The classification of 2-extendable edge-regular graphs with diameter 2

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Abstract

Let ℓ denote a non-negative integer. A connected graph Γ of even order at least $2\ell+2$ is *ℓ-extendable* if it contains a matching of size ℓ and if every such matching is contained in a perfect matching of Γ. A connected regular graph Γ is *edge-regular*, if there exists an integer λ such that every pair of adjacent vertices of Γ have exactly λ common neighbours. In this paper we classify 2-extendable edge-regular graphs of even order and diameter 2.

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1 Introductory remarks

Throughout this paper graphs are assumed to be finite and simple. We first recall the definition of ℓ-extendable graphs, introduced in 1980 by Plummer [18]. Let ℓ denote a non-negative integer and let Γ be a connected graph of even order at least $2\ell + 2$. It is said that Γ is *ℓ-extendable* if it contains a matching of size ℓ, and if every such matching is contained in a perfect matching of Γ. Otherwise, Γ is said to be *non-ℓ-extendable*. We remark that in his definition from 1980, Plummer did not require an ℓ-extendable graph to have order at least $2\ell + 2$. But it turns out that this additional assumption is convenient since in this case ℓ-extendability of Γ implies $(\ell - 1)$-extendability of Γ. Since its introduction in 1980, the family of ℓ-extendable graphs has been studied from various points of view, see for instance [1, 3, 4, 14, 16, 19, 20, 21, 22, 23].

Considerable attention was given to the study of extendability of various families of highly regular graphs. For example, the extendability of the well known family of strongly regular graphs was considered in [5, 12, 15]. Recall that a connected graph Γ is *strongly regular* with parameters (n, k, λ, μ) if Γ is a k-regular graph on n vertices such that any two adjacent (nonadjacent and distinct, respectively) vertices have exactly λ (μ, respectively) common neighbours. It was proved in [12] that each strongly regular graph of even order $n \geqslant 4$ is 1-extendable. Moreover, the results of [12, 15] imply the a strongly regular graph Γ of even order is not 2-extendable if and only if Γ is either the four cycle C_4 (of valency 2), or the Petersen graph (of valency 3), or the complete tripartite graph $K_{2,2,2}$ (of valency 4). The extendability of distance-regular graphs, which are a generalization of strongly regular graphs, was studied in [6].

The diameter of a connected strongly regular graph, which is not complete, is of course 2. Therefore, it seems natural to consider extendability of other regular graphs of diameter 2. For example, the extendability of Deza graphs, where we only insist that there exist two numbers such that for any pair of different vertices the number of their common neighbours equals one of those two numbers, was studied in [17]. It was proved that, apart from the above mentioned C_4, the Petersen graph and $K_{2,2,2}$, the only remaining non-2-extendable Deza graph of even order and diameter 2 is the complement of the Möbius ladder on eight vertices. Recently, the so-called *quasi-strongly regular graphs* with diameter 2 and grade 2 (see [9] for the definition of these graphs), which are 2-extendable, were classified in [1].

The most important facts, on which the proofs of the results from [1, 12, 15, 17] rely, are the regularity of the graph, the fact that its diameter equals 2 and the existence of the parameter λ. Therefore, when one studies 2-extendability of regular graphs, it seems natural to consider the class of the so-called edge-regular graphs with diameter 2. A graph Γ is called *edge-regular* with parameters (n, k, λ), if Γ is a k-regular graph on n vertices, such that any pair of adjacent vertices share λ common neighbours. The following problem was posed in [1]:

Problem 1. Classify the 2-extendable edge-regular graphs (of even order) and diameter 2.

In this paper we solve the above problem. The main result of this paper is the following theorem.

Theorem 2. *Let Γ be an edge-regular graph of diameter 2 with parameters (n, k, λ), where $k \geqslant 3$ and $n \geqslant 6$ is even. Then Γ is not 2-extendable if and only if it is isomorphic to one of the following graphs:*

- *the complete multipartite graph $K_{2,2,2}$ (which is strongly regular with parameters $(6, 4, 2, 4)$);*
- *the Petersen graph (which is strongly regular with parameters $(10, 3, 0, 1)$);*
- *the Möbius ladder on eight vertices (which is edge-regular with parameters $(8, 3, 0)$);*
- *the lexicographic product $C_5[2K_1]$ of the 5-cycle with the empty graph on two vertices (which is edge-regular with parameters $(10, 4, 0)$);*
- *a graph from Construction 10 (which is edge-regular with parameters $(4k - 2, k, 0)$ for some integer $k \geqslant 3$).*

2 Preliminaries

In this section we first fix some notation and then gather various results from the literature that will be used in the remainder of the paper.

Let Γ be a connected graph with vertex set $V = V(\Gamma)$. By the *order* of Γ we mean the cardinality of set V. Let $u, v \in V$. The distance between u and v will be denoted by $d(u, v)$. For a non-negative integer i, we denote by $N_i(v) = \{x \in V : d(x, v) = i\}$ the set of all vertices of Γ at distance i from v. We abbreviate $N_1(v)$ by $N(v)$. The *valency* of v is the cardinality of set $N(v)$. If all vertices of Γ have the same valency k, then we say that Γ is *regular* with valency k. The fact that the vertices u and v are adjacent in Γ will be denoted by $u \sim v$. For a subset $S \subseteq V$ we let $\Gamma - S$ be the subgraph of Γ induced on the set $V \setminus S$.

