Latin cubes with forbidden entries

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Abstract

We consider the problem of constructing Latin cubes subject to the condition that some symbols may not appear in certain cells. We prove that there is a constant $\gamma > 0$ such that if $n = 2^t$ and A is a 3-dimensional $n \times n \times n$ array where every cell contains at most γn symbols, and every symbol occurs at most γn times in every line of A, then A is *avoidable*; that is, there is a Latin cube L of order n such that for every $1 \leqslant i, j, k \leqslant n$, the symbol in position (i, j, k) of L does not appear in the corresponding cell of A.

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1 Introduction

Consider an $n \times n$ array A in which every cell (i, j) contains a subset $A(i, j)$ of the symbols in $[n] = \{1, \dots, n\}$. If every cell contains at most m symbols, and every symbol occurs at most m times in every row and column, then A is an (m, m, m)*-array*. Confirming a conjecture by Häggkvist [11], it was proved in [1] that there is a constant $c > 0$ such that if $m \leqslant cn$ and A is an (m, m, m)-array, then A is *avoidable*; that is, there is a Latin square L such that for every (i, j) the symbol in position (i, j) in L is not in $A(i, j)$ (see also [3, 2]). The purpose of this note is to prove an analogue of this result for Latin cubes of order $n = 2^t$.

In order to make this precise, we imagine a 3-dimensional array having layers stacked on top of each other; we shall refer to such a 3-dimensional array as a *cube*. Now, a cube has *lines* in three directions obtained from fixing two coordinates and allowing the third to vary. The lines obtained by varying the first, second, and third coordinates

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will be referred to respectively as *columns*, *rows*, and *files*. The first, second, and third coordinates themselves will be referred to as the indices of the rows, columns, and files.

A *Latin cube* L of order n on the symbols $\{1, \ldots, n\}$ is an $n \times n \times n$ cube such that each symbol in $\{1, \ldots, n\}$ appears exactly once in each row, column and file. The symbol in position (i, j, k) of L is denoted by $L(i, j, k)$. Latin cubes have been studied by a number of authors, both with respect to enumeration and e.g. extension from partial cubes. An extensive survey of early results can be found in [13].

An $n \times n \times n$ cube where each cell contains a subset of the symbols in the set $\{1, \ldots, n\}$ is called an (m, m, m, m)*-cube (of order* n*)* if the following conditions are satisfied:

(a) No cell contains a set with more than m symbols.

(b) Each symbol occurs at most m times in each row.

(c) Each symbol occurs at most m times in each column.

(d) Each symbol occurs at most m times in each file.

Let $A(i, j, k)$ denote the set of symbols in the cell (i, j, k) of A. If we simplify notation, and write $A(i, j, k) = q$ if the set of symbols in cell (i, j, k) of A is $\{q\}$, then a $(1, 1, 1, 1)$-cube is a *partial Latin cube*, and a *Latin cube* L is simply a $(1, 1, 1, 1)$-cube with no empty cell.

Given an (m, m, m, m)-cube A of order n, a Latin cube L of order n *avoids* A if there is no cell (i, j, k) of L such that $L(i, j, k) \in A(i, j, k)$; if there is such a Latin cube, then A is *avoidable*.

Problems on extending partial Latin cubes have been studied for a long time, with the earliest results appearing in the 1970s [6]; in the more recent literature we have [4, 5, 12, 8]. The more general problem of constructing Latin cubes subject to the condition that some symbols cannot appear in certain cells seems to be a hitherto quite unexplored line of research. Our main result is the following, which establishes an analogue of the main result of [3], which considered Latin squares, for Latin cubes.

Theorem 1. *There is a positive constant* γ *such that if* $t \geqslant 30$ *and* $m \leqslant \gamma 2^t$*, then any* (m, m, m, m)*-cube* A *of order* 2^t *is avoidable.*

The restriction on the order of the cube is not believed to be necessary, but as for Latin squares, general orders are expected to require far more technical proof (unless some completely new method is invented). Our proof establishes this result for a small value of γ which we believe to be far from the optimal one, much like the case for the similar results for Latin squares. We know from [7] that $\gamma \leqslant \frac{1}{3}$, since that is an upper bound for the corresponding result for Latin squares, and every $n \times n$ sub-array of an avoidable (m, m, m, m)-cube of order n must be avoidable (in the sense that there is an $n \times n$ Latin square that avoids this array). It would be interesting to see if this upper bound could be improved in the setting of Latin cubes.

Problem 2. For how small $\gamma' = \frac{m}{n}$ does there exist an unavoidable (m, m, m, m)-cube A of order n?

Since a Latin cube is a more highly structured object than a Latin square, we suspect that in an (m, m, m, m)-cube yielding an optimal value of γ' in Problem 2, all $n \times n$ sub-arrays are avoidable. Thus we believe it would be interesting to investigate Problem 2 for the particular case when every $n \times n$ sub-array of the (m, m, m, m)-cube is avoidable. We note that for this latter question, we have the similar bound $\gamma' \leqslant 1/3$, since there are $(\gamma' n, \gamma' n, \gamma' n, \gamma' n)$-cubes of order n with $\gamma' > 1/3$ that are unavoidable, although every $n \times n$ subarray is avoidable. Such an unavoidable cube can be constructed in the following way: Let S be a Latin cube of order $n/2$. From S we construct an $n/2 \times n/2 \times n/2$ cube B by putting symbols $1, \dots, \gamma' n$ in every cell of B such that the corresponding cell of S has an entry from $\{1, \dots, \gamma' n\}$; all other cells of B are empty. From B we construct a $(\gamma' n, \gamma' n, \gamma' n, \gamma' n)$-cube A of order n by taking two copies A_1 and A_2 of B and placing them in "opposite" corners of A, i.e., occupying disjoint rows, columns, and files; all other cells of A are empty. Now, if L is an $n \times n \times n$ Latin cube avoiding A, then in the subcube L_1 of L corresponding to A_1, there are at most $(\frac{1}{8} - \frac{\gamma'}{4})n^3$ cells with entries from $\{1, \dots, \gamma' n\}$. Therefore, there must be at least

$$\frac{\gamma' n^3}{4} - \left(\frac{1}{8} - \frac{\gamma'}{4}\right) n^3 = \left(\frac{\gamma'}{2} - \frac{1}{8}\right) n^3$$

cells in the subcube L_2 of L corresponding to A_2 with entries from $\{1, \dots, \gamma' n\}$. However, as for L_1, in L_2 there are at most $(\frac{1}{8} - \frac{\gamma'}{4})n^3$ cells with entries from $\{1, \dots, \gamma' n\}$. Hence, $\gamma' \leqslant 1/3$.

We may also note that the main result of this paper, as well as the problem of extending partial Latin cubes, can be recast as list edge coloring problems on the complete 3-uniform 3-partite hypergraph $K^3_{n,n,n}$. Problems on extending partial edge colorings for ordinary graphs have been studied to some extent, see e.g. [9, 10] and the references given there, but similar problems for hypergraphs remain mostly unexplored.

