

Algebraically grid-like graphs have large tree-width

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Abstract

By the Grid Minor Theorem of Robertson and Seymour, every graph of sufficiently large tree-width contains a large grid as a minor. Tree-width may therefore be regarded as a measure of ‘grid-likeness’ of a graph.

The grid contains a long cycle on the perimeter, which is the \mathbb{F}_2 -sum of the rectangles inside. Moreover, the grid distorts the metric of the cycle only by a factor of two. We prove that every graph that resembles the grid in this algebraic sense has large tree-width:

Let k, p be integers, γ a real number and G a graph. Suppose that G contains a cycle of length at least $2\gamma pk$ which is the \mathbb{F}_2 -sum of cycles of length at most p and whose metric is distorted by a factor of at most γ . Then G has tree-width at least k .

Mathematics Subject Classifications: 05C83, 05C12, 05C38

1 Introduction

The *edge space* of a graph $G = (V, E)$ is the vector space over the 2-element field \mathbb{F}_2 consisting of all maps $E \rightarrow \mathbb{F}_2$, with addition defined pointwise as $(f \oplus g)(e) = f(e) + g(e)$. Each such map $f : E \rightarrow \mathbb{F}_2$ may be identified with the set of all $e \in E$ for which $f(e) = 1$. Addition of maps then corresponds to symmetric difference of sets of edges, and we call $F_1 \oplus F_2 := F_1 \Delta F_2$ the \mathbb{F}_2 -sum of $F_1, F_2 \subseteq E$.

The *cycle space* of G is the subspace of the edge space which is generated by the (edge sets of) cycles of G . In this context, we do not formally distinguish between a cycle $C \subseteq G$ and its set of edges $E(C) \subseteq E$ and simply speak of the \mathbb{F}_2 -sum of cycles. See the book by Diestel [5] for more background on the cycle space.

For a positive integer n , the $(n \times n)$ -grid is the graph G_n whose vertices are all pairs (i, j) with $1 \leq i, j \leq n$, where two points are adjacent when they are at Euclidean

distance 1. The cycle C_n , which bounds the outer face in the natural drawing of G_n in the plane, has length $4(n - 1)$ and is the \mathbb{F}_2 -sum of the rectangles bounding the inner faces. This is by itself not a distinctive feature of graphs with large tree-width: The situation is similar for the n -wheel W_n , the graph consisting of a cycle D_n of length n and a vertex $x \notin D_n$ which is adjacent to every vertex of D_n . There, D_n is the \mathbb{F}_2 -sum of all triangles xyz for $yz \in E(D_n)$. Still, W_n only has tree-width 3.

The key difference is the fact that in the wheel, the metric of the cycle is heavily distorted: any two vertices of D_n are at distance at most two within W_n , even if they are far apart within D_n . In the grid, however, the distance between two vertices of C_n within G_n is at least half of their distance within C_n .

In order to incorporate this factor of two and to allow for more flexibility, we equip the edges of our graphs with lengths. For a graph G , a *length-function on G* is simply a map $\ell : E(G) \rightarrow \mathbb{R}_{>0}$. We then define the ℓ -length $\ell(H)$ of a subgraph $H \subseteq G$ as the sum of the lengths of all edges of H . This naturally induces a notion of distance between two vertices of G , where we define d_G^ℓ as the minimum ℓ -length of a path containing both. A subgraph $H \subseteq G$ is ℓ -geodesic if it contains a path of length $d_G^\ell(a, b)$ between any two vertices $a, b \in V(H)$.

When no length-function is specified, the notions of length, distance and geodesicity are to be read with respect to $\ell \equiv 1$ constant.

On the grid-graph G_n , consider the length-function ℓ which is equal to 1 on $E(C_n)$ and assumes the value 2 elsewhere. Then C_n is ℓ -geodesic of length $\ell(C_n) = 4(n - 1)$ and the sum of cycles of ℓ -length at most 8. We show that any graph which shares this algebraic feature has large tree-width.