Let Γ_1 and Γ_2 be graphs. The *lexicographic product* of Γ_1 with Γ_2, denoted by $\Gamma_1[\Gamma_2]$, is the graph whose vertex set is the cartesian product $V(\Gamma_1) \times V(\Gamma_2)$, with vertices (u_1, u_2) and (v_1, v_2) being adjacent if and only if either u_1 is adjacent with v_1 in Γ_1, or $u_1 = v_1$ and u_2 is adjacent with v_2 in Γ_2. We will denote the complete graph and the cycle on n vertices by K_n and C_n, respectively.

Let us now state some results from the literature that we will need in the remainder of the paper. The first is about all regular graphs of diameter 2.

Proposition 3. ([11, Theorem 1.3], [17, Theorem 2.2, Theorem 2.3]) *Let Γ be a regular graph of even order and diameter 2. Then Γ is both 0- and 1-extendable.*

The general idea of the proof of Theorem 2 is similar to the one used in [15] ([17], respectively) for strongly regular graphs (Deza graphs, respectively) of even order. A similar idea was recently employed in [1]. One of the key factors in all of these proofs is the classical result of Tutte from 1947 giving a necessary and sufficient condition for a graph to contain a perfect matching. To state it we first need to fix some additional notation. Connected components of a graph Γ will simply be called *components* of Γ. A component C of Γ is called *even* (*odd*, respectively), if the cardinality of C is even (odd, respectively). The number of odd components of Γ will be denoted by $o(\Gamma)$. We can now state the above mentioned result of Tutte.

Theorem 4. ([7, Theorem 2.2.1]) *A graph Γ has a perfect matching if and only if for every subset $S \subseteq V(\Gamma)$ we have $o(\Gamma - S) \leqslant |S|$.*

When dealing with ℓ-extendability, the following corollary of Theorem 4 is of use. The result was implicitly proved in [22, Theorem 2.2]. A short proof is given also in [1].

Proposition 5. *Let $\ell \geqslant 1$ be an integer and let Γ be a connected graph of order at least $2\ell + 2$ containing a perfect matching. Then the graph Γ is not ℓ-extendable if and only if it contains a subset S of vertices such that the subgraph $\Gamma(S)$ induced by S contains ℓ independent edges and $o(\Gamma - S) \geqslant |S| - 2\ell + 2$.*

When searching for the largest ℓ for which the graph is still ℓ-extendable one can use the following corollary (see [1, Corollary 2.5]), which will be the main ingredient of our proofs throughout the paper.

Corollary 6. *Let $\ell \geqslant 1$ be an integer and let Γ be an $(\ell - 1)$-extendable connected graph of order at least $2\ell + 2$. Then Γ is not ℓ-extendable if and only if it contains a subset S of vertices such that the subgraph induced by S contains ℓ independent edges and $o(\Gamma - S) = |S| - 2\ell + 2$.*

Recall that a graph Γ is *edge-regular* with parameters (n, k, λ) if Γ is a k-regular graph on n vertices such that any pair of adjacent vertices share λ common neighbours. Let now Γ denote an edge-regular graph with parameters (n, k, λ). Clearly we have that $\lambda \leqslant k-1$, and if $\lambda = k-1$, then Γ is a disjoint union of complete graphs on $k+1$ vertices. It is also well known and easy to see that Γ contains exactly $nk\lambda/6$ triangles. The following upper bound on the number of vertices of an edge-regular graph will be very useful in the rest of the paper.

Proposition 7. ([13, Proposition 2.4]) *Let Γ be an (n, k, λ) edge-regular graph with diameter 2. Then*

$$n \leqslant 1 + k + k(k - \lambda - 1). \tag{1}$$

It proves convenient to analyze the cubic graphs of diameter 2 separately. These are particularly easy to deal with since there is only a handful of graphs to consider (see [17, Theorem 3.1]).

Theorem 8. *The five graphs from Figure 1 are the only cubic graphs of diameter 2. The only 2-extendable graph among them is the complete bipartite graph $K_{3,3}$ and the only edge-regular graphs among them are the complete bipartite graph $K_{3,3}$, the Petersen graph and the Möbius ladder on eight vertices. As a consequence, the Petersen graph and the Möbius ladder on eight vertices are the only cubic non-2-extendable edge-regular graphs of diameter 2.*

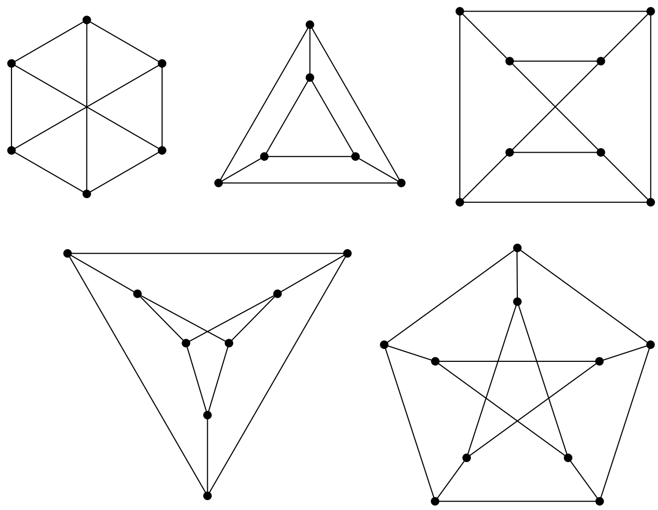

Figure 1: The five cubic graphs of diameter 2.

