Regularity and h-polynomials of edge ideals

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Abstract

For any two integers $d, r \geqslant 1$, we show that there exists an edge ideal $I(G)$ such that $\operatorname{reg}(R/I(G))$, the Castelnuovo-Mumford regularity of $R/I(G)$, is r, and $\deg h_{R/I(G)}(t)$, the degree of the h-polynomial of $R/I(G)$, is d. Additionally, if G is a graph on n vertices, we show that $\operatorname{reg}(R/I(G)) + \deg h_{R/I(G)}(t) \leqslant n$.

Mathematics Subject Classifications: 13D02, 13D40, 05C69, 05C70, 05E40

1 Introduction

Let I be a homogeneous ideal of the polynomial ring $R = k[x_1, \ldots, x_n]$ where k is a field. Associated to I is a graded minimal free resolution of the form

$$0 \to \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{p,j}(I)} \to \cdots \to \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{1,j}(I)} \to R \to R/I \to 0$$

where $R(-j)$ denotes the polynomial ring R with its grading twisted by j, and $\beta_{i,j}(I)$ is the i,j-th graded Betti number. This resolution encodes a number of important invariants of R/I. One such invariant is the *(Castelnuovo-Mumford) regularity* of I, which is defined by

$$\operatorname{reg}(R/I) = \max\{j - i \mid \beta_{i,j}(I) \neq 0\}.$$

The *Hilbert series* of R/I, that is, $H_{R/I}(t) = \sum_{j\in\mathbb{N}} \dim_k(R/I)_j t^j$, can also be read from this resolution; in particular, if $b_{i,i+j} = \beta_{i,i+j}(I)$, then (see [7, p. 100])

$$H_{R/I}(t) = \frac{\sum_i (-1)^i \left(\sum_j b_{i,i+j} t^{i+j}\right)}{(1-t)^n}.$$

This rational function may or may not be in lowest terms; when we rewrite $H_{R/I}(t)$ in lowest terms, the Hilbert-Serre theorem (see [1, Proposition 4.4.1]) says

$$H_{R/I}(t) = \frac{h_{R/I}(t)}{(1-t)^{\dim(R/I)}} \text{ with } h(t) \in \mathbb{Z}[t] \text{ and } h(1) \neq 0.$$

The polynomial $h_{R/I}$ is called the *h-polynomial* of R/I.

Given that $\operatorname{reg}(R/I)$ and $\deg h_{R/I}(t)$ are both derived from the graded minimal free resolution, one can ask if there is any relationship between these two invariants. For example, from [1, Lemma 4.1.3], it follows that if I has a *pure resolution* (for each i, there is at most one j such that $\beta_{i,i+j}(I) \neq 0$), then

$$\deg h_{R/I}(t) - \operatorname{reg}(R/I) = \dim(R/I) - \operatorname{depth}(R/I).$$

The first two authors initiated a comparison of these two invariants in [9, 10, 11]. It was shown in [9] that for all $r, d \geqslant 1$, there exists a monomial ideal such that $\operatorname{reg}(R/I) = r$ and $\deg(R/I) = d$; in [10], it shown that this monomial ideal could be taken to be a lexsegment monomial ideal. In both cases, the degrees of the minimal generators of I depend upon on r and/or d. However, if restrict our family of ideals, one might expect some restriction on the values of r and d. For example, it is shown in [11] that for $2 \leqslant r \leqslant d$, there exists a binomial edge ideal (see [8, 14]) J_G with $\operatorname{reg}(R/J_G) = r$ and $\deg h_{R/J_G}(t) = d$, and furthermore, [16, Theorem 2.1] says that $\deg h_{R/J_G}(t) = 1$ if $\operatorname{reg}(R/J_G) = 1$.

