The Canonical Join Complex

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Abstract

A canonical join representation is a certain minimal "factorization" of an element in a finite lattice L analogous to the prime factorization of an integer from number theory. The expression $\bigvee A = w$ is the canonical join representation of w if A is the unique lowest subset of L satisfying $\bigvee A = w$ (where "lowest" is made precise by comparing order ideals under containment). Canonical join representations appear in many familiar guises, with connections to comparability graphs and noncrossing partitions. When each element in L has a canonical join representation, we define the canonical join complex to be the abstract simplicial complex of subsets A such that $\bigvee A$ is a canonical join representation. We characterize the class of finite lattices whose canonical join complex is flag, and show how the canonical join complex is related to the topology of L.

Mathematics Subject Classifications: 05Exx; 06A07

1 Introduction

Let L be a finite lattice. In this paper we study a certain unique minimal "join factorization" of the elements of L called the canonical join representation. For an element $w \in L$, an expression $\bigvee A = w$ is called a join representation of w. A join representation $\bigvee A$ of w is irredundant if $\bigvee A' < w$, for each proper subset $A' \subset A$. Informally, the canonical join representation of w is the unique "lowest" irredundant join representation $\bigvee A$ of w. In this case, we also say that the set A is a canonical join representation. (We make the notion of "lowest" precise in Section 3.1.) There is an analogous factorization in terms of the meet operation called the ***canonical meet representation*** that is defined dually (by replacing "$\bigvee$" with "$\bigwedge$" and "lowest" with "highest" in the sentence above). The canonical join representation or canonical meet representation for a given element may not exist, as we discuss below for Figure 1. See Figure 3 for two additional examples. If each element in L has a canonical join representation then L is join-semidistributive. If both L and its dual are join-semidistributive, then we say that L is semidistributive. (See Section 3.1, and Theorem 16 in particular, for an equivalent definition.)

When L is join-semidistributive, we define the ***canonical join complex*** to be the abstract simplicial complex $\Gamma(L)$ whose faces are the subsets $A \subset L$ such that the join A is a canonical join representation. ([23, Proposition 2.2]) says that this is indeed a complex.) Recall that a simplicial complex is ***flag*** if it is the clique complex of its one-skeleton, or equivalently, its minimal non-faces have size equal to 2. Our main result is:

Theorem 1. *Suppose L is a finite join-semidistributive lattice. Then the canonical join complex of L is flag if and only if L is semidistributive.*

In other words, if each element in L admits a canonical join representation then $\Gamma(L)$ is flag if and only if each element *also* admits a canonical meet representation. In light of Theorem 1, we define the ***canonical join graph*** for L to be the one-skeleton of its canonical join complex. Canonical join representations and canonical join graphs appear in many familiar guises. See Section 2 for connections to comparability graphs and noncrossing partitions.

Below in Figure 1, we consider a finite join-semidistributive lattice L whose canonical join complex is not flag. Observe that each pair of atoms in this lattice is a face in the

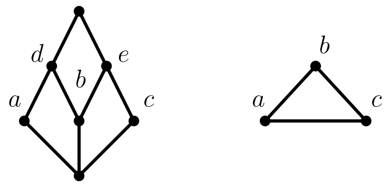

Figure 1: The canonical join complex is an empty triangle.

canonical join complex. Since the join of all three atoms is redundant (because we can remove b and obtain the same join), the canonical join complex is an empty triangle. Also, note that the bottom element $\hat{0}$ of this lattice does not have a canonical meet representation: Both $a \wedge e$ and $c \wedge d$ are meet-representations for $\hat{0}$ that are "as high as possible". This lattice also exhibits some unpleasant topological properties. Recall that the ***crosscut complex*** of L is the abstract simplicial complex whose faces are the subsets A' of atoms in L such that $\bigvee A' < \hat{1}$. A lattice is ***crosscut-simplicial*** if the crosscut complex for each interval is either a simplex or the boundary of a simplex. The *Crosscut Theorem* says that the order complex of a finite poset $\mathcal{P}$ is homotopy equivalent to its crosscut complex ([5, Theorem 10.8]). Therefore, if L is crosscut-simplicial then the order complex of each open interval (x, y) in L is either contractible or homotopy equivalent to a sphere with dimension two less than the number of atoms in $[x, y]$ (see also [17, Theorem 3.7]). In particular, $\mu(x, y) \in \{-1, 0, 1\}$.

Observe that the facets of the crosscut complex for the lattice L in Figure 1 are $\{a, b\}$ and $\{b, c\}$. Therefore, L is not crosscut-simplicial. By contrast, Hersh and Mészáros recently showed that a large class of finite semidistributive lattices—including the class of finite distributive lattices, the weak order on a finite Coxeter group, and the Tamari lattice ([17, Theorems 5.1, 5.3 and 5.5])—are crosscut-simplicial. Building on their work, McConville proved that if L is semidistributive, then it is crosscut-simplicial ([19, Theorem 3.1]). When each element in L has a canonical join representation, we prove that the converse is true.

Theorem 2. *Suppose that L is a finite join-semidistributive lattice. The following are equivalent:*

1. *The canonical join complex of L is flag.*
2. *L is crosscut-simplicial.*
3. *L is semidistributive.*

As an immediate corollary, we obtain the following topological obstruction to the flag property of the canonical join complex.

Corollary 3. *Suppose that L is a finite join-semidistributive lattice and its canonical join complex is flag. Then:*

1. *The order complex of each interval (x, y) in L is either contractible or homotopy equivalent to $\mathbb{S}^{d-2}$, where d is the number of atoms in $[x, y]$;*
2. *The Möbius function takes only the values $\{-1, 0, 1\}$ on the intervals of L.*

McConville showed in [19, Corollary 5.4] that if L is crosscut-simplicial then so is each of its lattice quotients. Because semidistributivity is preserved under taking sublattices and quotients when L is finite (see Section 4), we immediately obtain the following extension of McConville's result for finite join-semidistributive lattices.

Corollary 4. *Suppose that L is a finite join-semidistributive lattice that is crosscut-simplicial. Then each sublattice and quotient lattice of L is also crosscut-simplicial.*

Theorem 1 is surprising in part because its proof does not explicitly use the canonical meet representation of the elements in L. Instead, we make use of a local characterization of the canonical join representation in terms of the cover relations, and a bijection κ from the set of join-irreducible elements to the set of meet-irreducible elements in L. We obtain the following corollary:

Corollary 5. *Suppose that L is a finite semidistributive lattice. Then the bijection κ induces an isomorphism from the canonical join complex to the canonical meet complex of L.*

Using the isomorphism from Corollary 5, one obtains an operation on the canonical join complex that generalizes the operation of rowmotion on the set of antichains in a poset. See Remark 32.