Proof. That the five graphs from Figure 1 are the only cubic graphs of diameter 2 follows from [17, Theorem 3.1]. The other claims are easy to check. $\square$

The following result about tetravalent edge-regular graph with diameter 2 will also be useful.

Proposition 9. *Let Γ be an $(n, 4, \lambda)$ edge-regular graph of diameter 2 and with n even. Then either Γ is isomorphic to the complete tripartite graph $K_{2,2,2}$ (which is strongly regular with $\lambda = 2$) or $\lambda = 0$.*

Proof. If $\lambda = 3$, then Γ is a complete graph, contradicting the assumption on the diameter of Γ. If $\lambda = 2$, then it is not difficult to see that Γ is isomorphic to $K_{2,2,2}$ (see also the proof of [1, Lemma 3.2]). Assume finally that $\lambda = 1$. Since n is even, inequality (1) implies that $n \leqslant 12$. As the number of triangles contained in Γ equals $nk\lambda/6 = 2n/3$, we have that n is divisible by 3, and so $n \in \{6, 12\}$. It is easy to see that $n = 6$ is not possible, so $n = 12$. By [10, Corollary 6], there are exactly two $(12, 4, 1)$ edge-regular graphs, namely the line graphs of the cube graph and the Möbius ladder on eight vertices. However, one can check that both have diameter 3. This finishes the proof. $\square$

We finish this section with the following construction providing an infinite family of non-2-extendable edge-regular graphs of diameter 2.

Construction 10. *Let $k \geqslant 3$ be an integer and let $W = \{w_1, w_2, \ldots, w_{2k-2}\}$. Let $S_1, S_2, \ldots, S_k$ be pairwise distinct $(k-1)$-element subsets of W such that the following is satisfied:*

(1) for $1 \leqslant i < j \leqslant k$ we have $S_i \cap S_j \neq \emptyset$;

(2) for every 2-element subset T of W there exists i $(1 \leqslant i \leqslant k)$, such that either $T \subseteq S_i$ or $T \subseteq W \setminus S_i$.

Let Γ be the graph with vertex set $\{x_i \mid 1 \leqslant i \leqslant k\} \cup \{y_i \mid 1 \leqslant i \leqslant k\} \cup W$ and edge set $E_1 \cup E_2 \cup E_3$, where

$$E_1 = \{x_i y_i \mid 1 \leqslant i \leqslant k\},$$

$$E_2 = \bigcup_{1 \leqslant i \leqslant k} \{x_i w_j \mid j \in S_i\}, \qquad E_3 = \bigcup_{1 \leqslant i \leqslant k} \{y_i w_j \mid j \in W \setminus S_i\}.$$

Proposition 11. *Let Γ be as in Construction 10. Then Γ is a $(4k-2, k, 0)$ edge-regular graph of diameter 2 which is not 2-extendable.*

Proof. Note that each w_i is adjacent to precisely one of x_j, y_j for each $1 \leqslant j \leqslant k$ and to no other vertices. It is now clear that Γ is a k-valent graph of order $4k-2$ without triangles, and so it is a $(4k-2, k, 0)$ edge-regular graph. Condition (2) from Construction 10 ensures that the distance between any two vertices from W is 2. The fact that the $(k-1)$-element subsets S_i of W are pairwise distinct together with the condition (1) from Construction 10 imply that for any $1 \leqslant i < j \leqslant k$ the distance between any of x_i, y_i to any of x_j, y_j is also 2. Observe also that for $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant 2k-2$, exactly one of x_i, y_i is adjacent with w_j. As x_i and y_i are adjacent, it follows that $d(x_i, w_j) \leqslant 2$ and $d(y_i, w_j) \leqslant 2$. Thus, Γ is of diameter 2. Taking $S = \{x_i \mid 1 \leqslant i \leqslant k\} \cup \{y_i \mid 1 \leqslant i \leqslant k\}$ Proposition 3 and Corollary 6 imply that Γ is not 2-extendable. □

Remark 12. *Observe that a suitable collection of subsets $S_1, S_2, \ldots, S_k$ exists for every $k \geqslant 3$. Namely, one can take $S_i = \{w_i, w_k, w_{k+1}, \ldots, w_{2k-3}\}$ for $1 \leqslant i \leqslant k-1$ and $S_k = \{w_1, w_2, \ldots, w_{k-1}\}$. We would also like to point out that one can verify that in the case of $k = 3$, all of the graphs from Construction 10 are isomorphic to the Petersen graph (no matter what the subsets S_1, S_2 and S_3 are). Similarly, it can be shown that for $k = 4$ all graphs from Construction 10 are isomorphic. However, this does not hold in general. For instance, for $k = 5$ there are at least three nonisomorphic graphs that can be obtained via Construction 10.*

3 Odd components

Throughout this section let Γ be a non-2-extendable edge-regular graph with parameters (n, k, λ), n even, $k \geqslant 4$ and with diameter 2. As non-2-extendable strongly regular graphs were already classified, we can assume that Γ is not strongly regular. Recall that Γ is 1-extandable by Proposition 3, and so Corollary 6 implies that it contains a subset S of vertices such that the subgraph induced by S contains two independent edges and $o(\Gamma - S) = |S| - 2$. In this section we prove that all odd components of $\Gamma - S$ are singletons.

