. Levande [7] introduced the notion of Fishburn diagrams and proved that Fishburn diagrams are counted by Fishburn numbers, confirming a conjecture posed by Claesson and Linusson [2]. Jelínek [8] showed that some Fishburn triples are enumerated by Fishburn numbers.

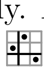
Zagier [14] and Bousquet-Mélou et al. [1] obtained the generating function of F_n, that is

$$\sum_{n\geqslant 0} F_n x^n = \sum_{n\geqslant 0} \Pi_{k=1}^{n}(1-(1-x)^k).$$

Kitaev and Remmel [9] extended the work and found the generating function for (2+2)-free posets when four statistics are taken into account. Levande [7] and Yan [13] independently presented a combinatorial proof of a conjecture of Kitaev and Remmel [9] concerning the generating function for the number of $(2+2)$-free posets.

Let us recall the notions of pattern avoiding permutations and Fishburn matrices before we state our main results. Let S_n be the symmetric group on n elements and $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation of S_n. We say that π contains the pattern  if there is a subsequence $\pi_i\pi_{i+1}\pi_j$ of π satisfying that $\pi_i + 1 = \pi_j < \pi_{i+1}$, otherwise we say that π avoids the pattern  . For example, the permutation 42513 contains the pattern  while the permutation 52314 avoids it.



The pattern  can be defined similarly. Let $S_n($$)$ be the set of $($$)$-avoiding permutations of $[n]$ and $S_n($$)$ be the set of $($$)$-avoiding permutations of $[n]$, respectively. These two sets are both enumerated by Fishburn numbers [1, 11]. In a permutation π, we say π_i is a left-to-right maximum (or LR-maximum) if π_i is larger than any element among $\pi_1, \pi_2, \ldots, \pi_{i-1}$. Let $LRMAX(\pi)$ denote the set of LR-maxima of π and let $LRmax(\pi)$ denote the number of LR-maxima of π. Analogously, we can define LR-minima, RL-maxima, RL-minima of a permutation π. Denote by $LRMIN(\pi)$, $RLMAX(\pi)$ and $RLMIN(\pi)$ the set of LR-minima, RL-maxima and RL-minima of π, their cardinalities being denoted by $LRmin(\pi)$, $RLmax(\pi)$ and $RLmin(\pi)$, respectively.

Fishburn matrices were introduced by Fishburn [6] to represent interval orders. A Fishburn matrix is an upper triangular matrix with nonnegative integers whose every row and every column contain at least one non-zero entry. The weight of a matrix is the sum of its entries. Similarly, the weight of a row (or a column) of a matrix is the sum of the entries in this row (or column). Denote by $\mathcal{M}_n$ the set of Fishburn matrices of weight n. For example,

$$\mathcal{M}_3 = \{(3), \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\}.$$

Given a matrix A, we use the term *cell* (i,j) of A to refer to the the entry in the i-th row and j-th column of A, and we let $A_{i,j}$ denote its value. We assume that the rows of a matrix are numbered from top to bottom and the columns of a matrix are numbered from left to right in which the topmost row is numbered by 1 and the leftmost column is numbered by 1. A cell (i,j) of a matrix A is said to be zero if $A_{i,j} = 0$. Otherwise, it is said to be *nonzero*. A row (or column) is said be zero if it contains no nonzero cells. Otherwise, it is said to be *nonzero* row (or column).

A cell (i,j) of a matrix A is a weakly north-east cell (or wNE-cell) if it is a nonzero cell and any other cell weakly north-east from it is a zero cell. More precisely, a nonzero cell (i,j) of a matrix A is a wNE-cell if $A_{s,t} = 0$ holds for all $s \leqslant i$ and $t \geqslant j$ and $(s,t) \neq (i,j)$.

Jelínek [8] posed the following conjecture.

Conjecture 1. (See [8], Conjecture 4.1) For every n, there is a bijection α between $S_n($$)$ and $\mathcal{M}_n$ satisfying that:

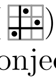
- LRmax(π) is the weight of the first row of $\alpha(\pi)$,
- RLmin(π) is the weight of the last column of $\alpha(\pi)$,
- RLmax(π) is the number of wNE-cells of $\alpha(\pi)$,
- LRmin(π) is the number of nonzero cells of $\alpha(\pi)$ belonging to the main diagonal, and
- $\alpha(\pi^{-1})$ is obtained from $\alpha(\pi)$ by transposing along the North-East diagonal.

By using generating functions, Jelínek [8] proved the following symmetric joint distribution on $\mathcal{M}_n$.

Theorem 2. (See [8], Theorem 3.7) *For any n, the number of wNE-cells and the weight of the first row have symmetric joint distribution on $\mathcal{M}_n$.*

Jelínek [8] also posed the following weaker conjecture which follows directly from Theorem 2 and Conjecture 1.