Theorem 1. *Let k be a positive integer and $r > 0$. Let G be a graph with rational-valued length-function ℓ . Suppose G contains an ℓ -geodesic cycle C with $\ell(C) \geq 2rk$, which is the \mathbb{F}_2 -sum of cycles of ℓ -length at most r . Then the tree-width of G is at least k .*

In Section 3, we prove a qualitative converse to Theorem 1, showing that graphs of large tree-width are ‘algebraically grid-like’.

The starting point of Theorem 1 was a similar result of Matthias Hamann and the author [2]. There, it is assumed that not only the fixed cycle C , but the whole cycle space of G is generated by short cycles.

Theorem 2 ([2, Corollary 3]). *Let k, p be positive integers. Let G be a graph whose cycle space is generated by cycles of length at most p . If G contains a geodesic cycle of length at least kp , then the tree-width of G is at least k .*

It should be noted that Theorem 2 is not implied by Theorem 1, as the constant factors are different. In fact, the proofs are also quite different, although Lemma 5 below was inspired by a similar parity-argument in [2].

It is tempting to think that, conversely, Theorem 1 could be deduced from Theorem 2 by adequate manipulation of the graph G , but we have not been successful with such attempts.

2 Proof of Theorem 1

The relation to tree-width is established via a well-known separation property of graphs of bounded tree-width, due to Robertson and Seymour [3].

Lemma 3 ([3]). *Let k be a positive integer, G a graph and $A \subseteq V(G)$. If the tree-width of G is less than k , then there exists $X \subseteq V(G)$ with $|X| \leq k$ such that every component of $G - X$ contains at most $\frac{1}{2}|A \setminus X|$ vertices of A .*

It is not hard to see that Theorem 1 can be reduced to the case where $\ell \equiv 1$. This case is treated in the next theorem.

Theorem 4. *Let k, p be positive integers. Let G be a graph containing a geodesic cycle C of length at least $4\lfloor p/2 \rfloor k$, which is the \mathbb{F}_2 -sum of cycles of length at most p . Then for every $X \subseteq V(G)$ of order at most k , some component of $G - X$ contains at least half the vertices of C .*

Proof of Theorem 1, assuming Theorem 4. Let \mathcal{D} be a set of cycles of ℓ -length at most r with $C = \bigoplus \mathcal{D}$. Since ℓ is rational-valued and the premise also holds for r' the maximum ℓ -length of a cycle in \mathcal{D} , we may assume that $r \in \mathbb{Q}$. Take an integer M so that rM and $\ell'(e) := M\ell(e)$ are natural numbers for every $e \in E(G)$.

Obtain the subdivision G' of G by replacing every $e \in E(G)$ by a path of length $\ell'(e)$. Denote by C', D' the subdivisions of C and D for every $D \in \mathcal{D}$, respectively. Then $C' = \bigoplus_{D \in \mathcal{D}} D'$ and $|C'| = M\ell(C) \geq 2(Mr)k$, while $|D'| = M\ell(D) \leq Mr$ for every $D \in \mathcal{D}$.

Assume for a contradiction that the tree-width of G was less than k . Since tree-width is invariant under subdivision, the tree-width of G' is also less than k . By Lemma 3, there exists a set $X \subseteq V(G')$ with $|X| \leq k$ such that every component of $G' - X$ contains at most $\frac{1}{2}|V(C') \setminus X|$ vertices of $V(C')$. Since C' is a connected subgraph of G' , X must contain at least one vertex of C' . By Theorem 4, there is a component K of $G' - X$ that contains at least half the vertices of C' . But then

$$\frac{1}{2}|C'| \leq |K \cap V(C')| \leq \frac{1}{2}|V(C') \setminus X| < \frac{1}{2}|C'|,$$

which is a contradiction. □

Our goal is now to prove Theorem 4. The proof consists of two separate lemmas. The first lemma involves separators and \mathbb{F}_2 -sums of cycles.