In Section 2 we give some definitions and preparatory lemmas, and in Section 3 we prove Theorem 1.

2 Definitions and properties of Boolean Latin cubes

In this section we give some definitions and collect essential properties of Boolean Latin cubes.

Let A be an $n \times n \times n$ cube. Given $i \in [n]$, *row layer* i in A is a set of n^2 cells $\{(i, j^*, k^*) : j^* \in [n], k^* \in [n]\}$; given $j \in [n]$, *column layer* j in A is a set of n^2 cells $\{(i^*, j, k^*) : i^* \in [n], k^* \in [n]\}$; given $k \in [n]$, *file layer* k in A is a set of n^2 cells $\{(i^*, j^*, k) : i^* \in [n], j^* \in [n]\}$. As mentioned above, by fixing two coordinates and varying the third, we obtain rows, columns and files of a $n \times n \times n$ cube. Formally we define a row of such a cube A as a set of cells $R_{i,k} = \{(i, j^*, k) : j^* \in [n]\}$, a column as the set $C_{j,k} = \{(i^*, j, k) : i^* \in [n]\}$, and files $F_{i,j} = \{(i, j, k^*) : k^* \in [n]\}$.

Definition 3. The *Boolean Latin square* of order 2^t is the Latin square with entries as in the addition table of $\mathbb{Z}_2^t$ with the elements of $\mathbb{Z}_2^t$ mapped to the integers $1, \dots, 2^t$.

A 4-*cycle* (or *intercalate*) in a Latin square L is a set of four cells

$$\{(i_1,j_1),(i_1,j_2),(i_2,j_1),(i_2,j_2)\}$$

such that $L(i_1,j_1)=L(i_2,j_2)$ and $L(i_1,j_2)=L(i_2,j_1)$. We note some important properties of Boolean Latin squares (cf. [3]).

Property 4. *Each cell in the $n\times n$ Boolean Latin square is in $n-1$ distinct 4-cycles. Permuting the rows, the columns or the symbols does not affect the number of 4-cycles that a cell is part of.*

Property 5. *A 4-cycle in the Boolean Latin square is uniquely determined by two cells; that is, if C is a 4-cycle and $(i_1,j_1),(i_1,j_2)\in C$, then $(i_2,j_1),(i_2,j_2)\in C$, where i_2 is the row such that $L(i_1,j_1)=L(i_2,j_2)$ and $L(i_1,j_2)=L(i_2,j_1)$.*

Property 6. *The intersection of two 4-cycles is either empty, or it contains 1 or 4 cells.*

Given an integer t, let a_i $(1\leqslant i\leqslant 2^t)$ be the ith smallest element of $\mathbb{Z}_2^t$. (For example, with $t=2$, $a_1=00, a_2=01, a_3=10, a_4=11$.) We define the *Boolean Latin cube* similarly as the Boolean Latin square.

Definition 7. The *Boolean Latin cube* B of order $n=2^t$ on the symbols $\{1,\ldots,n\}$ is an $n\times n\times n$ Latin cube such that $B(i,j,k)=x$ with $a_x=a_i+a_j+a_k$ (addition in $\mathbb{Z}_2^t$) for all $1\leqslant i,j,k\leqslant n$.

Definition 8. A *subcube of order* 2 (or just *subcube*) in a Latin cube L is a set of eight cells

$$\{(i_1,j_1,k_1),(i_1,j_2,k_1),(i_2,j_1,k_1),(i_2,j_2,k_1),(i_1,j_1,k_2),(i_1,j_2,k_2),(i_2,j_1,k_2),(i_2,j_2,k_2)\}$$

such that

$$L(i_1,j_1,k_1)=L(i_2,j_2,k_1)=L(i_1,j_2,k_2)=L(i_2,j_1,k_2)$$

and

$$L(i_1,j_2,k_1)=L(i_2,j_1,k_1)=L(i_1,j_1,k_2)=L(i_2,j_2,k_2).$$

Note that every row, column and file layer of the Boolean Latin cube is isotopic to a Boolean Latin square. For the Boolean Latin cube we have the following analogue of Property 4.

Property 9. *Each cell in the Boolean Latin cube of order n belongs to $n-1$ subcubes.*

Proof. Consider an arbitrary cell (i_1,j_1,k_1) of the Boolean Latin cube B which belongs to a 4-cycle $\mathfrak{c}_1=\{(i_1,j_1,k_1),\ (i_1,j_2,k_1),\ (i_2,j_1,k_1),\ (i_2,j_2,k_1)\}$ such that $B(i_1,j_1,k_1)=B(i_2,j_2,k_1)$ and $B(i_1,j_2,k_1)=B(i_2,j_1,k_1)$. There are $n-1$ 4-cycles $\mathfrak{c}_1$ in file layer k_1 containing (i_1,j_1,k_1), since by construction, the file layers of the Boolean Latin cube are isotopic to Boolean Latin squares; this also holds for row and column layers.

Now, by Property 6, the two cells (i_1, j_1, k_1) and (i_2, j_1, k_1) define a unique 4-cycle

$$\mathfrak{c}_2 = \{(i_1, j_1, k_1), (i_2, j_1, k_1), (i_1, j_1, k_2), (i_2, j_1, k_2)\}$$

in the column layer j_1 such that $B(i_1, j_1, k_1) = B(i_2, j_1, k_2)$ and $B(i_2, j_1, k_1) = B(i_1, j_1, k_2)$. By Definition 7,

$$a_{i_1} + a_{j_1} + a_{k_1} = a_{i_2} + a_{j_2} + a_{k_1} = a_{i_2} + a_{j_1} + a_{k_2}$$

and

$$a_{i_1} + a_{j_2} + a_{k_1} = a_{i_2} + a_{j_1} + a_{k_1} = a_{i_1} + a_{j_1} + a_{k_2}.$$

Hence, we have

$$a_{i_1} + a_{j_1} + a_{k_2} = a_{i_2} + a_{j_2} + a_{k_2} = a_{i_2} + a_{j_1} + a_{k_1}$$

and

$$a_{i_1} + a_{j_2} + a_{k_2} = a_{i_2} + a_{j_1} + a_{k_2} = a_{i_1} + a_{j_1} + a_{k_1};$$

or, in other words,

$$B(i_1, j_2, k_1) = B(i_2, j_1, k_1) = B(i_1, j_1, k_2) = B(i_2, j_2, k_2)$$

and

$$B(i_1, j_1, k_1) = B(i_2, j_2, k_1) = B(i_1, j_2, k_2) = B(i_2, j_1, k_2).$$

This implies that

$$\{(i_1, j_1, k_1), (i_1, j_2, k_1), (i_2, j_1, k_1), (i_2, j_2, k_1), (i_1, j_1, k_2), (i_1, j_2, k_2), (i_2, j_1, k_2), (i_2, j_2, k_2)\}$$

is a subcube; and so each cell in the Boolean Latin cube belongs to $n - 1$ subcubes. □