The starting point of this paper is to ask what happens if we restrict to edge ideals. Recall that if $G = (V(G), E(G))$ is a finite simple graph on $V(G) = \{x_1, \ldots, x_n\}$, then the *edge ideal* is the ideal $I(G) = (x_i x_j \mid \{x_i, x_j\} \in E) \subseteq R = k[x_1, \ldots, x_n]$. Our main result is the perhaps surprising fact that one can obtain the main result of [9] using only edge ideals (unlike [9, 10] where the degrees of the generators change, our generators always have degree two):

Theorem 1 (Theorem 4)**.** *Let $r, d \geqslant 1$ be integers. Then there is a finite simple graph G with $r = \operatorname{reg}(R/I(G))$ and $d = \deg h_{R/I(G)}(t)$.*

Our proof of Theorem 1 uses the following strategy. We show that if G is a graph with $\operatorname{reg}(R/I(G)) = r$ and $\deg h_{R/I(G)}(t) = d$, then one can construct a new graph G' with $\operatorname{reg}(R/I(G')) = r+1$ and $\deg h_{R/I(G')}(t) = d+1$. The proof of Theorem 1 then reduces to constructing graphs with $(\operatorname{reg}(R/I(G)), \deg h_{R/I(G)}(t)) = (1, d)$ or $(r, 1)$ for any integers $d, r \geqslant 1$.

Interestingly, $\operatorname{reg}(R/I(G))$ and $\deg h_{R/I(G)}(t)$ are related by the following inequality.

Theorem 2 (Theorem 13)**.** *Let G be a graph on n vertices. Then*

$$\operatorname{reg}(R/I(G)) + \deg h_{R/I(G)}(t) \leqslant n.$$

We provide examples to show that this bound is sharp. Note that Theorem 2 gives a new upper bound on the regularity of edge ideals, i.e., $\operatorname{reg}(R/I(G)) \leqslant n - \deg h_{R/I(G)}(t)$, which complements past research on the regularity of edge ideals (see [5, 6]).

2 Background

We recall the relevant graph theory and commutative algebra background. We continue to use the notation and terminology from the introduction.

Let $G = (V(G), E(G))$ be a finite simple graph on the vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$ consisting of unordered pairs of distinct elements of $V(G)$, that is, if $e \in E(G)$, then $e = \{x_i, x_j\}$ for some $i \neq j$. If G is clear, we write V, respectively E, for $V(G)$, respectively $E(G)$.

We say that there is a *path* between the vertices x_i and x_j if there is a collection of edges $\{e_1, e_2, \ldots, e_t\}$ such that $x_i \in e_1, x_j \in e_t$, and $e_\ell \cap e_{\ell+1} \neq \varnothing$ for all $\ell = 1, \ldots, t-1$. A graph G is *connected* if there is a path between every pair of vertices of G; otherwise, G is said to be *disconnected*. A *connected component* of G is a maximal connected subgraph.

Given any subset $W \subseteq V(G)$, the *induced subgraph* of G on W is the graph $G_W = (W, E(G_W))$ where $E(G_W) = \{e \in E(G) \mid e \subseteq W\}$. Given an $x \in V(G)$, the set of *neighbours* of x is the set $N(x) = \{y \mid \{x, y\} \in E(G)\}$.

A set of vertices $W \subseteq V$ is an *independent set* if for all $e \in E$, $e \nsubseteq W$. An independent set is a *maximal independent set* if it is maximal with respect to inclusion. We let $\alpha(G)$ denote the size of the largest maximal independent set. Using the independent sets, we can build a simplicial complex. In particular, the *independence complex* of G is the simplicial complex:

$$\operatorname{Ind}(G) = \{W \subseteq V \mid W \text{ is an independent set}\}.$$

Note that $\alpha(G)$ is the cardinality of the largest element in $\operatorname{Ind}(G)$.

A set of vertices $W \subseteq V$ is a *vertex cover* if for all $e \in E$, $e \cap W \neq \varnothing$. A vertex cover is a *minimal vertex cover* if it is minimal with respect to inclusion. We let $\beta(G)$ denote the size of the smallest minimal vertex cover. There is duality between independent sets and vertex covers; specifically, $W \subseteq V$ is an independent set if and only if $V \setminus W$ is a vertex cover. Consequently

$$\alpha(G) + \beta(G) = n. \tag{1}$$

A set of edges $\{e_1, \ldots, e_s\} \subseteq E$ is said to be a *matching* if none of the edges share a common vertex. We let $\alpha'(G)$ denote the size of the maximum matching in G. We then always have the following inequality:

$$\alpha'(G) \leqslant \beta(G). \tag{2}$$

Indeed, for any matching $\{e_1, \ldots, e_s\} \subseteq E$, any minimal vertex cover must contain at least one vertex from each e_i.