Conjecture 3. (See [8], Conjecture 4.2) For any n, LRmax and RLmax have symmetric joint distribution on $S_n($$)$.

The main objective of this paper is to establish a bijection between $S_n($$)$ and $\mathcal{M}_n$ which satisfies the former four items of Conjecture 1, thereby confirming Conjecture 3.

2 Bijection between permutations and ascent sequences

In this section, we shall construct a bijection θ between $S_n(\ )$ and $\mathcal{A}_n$, and show that the map θ proves the equidistribution of two 4-tuples of statistics.

Let π be a permutation in $S_n(\ )$ and let τ be the permutation obtained by deleting n from π. Then we have that τ is also a permutation in $S_n(\ )$. If not, we assume that $\tau_i\tau_{i+1}\tau_j$ is a pattern in τ. Since π is ( )-avoiding, we have $\pi_{i+1} = n$. Then $\pi_i\pi_{i+1}\pi_{j+1}$ forms a pattern in π, a contradiction. This property allows us to construct the permutation of $S_n(\ )$ inductively, starting from the empty permutation and adding a new maximal value at each step.

Let τ be a permutation in $S_{n-1}(\ )$. The positions where we can insert the element n into τ to obtain a ( )-avoiding permutation are called active sites. The site after the maximal entry n in π is always an active site. We label the active sites in π from right to left with $0, 1, 2$ and so on.

The bijection θ between $S_n(\ )$ and $\mathcal{A}_n$ can be defined recursively. Set $\theta(1) = 0$. Suppose that π is a permutation in $S_n(\ )$ which is obtained from τ by inserting the element n into the x_n-th active site of τ. Then we set $\theta(\pi) = x_1x_2\cdots x_{n-1}x_n$, where $x_1x_2\cdots x_{n-1} = \theta(\tau)$.

Example 4. The permutation 85231647 corresponds to the sequence 01102103 since it is obtained by the following insertions, where the subscripts indicate the labels of the active sites.

$$
\begin{aligned}
{}_1 1_0 &\xrightarrow{x_2=1} {}_2 2_1 1_0 \\
&\xrightarrow{x_3=1} {}_2 2\ 3_1 1_0 \\
&\xrightarrow{x_4=0} {}_2 2\ 3\ 1_1 4_0 \\
&\xrightarrow{x_5=2} {}_3 5_2 2\ 3\ 1_1 4_0 \\
&\xrightarrow{x_6=1} {}_3 5\ 2\ 3\ 1_2 6_1 4_0 \\
&\xrightarrow{x_7=0} {}_3 5\ 2\ 3\ 1_2 6\ 4_1 7_0 \\
&\xrightarrow{x_8=3} {}_4 8_3 5\ 2\ 3\ 1_2 6\ 4_1 7_0.
\end{aligned}
$$

Lemma 5. *Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in $S_n(\ )$ and $\theta(\pi) = x = x_1x_2\cdots x_n$. Then we have that*

$$s(\pi) = 2 + asc(x) \quad \text{and} \quad a(\pi) = x_n, \tag{1}$$

where $s(\pi)$ denotes the number of active sites of π and $a(\pi)$ denotes the label of the site located just after the entry n of π.

Proof. Suppose that π is obtained from τ by inserting the element n into the x_n-th active site of τ. Then we have $\theta(\tau) = x'$, where $x' = x_1x_2\cdots x_{n-1}$. For any entry i which is to the right of n, i is followed by an active site in π if and only if i is followed by an active site in τ. Since the site after n in π is always active, we obtain $a(\pi) = x_n$

Now let us focus on the equation $s(\pi) = 2 + asc(x)$. We will prove it by induction on n. It obviously holds for $n = 1$. Assume that it holds for $n - 1$. For any entry $i < n - 1$, i is followed by an active site in π if and only if i is followed by an active site in τ. The site after n in π is always an active site. Thus, to determine $s(\pi)$, the only question is whether the site after $n - 1$ is active. We need consider two cases.
Case 1: If $0 \leqslant x_n \leqslant a(\tau) = x_{n-1}$, then the entry n in π is to the right of $n - 1$. It follows that the site after $n-1$ is not an active cite in π. Since the site after $n-1$ is an active site in τ, we have that $s(\pi) = s(\tau)$. By the induction hypothesis, $s(\tau) = 2+asc(x') = 2+asc(x)$. Hence we deduce that $s(\pi) = 2 + asc(x)$.
Case 2: If $x_n > a(\tau) = x_{n-1}$, then the entry n in π is to the left of $n - 1$. It yields that the site after $n - 1$ is also an active cite in π. Hence $s(\pi) = s(\tau) + 1$. Since $x_n > x_{n-1}$, we have that $asc(x) = asc(x') + 1$. By the induction hypothesis, $s(\tau) = 2 + asc(x')$. Thus we have $s(\pi) = 2 + asc(x)$. This completes the proof. □

Theorem 6. *The map θ is a bijection between $S_n($$)$ and $\mathcal{A}_n$.*

Proof. We prove this conclusion by induction on n. It obviously holds for $n = 1$. Assume that θ is a bijection between $S_{n-1}($$)$ and $\mathcal{A}_{n-1}$.