The canonical join complex was first introduced in [23], in which Reading showed that it is flag for the special case of the weak order on the symmetric group (see Example 11). Recently, canonical join representations have played a role in the study of functorially finite torsion classes for the preprojective algebra of Dynkin-type W, when W is a simply laced Weyl group (see for example [13, 18]). In the forthcoming [4], the authors study the canonical join complex of *any* finite dimensional associative algebra Λ. Since the weak order on any finite Coxeter group W and the lattice of torsion classes for Λ of finite representation type are both examples of finite semidistributive lattices (see [8, Lemma 9] and [13, Theorem 4.5]), we obtain the following two applications of Theorem 1:

Corollary 6. *Suppose that W is a finite Coxeter group. Then the canonical join complex of the weak order on W is flag.*

Corollary 7. *Suppose that Λ is an associative algebra of finite representation type, and* $\text{tors}(\Lambda)$ *is its lattice of torsion classes ordered by containment. Then the canonical join complex of* $\text{tors}(\Lambda)$ *is flag.*

2 Motivation

Before we give the technical background for our main results, we describe several familiar examples in which the combinatorics of canonical join representations appear. We begin with an example from number theory and commutative algebra.

Example 8 (The divisibility poset)**.** It is often useful to give a canonical factorization of the elements in a set of equipped with some algebraic structure. A familiar example is the primary decomposition of an ideal in a Noetherian ring. The canonical join representation is the natural lattice-theoretic analogue. Indeed, when L is the the divisibility poset (whose elements are the positive integers ordered $r \leqslant s$ if and only if r divides s), the canonical join representation of $x \in L$ coincides with the primary decomposition of the ideal generated by x:

$$x = \bigvee \{p^d : p \text{ is prime and } p^d \text{ is the largest power of } p \text{ dividing } x\}.$$

Suppose that L is a finite lattice, such that each element in L admits a canonical join representation. One pleasant property of the canonical join representation (and its dual, the canonical meet representation) is that it "sees" the geometry of the Hasse diagram for L. Suppose that $w \in L$ has the canonical join representation $\bigvee A$. We will shortly prove that the elements of A are in bijection with the elements covered by w. So, the down-degree of w in the Hasse diagram for L is equal to the size of A. Specifically, we will prove the following proposition; see Lemma 19 and Proposition 21. (Similar constructions appear in the literature, for example see [10, Theorem 3.5] which gives essentially the same statement for free lattices.)

Proposition 9. *Suppose that $\bigvee A = w$ is the canonical join representation of w. Then, for each element y that is covered by w there is a corresponding element $j \in A$ such that*

$j \vee y = w$, and j is the unique minimal element in L with this property. The correspondence $y \mapsto j$ is a bijection.

With this proposition in mind, we consider the class of finite distributive lattices.

Example 10 (Finite distributive lattices)**.** Suppose that L is a finite distributive lattice. Recall that (see for example [27, Theorem 3.4.1]) L is isomorphic to the lattice $J(\mathcal{P})$ of order ideals of some finite poset $\mathcal{P}$ ordered by inclusion. Suppose that A is an antichain in $\mathcal{P}$. We write I_A for the order ideal generated by A (that is, the elements of A are the maximal elements of I_A). Dually, we write I^A for the order ideal $\{x \in \mathcal{P} : x \not\geq y \text{ for each } y \in A\}$. Equivalently, A is the set of minimal elements in $\mathcal{P} \setminus I^A$.

Observe that the order ideals covered by I_A are exactly of the form $I_{A\setminus\{y\}} = I_A \setminus \{y\}$, where $y \in A$. Since I_y is the smallest order ideal in $J(\mathcal{P})$ containing y, it follows immediately from Proposition 9 that the canonical join representation of I_A is $\bigcup\{I_y : y \in A\}$. (Dually, the canonical meet representation for the ideal I^A is $\bigcap\{I^y : y \in A\}$.) It follows that the canonical join graph of $J(\mathcal{P})$ is the incomparability graph of $\mathcal{P}$.

Example 11 (The Symmetric group and noncrossing arc diagrams)**.** Recently Reading gave an explicit combinatorial model for the canonical join complex of the weak order on the symmetric group S_n in terms of certain noncrossing arc diagrams. A ***noncrossing arc diagram*** is a diagram consisting of n nodes arranged vertically, together with a collection of curves called arcs that must satisfy certain compatibility conditions. In particular, the arcs in a noncrossing arc diagram do not intersect in their interiors (see [23] for details). Each diagram is determined by its combinatorial data: the endpoints of its arcs, and on which side (either left or right) each arc passes the nodes in the diagram.

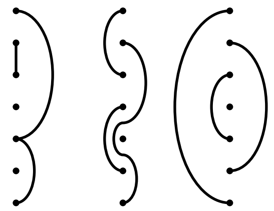

Figure 2: Some examples of noncrossing arc diagrams.

We say that two arcs are ***compatible*** if there is a noncrossing arc diagram that contains them. The following is a combination of [23, Corollary 3.4 and Corollary 3.6]. (In the statement of the Theorem, we take "a collection of arcs" to also mean a collection of noncrossing arc diagrams, each containing a single arc.)

Theorem 12. *There is a bijection δ from the set of join-irreducible permutations in S_n to the set of noncrossing arc diagrams on n nodes that contain precisely one arc. Moreover, a collection of arcs $\mathcal{E}$ corresponds to a face in the canonical join complex of S_n if and only if the arcs in $\mathcal{E}$ are pairwise compatible.*

Example 13 (The Tamari lattice and noncrossing partitions). We conclude our list of examples by considering the Tamari lattice. The Tamari lattice T_n is a finite semidistributive lattice (see for example [16, Theorem 3.5]), which can be realized as an ordering on the set of triangulations for a fixed convex polygon P. Recall that the simple associahedron is a convex polytope, whose faces are in bijection with the collections of pairwise noncrossing diagonals of P (see [11, Figure 3.5]). In particular, the Hasse diagram for T_n is an orientation for the one-skeleton of the associahedron. Since the number of factors in a canonical join representation (called the *canonical joinands*) for $w \in T_n$ is equal to the down-degree of w, we obtain the following result:

Proposition 14. *The f-vector for the canonical join complex of the Tamari lattice $\mathcal{T}_n$ is equal to the the h-vector of the rank $n-1$ associahedron. Specifically, the number of size-k faces in the canonical join complex is equal to the Narayana number*

$$N(n,k) = \frac{1}{n}\binom{n}{k+1}\binom{n}{k}.$$

The canonical join representation of $w \in T_n$ is essentially a noncrossing partition. It is well-known that the Tamari lattice T_n may be realized as the induced subposet of permutations avoiding the 312-pattern. In a noncrossing arc diagram, a ***right arc*** is an arc that does not pass to the left of any node. See the leftmost noncrossing diagram in Figure 2. It is a fact that a permutation avoids the 312-pattern if and only if its image under the bijection δ (from Theorem 12) is a noncrossing arc diagram consisting of only right arcs. (Indeed, $\Gamma(T_n)$ is isomorphic to the subcomplex of compatible arcs on n nodes induced by the set of right arcs. We will revisit this claim in Example 42.) The number $(k-1)$-dimensional faces in $\Gamma(T_n)$ is equal to the number of noncrossing arc diagrams with precisely k right arcs. Rotating such a diagram by a quarter-turn gives the familiar representation of a noncrossing partition as a bump diagram. The Narayana number $N(n,k)$ counts noncrossing partitions of $[n]$ with precisely k arcs, The Narayana number $N(n,k)$ counts noncrossing partitions of $[n]$ with precisely k arcs, which gives us the statement of the proposition. (See [23, Example 4.5] for details, and [25, Theorem 2.7] and the discussion following [25, Proposition 8.8] for a type-free discussion.)

3 Finite semidistributive lattices

3.1 Definitions

In this paper, we study only finite lattices. We write $\hat{0}$ for the unique smallest element in L and $\hat{1}$ for the unique largest element. A ***join representation*** of w is an expression $\bigvee A$ which evaluates to w in L. At times we will also refer to the set A as a join representation. We write $\text{cov}_\downarrow(w)$ for the set $\{y \in L : w \gtrdot y\}$. Similarly, we write $\text{cov}^\uparrow(w)$ for the set of upper covers of w. Recall that w is ***join-irreducible*** if $w = \bigvee A$ implies that $w \in A$. (In particular, the bottom element $\hat{0}$ is not join-irreducible, because it is equal to the empty join.) Since L is finite, w is join-irreducible when $\text{cov}_\downarrow(w)$ has exactly one element.

Meet-irreducible elements satisfy the dual condition. We write $\mathrm{Irr}(L)$ for the set of join-irreducible elements in L.

A join representation $\bigvee A$ of w is ***irredundant*** if $\bigvee A' < \bigvee A$ for each proper subset $A' \subset A$. Each irredundant join representation is an antichain in L. We say that a subset A of L ***join-refines*** a subset B if, for each element $a \in A$, there exists some element $b \in B$ such that $a \leqslant b$. Join-refinement defines a preorder on the subsets of L that is a partial order when restricted to antichains. Indeed, for antichains A and B, A join-refines B if and only if the order ideal generated by A is contained in the order ideal generated by B.

We write $\mathrm{ijr}(w)$ for the set of irredundant join representations of w. The ***canonical join representation*** of w in L is the unique minimal element, in the sense of join-refinement, of $\mathrm{ijr}(w)$, when such an element exists. We write $\mathrm{can}(w)$ for the canonical join representation of w. An element $j \in \mathrm{can}(w)$ is a ***canonical joinand*** for w. If $A = \mathrm{can}(w)$ for some element $w \in L$ then we say that $\bigvee A$ or the set A is a canonical join representation. It follows immediately from the definition that each canonical joinand of w is join-irreducible. Moreover, the canonical join representation of each join-irreducible element j exists and is equal to $\{j\}$. The canonical meet representation of w is defined dually (when it exists).

In Figure 3, we give two examples in which the canonical join representation of $\hat{1}$ does not exist.

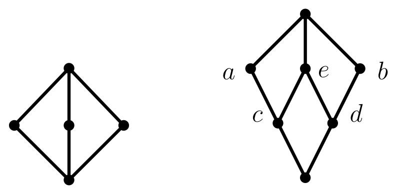

Figure 3: Two finite lattices whose top elements have no canonical join representation.

Example 15. In the modular lattice on the left, each pair of atoms is an irredundant join representation for the top element, and these are minimal elements in $\mathrm{ijr}(\hat{1})$ with respect to join-refinement. Since there is no *unique* minimal join representation, the canonical join representation for $\hat{1}$ does not exist. Arguing dually, we see that the canonical meet representation for the bottom element $\hat{0}$ does not exist either. In the lattice on the right, each element has a canonical meet representation. However, both $a \vee d$ and $b \vee c$ are minimal elements of $\mathrm{ijr}(\hat{1})$. Again, the canonical join representation of $\hat{1}$ does not exist.

In the lattice on the right in Figure 3, we observe the following failure of the distributive law: both $e \vee a$ and $e \vee b$ are equal to $\hat{1}$, but $e \vee (a \wedge b) = e$. (A similar failure is easily verified among the atoms of the modular lattice.) We will see that correcting for precisely this kind of failure of distributivity guarantees the existence of canonical join representations, when L is finite.

A lattice L is ***join-semidistributive*** if L satisfies the following implication for all x, y and z:

$$\text{If } x \vee y = x \vee z, \text{ then } x \vee (y \wedge z) = x \vee y \qquad (SD_\vee)$$

L is ***meet-semidistributive*** if it satisfies the dual condition:

$$\text{If } x \wedge y = x \wedge z, \text{ then } x \wedge (y \vee z) = x \wedge y \qquad (SD_\wedge)$$

A lattice is ***semidistributive*** if it is join-semidistributive and meet-semidistributive. The following result is the finite case of [10, Theorem 2.24], and it says that this definition of join-semidistributivity is equivalent to one given in the introduction.

Theorem 16. *Suppose that L is a finite lattice. Then L satisfies $SD_\vee$ if and only if each element in L has a canonical join representation. Dually, L satisfies $SD_\wedge$ if and only if each element in L has a canonical meet representation.*

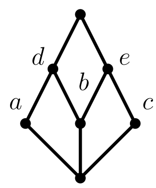

Figure 4: $\kappa(b)$ does not exist.

Assume that L is a finite join-semidistributive lattice, and let $j \in \text{Irr}(L)$. We write j_* for the unique element covered by j, and $\mathcal{K}(j)$ for the set of elements $a \in L$ such that $a \geqslant j_*$ and $a \not\geqslant j$. When it exists, we write $\kappa(j)$ for the unique maximal element of $\mathcal{K}(j)$. It is immediate that $\kappa(j)$ is meet-irreducible. Below, we quote [10, Theorem 2.56]:

Proposition 17. *A finite lattice L is meet-semidistributive if and only if $\kappa(j)$ exists for each join-irreducible element j in L.*

Example 18. Consider the lattice L shown in Figure 4. This is the same finite join-semidistributive lattice from the introduction. L is not meet-semidistributive because, for example, $b \wedge a = b \wedge c = \hat{0}$ but $b \wedge (a \vee c) = b$. Note that $\mathcal{K}(b) = \{a, c, \hat{0}\}$. Since there is no unique maximal element in $\mathcal{K}(b)$, we conclude that $\kappa(b)$ does not exist. Indeed, $\kappa(j)$ does not exist whenever $\bigvee \mathcal{K}(j) \geqslant j$. In particular, if $\bigvee \mathcal{K}(j) = \hat{1}$, then $\kappa(j)$ does not exist.