Finally, we will require the following bound on the regularity of $R/I(G)$.

Theorem 3 ([6, Theorem 6.7])**.** *For any finite simple graph G,* $\operatorname{reg}(R/I(G)) \leqslant \alpha'(G)$.

3 Main Theorem

In this section we will prove our main theorem:

Theorem 4. *Let $r, d \geqslant 1$ be integers. Then there is a finite simple graph G with $r = \operatorname{reg}(R/I(G))$ and $d = \deg h_{R/I(G)}(t)$.*

In order to show this theorem, we will prepare some lemmata.

Lemma 5 ([12, Lemma 3.2])**.** *Let $R_1 = k[x_1, \ldots, x_{n'}]$ and $R_2 = k[x_{n'+1}, \ldots, x_n]$ be polynomial rings over a field k. Let I_1, respectively I_2, be a nonzero homogeneous ideal of R_1, respectively R_2. We write R for $R_1 \otimes_k R_2 = k[x_1, \ldots, x_n]$ and regard $I_1 + I_2$ as a homogeneous ideal of R. Then*

$$\begin{aligned}\operatorname{reg}(R/I_1 + I_2) &= \operatorname{reg}(R_1/I_1) + \operatorname{reg}(R_2/I_2), \quad \textit{and} \\ H_{R/I_1+I_2}(t) &= H_{R_1/I_1}(t) \cdot H_{R_2/I_2}(t).\end{aligned}$$

By virtue of this lemma, one has:

Lemma 6. *Let G be a simple graph, and let $G_1, \ldots, G_\ell$ be the connected components of G. Then*

$$\operatorname{reg}(R/I(G)) = \sum_{i=1}^{\ell} \operatorname{reg}(R_i/I(G_i)), \quad \textit{and} \quad \deg h_{R/I(G)}(t) = \sum_{i=1}^{\ell} \deg h_{R_i/I(G_i)}(t),$$

where $R_i = k[x_j \mid j \in V(G_i)]$ for $i = 1, \ldots, \ell$, and $R = R_1 \otimes_k \cdots \otimes_k R_\ell$.

Remark 7. By Lemma 6, if G is graph with $\operatorname{reg}(R/I(G)) = r$ and $\deg h_{R/I(G)}(t) = d$, then the graph G' which is the disjoint union of G and a single edge on two new vertices $\{z_1, z_2\}$ has $\operatorname{reg}(R'/I(G')) = r + 1$ and $\deg h_{R'/I(G')}(t) = d + 1$ where $R' = R \otimes_k k[z_1, z_2]$. To prove Theorem 4 we need to show that for each $r \geqslant 1$, there exists a graph G with $\operatorname{reg}(R/I(G)) = r$ and $\deg h_{R/I(G)}(t) = 1$, and for each $d \geqslant 1$, there is a graph G with $\operatorname{reg}(R/I(G)) = 1$ and $\deg h_{R/I(G)}(t) = d$. We now work towards this goal.

Example 8. Let $d \geqslant 1$ be a positive integer and let $K_{d,d}$ be the complete bipartite graph, i.e., the graph with $V(K_{d,d}) = \{x_1, \ldots, x_d, y_1, \ldots, y_d\}$ and $E(K_{d,d}) = \{x_i y_j \mid 1 \leqslant i, j \leqslant d\}$. By virtue of Fröberg's Theorem [3, Theorem 1], one has $\operatorname{reg}(R/I(K_{d,d})) = 1$. In addition, the Hilbert series of $R/I(K_{d,d})$ can be computed from the graded minimal free resolution (e.g., see [13, Theorem 5.2.4]); in particular:

$$H_{R/I(K_{d,d})}(t) = \frac{-(1-t)^d + 2}{(1-t)^d}.$$

Hence $\deg h_{R/I(K_{d,d})}(t) = d$.

We now require the following graph construction. Let G be a simple graph on $V(G) = \{x_1, \dots, x_n\}$. For $S \subset V(G)$, the graph G^S is defined by

- $V(G^S) = V(G) \cup \{x_{n+1}\}$, where x_{n+1} is a new vertex; and
- $E(G^S) = E(G) \cup \{\{x_i, x_{n+1}\} \mid x_i \in S\}$.