We first show that θ is a map from $S_n($$)$ to $\mathcal{A}_n$. Let $\pi = \pi_1\pi_2\cdots\pi_n$ be a permutation in $S_n($$)$ which is obtained from τ by inserting a maximal entry n in the active site labeled by x_n in τ. Then $\theta(\pi) = x = x_1x_2\cdots x_n$, where $\theta(\tau) = x' = x_1x_2\cdots x_{n-1}$. To prove that $x \in \mathcal{A}_n$, it suffices to show that $x_n \leqslant asc(x') + 1$. Recall that the rightmost active site is labeled 0. Hence the leftmost active site in τ is labeled $s(\tau) - 1$. By the recursive description of the map θ, we have that $x_n \leqslant s(\tau) - 1$. From Lemma 5 we see that $s(\tau) = 2+asc(x')$. Thus we have $x_n \leqslant asc(x')+1$. Since x encodes the construction of π, θ is an injective map from $S_n($$)$ to $\mathcal{A}_n$.

It remains to show that θ is surjection. Let $y = y_1y_2\cdots y_n$ be an ascent sequence and $p = p_1p_2\cdots p_{n-1} = \theta^{-1}(y')$, where $y' = y_1y_2\cdots y_{n-1}$. From the definition of ascent sequence and Lemma 5, we have that $y_n \leqslant asc(y')+1 = s(p)-1$. Let q be the permutation obtained from p by inserting the maximal entry n into the active site labeled y_n in p. By the construction of the map θ, it can be easily seen that $\theta(q) = y$. This concludes the proof. □

Let $x = x_1x_2\cdots x_n$ be an ascent sequence in $\mathcal{A}_n$. The *modified ascent sequence* of x, denoted by $\hat{x}$, is defined by the following procedure:
for $i \in ASC(x)$
 for $j = 1, 2, \ldots, i - 1$
  if $x_j \geqslant x_{i+1}$ then $x_j := x_j + 1$.
For example, for $x = 01012213$, we have $ASC(x) = \{1, 3, 4, 7\}$ and $\hat{x} = 04012213$. Modified ascent sequences were introduced by Bousquet-Mélou et al., see more details in [1].

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n($$)$, let $l(\pi_i)$ be the largest label of the active site to the right of π_i and let $LMAXL(\pi)$ be the multiset of $l(\pi_i)$ when π_i ranges over all LR-maxima of π. That is

$$LMAXL(\pi) = \{l(\pi_i) \mid \pi_i \in LRMAX(\pi)\}.$$

Similarly, let

$$RMAXL(\pi) = \{l(\pi_i) \mid \pi_i \in RLMAX(\pi)\}.$$

For example, for $\pi = 42178536$, its active sites are labeled as ${}_4 4 2 1_3 7 8_2 5 3_1 6_0$. Then we have $RMAXL(\pi) = \{0, 2\}$ and $LMAXL(\pi) = \{2, 2, 3\}$.

For an ascent sequence $x = x_1 x_2 \cdots x_n$, let

$$ZERO(x) = \{i \mid x_i = 0\},$$

and

$$MAX(x) = \{i \mid x_i = asc(x_1 x_2 \cdots x_{i-1}) + 1\},$$

with their cardinalities being denoted by $zero(x)$ and $max(x)$ respectively.

For a sequence $x = x_1 x_2 \cdots x_n$, let

$$RMIN(x) = \{x_i \mid x_i < x_j \text{ for all } j > i\},$$

$$RMAX(x) = \{x_i \mid x_i \geqslant x_j \text{ for all } j > i\}.$$

It should be noted that the set $RMAX(x)$ is a multiset. Denote by $Rmin(x)$ and $Rmax(x)$ the cardinalities of the sets $RMIN(x)$ and $RMAX(x)$, respectively. For example, let $x = 01012201$. We have $RMIN = \{0, 1\}$, $RMAX = \{1, 2, 2\}$, $Rmin(x) = 2$ and $Rmax(x) = 3$.