Below we establish a bijection from the set $\text{cov}_\downarrow(w)$ to $\text{can}(w)$. A similar construction also appears in [10, Theorem 3.5]. Suppose that $w \in L$. For each $y \in \text{cov}_\downarrow(w)$, there is some element $j \in \text{can}(w)$ such that $y \vee j = w$ (because there is some element $j \in \text{can}(w)$ such that $j \not\leqslant y$). For this j, the set $\text{can}(w)$ join-refines $\{j, y\}$. Because $\text{can}(w)$ is an antichain, each $j' \in \text{can}(w) \setminus \{j\}$ satisfies $j' \leqslant y$. Therefore, j is the unique canonical joinand of w such that $y \vee j = w$. We define a map $\eta : \text{cov}_\downarrow(w) \to \text{can}(w)$ which sends y to the unique canonical joinand j such that $y \vee j = w$.

Lemma 19. *Suppose that L is a finite join-semidistributive lattice, and $w \in L$.*

1. *The map $\eta : \mathrm{cov}_\downarrow(w) \to \mathrm{can}(w)$ is a bijection and*
2. *for each $y \in \mathrm{cov}_\downarrow(w)$, we have $y \geqslant \bigvee \mathrm{can}(w) \setminus \{\eta(y)\}$ and $y \in \mathcal{K}(\eta(y))$.*

Proof. Suppose there exist distinct y and y' in $\mathrm{cov}_\downarrow(w)$ satisfying $\eta(y) = \eta(y')$. Then, $y \vee y' = w$, and $\mathrm{can}(w)$ does not join-refine $\{y, y'\}$ (because $\eta(y)$ is below neither y nor y'). That is a contradiction. Thus, η is injective. Suppose that $j \in \mathrm{can}(w)$. Since $\bigvee \mathrm{can}(w)$ is irredundant, $\bigvee(\mathrm{can}(w) \setminus \{j\}) < w$. So, there is a chain of cover relations

$$w \gtrdot \cdots \gtrdot \bigvee(\mathrm{can}(w) \setminus \{j\})$$

beginning with w and ending with $\bigvee(\mathrm{can}(w) \setminus \{j\})$. This chain passes through some $y \in \mathrm{cov}_\downarrow(w)$. Thus, there is some $y \in \mathrm{cov}_\downarrow(w)$ such that $y \geqslant \bigvee(\mathrm{can}(w) \setminus \{j\})$. Observe that $y \not\geqslant j$, otherwise $y \geqslant j \vee \bigvee(\mathrm{can}(w) \setminus \{j\})) = w$. We conclude that $j = \eta(y)$, and that η is a bijection.

We have already argued, in the paragraph above the statement of the proposition, that $y \geqslant \bigvee \mathrm{can}(w) \setminus \{\eta(y)\}$. To complete the proof, suppose that $y \vee \eta(y)_* = w$. Since, $\mathrm{can}(w)$ does not join-refine $\{y, \eta(y)_*\}$ (because $\eta(y) \not\leqslant \eta(y)_*$ and $\eta(y) \not\leqslant y$), we obtain a contradiction as above. We conclude that $y \vee \eta(y)_* < w$. Since y is covered by w, we have $y \vee \eta(y)_* = y$. Thus, $y \in \mathcal{K}(\eta(y))$, for each $y \in \mathrm{cov}_\downarrow(w)$. $\square$

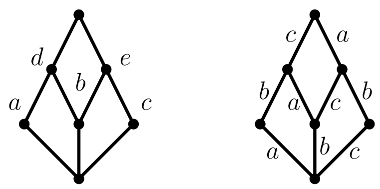

Figure 5: A demonstration of the map η.

Example 20. Consider the top element $\hat{1}$ in the finite join-semidistributive lattice from Example 18. The element $\hat{1}$ covers d (on the left) and e (on the right). The smallest element of the set $\{x \in L : x \vee d = \hat{1}\}$ is c. Thus, $\eta(d) = c$. Similarly, $\eta(e) = a$. We visualize the map η as an edge-labeling of the Hasse diagram of L, from which one can read off the canonical join representation of any element. See Figure 5.

As a consequence of Lemma 19, we obtain a proof of Proposition 9, which we restate here with the notation from of Lemma 19.

Proposition 21. *Suppose that L is a finite join-semidistributive lattice, and y is covered by w in L. Then, $\eta(y)$ is the unique minimal element of L such that $\eta(y) \vee y = w$.*

Proof. Suppose that $x \in L$ has $x \vee y = w$. Since $\mathrm{can}(w)$ join-refines $\{x, y\}$ and $\eta(y)$ and y are incomparable, we conclude that $\eta(y) \leqslant x$. $\square$

In fact, the previous proposition characterizes of finite join-semidistributive lattices. (Similar constructions exist; for example, see the proof of [1, Theorem 3-1.4].) Because the proof is similar to the proof of Lemma 19, we leave the details to the reader.

Proposition 22. *Suppose that L is a finite lattice. The following statements are equivalent:*

1. *For each $w \in L$ and each $y \in \mathrm{cov}_{\downarrow}(w)$, there is a unique minimal element $\eta(y) \in L$ satisfying $y \vee \eta(y) = w$.*

2. *L is join-semidistributive.*

Suppose that L is a finite join-semidistributive lattice, $j \in \mathrm{Irr}(L)$ and F is a canonical join representation. The next lemma, in particular, implies that $F \cup \{j\}$ is a canonical join representation if and only if $\bigvee F \vee j > \bigvee F \vee j_*$.

Lemma 23. *Suppose that L is a finite join-semidistributive lattice, $j \in \mathrm{Irr}(L)$ and $F \subset L \setminus \{j\}$ such that $x \not\geqslant j$, for each $x \in F$. Then:*

1. *j is a canonical joinand of $y \vee j$, for each $y \in \mathcal{K}(j)$;*

2. *j is a canonical joinand of $\bigvee F \vee j$ if and only if $\bigvee F \vee j > \bigvee F \vee j_*$.*

Proof. If $y = j_*$, then the first statement is obvious (because $\{j\}$ is the canonical join representation), so we assume that y and j are incomparable. We write w for the join $j \vee y$, and we write $A = \{j' \in \mathrm{can}(w) : j' \leqslant j\}$ and $A' = \{j' \in \mathrm{can}(w) : j' \leqslant y\}$. Because $\mathrm{can}(w)$ join-refines $\{j, y\}$, we have $A \cup A' = \mathrm{can}(w)$. Also, the set A is not empty because the join $y \vee j$ is irredundant. We want to show that $A = \{j\}$. Since j is join-irreducible, it is enough to show that $j = \bigvee A$. Since $y \geqslant \bigvee A'$, we see that $\bigvee A \vee y = j \vee y$. If $\bigvee A < j$, then $j_* \vee y = j \vee y$, and that is impossible because $y \in \mathcal{K}(j)$. We conclude that j is a canonical joinand of $y \vee j$.