We first argue that without loss of generality we can assume that all components of $\Gamma - S$ are odd. Indeed, suppose that $\Gamma - S$ has an even component C. Pick $x \in C$ and set $S' = S \cup \{x\}$. Observe that each component of $\Gamma - S$, different from C, is also a component of $\Gamma - S'$. The remaining components of $\Gamma - S'$ are the components of the subgraph of Γ, induced on $C \setminus \{x\}$. Since this set is of odd size, at least one of them is odd. Thus $o(\Gamma - S') \geqslant o(\Gamma - S) + 1 = |S| - 1 = |S'| - 2$. However, as Γ is 1-extendable and $o(\Gamma - S')$ and $|S'|$ are of the same parity, Proposition 5 implies that in fact $o(\Gamma - S') = |S'| - 2$ holds. Repeating this process of enlarging S until no even component of $\Gamma - S$ exists we can thus eliminate all even components of $\Gamma - S$. For the rest of the paper we can thus assume that $\Gamma - S$ has no even components. For future reference we also name the endvertices of the chosen two independent edges as follows.

Notation 13. *Let Γ be a non-2-extendable edge-regular graph with parameters (n, k, λ), n even, $k \geqslant 4$ and with diameter 2, which is not strongly regular. Pick a subset S of $V(\Gamma)$, such that $o(\Gamma - S) = |S| - 2$, that $\Gamma - S$ has no even components and that the subgraph induced by S contains 2 independent edges u_1v_1 and u_2v_2.*

Remark 14. *Since $|S| \geqslant 4$, we have that $o(\Gamma - S) \geqslant 2$. As Γ is of diameter 2, it thus follows that each $v \in V(\Gamma) \setminus S$ has at least one neighbour in S.*

Lemma 15. *With reference to Notation 13, each vertex in $\Gamma - S$ has at least two neighbours in S.*

Proof. Suppose on the contrary that there is a vertex v in an (odd) component C, which has just one neighbour in S. Denote this neighbour by u. Let $C_1, C_2, \ldots, C_{|S|-3}$ denote the other (odd) components of $\Gamma - S$ and let $m_i = |C_i|$ for $1 \leqslant i \leqslant |S| - 3$. Without loss of generality assume $m_1 \leqslant m_2 \leqslant \cdots \leqslant m_{|S|-3}$. As the diameter of Γ is 2, the unique neighbour u of v from S must be adjacent to all vertices in $C_1 \cup C_2 \cup \cdots \cup C_{|S|-3}$. Since u and v also have λ common neighbours (which are all in C), this implies

$$k \geqslant |N(u) \setminus S| \geqslant 1 + m_1 + m_2 + \cdots + m_{|S|-3} + \lambda. \tag{2}$$

On the other hand, for every vertex $w \in C_1$ we have $N(w) \subseteq C_1 \cup S$. Moreover, since u is adjacent to each vertex of C_1, w can have at most λ neighbours in C_1, implying

$$k \leqslant \lambda + |S|. \tag{3}$$

We now split our analysis into two cases.

Case 1: $m_1 \geqslant 3$. In this case, using (2) and (3), we get $|S| = 4$. But then $u \in \{u_1, u_2, v_1, v_2\}$ has at least one neighbour in S, and so $1 + \lambda + m_1 + 1 \leqslant k$. Using this together with $m_1 \geqslant 3$ and (3) we get $\lambda + 5 \leqslant \lambda + 4$, a contradiction.

Case 2: $m_1 = 1$. Denote the (unique) vertex in C_1 by w_1. As u and w_1 have λ common neighbours (which are all contained in S), u must have at least λ neighbours in S. Therefore, similarly as in (2), we get $k \geqslant 1 + 1 + m_2 + \cdots + m_{|S|-3} + 2\lambda$. As all neighbours of w_1 are clearly in S, we also have $k \leqslant |S|$, and so

$$|S| - 3 + m_{|S|-3} + 2\lambda \leqslant 1 + 1 + m_2 + \cdots + m_{|S|-3} + 2\lambda \leqslant k \leqslant |S|. \tag{4}$$

It follows that $\lambda \in \{0, 1\}$.

If $k = |S|$, then w_1 is a neighbour of u_1 and v_1, implying $\lambda = 1$. Then (4) implies $m_{|S|-3} = 1$, and so $k = |S| = 4$, since otherwise the unique element of $C_{|S|-3} \neq C_1$ is another common neighbour of $u_!$ and v_1. This contradicts Proposition 9.