Lemma 5. *Let G be a graph, $C \subseteq G$ a cycle and \mathcal{D} a set of cycles in G such that $C = \bigoplus \mathcal{D}$. Let \mathcal{R} be a set of disjoint vertex-sets of G such that for every $R \in \mathcal{R}$, $R \cap V(C)$ is either empty or induces a connected subgraph of C . Then either some $D \in \mathcal{D}$ meets two distinct $R, R' \in \mathcal{R}$ or there is a component Q of $G - \bigcup \mathcal{R}$ with $V(C) \subseteq V(Q) \cup \bigcup \mathcal{R}$.*

Proof. Suppose that no $D \in \mathcal{D}$ meets two distinct $R, R' \in \mathcal{R}$. Then C has no edges between the sets in \mathcal{R} , since any such edge would have to lie in at least one $D \in \mathcal{D}$. Let $Y := \bigcup \mathcal{R}$ and let \mathcal{Q} be the set of components of $G - Y$.

We claim that for all $Q \in \mathcal{Q}, R \in \mathcal{R}$ and $D \in \mathcal{D}$, the number of edges of D between Q and R is even. This is trivial if D has no edges between Q and R . Otherwise, D meets R and thus cannot meet $Y \setminus R$. Therefore, all edges of D between Q and $V(G) \setminus Q$ must join Q to R . As D is a cycle, it has an even number of edges between Q and $V(G) \setminus Q$ and thus between Q and R . This finishes the proof of the claim.

Now, as $C = \bigoplus \mathcal{D}$, we find that that for all $Q \in \mathcal{Q}, R \in \mathcal{R}$, the number of edges of C between Q and R is

$$e_C(Q, R) \equiv \sum_{D \in \mathcal{D}} e_D(Q, R) \equiv 0 \pmod{2}.$$

For every $R \in \mathcal{R}$ which intersects C , there are precisely two edges of C between R and $V(C) \setminus R$, because $R \cap C$ is connected. As mentioned above, C contains no edges between R and $Y \setminus R$, so both edges join R to $V(G) \setminus Y$. But C has an even number of edges between R and each component of $V(G) \setminus Y$, so it follows that both edges join R to the same $Q \in \mathcal{Q}$.

Since every component of $C - (C \cap Y)$ is contained in a component of $G - Y$, it follows that there is a $Q \in \mathcal{Q}$ containing all vertices of C not contained in Y . \square

To deduce Theorem 4, we want to apply Lemma 5 to a suitable family \mathcal{R} with $\bigcup \mathcal{R} \supseteq X$ to deduce that some component of $G - X$ contains many vertices of C . Here, \mathcal{D} consists of cycles of length at most ℓ , so if the sets in \mathcal{R} are at pairwise distance $> \lfloor \ell/2 \rfloor$, then no $D \in \mathcal{D}$ can pass through two of them. The next lemma ensures that we can find such a family \mathcal{R} with a bound on $|\bigcup \mathcal{R}|$, when the cycle C is geodesic.

Lemma 6. *Let d be a positive integer, G a graph, $X \subseteq V(G)$ and $C \subseteq G$ a geodesic cycle. Then there exists a family \mathcal{R} of disjoint sets of vertices of G with $X \subseteq \bigcup \mathcal{R} \subseteq X \cup V(C)$ and $|V(C) \cap \bigcup \mathcal{R}| \leq 2d|X|$ such that for each $R \in \mathcal{R}$, the set $V(C) \cap R$ induces a (possibly empty) connected subgraph of C and the distance between any two sets in \mathcal{R} is greater than d .*

Proof. Let $Y \subseteq V(G)$ and $y \in Y$. For $j \geq 0$, let $B_Y^j(y)$ be the set of all $z \in Y$ at distance at most jd from y . Since $|B_Y^0(y)| = 1$, there is a maximum number j for which $|B_Y^j(y)| \geq 1 + j$, and we call this $j = j_Y(y)$ the *range of y in Y* . Observe that every $z \in Y \setminus B_Y^{j_Y(y)}$ has distance greater than $(j_Y(y) + 1)d$ from y .