If $\bigvee F \vee j > \bigvee F \vee j_*$, then $\bigvee F \vee j_* \in \mathcal{K}(j)$. We conclude that j is a canonical joinand of $\bigvee F \vee j$. The remaining direction of the second item is straightforward to verify. □

We close this subsection by quoting the following easy proposition (for example see [23, Proposition 2.2]), which says that the canonical join complex is indeed a simplicial complex.

Proposition 24. *Suppose L is a finite lattice, and A is a canonical join representation in L. Then each proper subset of A is also a canonical join representation.*

3.2 The flag property

In this section we prove Theorem 1. We begin by presenting the key arguments in one direction the proof: If L is a finite semidistributive lattice, then its canonical join complex is flag. Most of the work is done in the following two lemmas.

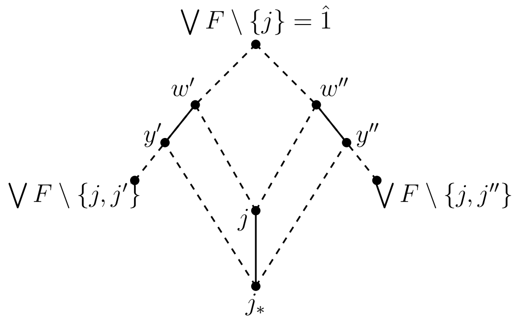

Figure 6: An illustration of the proof of Lemma 25. Dashed lines represent order relations in L while solid lines represent cover relations.

Lemma 25. *Suppose that L is a finite semidistributive lattice, and F is a subset of* $\mathrm{Irr}(L)$ *such that $|F| \geqslant 3$ and each proper subset of F is a face in $\Gamma(L)$. Then, for each distinct j and j' in F, the elements $\bigvee(F \setminus \{j\})$ and $\bigvee(F \setminus \{j'\})$ are incomparable.*

Proof. Without loss of generality we assume that $\bigvee F = \hat{1}$. (If $\bigvee F = x$ and $x < \hat{1}$ then restrict to the interval $[\hat{0}, x]$. Observe that this interval is also a finite semidistributive lattice, and the argument below applies.) Suppose there exists distinct $j, j' \in F$ such that $\bigvee(F \setminus \{j\}) \geqslant \bigvee(F \setminus \{j'\})$. Observe that

$$\bigvee(F \setminus \{j\}) = \Big(\bigvee(F \setminus \{j\})\Big) \vee \Big(\bigvee(F \setminus \{j'\})\Big) = \bigvee F = \hat{1}.$$

Since F has at least three elements, there exists $j'' \in F \setminus \{j, j'\}$. We write w' for $\bigvee(F \setminus \{j'\})$ and w'' for $\bigvee(F \setminus \{j''\})$. See Figure 6. By assumption, j is a canonical joinand for both w' and w''. Lemma 19 implies that there exists $y' \in \mathrm{cov}_\downarrow(w')$ and $y'' \in \mathrm{cov}_\downarrow(w'')$ such that $y', y'' \in \mathcal{K}(j)$. Moreover, $y' \geqslant \bigvee(F \setminus \{j, j'\})$ and similarly $y'' \geqslant \bigvee(F \setminus \{j, j''\})$.

So, we have: $y' \vee y'' \geqslant (\bigvee(F \setminus \{j, j'\})) \vee (\bigvee(F \setminus \{j, j''\})) = \bigvee(F \setminus \{j\})$. Since $\bigvee(F \setminus \{j\}) = \hat{1}$, we conclude that $\bigvee \mathcal{K}(j) = \hat{1}$. This implies that there is no unique maximal element in $\mathcal{K}(j)$. Thus $\kappa(j)$ does not exist, contradicting Proposition 17. □

Lemma 26. *Suppose that L is a finite join-semidistributive lattice, and F is a subset of* $\mathrm{Irr}(L)$ *satisfying the following conditions: First, $|F| \geqslant 3$; second, each proper subset of F is a face in $\Gamma(L)$; third, $\bigvee F$ is irredundant; fourth, F is not a face in $\Gamma(L)$. Then there exists $j \in F$ such that $\kappa(j)$ does not exist.*

Proof. Without loss of generality, we assume that $\bigvee F = \hat{1}$. If $F \subseteq \mathrm{can}(\hat{1})$ then F join-refines $\mathrm{can}(\hat{1})$. But, by definition of canonical join representation, $\mathrm{can}(\hat{1})$ is the smallest irredundant join representation of $\hat{1}$ (ordered by join-refinement). Thus, $\mathrm{can}(\hat{1}) = F$. That is impossible, because we assumed that $F \notin \Gamma(L)$. Therefore, there exists some $j \in F$ such that $j \notin \mathrm{can}(\hat{1})$.

Lemma 23 implies that $\bigvee(F \setminus \{j\}) \vee j_* = \hat{1}$. Let j' and j'' be distinct elements in $F \setminus \{j\}$. By our assumption, $\bigvee F \setminus \{j'\}$ is the canonical join representation of some element in w' in L. By Lemma 19, there exists $y' \in \mathrm{cov}(w')$ such that $\eta(y') = j$. In particular,

$y' \in \mathcal{K}(j)$ and $y' \geqslant \bigvee(F \setminus \{j, j'\})$. By the same argument, there is an element y'' covered by $\bigvee F \setminus \{j''\}$ such that $y'' \in \mathcal{K}(j)$ and $y'' \geqslant \bigvee(F \setminus \{j, j''\})$. Because y' and y'' are both members of $\mathcal{K}(j)$, we have $y', y'' \geqslant j_*$. Therefore, $y' \vee y'' \geqslant \bigvee(F \setminus \{j\}) \vee j_* = \hat{1}$. It follows that $\bigvee \mathcal{K}(j) = \hat{1}$, and thus $\kappa(j)$ does not exist. Again, this contradicts Proposition 17. □