Theorem 7. *For any* $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n($  $)$ *and* $x = x_1 x_2 \cdots x_n \in \mathcal{A}_n$ *with* $\theta(\pi) = x$*, we have*

(1) $RLMIN(\pi) = ZERO(x)$*;*

(2) $LRMIN(\pi) = MAX(x)$*;*

(3) $RMAXL(\pi) = RMIN(x)$*;*

(4) $LMAXL(\pi) = RMAX(\hat{x})$*.*

Proof. We will prove points (1)-(4) by induction on n. It is easily checked that the statement holds for $n = 1$. Assume that it also holds for some $n - 1$ with $n \geqslant 2$. Let τ be the permutation which is obtained from π by deleting the largest entry n in π. Then we have that $x' = x_1 x_2 \cdots x_{n-1} = \theta(\tau)$. From the construction of the bijection θ and the induction hypothesis, one can easily verify that

$$RLMIN(\pi) = \begin{cases} RLMIN(\tau) \cup \{n\} = ZERO(x') \cup \{n\} = ZERO(x) & \text{if } x_n = 0, \\ RLMIN(\tau) = ZERO(x') = ZERO(x) & \text{otherwise }, \end{cases}$$

$$LRMIN(\pi) = \begin{cases} LRMIN(\tau) = MAX(x') = MAX(x) & \text{if } x_n \leqslant asc(x'), \\ LRMIN(\tau) \cup \{n\} = MAX(x') \cup \{n\} = MAX(x) & \text{if } x_n = asc(x') + 1, \end{cases}$$

and

$$\begin{aligned} RMAXL(\pi) &= \{i \mid i \in RMAXL(\tau), i < x_n\} \cup \{x_n\} \\ &= \{i \mid i \in RMIN(x'), i < x_n\} \cup \{x_n\} \\ &= RMIN(x). \end{aligned}$$

For point (4), we consider two cases. If $x_n \leqslant x_{n-1}$, then n is to the right of $n-1$ in π. Notice that all the LR-maxima in τ are also LR-maxima in π. One can easily check that $LMAXL(\pi) = LMAXL(\tau) \cup \{x_n\}$ and $RMAX(\hat{x}) = RMAX(\hat{x'}) \cup \{x_n\}$. By the induction hypothesis, we have

$$LMAXL(\pi) = LMAXL(\tau) \cup \{x_n\} = RMAX(\hat{x'}) \cup \{x_n\} = RMAX(\hat{x}).$$

If $x_n > x_{n-1}$, then n is to the left of $n-1$ in π. In this case, τ_i is a LR-maximum in π if and only if τ_i is a LR-maximum in τ and $l(\tau_i) \geqslant x_n$. After the inserting n into τ, $l(\tau_i)$ is increased by 1 if τ_i is also a LR-maximum in π. Hence we have that

$$LMAXL(\pi) = \{i+1 \mid i \in LMAXL(\tau), i \geqslant x_n\} \cup \{x_n\}.$$

From the definition of the modified ascent sequence, it follows that

$$RMAX(\hat{x}) = \{i+1 \mid i \in RMAX(\hat{x'}), i \geqslant x_n\} \cup \{x_n\}.$$

By the induction hypothesis, we immediately deduce that $LMAXL(\pi) = RMAX(\hat{x})$ as desired. This completes the proof. □

Combining Theorems 6 and 7, we are led to the following result.

Theorem 8. *The map θ is a bijection between $S_n($**$)$ and $\mathcal{A}_n$. Moreover, for any $\pi \in S_n($**$)$ and $x \in \mathcal{A}_n$ with $\theta(\pi) = x$, we have*

$$(RLmin, LRmin, RLmax)\pi = (zero, max, Rmin)x$$

and $LRmax(\pi) = Rmax(\hat{x})$.

3 Bijection between ascent sequences and Fishburn matrices

The main objective of this section is to establish a bijection ϕ between $\mathcal{A}_n$ and $\mathcal{M}_n$. To this end, we will define a removal operation and an addition operation on the matrices of $\mathcal{M}_n$.

Given a matrix A in $\mathcal{M}_n$, let $dim(A)$ denote the number of rows of the matrix A and let $index(A)$ denote the smallest value of i such that $A_{i,dim(A)} > 0$. Denote by $rsum_i(A)$ and $csum_i(A)$ the sum of the entries in row i and column i of A, respectively. We define a removal operation f on a given matrix $A \in \mathcal{M}_n$ as follows.

(Rem1) If $rsum_{index(A)}(A) > 1$, let $f(A)$ be the matrix A with the entry $A_{index(A),dim(A)}$ reduced by 1.

(Rem2) If $rsum_{index(A)}(A) = 1$ and $index(A) = dim(A)$, then let $f(A)$ be the matrix A with row $dim(A)$ and column $dim(A)$ removed.

(Rem3) If $rsum_{index(A)}(A) = 1$ and $index(A) < dim(A)$, then we construct $f(A)$ in the following way. Let S be the set of indices j such that $j \geqslant index(A)$ and column j contains at least one nonzero entry above row $index(A)$. Suppose that $S = \{c_1, c_2, \ldots, c_\ell\}$ with $c_1 < c_2 \ldots < c_\ell$. Clearly we have $c_1 = index(A)$. Let $c_{\ell+1} = dim(A)$. For all $1 \leqslant i < index(A)$ and $1 \leqslant j \leqslant \ell$, move all the entries in the cell (i, c_j) to the cell (i, c_{j+1}). Simultaneously delete row $index(A)$ and column $index(A)$.