Property 10. *Let (i_1, j_1, k_1), (i_2, j_2, k_2), (i_3, j_3, k_3) be three cells in the Boolean Latin cube B such that $(i_1 - i_2)(j_1 - j_2)(k_1 - k_2) \neq 0$, $(i_1 - i_3)(j_1 - j_3)(k_1 - k_3) \neq 0$ and $(i_2 - i_3)(j_2 - j_3)(k_2 - k_3) \neq 0$. If (i_1, j_1, k_1) and (i_2, j_2, k_2) both are in a subcube $\mathcal{C}_1$, and (i_1, j_1, k_1) and (i_3, j_3, k_3) are in a subcube $\mathcal{C}_2$, then (i_2, j_2, k_2) and (i_3, j_3, k_3) are in a subcube $\mathcal{C}_3$.*

Proof. Assume $B(i_1, j_1, k_1) = x$, $B(i_2, j_2, k_2) = y$, $B(i_3, j_3, k_3) = z$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are subcubes, we have that $B(i_2, j_2, k_1) = B(i_1, j_1, k_1) = B(i_3, j_3, k_1)$, i.e, $a_{i_2} + a_{j_2} + a_{k_1} = a_{i_1} + a_{j_1} + a_{k_1} = a_{i_3} + a_{j_3} + a_{k_1}$. It follows that $a_{i_2} + a_{j_2} = a_{i_3} + a_{j_3}$, so $a_{i_2} + a_{j_2} + a_{k_2} = a_{i_3} + a_{j_3} + a_{k_2}$, which implies that $B(i_3, j_3, k_2) = B(i_2, j_2, k_2) = y$. Similarly, we have $B(i_3, j_2, k_3) = B(i_2, j_3, k_3) = B(i_2, j_2, k_2) = y$ and $B(i_3, j_2, k_2) = B(i_2, j_3, k_2) = B(i_2, j_2, k_3) = B(i_3, j_3, k_3) = z$, which implies that (i_2, j_2, k_2) and (i_3, j_3, k_3) are two cells of a subcube

$$\mathcal{C}_3 = \{(i_2, j_2, k_2), (i_2, j_3, k_2), (i_3, j_2, k_2), (i_3, j_3, k_2), (i_2, j_2, k_3), (i_2, j_3, k_3), (i_3, j_2, k_3), (i_3, j_3, k_3)\}.$$

□

Property 11. *The intersection of two subcubes in a Latin cube is either empty, or it contains* 1 *or* 8 *cells.*

Proof. Assume that the intersection of two given subcubes contains at least 2 cells. If these 2 cells lie in a 4-cycle of a layer of the Latin cube, then by Property 6, this 4-cycle belongs to the intersection of two subcubes. But each 4-cycle defines a unique subcube, which implies that the intersection of the two subcubes contains 8 cells. If not, these 2 cells must have distinct row, column and file coordinates, so if we denote these two cells by (i_1, j_1, k_1) and (i_2, j_2, k_2), respectively, then $i_1 \neq i_2$, $j_1 \neq j_2$, $k_1 \neq k_2$. Hence, the intersection of the two subcubes must be the 8 cells (i_1, j_1, k_1), (i_1, j_2, k_1), (i_2, j_1, k_1), (i_2, j_2, k_1), (i_1, j_1, k_2), (i_1, j_2, k_2), (i_2, j_1, k_2), (i_2, j_2, k_2). □

Definition 12. Given a subcube

$$\mathcal{C} = \{(i_1, j_1, k_1), (i_1, j_2, k_1), (i_2, j_1, k_1), (i_2, j_2, k_1), (i_1, j_1, k_2), (i_1, j_2, k_2), (i_2, j_1, k_2), (i_2, j_2, k_2)\}$$

in a Latin cube L, a *swap on* $\mathcal{C}$ (or simply a *swap*) denotes the transformation $L \to L'$ which retains the content of all cells of L except that if

$$L(i_1, j_1, k_1) = L(i_2, j_2, k_1) = L(i_1, j_2, k_2) = L(i_2, j_1, k_2) = x_1$$

and

$$L(i_1, j_2, k_1) = L(i_2, j_1, k_1) = L(i_1, j_1, k_2) = L(i_2, j_2, k_2) = x_2$$

then

$$L'(i_1, j_1, k_1) = L'(i_2, j_2, k_1) = L'(i_1, j_2, k_2) = L'(i_2, j_1, k_2) = x_2$$

and

$$L'(i_1, j_2, k_1) = L'(i_2, j_1, k_1) = L'(i_1, j_1, k_2) = L'(i_2, j_2, k_2) = x_1.$$

Property 13. *Consider an arbitrary column* $\{(i_1, j_1, k_1), \ldots, (i_n, j_1, k_1)\}$ *of a Boolean Latin cube* B *of order* n. *For any* k_2 *(*j_2*), there exists a unique* j_2 *(*k_2*), such that* $B(x, j_1, k_1) = B(x, j_2, k_2)$ *for every* $x \in \{1, \ldots, n\}$.

Proof. For any k_2, we can choose j_2 satisfying $a_{j_2} = a_{j_1} + a_{k_1} - a_{k_2}$, and for any j_2, we can choose k_2 satisfying $a_{k_2} = a_{j_1} + a_{k_1} - a_{j_2}$. □

Evidently, all rows and files of a Boolean Latin cube have corresponding properties.

Property 14. *Let* B *be a Boolean Latin cube of order* n, $\mathfrak{b}$ *an arbitrary symbol in* B, *and* S_1 *be the set of cells of* B *in the first row layer which contain* $\mathfrak{b}$. *For any row layer* i, *the set of cells* S_i *of* B *in row layer* i *which have the same column and file coordinates as cells in* S_1 *all contain the same symbol.*

Proof. Assume that $(i, j_1, k_1) \in S_i$ and $B(i, j_1, k_1) = x$, and consider an arbitrary cell $(i, j_2, k_2) \in S_i$. By definition, there are two cells $(1, j_1, k_1)$ and $(1, j_2, k_2)$ such that $B(1, j_1, k_1) = B(1, j_2, k_2) = \mathfrak{b}$, that is, $a_1 + a_{j_1} + a_{k_1} = a_1 + a_{j_2} + a_{k_2}$. This implies that $a_i + a_{j_1} + a_{k_1} = a_i + a_{j_2} + a_{k_2}$, which means that $B(i, j_2, k_2) = B(i, j_1, k_1) = x$. Hence, all cells in S_i contain the same symbol. □

Note that all column and file layers of B have the same property.

The following simple observation enables us to permute layers and symbols in a Latin cube.

Property 15. *If L is a Latin cube, then the cube obtained by permuting the row layers, the column layers, the file layers and/or the symbols of L is a Latin cube.*

For Boolean Latin cubes an even stronger property holds. If a Latin cube L' is obtained from another Latin cube L by permuting row/column/file layers and/or symbols of L, then we say that L and L' are *isotopic*. Henceforth, all Latin cubes have order n.