Starting with $X_1 := X$, repeat the following procedure for $k \geq 1$. If X_k does not meet C , terminate the process. Otherwise, pick an $x_k \in X_k \cap V(C)$ of maximum range in X_k . Let $j_k := j_{X_k}(x_k)$ and $B_k := B_{X_k}^{j_k}(x_k)$. Define $X_{k+1} := X_k \setminus B_k$ and repeat.

Since the size of X_k decreases in each step, there is a smallest integer m for which $X_{m+1} \cap V(C)$ is empty, at which point the process terminates. By construction, the distance between B_k and X_{k+1} is greater than d for each $k \leq m$. For each k satisfying $1 \leq k \leq m$, there are two edge-disjoint paths $P_k^1, P_k^2 \subseteq C$, starting at x_k and each of length at most $j_k d$, so that $B_k \cap V(C) \subseteq S_k := P_k^1 \cup P_k^2$, because C is geodesic. Choose these paths minimal, so that the endvertices of S_k lie in B_k . Note that every vertex of S_k has distance at most $j_k d$ from x_k . Therefore, the distance between $R_k := B_k \cup S_k$ and X_{k+1} is greater than d .

We claim that the distance between R_k and $R_{k'}$ is greater than d for any $k < k'$. Since $B_{k'} \subseteq X_{k+1}$, it is clear that every vertex of $B_{k'}$ has distance greater than d from R_k . Take a vertex $q \in S_{k'} \setminus B_{k'}$ and assume for a contradiction that its distance to R_k was at most d . Then the distance between x_k and q is at most $(j_k + 1)d$. Let $a, b \in B_{k'}$ be the endvertices of $S_{k'}$. If $x_k \notin S_{k'}$, then one of a and b lies on the shortest path from x_k to q within C . Since C is geodesic, that endvertex then has distance at most $(j_k + 1)d$ from x_k . But then, since j_k is the range of x_k in X_k , that vertex would already lie in B_k , a contradiction. Suppose now that $x_k \in S_{k'}$. Then x_k lies on one of $P_{k'}^1$ or $P_{k'}^2$, so the distance between x_k and $x_{k'}$ is at most $j_{k'}d$. Since $x_{k'} \in X_k \cap V(C)$, it follows from our choice of x_k that

$$j_k = j_{X_k}(x_k) \geq j_{X_k}(x_{k'}) \geq j_{X_{k'}}(x_{k'}) = j_{k'},$$

where the second inequality follows from the fact that $X_{k'} \subseteq X_k$ and $j_Y(y) \geq j_{Y'}(y)$ whenever $Y \supseteq Y'$. But then $x_{k'} \in B_k$, a contradiction. This finishes the proof of the claim.

Finally, let $\mathcal{R} := \{R_k : 1 \leq k \leq m\} \cup \{X_{m+1}\}$. The distance between any two sets in \mathcal{R} is greater than d . For $k \leq m$, $R_k \cap V(C) = S_k$ is a connected subgraph of C , while $X_{m+1} \cap V(C)$ is empty. Moreover,

$$\begin{aligned} |V(C) \cap \bigcup \mathcal{R}| &= \sum_{k=1}^m |S_k| \leq \sum_{k=1}^m (1 + 2j_k d) \leq \sum_{k=1}^m (1 + 2(|B_k| - 1)d) \\ &\leq \sum_{k=1}^m 2|B_k|d \leq 2d|X|. \end{aligned} \quad \square$$

Proof of Theorem 4. Let $X \subseteq V(G)$ be of order at most k and let $d := \lfloor p/2 \rfloor$. By Lemma 6, there exists a family \mathcal{R} of disjoint sets of vertices of G with $X \subseteq \bigcup \mathcal{R} \subseteq X \cup V(C)$ and $|V(C) \cap \bigcup \mathcal{R}| \leq 2dk$ so that for each $R \in \mathcal{R}$, the set $R \cap V(C)$ induces a (possibly empty) connected subgraph of C and the distance between any two sets in \mathcal{R} is greater than d .