Lemma 9. *Let G be a graph and let $S \subset V(G)$. Assume that*

- $\dim R/I(G) \geqslant 2$ *and* $h_{R/I(G)}(t) = 1 + h_1 t + h_2 t^2$*;*
- $\operatorname{reg}(R/I(G)) \geqslant 2$*;*
- $|S| = |V(G)| - \dim R/I(G) + 2$*; and*
- *For any $u \in V(G) \setminus S$, there exists $u' \in S$ such that $\{u, u'\} \in E(G)$.*

Then

$$H_{R'/I(G^S)}(t) = \frac{1 + (h_1+1)t + (h_2-1)t^2}{(1-t)^{\dim R/I(G)}} \quad \textit{and} \quad \operatorname{reg}\left(R'/I(G^S)\right) = r,$$

where $R' = R \otimes_k k[x_{n+1}]$.

Proof. By the assumptions and the definition of G^S, we have $I(G^S) + (x_{n+1}) = (x_{n+1}) + I(G)$, and $I(G^S) : (x_{n+1}) = (x_i \mid x_i \in S)$. Hence $R'/(I(G^S) + (x_{n+1})) \cong R/I(G)$ and $R'/(I(G^S) : (x_{n+1})) \cong k[x_i \mid x_i \notin S] \otimes_k k[x_{n+1}]$. Thus, by the additivity of Hilbert series on the short exact sequence

$$0 \to \left(R'/(I(G^S) : (x_{n+1}))\right)(-1) \xrightarrow{\times x_{n+1}} R'/I(G^S) \to R'/(I(G^S) + (x_{n+1})) \to 0,$$

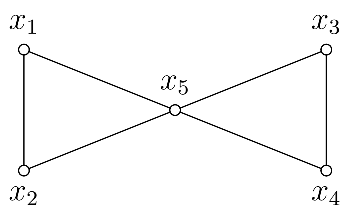
we have

$$\begin{aligned} H_{R'/I(G^S)}(t) &= H_{R'/(I(G^S)+(x_{n+1}))}(t) + t \cdot H_{R'/(I(G^S):(x_{n+1}))}(t) \\ &= H_{R/I(G)}(t) + \frac{t}{(1-t)^{|V(G)|-|S|+1}} \\ &= \frac{1 + h_1 t + h_2 t^2}{(1-t)^{\dim R/I(G)}} + \frac{t}{(1-t)^{\dim R/I(G) - 1}} \\ &= \frac{1 + (h_1+1)t + (h_2-1)t^2}{(1-t)^{\dim R/I(G)}}. \end{aligned}$$

Furthermore, we have $\operatorname{reg}\left(R'/I(G^S)\right) = r$ by virtue of [2, Lemma 2.10]. □

Example 10. Let G be the two disjoint edges $\{x_1, x_2\}$ and $\{x_3, x_4\}$ and $S = V(G)$. Then $G^S = G_{\text{ribbon}}$ where G_{ribbon} is the following graph:

$G_{\text{ribbon}} =$

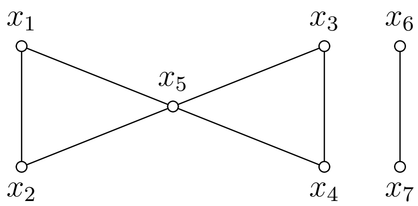


Since $I(G) = (x_1x_2, x_3x_4)$ is a complete intersection, we have $H_{R/I(G)}(t) = \dfrac{1+2t+t^2}{(1-t)^2}$ and $\operatorname{reg}(R/I(G)) = 2$. Hence, by applying Lemma 9, one has

$$H_{R'/I(G_{\text{ribbon}})}(t) = \frac{1+3t}{(1-t)^2} \quad \text{and} \quad \operatorname{reg}(R'/I(G_{\text{ribbon}})) = 2.$$

So $\deg h_{R'/I(G_{\text{ribbon}})}(t) = 1$.