Therefore, $|S| > k$, and so (4) implies $\lambda = 0$. Consequently at least one of u_1, v_1 and at least one of u_2, v_2 is not adjacent to w_1, and so $k \leqslant |S| - 2$. But then (4) implies $k = |S| - 2$ and $m_1 = m_2 = \cdots = m_{|S|-3} = 1$. For each $1 \leqslant i \leqslant |S| - 3$ let w_i be the unique element of C_i. Now, $N(u) = \{v, w_1, \dots, w_{|S|-3}\}$. As the diameter of Γ is 2, v must be adjacent to all vertices of $C \setminus \{v\}$, and so $|C| = k$. Since $\lambda = 0$, this shows that every vertex of $C \setminus \{v\}$ has $k - 1$ neighbours in S. Therefore, there are $(k-1)^2 + 1$ edges between C and S. As each of w_i ($1 \leqslant i \leqslant |S| - 3$) has all of its neighbours in S, there are $(|S|-3)k + (k-1)^2 + 1 = 2k^2 - 3k + 2$ edges between $V \setminus S$ and S. On the other hand, as there are at least two edges in S there are at most $k|S| - 4 = k^2 + 2k - 4$ edges between S and $V \setminus S$. Therefore $2k^2 - 3k + 2 \leqslant k^2 + 2k - 4$, implying that $(k-3)(k-2) \leqslant 0$. But this contradicts $k \geqslant 4$. □

The proof of the following result is similar to the proof of [1, Lemma 4.3] and is therefore omitted.

Lemma 16. *With reference to Notation 13, suppose C is a component of $\Gamma - S$ which is not a singleton. Then there are at least $3k/2$ edges between C and S. In particular, for each component C of $\Gamma - S$ there are at least k edges between C and S.*

Proposition 17. *With reference to Notation 13, $\Gamma - S$ has at most one component with cardinality at least 3.*

Proof. Suppose that $\Gamma - S$ has two odd components C_1, C_2 with cardinalities at least 3. Without loss of generality we can assume that $m_1 = |C_1| \leqslant m_2 = |C_2|$. Let t denote the number of edges between S and $V \setminus S$. Since S contains at least two (independent) edges, we have that

$$t \leqslant k|S| - 4. \tag{5}$$

Suppose first that $k \geqslant m_1 + 2$. As each vertex of C_1 has at least $k - (m_1 - 1)$ neighbours in S, there are at least $km_1 - m_1(m_1 - 1)$ edges between C_1 and S. By Lemma 16 there are at least $3k/2$ edges between C_2 and S and at least k edges between any of the remaining $|S| - 4$ components of $\Gamma - S$ and S, which implies

$$t - (k|S| - 4) \geqslant km_1 - m_1(m_1 - 1) + \frac{3k}{2} + k(|S| - 4) - (k|S| - 4) =$$

$$k(m_1 - \frac{5}{2}) - m_1(m_1 - 1) + 4 \geqslant (m_1 + 2)(m_1 - \frac{5}{2}) - m_1(m_1 - 1) + 4 = \frac{m_1}{2} - 1 > 0.$$

But this contradicts (5), and so we have that $k \leqslant m_1 + 1$.

Recall that, by Lemma 15, each vertex of $C_1 \cup C_2$ has at least two neighbours in S. Using this and Lemma 16 we find that

$$k|S| - 4 \geqslant t \geqslant 2m_1 + 2m_2 + k(|S| - 4) \geqslant 4m_1 + k(|S| - 4). \tag{6}$$

This implies $k \geqslant m_1 + 1$, and therefore $k = m_1 + 1$. But then equality holds in (6), and so $k|S| - 4 = t = 2m_1 + 2m_2 + k(|S| - 4) = 4m_1 + k(|S| - 4)$. This implies that there are exactly two edges in S, that $m_1 = m_2$, that each vertex in $C_1 \cup C_2$ has exactly two neighbours in S, and that all of the remaining components of $\Gamma - S$ are singletons. It follows that each vertex of C_1 (C_2, respectively) has exactly $m_1 - 1$ neighbours in C_1 (C_2, respectively), which implies that the subgraphs induced on C_1 and C_2 are complete graphs. Consequently, $\lambda \geqslant m_1 - 2 \geqslant 1$, and so Proposition 9 implies $k > 4$, yielding $m_1 \geqslant 5$. Since $\lambda \leqslant k - 1 = m_1$ and Γ is not a complete graph, it follows that $\lambda \in \{m_1 - 1, m_1 - 2\}$.

Recall that by our assumption Γ is not strongly regular. It follows from [2, Theorem 1.4.3(ii)] that

$$\lambda < k + \frac{1}{2} - \sqrt{2k+2} = m_1 + \frac{3}{2} - \sqrt{2m_1 + 4}.$$

Since $m_1 \geqslant 5$ we have that $m_1 + \frac{3}{2} - \sqrt{2m_1+4} < m_1 - 2$, and so $\lambda < m_1 - 2$, a contradiction. $\square$

Proposition 18. *With reference to Notation 13, all components of $\Gamma - S$ are singletons.*

Proof. Suppose on the contrary that $\Gamma - S$ has a component with cardinality at least 3. By Proposition 17, $\Gamma - S$ has exactly one such component. Denote the vertices of the singleton components by $w_1, \ldots, w_{|S|-3}$ and denote the (odd) component with cardinality $m = 2r + 1 \geqslant 3$ by C. Let s denote the number of edges contained in S and let t denote the number of edges between S and C. Counting the number of edges between S and $V(\Gamma) \setminus S$ in two ways we get

$$k|S| - 2s = k(|S| - 3) + t. \tag{7}$$

By Lemma 16 we have that $t \geqslant 3k/2$, and so (7) implies $s \leqslant 3k/4$.