Example 9. Let A, B, C be the following three Fishburn matrices:

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}; \quad C = \begin{pmatrix} 2 & 4 & 1 & 3 & 0 \\ 0 & 5 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

For Matrix A, rule (Rem1) is applied since $rsum_{index(A)}(A) = 3$ and

$$f(A) = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

For Matrix B, since $rsum_{index(B)}(B) = 1$ and $index(B) = dim(B)$, rule (Rem2) is applied and

$$f(B) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

For matrix C, since $rsum_{index(C)}(C) = 1$ and $index(C) < dim(C)$, rule (Rem3) is applied. It is easy to check that $S = \{3, 4\}$, and thus we have

$$f(C) = \begin{pmatrix} 2 & 4 & 1 & 3 \\ 0 & 5 & 2 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

The following lemma shows that the removal operation on a Fishburn matrix of $\mathcal{M}_n$ will yield a Fishburn matrix in $\mathcal{M}_{n-1}$.

Lemma 10. *Let $n \geqslant 2$ be an integer and $A \in \mathcal{M}_n$, then we have that $f(A) \in \mathcal{M}_{n-1}$.*

Proof. It is easily seen that for any removal operation applied on the matrix A, the weight of $f(A)$ is one less than the weight of A. It is trivial to check that there exists no zero columns or rows in $f(A)$. Moreover, the removal operation also preserves the property of being upper-triangular. Thus, $f(A) \in \mathcal{M}_{n-1}$. This completes the proof. □

Lemma 10 tells us that for any $A \in \mathcal{M}_n$, after n applications of the removal operation f to A, we will get a sequence of Fishburn matrices, say $A^{(1)}, A^{(2)}, \ldots, A^{(n)}$, where $A^{(k-1)} = f(A^{(k)})$ for all $1 < k \leqslant n$ and $A^{(n)} = A$. Define $\psi(A) = x = x_1 x_2 \ldots x_n$ where $x_k = index(A^{(k)}) - 1$.

We now define an addition operation g on a Fishburn matrix which is shown to be the inverse of the removal operation later. Given a matrix $A \in \mathcal{M}_n$ and $i \in [0, dim(A)]$, We construct a matrix $g(A, i)$ in the following manner.

(Add1) If $0 \leqslant i \leqslant index(A) - 1$, then let $g(A, i)$ be the matrix obtained from A by increasing the entry in the cell $(i + 1, dim(A))$ by 1.

(Add2) If $i = dim(A)$, then let $g(A, i)$ be the matrix $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

(Add3) If $index(A) \leqslant i < dim(A)$, then we construct $g(A, i)$ in the following way. In A, insert a new (empty) row between rows i and $i+1$, and insert a new (empty) column between columns i and $i + 1$. Let the new row be filled with all zeros except for the rightmost cell which is filled with a 1. Denote by A' the resulting matrix. Let T be the set of indices j such that $j \geqslant i + 1$ and column j contains at least one nonzero cell above row $i + 1$. Suppose that $T = \{c_1, c_2, \ldots, c_\ell\}$. Clearly we have $c_\ell = dim(A')$. Let $c_0 = i+1$. For all $1 \leqslant a \leqslant i$ and $1 \leqslant b \leqslant \ell$, move all the entries in the cell (a, c_b) to the cell (a, c_{b-1}), and fill all the cells which are in column $dim(A')$ and above row $i + 1$ with zeros.

Example 11. Consider the matrix

$$A = \begin{pmatrix} 2 & 4 & 0 & 3 \\ 0 & 5 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Obviously, we have $dim(A) = 4$ and $index(A) = 1$. For $i = 0$, since $i \leqslant index(A) - 1$, rule (Add1) applies and we get

$$g(A, 0) = \begin{pmatrix} 2 & 4 & 0 & 4 \\ 0 & 5 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

For $i = 4$, since $i = dim(A)$, rule (Add2) applies and we get

$$g(A, 4) = \begin{pmatrix} 2 & 4 & 0 & 3 & 0 \\ 0 & 5 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $i = 1$, since $index(A) \leqslant i < dim(A)$, rule (Add3) applies and we get

$$A' = \begin{pmatrix} 2 & \mathbf{0} & 4 & 0 & 3 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ 0 & \mathbf{0} & 5 & 0 & 2 \\ 0 & \mathbf{0} & 0 & 1 & 3 \\ 0 & \mathbf{0} & 0 & 0 & 2 \end{pmatrix},$$

where the new inserted row and column are illustrated in bold. Then we have $T = \{3, 5\}$. Finally, we get

$$g(A, 1) = \begin{pmatrix} 2 & 4 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

By similar arguments as in the proof of Lemma 10, one can easily verify that the addition operation will also yield a Fishburn matrix.