Let \mathcal{D} be a set of cycles of length at most p with $C = \bigoplus \mathcal{D}$. Then no $D \in \mathcal{D}$ can meet two distinct $R, R' \in \mathcal{R}$, since the diameter of D is at most d . By Lemma 5, there is a component Q of $G - \bigcup \mathcal{R}$ which contains every vertex of $C \setminus \bigcup \mathcal{R}$. This component is connected in $G - X$ and therefore contained in some component Q' of $G - X$, which then satisfies

$$|Q' \cap V(C)| \geq |C| - |V(C) \cap \bigcup \mathcal{R}| \geq |C| - 2dk.$$

Since $|C| \geq 4dk$, the claim follows. □

3 Remarks

We have described the content of Theorem 1 as an *algebraic* criterion for a graph to have large tree-width. The reader might object that the cycle C being ℓ -geodesic is a metric property and not an algebraic one. Karl Heuer has pointed out to us, however,

that geodecity of a cycle can be expressed as an algebraic property after all. This is a consequence of a more general lemma of Gollin and Heuer [1], which allowed them to introduce a meaningful notion of geodecity for cuts.

Proposition 7 ([1]). *Let G be a graph with length-function ℓ and $C \subseteq G$ a cycle. Then C is ℓ -geodesic if and only if there do not exist cycles D_1, D_2 with $\ell(D_1), \ell(D_2) < \ell(C)$ such that $C = D_1 \oplus D_2$.*

Finally, we would like to point out that Theorem 1 does not only offer a ‘one-way criterion’ for large tree-width, but that it has a qualitative converse. First, we recall the Grid Minor Theorem of Robertson and Seymour [4], phrased in terms of walls. For a positive integer t , an *elementary t -wall* is the graph obtained from the $2t \times t$ -grid as follows. Delete all edges with endpoints $(i, j), (i, j + 1)$ when i and j have the same parity. Delete the two resulting vertices of degree one. A *t -wall* is any subdivision of an elementary t -wall. Note that the $(2t \times 2t)$ -grid has a subgraph isomorphic to a t -wall.

Theorem 8 (Grid Minor Theorem [4]). *For every $t \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that every graph of tree-width at least k contains a t -wall.*

Here, then, is our qualitative converse to Theorem 1, showing that the algebraic condition in the premise of Theorem 1 in fact captures tree-width.

Corollary 9. *For every $L \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ such that for every graph G the following holds. If G has tree-width at least k , then there exists a rational-valued length-function on G such that G contains an ℓ -geodesic cycle C with $\ell(C) \geq L$ which is the \mathbb{F}_2 -sum of cycles of ℓ -length at most 1.*

Proof. Let $s := 6L$. By the Grid Minor Theorem, there exists an integer k such that every graph of tree-width at least k contains an s -wall. Suppose G is a graph of tree-width at least k . Let W be an elementary s -wall so that G contains some subdivision W' of W , where each $e \in E(W)$ has been replaced by some path $P^e \subseteq G$ of length $m(e)$. The outer cycle C of W satisfies $d_C(u, v) \leq 3d_W(u, v)$ for all $u, v \in V(C)$. Moreover, C is the \mathbb{F}_2 -sum of cycles of length at most six.

Define a length-function ℓ on G as follows. Let $e \in E(G)$. If $e \in P^f$ for $f \in E(C)$, let $\ell(e) := 1/m(f)$. Then $\ell(P^f) = 1$ for every $f \in E(C)$. If $e \in P^f$ for $f \in E(W) \setminus E(C)$, let $\ell(e) := 3/m(f)$. Then $\ell(P^f) = 3$ for every $f \in E(W) \setminus E(C)$. If $e \notin E(W')$, choose $\ell(e)$ large enough so that $\ell(e) > \ell(W')$, for example take $\ell(e) := 10s^3$.

It is easy to see that the subdivision $C' \subseteq G$ of C is ℓ -geodesic in G . It has length $\ell(C') = |C| \geq 3s$ and is the \mathbb{F}_2 -sum of the subdivisions of 6-cycles of W . Each of these has ℓ -length 18. Rescaling all lengths by a factor of $1/18$ yields the desired result. \square

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