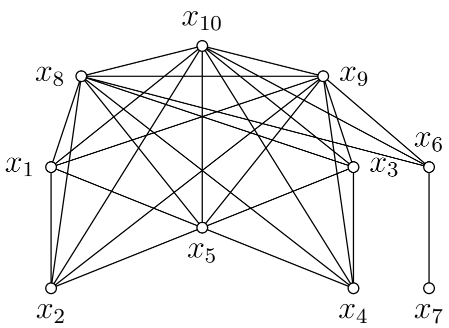
Example 11. Let G_0 be the union of G_{ribbon} and a disjoint edge $\{x_6, x_7\}$:

$G_0 =$ 

Then $H_{R/I(G_0)}(t) = \dfrac{1+3t}{(1-t)^2} \cdot \dfrac{1+t}{1-t} = \dfrac{1+4t+3t^2}{(1-t)^3}$ and $\operatorname{reg}(R/I(G_0)) = 2+1 = 3$ by virtue of Lemma 6 and Example 10. Now we set $S_i = V(G_i) \setminus \{x_7\}$ and $G_{i+1} = G_i^{S_i}$ for $i = 0, 1, 2$. Then, by using Lemma 9 repeatedly, one has

$$H_{R'/I(G_3)}(t) = \frac{1+7t}{(1-t)^3} \quad \text{and} \quad \operatorname{reg}\left(R'/I(G_3)\right) = 3,$$

where $R' = k[x_1, \ldots, x_{10}]$ and G_3 is the following graph:

$G_3 =$ 

Lemma 9 says that, given $r \geqslant 2$, we can construct a graph G' with $\deg h_{R/I(G')}(t) = 1$ and $\operatorname{reg}(R/I(G')) = r$ from a graph G for which $\deg h_{R/I(G)}(t) = 2$ and $\operatorname{reg}\left(R/I(G)\right) = r$, provided the hypotheses of Lemma 9 are met. We use this idea in the next lemma.

Lemma 12. *Given an integer $r \geqslant 3$, we put*

$$Y_r = \{y_{1,1}, y_{2,1} \ldots, y_{r-2,1}, y_{1,2}, y_{2,2}, \ldots, y_{r-2,2}\},$$

$$Z_r = \bigcup_{i=1}^{r-2} \left\{ z_1^{(i)}, z_2^{(i)}, \ldots, z_{2^{i+1}-1}^{(i)} \right\}$$

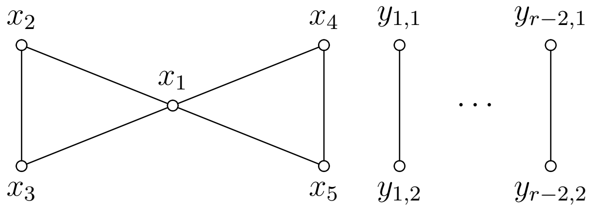
and

$$X = \{x_1, x_2, x_3, x_4, x_5\}.$$

Let $G^{(r)}$ be the graph on $X \cup Y_r \cup Z_r$ such that

- *the induced subgraph $G^{(r)}_{X,Y_r}$ is the following:*

$$G^{(r)}_{X,Y_r} =$$



- *the induced subgraph $G^{(r)}_{Z_r}$ is a complete graph, i.e., all vertices are adjacent; and*
- *for all $1 \leqslant i \leqslant r-2$ and for all $1 \leqslant j \leqslant 2^{i+1}-1$,*

$$N_G\left(z_j^{(i)}\right) = X \cup \{y_{1,1}, y_{2,1}, \ldots, y_{r-2,1}\} \cup Z_r \setminus \left\{z_j^{(i)}\right\}.$$

Let $R^{(r)} = k\left[\{X \cup Y_r \cup Z_r\}\right]$ be the polynomial ring over k whose variables equal to $X \cup Y_r \cup Z_r$. Then

1. $H_{R^{(r)}/I(G^{(r)})}(t) = \dfrac{1 + (2^r - 1)t}{(1-t)^r}$*, that is,* $\deg h_{R^{(r)}/I(G^{(r)})}(t) = 1$*, and*
2. $\operatorname{reg}\left(R^{(r)}/I(G^{(r)})\right) = r$.

Proof. We prove this lemma by induction on $r \geqslant 3$. The graph of Example 11 is $G^{(3)}$; we showed that $H_{R^{(3)}/I(G^{(3)})}(t) = \dfrac{1+7t}{(1-t)^3}$ and $\operatorname{reg}\left(R^{(3)}/I(G^{(3)})\right) = 3$.