Lemma 12. *For any matrix $A \in \mathcal{M}_{n-1}$ and $i \in [0, dim(A)]$, we have that $g(A, i) \in \mathcal{M}_n$.*

We now define a map ϕ from $\mathcal{A}_n$ to $\mathcal{M}_n$ recursively as follows. Given an ascent sequence $x = x_1x_2\ldots, x_n$, we define $A^{(1)} = (1)$ and $A^{(k)} = g(A^{(k-1)}, x_k)$ for all $1 < k \leqslant n$. Set $\phi(x) = A^{(n)}$.

Next we aim to show that the map ϕ is well defined and has the following desired properties.

Lemma 13. *For any $x = x_1x_2\cdots x_n \in \mathcal{A}_n$, we have $\phi(x) \in \mathcal{M}_n$ satisfying $dim(\phi(x)) = asc(x) + 1$ and $index(\phi(x)) = x_n + 1$.*

Proof. We will prove by induction on n. It is trivial to check that the statement holds for $n = 1$. Assume that it also holds for $n - 1$, that is,

$$\phi(x') \in \mathcal{M}_{n-1},\ dim(\phi(x')) = asc(x') + 1 \text{ and } index(\phi(x')) = x_{n-1} + 1,$$

where $x' = x_1x_2\cdots x_{n-1}$. Since $0 \leqslant x_n \leqslant asc(x') + 1 = dim(\phi(x'))$, from Lemma 12 we see that $\phi(x) = g(\phi(x'), x_n) \in \mathcal{M}_n$. From the construction of the addition operation, one can easily verify that $index(\phi(x)) = x_n + 1$ and

$$dim(\phi(x)) = \begin{cases} dim(\phi(x')) = asc(x') + 1 = asc(x) + 1 & \text{if } x_n \leqslant x_{n-1}, \\ dim(\phi(x')) + 1 = asc(x') + 2 = asc(x) + 1 & \text{if } x_n > x_{n-1}. \end{cases}$$

The result follows. □

For a matrix A, let $NE(A) = \{i - 1 | \text{ row } i \text{ contains a wNE-cell }\}$ and let $ne(A)$ denote the number of wNE-cells of A. Denote by $diag(A)$ the number of nonzero cells belonging to the main diagonal of A. Let $LAST(A)$ be the multiset of integers such that there are exactly c occurrences of i if and only if $A_{i+1,dim(A)} = c$ and $c > 0$.

Theorem 14. *For any $x = x_1x_2\cdots x_n \in \mathcal{A}_n$ and $A \in \mathcal{M}_n$ with $A = \phi(x)$, we have the following relations.*

(1) $zero(x) = rsum_1(A)$;

(2) $max(x) = diag(A)$;

(3) $RMIN(x) = NE(A)$;

(4) $RMAX(\hat{x}) = LAST(A)$;

(5) $Rmin(x) = ne(A)$;

(6) $Rmax(\hat{x}) = csum_{dim(A)}(A)$.

Proof. Points (5) and (6) follow directly from points (3) and (4). Now we verify points (1)-(4) by induction on n. Clearly, the statement holds for $n = 1$. Assume that it also holds for any some $n-1$ with $n \geqslant 2$. Let $x' = x_1x_2\cdots x_{n-1}$ and $B = \phi(x')$. Recall that $A = g(B, x_n)$. From the definition of the addition operation g and the induction hypothesis, it is not difficult to verify that

$$rsum_1(A) = \begin{cases} rsum_1(B) + 1 = zero(x') + 1 = zero(x) & \text{if } x_n = 0, \\ rsum_1(B) = zero(x') = zero(x) & \text{otherwise }, \end{cases}$$

and

$$diag(A) = \begin{cases} diag(B) = max(x') = max(x) & \text{if } x_n \leqslant asc(x'), \\ diag(B) + 1 = max(x') + 1 = max(x) & \text{if } x_n = asc(x') + 1. \end{cases}$$

For point (3), from the construction of the addition operation g, we see that the cell $(x_n + 1, dim(A))$ is always a wNE-cell. Moreover, there is a wNE-cell in row i of A if and only if there is a wNE-cell in row i of B and $i < x_n + 1$. This yields that

$$\begin{aligned} NE(A) &= \{i \mid i \in NE(B), i < x_n\} \cup \{x_n\} \\ &= \{i \mid i \in RMIN(x'), i < x_n\} \cup \{x_n\} \\ &= RMIN(x). \end{aligned}$$

For point (4), we have two cases.