Pick $i \in \{1, 2, \ldots, |S| - 3\}$ and observe that $N(w_i) \subseteq S$. Note also that there are exactly $k\lambda/2$ edges contained in $N(w_i)$, which implies $k\lambda/2 \leqslant s \leqslant 3k/4$. This shows that $\lambda \in \{0, 1\}$.

Case 1: $\lambda = 0$. Since Γ is triangle free, so is the subgraph of Γ induced on C. By the well-known result of Mantel (see [7, Theorem 7.1.1] for a more general Turán's theorem), there are at most $\lfloor m^2/4 \rfloor = r(r+1)$ edges contained in C, and so $t \geqslant km - 2r(r+1)$. Thus (7) gives

$$k|S| - 2s \geqslant k(|S| - 3) + k(2r + 1) - 2r(r + 1),$$

implying

$$r(r+1) \geqslant k(r-1) + s \geqslant k(r-1) + 2. \tag{8}$$

But this shows that $k(r-1) \leqslant r(r+1) - 2 = (r+2)(r-1)$, and so either $r = 1$ or $k \leqslant r + 2 = (m+3)/2$. In the latter case $m \geqslant 2k - 3$. As every vertex of C sends at least two edges to S, this implies that $t \geqslant 2m \geqslant 4k - 6$, and so (7) gives

$$k|S| - 2s \geqslant k(|S| - 3) + 4k - 6 = k|S| + k - 6.$$

This implies $k \leqslant 6 - 2s$, and so $s \geqslant 2$ forces $k \leqslant 2$, a contradiction. Suppose then that $r = 1$. By (8) we get $s = 2$, and so (7) implies $t = 3k - 4$. Moreover, since $\lambda = 0$ the connected component C must be the 2-path, say (z_1, z_2, z_3). Furthermore, since Γ is of diameter 2, each of the vertices w_i, $1 \leqslant i \leqslant |S| - 3$ is adjacent to each vertex from $S' = S \setminus \{u_1, v_1, u_2, v_2\}$ (recall that $s = 2$) and to precisely one of u_1, v_1 and to one of u_2, v_2. Therefore, $k = |S| - 2$. It follows that the number t' of edges between C and S' is $k(|S| - 4) - (k - 2)(|S| - 3) = k - 2$. On the other hand, since $\lambda = 0$, each of z_1 and z_3 has at least $k - 2$ neighbours in S' and z_2 has at least $k - 4$ neighbours in S'. Therefore, $k - 2 = t' \geqslant 2(k - 3) + k - 4 = 3k - 10$, implying $k = 4$. For each of the two vertices of S' to be at distance at most 2 from each of z_1, z_2, z_3 we must have that z_2 is adjacent to both vertices from S' (recall that each w_i, $1 \leqslant i \leqslant 3$ is adjacent to both vertices of S'). Thus z_1 must have three neighbours in $\{u_1, v_1, u_2, v_2\}$, contradicting $\lambda = 0$.

Case 2: $\lambda = 1$. By Proposition 9 we have that $k \geqslant 5$, and so $N(w_1) \subseteq S$, implying that $|S| \geqslant 5$. Consider $N(w_1) \cup N(w_2) \subseteq S$. Since $\lambda = 1$ there are $k/2$ edges within $N(w_1)$ and $k/2$ edges within $N(w_2)$. But no two vertices in $N(w_1) \cap N(w_2)$ can be adjacent (otherwise $\lambda > 1$), and so there are at least $k/2 + k/2 = k$ edges within $N(w_1) \cup N(w_2)$. Hence $s \geqslant k$, contradicting $s \leqslant 3k/4$. □

4 Proof of the main theorem

In this section we finally prove our main theorem. With reference to Notation 13, recall that by Proposition 18, $\Gamma - S$ is an empty graph on $|S| - 2$ which we denote by $w_1, \ldots, w_{|S|-2}$. We start with the following result.

Proposition 19. *With reference to Notation 13, there are exactly k edges contained in S. Furthermore, $\lambda = 0$.*

Proof. Let s denote the number of edges contained in S. Counting the edges between S and $V(\Gamma) \setminus S$ in two ways we find that

$$k|S| - 2s = k(|S| - 2) = k|S| - 2k,$$

and so $s = k$. Pick $i \in \{1, 2, \ldots, |S| - 2\}$ and observe that $N(w_i) \subseteq S$. The number of edges within $N(w_i)$ is $k\lambda/2$, and so $\lambda \in \{0, 1, 2\}$. We assume that $\lambda \neq 0$ and obtain a contradiction. Since Γ was assumed not to be strongly regular, Proposition 9 implies that $k \geqslant 5$.