Property 16. *If L is isotopic to a Boolean Latin cube, then any cell of L is in $n-1$ subcubes. Moreover, Property 10, 13, and 14 hold for L.*

In the following we shall define some sets of cells in Latin cubes that are isotopic to Boolean Latin cubes.

Definition 17. Let L be a Latin cube that is isotopic to a Boolean Latin cube. A *row block* of L is a set of n rows $R_{i,k}$ such that for every pair of rows $R_{i_1,k_1} = \{(i_1, j, k_1) : j \in [n]\}$ and $R_{i_2,k_2} = \{(i_2, j, k_2) : j \in [n]\}$ in this set, $B(i_1, x, k_1) = B(i_2, x, k_2)$ for every $x \in \{1, \dots, n\}$. It is obvious that there are n row blocks in total. *Column blocks* and *file blocks* are defined similarly.

Property 18. *If*

$$\mathcal{C} = \{(i_1, j_1, k_1), (i_1, j_2, k_1), (i_2, j_1, k_1), (i_2, j_2, k_1), (i_1, j_1, k_2), (i_1, j_2, k_2), (i_2, j_1, k_2), (i_2, j_2, k_2)\}$$

is a subcube in a Latin cube L that is isotopic to a Boolean cube, then the two rows R_{i_1,k_1} and R_{i_2,k_2} are in the same row block, as are also the two rows R_{i_2,k_1} and R_{i_1,k_2}.

Note that a similar property holds for columns blocks and file blocks.

Definition 19. If L is a Latin cube that is isotopic to a Boolean Latin cube, a *transversal-set* $\mathfrak{t}$ of L is a set of n cells that satisfy the following

- no two cells in $\mathfrak{t}$ are in the same row/column/file;
- no two cells in $\mathfrak{t}$ contain the same symbol;
- for any two cells in $\mathfrak{t}$, there is a unique subcube that contain these cells.

Note that by Property 10, a transversal-set is well-defined, and every row block, column block and file block contains exactly n disjoint transversal-sets.

Based on Property 14, we make the following definition.

Definition 20. A *symbol-row block* of a Latin cube L that is isotopic to a Boolean Latin cube is a set $\mathfrak{s}$ of n^2 cells satisfying that

- all cells of $\mathfrak{s}$ that are in the same row layer contain the same symbol, and
- for every cell of $\mathfrak{s}$, there are $n-1$ other cells that have the same column and file coordinate.

Symbol-column blocks and *symbol-file blocks* are defined similarly.

An intersection between a symbol-row block and a row layer (or a symbol-column block and a column layer, or a symbol-file block and a file layer) is called a *symbol-set*. It is obvious that all cells in a symbol-set contain the same symbol, and that each row layer, column layer, file layer, symbol-row block, symbol-column block, and symbol-file block contains n symbol-sets.

Definition 21. A *symbol block* of a Latin cube L is a set of n^2 cells such that all these cells contain the same symbol.

Note that a Latin cube that is isotopic to a Boolean Latin cube contains n symbol blocks in total, and for each symbol block, there are three different ways to divide this symbol block to n disjoint symbol-sets (group the symbol sets based on the row layers, the column layers or the file layers).

Given an $n \times n \times n$ cube A where each cell contains a subset of the symbols in $\{1,\dots,n\}$, and a Latin cube L of order n that does not avoid A, we say that those cells (i,j,k) of L where $L(i,j,k) \in A(i,j,k)$ are *conflict cells of L with A* (or simply *conflicts* of L). An *allowed subcube* of L is a subcube

$$\mathcal{C} = \{(i_1,j_1,k_1),(i_1,j_2,k_1),(i_2,j_1,k_1),(i_2,j_2,k_1),(i_1,j_1,k_2),(i_1,j_2,k_2),(i_2,j_1,k_2),(i_2,j_2,k_2)\}$$

in L such that swapping on $\mathcal{C}$ yields a Latin cube L' where none of (i_1,j_1,k_1), (i_1,j_2,k_1), (i_2,j_1,k_1), (i_2,j_2,k_1), (i_1,j_1,k_2), (i_1,j_2,k_2), (i_2,j_1,k_2), (i_2,j_2,k_2) is a conflict.

3 Proof of the main theorem

In this section we prove Theorem 1. Our basic proof strategy is similar to the one in [3, 1]; however, due to the extra dimension in a Latin cube, our arguments are considerably more involved and somewhat technical. Our starting point in the proof is the Boolean Latin cube; we permute its row layers, column layers, file layers and symbols so that the resulting Latin cube does not have too many conflicts with a given (m,m,m,m)-cube A. After that, we find a set of allowed subcubes such that each conflict belongs to one of them, with no two of the subcubes intersecting, and swap on those subcubes.

The proof of Theorem 1 involves a number of parameters:

$$\alpha, \gamma, \kappa, \epsilon, \theta,$$

and a number of inequalities that they must satisfy. For the reader's convenience, explicit choices for which the proof holds are presented here:

$$\alpha = 1 - 38 \times 2^{-25}, \gamma = 2^{-25}, \kappa = 6 \times 2^{-25}, \epsilon = 2^{-6}, \theta = 2^{-12}. \tag{1}$$

By an example of unavoidable $(\lfloor\frac{n}{3}\rfloor+1, \lfloor\frac{n}{3}\rfloor+1, \lfloor\frac{n}{3}\rfloor+1)$-arrays in [7], the value of γ for which Theorem 1 holds cannot exceed $\frac{1}{3}$. Thus, since the numerical value of γ for which the theorem holds is not anywhere near what we expect to be optimal, we have not put an effort into choosing optimal values for these parameters. Moreover, for simplicity of notation, we shall omit floor and ceiling signs whenever these are not crucial.

We shall establish that our main theorem holds by proving two lemmas.

Lemma 22. *Let α, γ, κ be constants and $n = 2^t$ such that*

$$\Big(7n^2 \frac{(\gamma n)^{\kappa n}}{(\kappa n)!} + 3n^3 \frac{(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!}\Big) < 1.$$

For any $(\gamma n, \gamma n, \gamma n, \gamma n)$-cube A of order n there is a quadruple of permutations $\sigma = (\tau_1, \tau_2, \tau_3, \tau_4)$ of the row layers, the column layers, the file layers and the symbols of the Boolean Latin cube B of order n, respectively, such that applying σ to B, we obtain a Latin cube L satisfying the following:

(a) No row in L contains more than κn conflicts with A.

(b) No column in L contains more than κn conflicts with A.

(c) No file in L contains more than κn conflicts with A.

(d) No symbol-set in L contains more than κn conflicts with A.

(e) No transversal-set in L contains more than κn conflicts with A.

(f) Each cell of L belongs to at least αn allowed subcubes.

Proof. Let X_a, X_b, X_c, X_d, X_e and X_f be the number of permutations which do not fulfill the conditions (a), (b), (c), (d), (e) and (f), respectively. Let X be the number of permutations satisfying the six conditions (a), (b), (c), (d), (e) and (f). There are $(n!)^4$ ways to permute the row layers, the column layers, the file layers and the symbols, so we have

$$X \geqslant (n!)^4 - X_a - X_b - X_c - X_d - X_e - X_f.$$

We shall prove that X is greater than 0.