Proof of one direction of Theorem 1. We show that if L is semidistributive, then its canonical join complex is flag. Suppose that $F \subset \mathrm{Irr}(L)$ such that $|F| \geqslant 3$ and each proper subset of F is a face of the canonical join complex. Without loss of generality, assume that $\bigvee F = \hat{1}$. Lemma 25 says that for each distinct j and j' in F, the joins $\bigvee(F \setminus \{j\})$ and $\bigvee(F \setminus \{j'\})$ are incomparable. Thus, we have

$$\bigvee(F \setminus \{j\}) < \Big(\bigvee(F \setminus \{j\})\Big) \vee \Big(\bigvee(F \setminus \{j'\})\Big) = \bigvee F.$$

We conclude that $\bigvee F$ is irredundant. Lemma 26 implies that F is a face of the canonical join complex. □

We now turn to the other direction of Theorem 1. In the following lemmas we will assume that L is a finite join-semidistributive lattice that fails $SD_\wedge$. By Proposition 17, there is some $j \in \mathrm{Irr}(L)$ such that $\kappa(j)$ does not exist. Our goal is to construct a set $A \subseteq \mathrm{Irr}(L)$ such that $A \cup \{j\}$ is a "hollow face" in the canonical join complex. More precisely, the set A must satisfy the following conditions. (NF stands for "not-flag".)

(NF1) $A \cup \{j\}$ is *not* a face in $\Gamma(L)$.

(NF2) Each pair of elements in $A \cup \{j\}$ *is* a face in $\Gamma(L)$.

The essential idea is that among all of the subsets of $\mathrm{Irr}(L)$ satisfying (NF1), a set A chosen as low as possible in L will also satisfy (NF2). For us, "as low as possible" means that A is chosen to be minimal in join-refinement. The argument is somewhat delicate because join-refinement is a preorder, not a partial order, on subsets of L. So, we must take extra care to compare only antichains $Y \subseteq \mathrm{Irr}(L)$ satisfying (NF1). To further emphasize this point, we write $A \lll B$ when A join-refines B, for *antichains* A and B. We write $A(j)$ for the collection of antichains $Y \subseteq L \setminus \{j\}$ satisfying $Y \cup \{j\}$ is an antichain. We write $E(j)$ for the set of elements $j' \in \mathrm{Irr}(L) \setminus \{j\}$ such that $j' \vee j$ is a canonical join representation. When it is possible, we suppress j, and simply write E.

Lemma 27. *Suppose that L is a finite join-semidistributive lattice and $j \in \mathrm{Irr}(L)$ such that $\kappa(j)$ does not exist. Let E be the set of $j' \in \mathrm{Irr}(L) \setminus \{j\}$ such that $j' \vee j$ is a canonical join representation. Then:*

1. $\bigvee E \vee j = \bigvee E \vee j_*$;
2. *There exists a nonempty antichain Y in $A(j)$ such that $\bigvee Y \vee j = \bigvee Y \vee j_*$.*

Proof. Assume that $\bigvee E \vee j > \bigvee E \vee j_*$. The second item of Lemma 23 says that j is a canonical joinand of $\bigvee E \vee j$. Also, for each element a in $\mathcal{K}(j)$, j is a canonical joinand of $a \vee j$. Write $\bigvee E' \vee j$ for the canonical join representation of $a \vee j$. Proposition 24 implies that $j' \vee j$ is a canonical join representation for each $j' \in E'$. Thus, $E' \subseteq E$. Thus, $a \vee j \leqslant \bigvee E \vee j$, and in particular $a \leqslant \bigvee E \vee j$.

Because j is a canonical joinand of $\bigvee E \vee j$, Lemma 19 implies that there is a unique element y covered by $\bigvee E \vee j$ such that $\eta(y) = j$. By definition of η, we have $y \vee j = \bigvee E \vee j$. The second assertion of Lemma 19 says that $y \in \mathcal{K}(j)$. If $a \nleqslant y$, then $y \vee a = \bigvee E \vee j$. Proposition 21 implies that $j \leqslant a$, contradicting the fact that $a \in \mathcal{K}(j)$. We conclude that $a \leqslant y$. We have proved that $y = \kappa(j)$, contradicting our hypothesis. Thus, we must have $\bigvee E \vee j = \bigvee E \vee j_*$.

If E is empty then the only face in $\Gamma(L)$ that contains j is the singleton $\{j\}$. Lemma 3.9 says that for each $a \in \mathcal{K}(j)$, the element $a \vee j$ has j as a canonical joinand. Therefore, when E is empty, $\mathcal{K}(j) = \{j_*\}$. This would contradict our assumption that $\kappa(j)$ does not exist. We conclude that E is nonempty. The antichain of maximal elements in $Y \subseteq E$ satisfies $\bigvee Y = \bigvee E$. Thus, $\bigvee Y \vee j = \bigvee Y \vee j_*$. Also, for each $j' \in Y$, we know that $\{j', j\}$ is an antichain because $\bigvee \{j', j\}$ a canonical join representation. Therefore, $Y \cup \{j\}$ is an antichain. We conclude that $Y \in A(j)$. This proves the second assertion of the lemma. □

Lemma 27 says that the collection of antichains Y in $A(j)$ satisfying

$$\bigvee Y \vee j = \bigvee Y \vee j_* \qquad (NC)$$

is nonempty. (Actually, we have shown something stronger: The collection of antichains $Y \subseteq E(j)$ that satisfy (NC) is nonemtpy.) We write (NC) for "not-canonical" because the second item Lemma 23 implies that $\bigvee Y \vee j$ is not a canonical join representation.

Suppose that $B \in A(j)$ is minimal in join-refinement among all antichains in $A(j)$ that satisfy NC. (Note that B is not necessarily unique.) The next lemma is the difficult part of the proof of the remaining direction of Theorem 1. We argue that $(B \setminus \{b\}) \cup \{j\}$ is a face in the canonical join complex for each $b \in B$. Thus, if B has at least three elements, then $B \cup \{j\}$ is the "hollow face" that we want to construct. We will deal with the case where $|B| \leqslant 2$ in Lemma 29 and Lemma 30.

Lemma 28. *Suppose that L is a finite join-semidistributive lattice and $j \in \mathrm{Irr}(L)$ such that $\kappa(j)$ does not exist. Among all antichains in $A(j)$ that satisfy (NC), let B be minimal in join-refinement. Then $(B \setminus \{b\}) \cup \{j\}$ is a canonical join representation, for each $b \in B$.*

Proof. We begin by pointing out two easy observations about the join-refinement relation. (Note that the second observation, (JR2), may fail if $S \cup \{x\}$ and $T \cup \{x\}$ are not antichains.)

(JR1) For any pair of subsets S and T, if S join-refines T then each subset $S' \subseteq S$ also join-refines T.