Case 1: $\lambda = 2$. Let $1 \leqslant i \leqslant |S| - 2$. By the above argument there are then k edges within $N(w_i)$, and so $s = k$ implies that no $x \in S \setminus N(w_i)$ can have a neighbour in S. But then $d(x, w_i) > 2$, a contradiction, which shows that $S = N(w_i)$ for all $1 \leqslant i \leqslant |S| - 2$. But then all of the $|S| - 2 = k - 2$ vertices w_i are common neighbours of u_1 and v_1. As $\lambda = 2$, this implies $k \leqslant 4$, a contradiction.

Case 2: $\lambda = 1$. We claim that every vertex of S has at least two neighbours in S. If x is a vertex of S which has no neighbours in S, then x can not have any common neighbours with any of the vertices w_i ($1 \leqslant i \leqslant |S|-2$), contradicting $\lambda = 1$. Suppose then that x is a vertex of S which has exactly one neighbour, say y, in S. Then x is adjacent with $k-1$ vertices in $\{w_i \mid 1 \leqslant i \leqslant |S|-2\}$. But if x is adjacent with some w_i, then w_i and x must have a common neighbour, and so this common neighbour must be y. Therefore, y is adjacent to the same $k-1$ vertices in $\{w_i \mid 1 \leqslant i \leqslant |S|-2\}$ as is x. But this shows that x and y have $k-1$ common neighbours, which forces $1 = \lambda \geqslant k-1$, a contradiction. This proves our claim that every vertex of S has at least two neighbours in S. As there are exactly k edges contained in S and $|S| \geqslant k$, this shows that every vertex of S has exactly two neighbours in S and $k = |S|$. Consequently, $N(w_i) = S$ for $1 \leqslant i \leqslant |S|-2$. As $|S| = k \geqslant 5$, this shows that adjacent vertices u_1, v_1 have at least three common neighbours (namely w_1, w_2 and w_3), contradicting $\lambda = 1$. This completes the proof. □

Our analysis of the remaining possibility $\lambda = 0$ is done separately for $k = 4$ and $k \geqslant 5$.

Proposition 20. *With reference to Notation 13, assume $k = 4$. Then either Γ is isomorphic to the lexicographic product $C_5[2K_1]$ of the 5-cycle with the empty graph on two vertices, or the subgraph of Γ induced on S is the disjoint union of four copies of the complete graph K_2.*

Proof. Recall that by Proposition 19 there are exactly 4 edges within S. By (1) we have that $n \leqslant 16$. Moreover, $n = 16$ is not possible by [8], and so $n \leqslant 14$. If $|S| \leqslant 5$, then w_1 has at least three neighbours in $\{u_1, v_1, u_2, v_2\}$, contradicting $\lambda = 0$. As $n = 2|S|-2 \geqslant 14$, this implies $6 \leqslant |S| \leqslant 8$.

Case 1: $|S| = 6$. Let $S = \{u_1, v_1, u_2, v_2, x, y\}$. As $\lambda = 0$, every w_i ($1 \leqslant i \leqslant 4$) is adjacent with with exactly one of the vertices u_1, v_1, with exactly one of the vertices u_2, v_2, and to both x and y. Therefore, x and y have no neighbours within S. Moreover, as there are 4 edges in S and $\lambda = 0$, we can assume that u_1 is adjacent with u_2 and that v_1 is adjacent with v_2. But it is now not difficult to see that $\Gamma \cong C_5[2K_1]$.

Case 2: $|S| = 7$. Let $S = \{u_1, v_1, u_2, v_2, x, y, z\}$. Note that since $|S|-2 = 5$ and $\lambda = 0$, at least one of u_1, v_1 can have at most two neighbours in $V(\Gamma) \setminus S$, and so has at least one neighbour in $S \setminus \{u_1, v_1\}$. Similarly, at least one of u_2, v_2 must have at least one neighbour in $S \setminus \{u_2, v_2\}$. But as there are exactly 4 edges within S, this implies that at least one of x, y, z, say x, has all neighbours in $\{w_i \mid 1 \leqslant i \leqslant 5\}$. It follows that the unique vertex of $\{w_i \mid 1 \leqslant i \leqslant 5\} \setminus N(x)$ is at distance at least 3 from x, a contradiction.

Case 3: $|S| = 8$. Similarly as in Case 2 above we can show that for the diameter of Γ to be 2, every vertex of S must have at least one neighbour in S. But as there are exactly 4 edges within S, this implies that the subgraph of Γ induced on S is the disjoint union of four copies of the complete graph K_2. □

Proposition 21. *With reference to Notation 13, assume $k \geqslant 5$. Then the subgraph of Γ induced on S is the disjoint union of k copies of the complete graph K_2.*

Proof. By Proposition 19 we have that $\lambda = 0$, and so $N(w_i)$ is an independent set for each $1 \leqslant i \leqslant |S|-2$. We first claim that each vertex of S has at least one neighbour in

S. If some $v \in S$ has no neighbours in S, then it must be adjacent to each w_i (otherwise the diameter of Γ would be at least 3), and so $k = |S| - 2$. Denote the two vertices of $S \setminus N(w_1)$ by x and y. As $N(w_1)$ is an independent set, each of the k edges within S is incident with at least one of x and y. As x, y are not adjacent with w_1, each of them has a neighbour in $N(w_1)$. Pick $z \in N(w_1) \cap N(x)$.