- To estimate X_a, assume that for any fixed permutation (τ_1, τ_3, τ_4) of the row layers, the file layers and the symbols, at most N_a choices of a permutation τ_2 of the column layers yield a quadruple $(\tau_1, \tau_2, \tau_3, \tau_4)$ of permutations that break condition (a); so $X_a \leqslant n!n!n!N_a$.

  Let R be a fixed row chosen arbitrarily; we count the number of ways a permutation τ_2 of the column layers can be constructed so that (a) does not hold on row R. Let S be a set of size κn of column layers of A. There are $\binom{n}{\kappa n}$ ways to choose S. In order to have a conflict at cell (i, j, k) of R, the column layers should be permuted in such a way that in the resulting Latin cube L, $L(i, j, k) \in A(i, j, k)$. Since $|A(i, j, k)| \leqslant \gamma n$, there are at most $(\gamma n)^{\kappa n}$ ways to choose which column layers of B are mapped by τ_2 to column layers in S so that all cells on row R that are in S are conflicts. The rest of the column layers can be arranged in any of the $(n-\kappa n)!$ possible ways. In total this gives at most

$$\binom{n}{\kappa n}(\gamma n)^{\kappa n}(n-\kappa n)! = \frac{n!(\gamma n)^{\kappa n}}{(\kappa n)!}$$

permutations τ_2 that do not satisfy condition (a) on row R. There are n^2 rows in B, so we have

$$N_a \leqslant n^2 \frac{n!(\gamma n)^{\kappa n}}{(\kappa n)!}$$

and

$$X_a \leqslant n!n!n!N_a \leqslant n^2(n!)^4 \frac{(\gamma n)^{\kappa n}}{(\kappa n)!}.$$

An analogous argument gives the same bound for X_b, X_c, so in total, we have that

$$X_a + X_b + X_c \leqslant 3n^2(n!)^4 \frac{(\gamma n)^{\kappa n}}{(\kappa n)!}.$$

- To estimate X_d, assume that for any fixed permutation (τ_1, τ_3, τ_4) of the row layers, the file layers and the symbols, at most N_d choices of a permutation τ_2 of the column layers give a quadruple $(\tau_1, \tau_2, \tau_3, \tau_4)$ of permutations that break condition (d); then $X_d \leqslant n!n!n!N_d$.

  Let $\mathfrak{b}$ be a fixed symbol chosen arbitrarily; we count the number of ways a permutation τ_2 of the column layers can be constructed so that (d) does not hold for $\mathfrak{b}$ in a given row layer. Let R_L be a fixed row layer; there are n cells containing $\mathfrak{b}$ in R_L and these cells belong to n different column layers since B is a Boolean Latin cube. Let S be a set of size κn of column layers of A; there are $\binom{n}{\kappa n}$ ways to choose S. Since in A, each symbol occurs at most γn times in each row, there are at most $(\gamma n)^{\kappa n}$ ways to choose which column layers of B are mapped by τ_2 to column layers in S so that all cells containing $\mathfrak{b}$ on row layer R_L that are in S are conflicts. The rest of the column layers can be arranged in any of the $(n-\kappa n)!$ possible ways. In total this gives at most

$$\binom{n}{\kappa n}(\gamma n)^{\kappa n}(n-\kappa n)! = \frac{n!(\gamma n)^{\kappa n}}{(\kappa n)!}$$

  permutations τ_2 such that in the resulting Latin cube L, symbol $\mathfrak{b}$ appears in more than κn conflicts in the row layer R_L. There are n different row layers, n different column layers and n different file layers in B, so we deduce that there are at most $3n\frac{n!(\gamma n)^{\kappa n}}{(\kappa n)!}$ permutations τ_2 that do not satisfy condition (d) on symbol $\mathfrak{b}$. There are n symbols in B, so we have

$$N_d \leqslant 3n^2 \frac{n!(\gamma n)^{\kappa n}}{(\kappa n)!}.$$

and

$$X_d \leqslant 3n^2(n!)^4\frac{(\gamma n)^{\kappa n}}{(\kappa n)!}.$$

- To estimate X_e, assume that for any fixed permutation (τ_1, τ_2, τ_3) of the row layers, the column layers, the file layers, at most N_e choices of a permutation τ_4 of the symbols give a quadruple $(\tau_1, \tau_2, \tau_3, \tau_4)$ of permutations that break condition (e); so $X_e \leqslant n!n!n!N_e$.

  Let T be a fixed transversal-set chosen arbitrarily; we count the number of ways a permutation τ_4 of the symbols can be constructed so that (e) does not hold on the set T. Let S be a set of size κn of cells of T; there are $\binom{n}{\kappa n}$ ways to choose S. In order to have a conflict at cell (i, j, k) of T, the symbols should be permuted in such a way that in the resulting Latin cube L, $L(i, j, k) \in A(i, j, k)$. Since $|A(i, j, k)| \leqslant \gamma n$, there are at most $(\gamma n)^{\kappa n}$ ways to choose which symbols of B are mapped by τ_4 to cells in S so that all cells in S are conflicts. The rest of the symbols can be arranged in any of the $(n - \kappa n)!$ possible ways. In total this gives at most

$$\binom{n}{\kappa n}(\gamma n)^{\kappa n}(n-\kappa n)! = \frac{n!(\gamma n)^{\kappa n}}{(\kappa n)!}$$

  permutations τ_4 that do not satisfy condition (e) on the transversal-set T. There are n^2 transversal-sets in B, so we have

$$N_e \leqslant n^2\frac{n!(\gamma n)^{\kappa n}}{(\kappa n)!},$$

  and so

$$X_e \leqslant n!n!n!N_e \leqslant n^2(n!)^4\frac{(\gamma n)^{\kappa n}}{(\kappa n)!}.$$

- To estimate X_f, assume that for any fixed permutation (τ_2, τ_4) of the column layers and the symbols at most N_f choices of a pair (τ_1, τ_3) of the row layers and the file layers yield a quadruple $(\tau_1, \tau_2, \tau_3, \tau_4)$ of permutations that break condition (f), then $X_f \leqslant n!n!N_f$.

  Let (i_1, j_1, k_1) be an arbitrary fixed cell of A. Each subcube $\mathcal{C}$ containing (i_1, j_1, k_1) is uniquely determined by the value of $j_2 \neq j_1$ where $(i_1, j_2, k_1) \in \mathcal{C}$; so a pair of permutations (τ_1, τ_3) satisfy that the quadruple $(\tau_1, \tau_2, \tau_3, \tau_4)$ adds to X_f if and only if there are more than $(1-\alpha)n$ choices for j_2 so that the swap along $\mathcal{C}$ is not allowed. We shall count the number of ways of choosing (τ_1, τ_3) so that this holds.