Assume $r > 3$. Let G' be the union of $G^{(r-1)}$ and a disjoint edge $\{y_{r-2,1}, y_{r-2,2}\}$. Let $R' = R^{(r-1)} \otimes_k k[y_{r-2,1}, y_{r-2,2}]$. Then

$$\begin{aligned} H_{R'/I(G')}(t) &= H_{R^{r-1}/I(G^{(r-1)})}(t) \cdot \frac{1+t}{1-t} = \frac{1 + (2^{r-1}-1)t}{(1-t)^{r-1}} \cdot \frac{1+t}{1-t} \\ &= \frac{1 + 2^{r-1}t + (2^{r-1}-1)t^2}{(1-t)^r} \end{aligned}$$

and

$$\operatorname{reg}(R'/I(G')) = \operatorname{reg}\left(R^{(r-1)}/I(G^{(r-1)})\right) + 1 = r - 1 + 1 = r$$

by the induction hypothesis and Lemma 9.

Let $S_0 = X \cup \{y_{1,1}, y_{1,2}, \ldots, y_{r-2,1}\} \cup Z_{r-1}$. Then $|S_0| = r + 3 + |Z_{r-1}|$ and

$$\begin{aligned} |V(G')| - \dim R'/I(G') + 2 &= |X| + |Y_{r-1}| + |Z_{r-1}| + 2 - r + 2 \\ &= 5 + 2(r-3) + |Z_{r-1}| + 4 - r \\ &= r + 3 + |Z_{r-1}|. \end{aligned}$$

Hence, by virtue of Lemma 9, one has

$$H_{R_0/I(G_0)}(t) = \frac{1 + (2^{r-1}+1)t + (2^{r-1}-2)t^2}{(1-t)^r} \quad \text{and} \quad \operatorname{reg}(R_0/I(G_0)) = r,$$

where $R_0 = R' \otimes_k k\left[z_1^{(r-2)}\right]$, $G_0 = (G')^{S_0}$, and $V(G_0) = V(G') \cup \left\{z_1^{(r-2)}\right\}$.

Now, for each $1 \leqslant j \leqslant 2^{r-1} - 2$, we define R_j, S_j and G_j inductively:

- $R_j = R_{j-1} \otimes_k k\left[z_{j+1}^{(r-2)}\right]$;
- $S_j = S_{j-1} \cup \left\{z_{j+1}^{(r-2)}\right\}$; and
- $G_j = G_{j-1}^{S_j}$.

Then $R_{2^{r-1}-2} = R^{(r)}$, $G_{2^{r-1}-2} = G^{(r)}$, and one has

$$H_{R^{(r)}/I\left(G^{(r)}\right)}(t) = \frac{1 + (2^r - 1)t}{(1-t)^r} \quad \text{and} \quad \operatorname{reg}\left(R^{(r)}/I(G^{(r)})\right) = r$$

by using Lemma 9 repeatedly. □

We are now in a position to finish the proof of Theorem 4.

Proof (of Theorem 4). We discuss each of the following three cases.

Case 1. Suppose that $1 \leqslant r \leqslant d$. Let G be the union of $K_{d-r+1,d-r+1}$ and $(r-1)$ disjoint edges. By virtue of Lemma 6 and Example 8, one has

$$\operatorname{reg}(R/I(G)) = 1 + (r-1) = r \quad \text{and} \quad \deg h_{R/I(G)}(t) = (d-r+1) + (r-1) = d.$$

Case 2. Suppose that $r, d \geqslant 1$ are integers with $r - d = 1$. Let G be the union of G_{ribbon} and $(r-2)$ disjoint edges. By virtue of Lemma 6 and Example 10, one has

$$\operatorname{reg}(R/I(G)) = 2 + (r-2) = r, \quad \text{and} \quad \deg h_{R/I(G)}(t) = 1 + (r-2) = r - 1 = d.$$

Case 3. Suppose that $r, d \geqslant 1$ are integers with $r - d > 1$. Let G be the union of the graph $G^{(r-d+1)}$ of Lemma 12 and $(d-1)$ disjoint edges. By virtue of Lemma 6 and 12, one has

$$\operatorname{reg}(R/I(G)) = (r-d+1) + (d-1) = r, \quad \text{and} \quad \deg h_{R/I(G)}(t) = 1 + (d-1) = d. \quad \square$$