If $x_n \leqslant x_{n-1} = index(B) - 1$, then rule (Add1) applies. It is trivial to check that $RMAX(\hat{x}) = RMAX(\hat{x'}) \cup \{x_n\}$ and $LAST(A) = LAST(B) \cup \{x_n\}$.

If $x_n > x_{n-1} = index(B) - 1$, then either rule (Add2) or rule (Add3) applies. It is not difficult to verify that

$$RMAX(\hat{x}) = \{i + 1 \mid i \in RMAX(\hat{x'}), i \geqslant x_n\} \cup \{x_n\},$$

and

$$LAST(A) = \{i + 1 \mid i \in LAST(B), i \geqslant x_n\} \cup \{x_n\}.$$

By the induction hypothesis, we have concluded that $RMAX(\hat{x}) = LAST(A)$. This completes the proof. ☐

Lemma 15. *For any $x = x_1x_2 \cdots x_n \in \mathcal{A}_n$, we have $\psi(\phi(x)) = x$.*

Proof. Suppose that we get a sequence of matrices $A^{(1)}, A^{(2)}, \ldots, A^{(n)}$ when we apply the map ϕ to x, where $A^{(1)} = (1)$ and $A^{(k)} = g(A^{(k-1)}, x_k)$ for all $1 < k \leqslant n$. Similarly, suppose that when we apply the map ψ to $\phi(x)$, we get a sequence $y = y_1y_2 \cdots y_n$ and a sequence of matrices $B^{(1)}, B^{(2)}, \ldots, B^{(n)}$, where $B^{(n)} = \phi(x)$, $B^{(k)} = f(B^{(k+1)})$ for all $1 \leqslant k < n$, and $y_k = index(B^{(k)}) - 1$. Lemma 13 ensures that $index(A^{(k)}) = x_k + 1$. In order to prove $x = y$, it suffices to show that $A^{(k)} = B^{(k)}$ for all $1 \leqslant k \leqslant n$. We proceed to prove this assertion by induction on n. Clearly, we have $B^{(n)} = \phi(x) = A^{(n)}$. Assume that we have $A^{(j)} = B^{(j)}$ for all $j \geqslant k+1$. In the following we aim to show that $A^{(k)} = B^{(k)}$. By the induction hypothesis, it suffices to show that $f(A^{(k+1)}) = A^{(k)}$. We have three cases.

Let us assume that $0 \leqslant x_{i+1} < index(A^{(k)})$. Then rule (Add1) applies and $A^{(k+1)}$ is simply a copy of $A^{(k)}$ with the entry in the cell $(x_{i+1}+1, dim(A^{(k)}))$ increased by one. Clearly, we have $dim(A^{(k)}) = dim(A^{k+1})$, $index(A^{(k+1)}) = x_{i+1} + 1$ and $rsum_{x_{i+1}+1}(A^{(k+1)}) > 1$. So rule (Rem1) applies and $f(A^{(k+1)})$ is obtained from $A^{(k+1)}$ by decreasing the the entry in the cell $(x_{i+1} + 1, dim(A^{(k+1)}))$ by one. Thus we have $f(A^{(k+1)}) = A^{(k)}$.

Next assume that $x_{i+1} = dim(A^{(k)})$. Then rule (Add2) applies and $A^{(k+1)} = \begin{pmatrix} A^{(k)} & 0 \\ 0 & 1 \end{pmatrix}$. In this case, we have $index(A^{(k+1)}) = x_{i+1} + 1 = dim(A^{(k+1)})$ and $rsum_{x_{i+1}+1}(A^{(k+1)}) = 1$. So rule (Rem2) applies and $f(A^{(k+1)})$ is obtained from $A^{(k+1)}$ by removing column $dim(A^{(k+1)})$ and row $dim(A^{(k+1)})$. Thus we have $f(A^{(k+1)}) = A^{(k)}$.