  Let us first note that there are at most $2\gamma n$ choices for a pair (i_x, k_x), where $i_1 = \tau_1(i_x)$, $k_1 = \tau_3(k_x)$, that yield a subcube $\mathcal{C}$ in L that is not allowed because of a conflict in the row with row index i_1 in file layer k_1; that is, after swapping on $\mathcal{C}$, we have a conflict cell on the row with row index i_1 in file layer k_1. This follows from the fact that there are γn choices for j_2 such that $A(i_1, j_2, k_1)$ contains $L(i_1, j_1, k_1)$, and since $|A(i_1, j_1, k_1)| \leqslant \gamma n$, we have γn choices for j_2 so that $L(i_1, j_2, k_1) \in A(i_1, j_1, k_1)$. So for a permutation (τ_1, τ_3) to contribute to N_f, (τ_1, τ_3) must be such that at least $(1 - \alpha - 2\gamma)n$ subcubes containing the cell (i_1, j_1, k_1) are not allowed because of restrictions on rows of A that are distinct from row i_1 in file layer k_1. Since each subcube $\mathcal{C}$ containing (i_1, j_1, k_1) has cells from three other rows, this implies that at least $(1 - \alpha - 2\gamma)n/3$ subcubes cannot be allowed because of conflicts appearing in one of these rows.

Now, there are n^2 ways to choose a row layer i_x and a file layer k_x so that $i_1 = \tau_1(i_x)$ and $k_1 = \tau_3(k_x)$; we fix such a row layer i_x and file layer k_x. Next, let N_{f_1} be the number of pairs of permutations (τ_1, τ_3) such that at least $(1 - \alpha - 2\gamma)n/3$ subcubes containing (i_1, j_1, k_1) are not allowed because swapping yields conflicts in cells in file layer k_1 that are not contained in row layer i_1. Let us first note that there are $(n - 1)!$ ways to permute the remaining file layers of B. Consider a fixed permutation τ_3 of the file layers; we count the number of permutations τ_1 of the row layers such that the pair (τ_1, τ_3) contributes to N_{f_1}. Let S be a set of columns, $(|S| = (1 - \alpha - 2\gamma)n/3)$, such that for every column $C_{j_2,k_2} \in S$, there is a unique i_2 satisfying that

$$\begin{aligned}\mathcal{C} = \{&(i_1, j_1, k_1), (i_1, j_2, k_1), (i_2, j_1, k_1), (i_2, j_2, k_1),\\ &(i_1, j_1, k_2), (i_1, j_2, k_2), (i_2, j_1, k_2), (i_2, j_2, k_2)\}\end{aligned}$$

is a subcube and this subcube is not allowed because of conflicts arising in row i_2 in file layer k_1. There are $\binom{n-1}{(1-\alpha-2\gamma)n/3}$ ways to choose S. Fix a column $C_{j_2,k_2} \in S$; in column j_1 of file layer k_1 of A, there are at most γn cells containing $L(i_1, j_1, k_1)$ and in the column j_2 in file layer k_1 of A, there are at most γn cells containing $L(i_1, j_2, k_1)$, so there are up to $2\gamma n$ choices for $\tau_1^{-1}(i_2)$ in B that would make $\mathcal{C}$ disallowed because of conflicts arising in rows distinct from i_1 in the file layer k_1.

Since every column in S yields a unique row index, S determines τ_1 on $(1-\alpha-2\gamma)n/3$ row layers. The remaining row layers can be permuted in $(n-1-(1-\alpha-2\gamma)n/3)!$ ways. This implies that the total number of permutations τ_1 that yield at least $(1 - \alpha - 2\gamma)n/3$ subcubes that are not allowed because of conflicts appearing in file layer k_1 that are not contained in row layer i_1 is bounded from above by

$$\begin{aligned}&\binom{n-1}{(1-\alpha-2\gamma)n/3}(2\gamma n)^{(1-\alpha-2\gamma)n/3}(n - 1 - (1 - \alpha - 2\gamma)n/3)!\\ &\qquad = \frac{(n-1)!(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!}.\end{aligned}$$

Hence, $N_{f_1} \leqslant (n-1)!\dfrac{(n-1)!(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!}$.

Let N_{f_2} be the number of pairs of permutations (τ_1, τ_3) such that at least $(1 - \alpha - 2\gamma)n/3$ subcubes containing (i_1, j_1, k_1) are not allowed because swapping on them yields conflicts in rows contained in the row layer i_1 but not in file layer k_1. There are $(n-1)!$ ways to permute the remaining row layers of B. We consider a fixed permutation τ_1 of the row layers and count the number of permutations τ_3 of the file layers such that the pair (τ_1, τ_3) contributes to N_{f_2}. Let S be a set of files, $(|S| = (1-\alpha-2\gamma)n/3)$, such that for every file $F_{i_2,j_2} \in S$, there is a unique k_2 satisfying that

$$\mathcal{C} = \{(i_1,j_1,k_1),(i_1,j_2,k_1),(i_2,j_1,k_1),(i_2,j_2,k_1),\\ (i_1,j_1,k_2),(i_1,j_2,k_2),(i_2,j_1,k_2),(i_2,j_2,k_2)\}$$

is a subcube and this subcube is not allowed because of conflicts arising in cells in row layer i_1 that are not in file layer k_1. There are $\binom{n-1}{(1-\alpha-2\gamma)n/3}$ ways to choose S. Fix a file $F_{i_2,j_2} \in S$; in the file F_{i_1,j_1} of A, there are at most γn cells containing $L(i_1,j_1,k_1)$ and in the file F_{i_1,j_2} of A, there are at most γn cells containing $L(i_1,j_2,k_1)$, so there are up to $2\gamma n$ choices for $\tau_3^{-1}(k_2)$ in B that would make $\mathcal{C}$ disallowed because of possible conflicts in row layer i_1 that are not in file layer k_1.

As before, S determines how τ_3 acts on $(1-\alpha-2\gamma)n/3)$ file layers, and the remaining file layers can be permuted in $(n-1-(1-\alpha-2\gamma)n/3)!$ ways. This implies that the total number of permutations τ_3 with not enough allowed subcubes due to the fact that swapping yield conflicts in rows contained in the row layer i_1 but not in file layer k_1 is bounded from above by

$$\binom{n-1}{(1-\alpha-2\gamma)n/3}(2\gamma n)^{(1-\alpha-2\gamma)n/3}(n-1-(1-\alpha-2\gamma)n/3)! \\ = \frac{(n-1)!(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!}.$$

Hence, $N_{f_2} \leqslant (n-1)!\dfrac{(n-1)!(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!}$.