Assume first that x and y are not adjacent. Then at least one of x, y has at most $k/2$ neighbours in S, and consequently it has at least $k/2$ neighbours in $\{w_i \mid 2 \leqslant i \leqslant |S|-2\}$ (recall that x, y are not adjacent with w_1). Without loss of generality we can assume that this vertex is x. As z can not be adjacent with any other neighbour of w_1, it has at least $k-3$ neighbours in $\{w_i \mid 2 \leqslant i \leqslant |S|-2\}$. But since z and x have no common neighbours, this implies that

$$k - 1 = |S| - 3 = |\{w_i \mid 2 \leqslant i \leqslant |S| - 2\}| \geqslant (k-3) + \frac{k}{2},$$

forcing $k \leqslant 4$, a contradiction.

Assume now that x and y are adjacent. Similarly as above we can show that at least one of them, say x, has at least $(k-1)/2$ neighbours in $\{w_i \mid 2 \leqslant i \leqslant |S|-2\}$. But as z can not be adjacent with y, this implies that it has $k-2$ neighbours in $\{w_i \mid 2 \leqslant i \leqslant |S|-2\}$, and so

$$k - 1 = |S| - 3 = |\{w_i \mid 2 \leqslant i \leqslant |S| - 2\}| \geqslant (k-2) + \frac{k-1}{2},$$

implying that $k \leqslant 3$, a contradiction. This proves our claim that each vertex in S has at least one neighbour in S.

If a vertex $x \in S$ has all neighbours in S, then for $S_1 = S \setminus \{x\}$ we have that $o(\Gamma - S_1) = |S| - 1 = |S_1|$. But then Proposition 5 implies that Γ is not 1-extendable, contradicting Proposition 3. Therefore, every vertex of S has at least one neighbour in $V(\Gamma) \setminus S$.

To complete the proof pick any $x \in S$. By the above comments there exists w_i such that $x \in N(w_i)$. As every vertex of $N(w_i)$ has at least one neighbour in S, and this neighbour can not be in $N(w_i)$, the fact that there are exactly k edges within S implies that each of the k vertices of $N(w_i)$ is incident with exactly one edge whose other endvertex is in S, and so x has a unique neighbour in S. It follows that each vertex of S has exactly one neighbour in S, and so $\Gamma(S)$ is the disjoint union of k copies of the complete graph K_2. $\square$

Proposition 22. *Let Γ be a $(n, k, 0)$ edge-regular graph of diameter 2 with n even and $k \geqslant 3$. Suppose there exists a subset S of $V(\Gamma)$, such that $\Gamma(S)$ is the disjoint union of k copies of the complete graph K_2, and such that $\Gamma - S$ consists of $|S| - 2$ singleton components. Then Γ is isomorphic to a graph from Construction 10.*

Proof. Note that $|S| = 2k$. Denote the vertices of S by x_i, y_i $(1 \leqslant i \leqslant k)$ such that $x_i \sim y_i$ for each $1 \leqslant i \leqslant k$, and let $W = \{w_1, w_2, \ldots, w_{2k-2}\} = V(\Gamma) \setminus S$. For $1 \leqslant i \leqslant k$ define

$$S_i = N(x_i) \cap W = N(x_i) \setminus \{y_i\}.$$

Note that $|S_i| = k-1$. As $\lambda = 0$, it is clear that $N(y_i) \setminus \{x_i\} = W \setminus S_i$. Pick arbitrary $1 \leqslant i < j \leqslant k$. As the diameter of Γ is 2 and x_i, x_j are not adjacent, they must have a common neighbour, and so $S_i \cap S_j \neq \emptyset$. Similarly, as x_i, y_j are not adjacent, we have $S_i \cap (W \setminus S_j) \neq \emptyset$, and so $S_i \neq S_j$. Finally, pick $w_s, w_t \in W$, such that $s \neq t$. Note that w_s, w_t are not adjacent, and so they must have a common neighbour. If x_i is a common neighbour of w_s and w_t then $\{w_s, w_t\} \subseteq S_i$. If however y_i is a common neighbour of w_s and w_t, then $\{w_s, w_t\} \subseteq W \setminus S_i$. This shows that Γ is isomorphic to a graph from Construction 10. □

We can now finally prove Theorem 2.

Proof. It is easy to check that each of the four graphs from the first four items of the statement of Theorem 2 is a non-2-extendable edge-regular with diameter 2. The last item is covered by Proposition 11. Conversely, assume that Γ is a non-2-extendable (n, k, λ) edge-regular graph of diameter 2, where $k \geqslant 3$ and $n \geqslant 6$ is even. If $k = 3$, then Theorem 8 implies that Γ is isomorphic either to the Petersen graph, or to the Möbius ladder on eight vertices. If $k \geqslant 4$ and Γ is strongly regular, then the results from [12, 15] imply that Γ is isomorphic to the complete multipartite graph $K_{2,2,2}$. If $k \geqslant 4$ and Γ is not strongly regular, then it follows from Propositions 20, 21 and 22 that Γ is either isomorphic to the lexicographic product $C_5[2K_1]$ or to one of the graphs from Construction 10. □

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