(JR2) Suppose that $S \cup \{x\}$ and $T \cup \{x\}$ are antichains. Then, $S \cup \{x\} \ll T \cup \{x\}$ if and only if $S \ll T$.

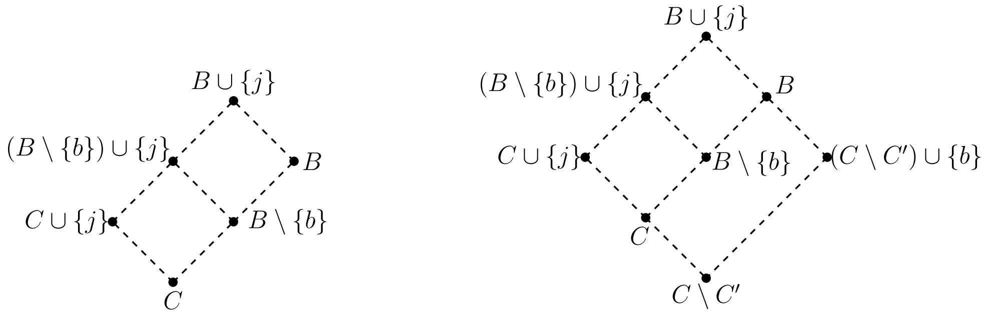

Figure 7: Some order relations in the join-refinement order for L.

In particular, (JR1) implies that $B \setminus \{b\} \lessdot B$. Thus, $\bigvee (B \setminus \{b\}) \vee j_* < \bigvee (B \setminus \{b\}) \vee j$. The second item of Lemma 23 says that j is a canonical joinand of $\bigvee (B \setminus \{b\}) \vee j$. We write $C \cup \{j\}$ for the canonical join representation of $\bigvee (B \setminus \{b\}) \vee j$, where $j \notin C$. We claim that $C \cup \{b\} = B$. In Figure 7, the lefthand poset depicts the relationship between C, B, and $B \setminus \{b\}$ in the join-refinement order. In the figure, we have $C \cup \{j\} \lessdot (B \setminus \{b\}) \cup \{j\}$, because $C \cup \{j\}$ is the canonical join representation for $\bigvee (B \setminus \{b\}) \vee j$. By (JR2), we have $C \lessdot B \setminus \{b\}$.

We make two observations that follow immediately from the second item of Lemma 23. First, we observe that j *not* a canonical joinand of $\bigvee B \vee j$. Thus,

$$\bigvee C \vee j = \bigvee (B \setminus \{b\}) \vee j < \bigvee B \vee j. \tag{1}$$

Second, we observe that:

$$\bigvee (C \cup \{b\}) \vee j = \bigvee (C \cup \{b\}) \vee j_*. \tag{2}$$

Indeed, if $\bigvee (C \cup \{b\}) \vee j_* < \bigvee (C \cup \{b\}) \vee j$ then the second item of Lemma 23 says that j is a canonical joinand of $\bigvee (C \cup \{b\}) \vee j = \bigvee B \vee j$. We have just noted that j is a *not* a canonical joinand for $\bigvee B \vee j$.

If $C \cup \{b\}$ is an antichain, then applying (JR2) to the relation $C \lessdot B \setminus \{b\}$, we have $C \cup \{b\} \lessdot B$. Thus, we have $C \cup \{b\}$ is an antichain in $A(j)$ that satisfies (NC) and join-refines B. By minimality of B, we conclude that $C \cup \{b\} = B$ as desired. So, we assume that $C \cup \{b\}$ is not an antichain. The inequality in Equation (1) implies that there exists no $c \in C$ with $b \leqslant c$. We write C' for the set $\{c \in C : c < b\}$.

We make three easy observations: First, $(C \setminus C') \cup \{b\}$ is member of $A(j)$. Second, applying (JR1) to the relation $C \lessdot B \setminus \{b\}$, we have that $C \setminus C' \lessdot B \setminus \{b\}$. By (JR2), we conclude that $(C \setminus C') \cup \{b\} \lessdot B$. We depict these relations in the righthand poset in Figure 7. Third,

$$\bigvee ((C \setminus C') \cup \{b\}) \vee j = \bigvee (C \cup \{b\}) \vee j = \bigvee (C \cup \{b\}) \vee j_* = \bigvee ((C \setminus C') \cup \{b\}) \vee j_*,$$

where the first and third equalities follow from the fact that $\bigvee (C \cup \{b\})$ is equal to $\bigvee (C \setminus C') \cup \{b\}$, and the middle equality is (2).

Therefore, the set $(C \setminus C') \cup \{b\}$ is an antichain in $A(j)$ that satisfies (NC) and join-refines B. By the minimality of B, we have $B = (C \setminus C') \cup \{b\}$. Since $C \lll B \setminus \{b\}$, we have that C join-refines its proper subset $C \setminus C'$. That is a impossible (because C is an antichain). Thus, C' is empty. We have proved that $C \cup \{b\} = B$.

To conclude, we defined the set C so that $\bigvee C \vee j$ is the canonical join representation of $\bigvee B \setminus \{b\} \cup \{j\}$. We then proved that $C \cup \{b\} = B$. Thus, $C = B \setminus \{b\}$. Therefore, $\bigvee B \setminus \{b\} \cup \{j\}$ is a canonical join representation, as desired. □

Our candidate for a "hollow face" in $\Gamma(L)$ is the antichain $B \cup \{j\}$ from Lemma 28. As we have noted, if B has at least three elements then B satisfies both (NF1) and (NF2).

Suppose that $B = \{b_1, b_2\}$. By Lemma 28, $\{j, b_i\}$ is a canonical join representation, for $i = 1, 2$. (Thus, B is minimal in join-refinement among the antichains *in* $E(j)$ that satisfy (NC).) The next lemma, in particular, implies that $\{b_1, b_2\}$ is a canonical join representation.

Lemma 29. *Suppose that L is a finite join-semidistributive lattice and $j \in \mathrm{Irr}(L)$ such that $\kappa(j)$ does not exist. Among all antichains in $A(j)$ that satisfy (NC), let B be minimal in join-refinement. Suppose that B has at least two elements. Then each pair of elements in $B \cup \{j\}$ is a face in the canonical join complex.*

Proof. If B has three or more elements, then the statement follows from Lemma 28 and Proposition 24. Assume that B has two elements, b_1 and b_2. By Lemma 28, we have $\{j, b_i\}$ is a canonical join representation, for $i = 1, 2$. Consider $\{b_1, b_2\}$. We will argue that b_1 is a canonical joinand of $b_1 \vee b_2$, and complete the proof by symmetry.