Let N_{f_3} be the number of pairs of permutations (τ_1, τ_3) such that at least $(1-\alpha-2\gamma)n/3$ subcubes $\mathcal{C}$ containing (i_1,j_1,k_1) are not allowed because swapping on them yields conflicts in cells which lie in row and file layers distinct from i_1 and k_1, respectively. There are $(n-1)!$ ways to permute the remaining file layers of B. Consider a fixed permutation τ_3 of the file layers; we count the number of permutations τ_1 of the row layers such that the pair (τ_1, τ_3) contributes to N_{f_3}. Let S be a set of columns $(|S| = (1-\alpha-2\gamma)n/3)$, such that for every column $C_{j_2,k_2} \in S$, there is a unique i_2 satisfying that

$$\mathcal{C} = \{(i_1,j_1,k_1),(i_1,j_2,k_1),(i_2,j_1,k_1),(i_2,j_2,k_1),\\ (i_1,j_1,k_2),(i_1,j_2,k_2),(i_2,j_1,k_2),(i_2,j_2,k_2)\}$$

is a subcube which is not allowed because swapping yields conflicts in cells in row i_2 in file layer k_2. There are $\binom{n-1}{(1-\alpha-2\gamma)n/3}$ ways to choose S. Fix a column $C_{j_2,k_2} \in S$; in the column C_{j_2,k_2} of A, there are at most γn cells containing symbol $L(i_1,j_1,k_1)$, and in the column C_{j_1,k_2} of A, there are at most γn cells containing $L(i_1, j_2, k_1)$; so there are up to $2\gamma n$ choices for $\tau_1^{-1}(i_2)$ in B that would make $\mathcal{C}$ disallowed because swapping yields conflicts in cells which lie in row and file layers distinct from i_1 and k_1, respectively.

The set S determines how τ_1 acts on $(1-\alpha-2\gamma)n/3$ row layers. The remaining row layers can be permuted in $(n-1-(1-\alpha-2\gamma)n/3)!$ ways. This implies that the total number of permutations τ_1 with too few allowed subcubes because of conflicts arising in cells in row and file layers distinct from i_1 and k_1 is bounded from above by

$$\binom{n-1}{(1-\alpha-2\gamma)n/3}(2\gamma n)^{(1-\alpha-2\gamma)n/3}(n-1-(1-\alpha-2\gamma)n/3)! \\ = \frac{(n-1)!(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!}.$$

Hence, $N_{f_3} \leqslant (n-1)!\dfrac{(n-1)!(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!}$.

The Boolean Latin cube contains n^3 cells in total, so

$$N_f \leqslant n^3(n^2N_{f_1} + n^2N_{f_2} + n^2N_{f_3}) \leqslant 3n^3(n!)^2\frac{(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!}$$

and

$$X_f \leqslant (n!)^2N_f \leqslant 3n^3(n!)^4\frac{(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!}$$

Summing up, we conclude that

$$X \geqslant (n!)^4 - 7n^2(n!)^4\frac{(\gamma n)^{\kappa n}}{(\kappa n)!} - 3n^3(n!)^4\frac{(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!} \\ \geqslant (n!)^4\Big(1 - 7n^2\frac{(\gamma n)^{\kappa n}}{(\kappa n)!} - 3n^3\frac{(2\gamma n)^{(1-\alpha-2\gamma)n/3}}{((1-\alpha-2\gamma)n/3)!}\Big)$$

By (1), X is strictly greater than 0, provided that n is large enough. □

Lemma 23. *Let L be a Latin cube that is isotopic to a Boolean Latin cube, and let A be an (m,m,m,m)-cube; both of order n. Furthermore, let $\alpha, \gamma, \kappa, \theta, \epsilon$ be constants, $n = 2^t$ such that $\epsilon n \geqslant 3$ and*

$$\alpha n - 21\kappa n - 7\epsilon n - \frac{84\kappa}{\epsilon}n - \frac{21\theta}{\epsilon}n - \frac{80\kappa}{\theta}n - 28 > 0.$$

If L has the following properties:

(a) no row in L contains more than κn conflicts with A;

(b) no column in L contains more than κn conflicts with A;

(c) no file in L contains more than κn conflicts with A;

(d) no symbol-set in L contains more than κn conflicts with A;

(e) no transversal-set in L contains more than κn conflicts with A;

(f) each cell of L belongs to at least αn allowed subcubes;

then there is a set of disjoint allowed subcubes such that each conflict of L belongs to one of them. Thus, by performing a number of swaps on subcubes in L, we obtain a Latin cube L' that avoids A.

Proof. For constructing L' from L, we will perform a number of swaps on subcubes, and we shall refer to this procedure as *S-swap*. We are going to construct a set S of disjoint allowed subcubes such that each conflict of L with A belongs to one of them. A cell that belongs to a subcube in S is called *used* in S-swap. Since no row in L contains more than κn conflicts with A, there are at most κn^3 conflicts in L, which implies that the total number of cells that are used in S-swap is at most $8\kappa n^3$.

A row layer, a column layer, a file layer, a row block, a column block, a file block, a symbol block, a symbol-row block, a symbol-column block, or a symbol-file block is *overloaded* if such a layer or block contains at least θn^2 cells that are used in S-swap; note that no more than $\dfrac{8\kappa n^3}{\theta n^2} = \dfrac{8\kappa}{\theta} n$ layers or blocks of each type are S-overloaded. A row, a column, a file, a transversal-set, or a symbol-set is *overloaded* if this row, column, file, transversal-set or symbol-set contains at least ϵn cells that are used in S-swap.

Using these facts, let us now construct our set S by steps; at each step we consider a conflict cell (i_1, j_1, k_1) and include an allowed subcube containing (i_1, j_1, k_1) in S. Initially, the set S is empty.

So let us consider a conflict cell (i_1, j_1, k_1) in L; there are at least αn allowed subcubes containing (i_1, j_1, k_1). We choose an allowed subcube

$$\mathcal{C} = \{(i_1, j_1, k_1), (i_1, j_2, k_1), (i_2, j_1, k_1), (i_2, j_2, k_1), (i_1, j_1, k_2), (i_1, j_2, k_2), (i_2, j_1, k_2), (i_2, j_2, k_2)\}$$

that satisfies the following:

(1) The row layer i_2, the column layer j_2, the file layer k_2, the row block containing the row R_{i_2,k_1}, the column block containing the column C_{j_2,k_1}, the file block containing the file F_{i_1,j_2}, the symbol-row block containing two cells (i_1, j_2, k_1) and (i_1, j_1, k_2), the symbol-column block containing two cells (i_2, j_1, k_1) and (i_1, j_1, k_2), the symbol-file block containing two cells (i_1, j_2, k_1) and (i_2, j_1, k_1), the symbol block containing symbol $L(i_1, j_2, k_1)$ are not overloaded. This eliminates at most $\dfrac{10 \times 8\kappa}{\theta} n = \dfrac{80\kappa}{\theta} n$ choices.