If $index(A^{(k)}) \leqslant x_{i+1} < dim(A^{(k)})$, then rule (Add3) applies and $A^{(k+1)}$ is obtained from $A^{(k)}$ in the following way. First we insert a new (empty) row between rows x_{i+1} and $x_{i+1} + 1$, and insert a new (empty) column between columns x_{i+1} and $x_{i+1} + 1$. Let the new row be filled with all zeros except for the rightmost cell which is filled with a 1. Denote by A' the resulting matrix. Let T be the set of indices j such that $j \geqslant x_{i+1} + 1$ and column j contains at least one nonzero cell above row $x_{i+1} + 1$. Suppose that $T = \{c_1, c_2, \ldots, c_\ell\}$ with $c_1 < c_2 < \ldots < c_\ell$. Let $c_0 = x_{i+1} + 1$. For all $1 \leqslant a \leqslant x_{i+1}$ and $1 \leqslant b \leqslant \ell$, move all the entries in the cell (a, c_b) to the cell (a, c_{b-1}), and fill all the cells in column $dim(A')$ and above row $x_{i+1} + 1$ with zeros. It is easy to check that $dim(A^{(k+1)}) = dim(A^{(k)}) + 1$, $index(A^{(k+1)}) = x_{i+1} + 1$ and $rsum_{x_{i+1}+1}(A^{(k+1)}) = 1$. So rule (Rem3) applies and $f(A^{(k+1)})$ is obtained from $A^{(k+1)}$ by the following procedure. Let S be the set of indices j such that $j \geqslant x_{i+1} + 1$ and column j contains at least one nonzero entry above row $x_{i+1} + 1$. It is not difficult to check that $S = \{c_0, c_1, c_2, \ldots, c_{\ell-1}\}$. Let $c_\ell = dim(A^{(k+1)})$. For all $1 \leqslant a < x_{i+1} + 1$ and $1 \leqslant b \leqslant \ell - 1$, move all the entries in the cell (a, c_b) to the cell (a, c_{b+1}). Simultaneously delete row $x_{i+1} + 1$ and column $x_{i+1} + 1$. These operations simply reverse the construction of $A^{(k+1)}$ from $A^{(k)}$, and therefore $f(A^{(k+1)}) = A^{(k)}$. This completes the proof. □

Theorem 16. *The map ϕ is a bijection between $\mathcal{A}_n$ and $\mathcal{M}_n$. Moreover, for any $x \in \mathcal{A}_n$ and $A \in \mathcal{M}_n$ with $\phi(x) = A$, we have*

$$(zero, max, Rmin)x = (rsum_1, diag, ne)A$$

and $Rmax(\hat{x}) = csum_{dim(A)}(A)$.

Proof. By Theorem 14, it remains to show that ϕ is a bijection. Lemma 15 tells us that if $\phi(x) = \phi(y)$ then we have $x = y$ for any $x, y \in \mathcal{A}_n$, and thus ϕ is injective. And, by cardinality reasons, it follows that ϕ is bijective. This completes the proof. □

Remark 17. Dukes and Parviainen [3] defined a bijection Γ between $\mathcal{A}_n$ and $\mathcal{M}_n$, and showed that the bijection Γ proves the equidistribution of two triples of statistics, that is,

$$(zero, max)x = (rsum_1, diag)\Gamma(x)$$

and $Rmax(\hat{x}) = csum_{dim(\Gamma(x))}\Gamma(x)$. But unlike our bijection ϕ, the bijection Γ does not transform $Rmin$ to ne. Our bijection ϕ is constructed in the sprit of Γ, and the two bijections are different from each other in the definition of rule (Add3) of the addition operation.

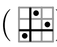
Combining Theorems 2 and 16, we are led to the following symmetric joint distribution on ascent sequences.

Corollary 18. *For any* n*, the statistics zero and Rmin have symmetric joint distribution on* $\mathcal{A}_n$*.*

Given a matrix $A \in \mathcal{M}_n$, the *flip* of A, denoted by $\mathcal{F}(A)$, is the matrix obtained from A by transposing along the North-East diagonal. It is not difficult to check that for any $A \in \mathcal{M}_n$, we have $\mathcal{F}(A) \in \mathcal{M}_n$ satisfying that

$$(rsum_1, diag, ne, csum_{dim(A)})A = (csum_{dim(\mathcal{F}(A))}, diag, ne, rsum_1)\mathcal{F}(A).$$

In view of Theorems 8 and 16, we are led to the following result, confirming the former four items of Conjecture 1.

Theorem 19. *The map* $\alpha = \mathcal{F} \cdot \phi \cdot \theta$ *is a bijection between* $S_n($$)$ *and* $\mathcal{M}_n$ *satisfying that:*

- *LRmax*(π) *is the weight of the first row of* $\alpha(\pi)$*,*
- *RLmin*(π) *is the weight of the last column of* $\alpha(\pi)$*,*
- *RLmax*(π) *is the number of wNE-cells of* $\alpha(\pi)$*,*
- *LRmin*(π) *is the number of nonzero cells of* $\alpha(\pi)$ *belonging to the main diagonal.*

Remark 20. It should be noted that our bijection α does not verify the last item of Conjecture 1. For example, let $\pi = 85231647$. Then we have $\pi^{-1} = 53472681$, $\theta(\pi) = x = 01102103$ and $\theta(\pi^{-1}) = y = 01223131$. It is easy to check that $asc(x) = 3$ and $asc(y) = 4$. By Lemma 13, we have $dim(\phi(x)) = 4$ and $dim(\phi(y)) = 5$. This implies that the resulting matrices $\alpha(\pi)$ and $\alpha(\pi^{-1})$ have different dimensions, and thus $\alpha(\pi^{-1}) \neq \mathcal{F}(\alpha(\pi))$.

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