4 Comparing the regularity and h-polynomial for fixed n

Theorem 4 shows that for all $(r, d) \in \mathbb{N}^2_{\geqslant 1}$, there exists a finite simple graph G with $\big(\operatorname{reg}(R/I(G)), \deg h_{R/I(G)}(t)\big) = (r, d)$. However, if we fix $n = |V(G)|$, then the regularity of $R/I(G)$ and the degree of the h-polynomial must also satisfy the following inequality:

Theorem 13. *Let G be a finite simple graph on n vertices. Then*

$$\deg h_{R/I(G)}(t) + \operatorname{reg}(R/I(G)) \leqslant n.$$

Proof. Via the Stanley-Reisner correspondence, the edge ideal $I(G)$ is associated to the independence complex $\operatorname{Ind}(G)$. The Hilbert series of $R/I(G)$ can then be expressed as

$$H_{R/I(G)}(t) = \sum_{i=0}^{d} \frac{f_{i-1}t^i}{(1-t)^i}$$

(see [7, Theorem 6.2.1]) where f_{j-1} is the number of independent sets of cardinality j in G (in other words, this in the number of faces of $\operatorname{Ind}(G)$ of dimension $j-1$). In particular, $d = \alpha(G)$, the size of the largest independent set. It follows that $\deg h_{R/I(G)}(t) \leqslant \alpha(G)$. By combining Theorem 3 and the inequality (2), we have the bound $\operatorname{reg}(R/I(G)) \leqslant \alpha'(G) \leqslant \beta(G)$. Thus

$$\deg h_{R/I(G)}(t) + \operatorname{reg}(R/I(G)) \leqslant \alpha(G) + \beta(G) = n,$$

as desired, where the last equality is (1). □

Remark 14. For an alternative proof, [15, Corollary 4.3] can be used to show that $\deg h_{R/I(G)}(t) \leqslant (n - \beta(G))$.

Example 15. The upper bound of Theorem 13 is sharp. In fact, we can give two families of graphs such that the equality $\deg h_{R/I(G)}(t) + \operatorname{reg}(R/I(G)) = n$ holds. For the first family, let $n = 2m$ and let G be the union of m disjoint edges. Then $\deg h_{R/I(G)}(t) = \operatorname{reg}(R/I(G)) = m$. For the second family, let $G = K_{1,n-1}$ be the star graph. Then $\deg h_{R/I(G)}(t) = n - 1$ and $\operatorname{reg}(R/I(G)) = 1$.

Remark 16. We end with an observation based upon our computer experiments. For any graph G with at least one edge, we have $\operatorname{reg}(R/I(G)) \geqslant 1$ and $\deg(R/I(G)) \geqslant 1$. If we fix an $n = |V(G)|$, it is natural to ask if we can describe all pairs $(r, d) \in \mathbb{N}^2_{\geqslant 1}$ for which there is a graph G on n vertices with $r = \operatorname{reg}(R/I(G))$ and $d = \deg h_{R/I(G)}(t)$. Theorem 13 implies that $r + d \leqslant n$. Furthermore, note that $\alpha'(G) \leqslant \lfloor \frac{n}{2} \rfloor$, so we must also have $r \leqslant \lfloor \frac{n}{2} \rfloor$ by Theorem 3.

However, these inequalities are not enough to desribe all the pairs (r, d) that may be realizable. For example, when $n = 9$, we computed $\big(\operatorname{reg}(R/I(G)), \deg h_{R/I(G)}(t)\big)$ for all 274668 graphs on nine vertices. We observed that for all such G,

$$\big(\operatorname{reg}(R/I(G)), \deg h_{R/I(G)}(t)\big) \notin \{(3,1), (4,1), (4,2)\}$$

even though these tuples satisfy the inequalities $r + d \leqslant 9$ and $r \leqslant 4$. A similar phenomenon was observed for other n, thus suggesting the existence of another bound relating $\operatorname{reg}(R/I(G))$ and $\deg h_{R/I(G)}(t)$ for a fixed n.

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