    With this strategy for including subcubes in S, after completing the construction of S, every layer (or block) contains at most $4\kappa n^2 + (\theta n^2 - 1) + 4$ cells that are used in S-swap. Hence, the number of overloaded rows (overloaded columns, overloaded files, overloaded transversal-sets or overloaded symbol-sets) in each layer (or block) is at most $\dfrac{4\kappa n^2+\theta n^2+3}{\epsilon n} \leqslant \dfrac{4\kappa+\theta}{\epsilon}n+1$. Note that here the statement "each symbol block contains at most $\dfrac{4\kappa+\theta}{\epsilon}n+1$ overloaded symbol-sets" is to be taken with respect to either row layers, column layers or file layers, i.e., when we consider the n different symbol sets of a given symbol block belonging to n different row layers (or n different column layers or n different file layers), the number of overloaded such symbol-sets is at most $\dfrac{4\kappa+\theta}{\epsilon}n+1$.

(2) Some rows, columns, files, transversal-sets, symbol-sets are not overloaded as the following:

(2a) The columns C_{j_2,k_1}, C_{j_1,k_2}, C_{j_2,k_2} are not overloaded; this condition eliminates at most $\dfrac{12\kappa+3\theta}{\epsilon}n+3$ choices since in the file layer k_1 (which contains the column C_{j_2,k_1}) and in the column layer j_1 (which contains the column C_{j_1,k_2}) and in the column block which contains the column C_{j_1,k_1} (which also contains the column C_{j_2,k_2}), there are in total at most $\dfrac{4\kappa+\theta}{\epsilon}n+1$ overloaded columns. Similarly, we need that the rows R_{i_2,k_1}, R_{i_1,k_2}, R_{i_2,k_2} and the files F_{i_1,j_2}, F_{i_2,j_1}, F_{i_2,j_2} are not overloaded; this eliminates at most $\dfrac{24\kappa+6\theta}{\epsilon}n+6$ choices.

(2b) The transversal-set $\mathfrak{t}_1$ containing (i_2,j_1,k_1) and (i_1,j_2,k_2) is not overloaded; this eliminates at most $\dfrac{4\kappa+\theta}{\epsilon}n+1$ choices, since in the column block which contains the column C_{j_1,k_1} (which also contains the transversal-set $\mathfrak{t}_1$), there are at most $\dfrac{4\kappa+\theta}{\epsilon}n+1$ overloaded transversal-sets. Similarly, we need that the transversal-set containing (i_1,j_2,k_1) and (i_2,j_1,k_2), and the transversal-set containing (i_2,j_2,k_1) and (i_1,j_1,k_2) are not overloaded; this eliminates at most $\dfrac{8\kappa+2\theta}{\epsilon}n+2$ choices.

(2c) The symbol-set $\mathfrak{s}_1$ containing (i_2,j_1,k_2) and (i_1,j_2,k_2) is not overloaded; this eliminates at most $\dfrac{4\kappa+\theta}{\epsilon}n+1$ choices, since in the symbol block which contains (i_1,j_1,k_1) (which also contains the symbol-set $\mathfrak{s}_1$), there are at most $\dfrac{4\kappa+\theta}{\epsilon}n+1$ overloaded symbol-sets. Similarly, we need that the symbol-set containing the cells (i_2,j_2,k_1) and (i_1,j_2,k_2), and the symbol-set containing (i_2,j_2,k_1) and (i_2,j_1,k_2) are not overloaded, this eliminates at most $\dfrac{8\kappa+2\theta}{\epsilon}n+2$ choices.

(2d) The symbol-set $\mathfrak{s}_2$ containing (i_1,j_2,k_1) and (i_2,j_1,k_1), and the symbol-set $\mathfrak{s}_3$ containing (i_1,j_2,k_1) and (i_2,j_2,k_2) are not overloaded. This requirement eliminates at most $\dfrac{8\kappa+2\theta}{\epsilon}n+2$ choices, since in the file layer k_1 (which contains the symbol-set $\mathfrak{s}_2$), and in the symbol-column block which contains (i_1, j_1, k_1) (which also contains symbol-set $\mathfrak{s}_3$), there are at most $\frac{4\kappa+\theta}{\epsilon}n+1$ overloaded symbol-sets. Similarly, we need that the symbol-set containing (i_1, j_1, k_2) and (i_1, j_2, k_1), the symbol-set containing (i_1, j_1, k_2) and (i_2, j_2, k_2), the symbol-set containing (i_2, j_1, k_1) and (i_1, j_1, k_2), the symbol-set containing (i_2, j_1, k_1) and (i_2, j_2, k_2) are not overloaded. This eliminates at most $\frac{16\kappa+4\theta}{\epsilon}n+4$ choices.

So in total, this eliminates at most $\frac{84\kappa+21\theta}{\epsilon}n+21$ choices. Note that with this strategy for including subcubes in S, after completing the construction of S, every row, column, file, transversal-set, and symbol-set contains at most $2\kappa n+(\epsilon n-1)+2$ or $2\kappa n+\epsilon n+1$ cells that are used in S-swap.

(3) Except for (i_1, j_1, k_1), none of the cells in $\mathcal{C}$ are conflicts or used before in S-swap.

(3a) The cell (i_2, j_1, k_1) is not a conflict and has not been used before in S-swap; this eliminates at most $3\kappa n+\epsilon n+1$ choices since the column C_{j_1,k_1} contains at most κn conflict cells and at most $2\kappa n+\epsilon n+1$ cells that are used in S-swap. Similarly, we need that the cell (i_1, j_2, k_1) and the cell (i_1, j_1, k_2) are not conflicts and has not used before in S-swap; in total, this eliminates at most $6\kappa n+2\epsilon n+2$ choices.

(3b) The cell (i_1, j_2, k_2) is not a conflict and has not been used before in S-swap. This eliminates at most $3\kappa n+\epsilon n+1$ choices, since in the symbol-set in row layer i_1 that contains the cell (i_1, j_1, k_1), there are at most κn conflict cells and at most $2\kappa n+\epsilon n+1$ cells that have been used in S-swap. Similarly, we need that the cell (i_2, j_1, k_2) and the cell (i_2, j_2, k_1) are not conflicts and has not been used before in S-swap; in total, this eliminates at most $6\kappa n+2\epsilon n+2$ choices.

(3c) The cell (i_2, j_2, k_2) is not a conflict and has not been used before in S-swap. This eliminates at most $3\kappa n+\epsilon n+1$ choices since in the transversal-set containing the cell (i_1, j_1, k_1), there are at most κn conflict cells and at most $2\kappa n+\epsilon n+1$ cells that are used in S-swap.

So in total, this eliminates at most $21\kappa n+7\epsilon n+7$ choices.

It follows that we have at least

$$\alpha n-21\kappa n-7\epsilon n-\frac{84\kappa}{\epsilon}n-\frac{21\theta}{\epsilon}n-\frac{80\kappa}{\theta}n-28$$

choices for an allowed subcube $\mathcal{C}$ which contains (i_1, j_1, k_1). By (1), this expression is greater than zero if n is large enough, so we can conclude that there is a subcube satisfying these conditions. Thus we may construct the set S by iteratively adding disjoint allowed subcubes such that each subcube contains a conflict cell.

After this process terminates, we have a set S of disjoint subcubes; we swap on all subcubes in S to obtain the Latin cube L'. Hence, we conclude that we can obtain a Latin cube L' that avoids A. □

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