Assume that $(b_1)_* \vee b_2 = b_1 \vee b_2$. Because $B = \{b_1, b_2\}$ satisfies (NC), we know that $b_1 \vee b_2 \vee j = b_1 \vee b_2 \vee j_*$. Therefore $(b_1)_* \vee b_2 \vee j = (b_1)_* \vee b_2 \vee j_*$. If $(b_1)_* \leqslant j_*$, then we have $b_2 \vee j = b_2 \vee j_*$, contradicting Lemma 23. By the same reasoning, $(b_1)_* \nleqslant b_2$. Also, $j \nleqslant (b_1)_*$ because b_1 and j are incomparable. Similarly, $b_2 \not< (b_1)_*$. Thus, $\{(b_1)_*, b_2\}$ is an antichain in $A(j)$ that satisfies (NC) and join-refines $\{b_1, b_2\}$. But this contradicts our hypothesis, which says that B is minimal in join-refinement among all such antichains. Thus $(b_1)_* \vee b_2 < b_1 \vee b_2$. Lemma 23 says that b_1 is a canonical joinand of $b_1 \vee b_2$. □

Finally, we turn to the case where B is a singleton. This turns out to be a non-issue. The next lemma says that we can always find such an antichain in $A(j)$ with at least two elements.

Lemma 30. *Suppose that L is a finite join-semidistributive lattice and $j \in \mathrm{Irr}(L)$ such that $\kappa(j)$ does not exist. Then there exists an antichain $A \in A(j)$ satisfying:*

1. *A has at least two elements; and*

2. *A is minimal in join-refinement among all antichains in $A(j)$ that satisfy (NC).*

Proof. Recall that $E(j)$ is the set of $j' \in \mathrm{Irr}(L) \setminus \{j\}$ such that $j' \vee j$ is a canonical join representation. Take A to be a nonempty antichain that is minimal in join-refinement among all antichains $Y \subseteq E(j)$ that satisfy (NC). Lemma 27 implies that such an antichain A exists. For each $a \in A$, we have $a \vee j$ is a canonical join representation. Since A satisfies (NC), the second item of Lemma 23 implies that $\bigvee A \vee \{j\}$ is not a canonical join representation. Thus, A has at least two elements.

Now we prove that A is minimal in join-refinement among all antichains in $A(j)$ that satisfy (NC). Suppose that $B \in A(j)$ satisfies (NC), and $B \lll A$. Without loss of generality, assume that B is minimal in join-refinement with this property. If B has two or more elements, then Lemma 28 implies that B is a subset of $E(j)$. Therefore, $B = A$. Thus we can assume that $B = \{b\}$. Since B join-refines A, there is some $a \in A$ such that $b \leqslant a$.

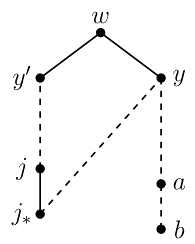

Figure 8: An illustration of the argument for Lemma 30. Dashed lines represent order relations in L while solid lines represent cover relations.

Write w for $a \vee j$. Since $a \vee j$ is the canonical join representation of w, Lemma 19 implies that $\mathrm{cov}_{\downarrow}(w)$ has precisely two elements, y and y'. Let $\eta(y) = j$ and $\eta(y') = a$, so that $y \in \mathcal{K}(j)$ and $y \geqslant a$. See Figure 8. Thus, we have $b \leqslant a \leqslant y$. On the one hand, $(b \vee j) \vee y = (b \vee j_*) \vee y = y$. On the other hand, $b \vee (j \vee y) = b \vee w = w$. By this contradiction, we have proved the result. $\square$

Finally, we complete the proof of the main result.

Proof of the remaining direction of Theorem 1. Now we argue that if L is a finite join-semidistributive lattice and the canonical join complex of L is flag, then L is semidistributive. By Proposition 17, it is enough to show that for each $j \in \mathrm{Irr}(L)$ the element $\kappa(j)$ exists.

Suppose $j \in \mathrm{Irr}(L)$ and $\kappa(j)$ does not exist. Among all nonempty antichains in $A(j)$ that satisfy (NC), let A be minimal in join-refinement, and choose A so that it has at least two elements. Lemma 30 says that such an antichain A exists. Lemma 29 says that each pair of elements in $A \cup \{j\}$ is face in the canonical join complex. Finally, the second item of Lemma 23 implies that $A \cup \{j\}$ is not face of the canonical join complex. But if the $\Gamma(L)$ were flag, then $A \cup \{j\}$ must be a face (because each pair of elements in $A \cup \{j\}$ is a face). We have reached a contradiction to our hypothesis that the canonical join complex is flag. By this contradiction, we conclude that L is semidistributive. $\square$

Suppose that m is meet-irreducible and write m_* for the unique element covering m. When it exists, let $\kappa_*(m)$ be the smallest element $j \in L$ with $j \leqslant m_*$ and $j \not\leqslant m$. It is immediate that $\kappa_*(m)$ is join-irreducible. Proposition 17, applied to the dual lattice, says that L is meet-semidistributive if and only if $\kappa_*(m)$ exists for each meet-irreducible element m. In fact, L is semidistributive if and only if κ is a bijection, with inverse map κ_*; this is the finite case of [10, Corollary 2.55]. (Recall that Theorem 16 says that each element in L has a canonical meet representation if and only if L is meet-semidistributive.)

Corollary 31. *Suppose that L is a finite meet-semidistributive lattice. Then, the canonical meet complex for L is flag if and only if L is semidistributive.*

Next, we prove Corollary 5 by showing that the bijection κ taking a join-irreducible element j to $\kappa(j)$ induces an isomorphism from the canonical join complex of L to the canonical meet complex of L.

Proof of Corollary 5. Corollary 31 says that the canonical meet complex of L is flag, so it is enough to show that κ bijectively maps edges of the canonical join complex to edges of the canonical meet complex. Suppose that $\{j_1, j_2\}$ is a face of the canonical join complex, and write m_1 for $\kappa(j_1)$ and m_2 for $\kappa(j_2)$. Suppose that $m_1 \wedge m_2 = (m_1)_* \wedge m_2$. Lemma 19 implies that there exists some $y \in \text{cov}_\downarrow(j_1 \vee j_2)$ satisfying : $j_1 \leqslant y \leqslant \kappa(j_2) = m_2$. (See Figure 9 for an illustration.) Since $j_1 \leqslant (m_1)_*$, we conclude that $j_1 \leqslant (m_1)_* \wedge m_2 = m_1 \wedge m_2$. We see that $j_1 \leqslant m_1$ and that is a contradiction. Therefore, $(m_1)_* \wedge m_2 > m_1 \wedge m_2$. By the dual statement of Lemma 23, we conclude that m_1 is a canonical meetand of $m_1 \wedge m_2$, and by symmetry m_2 is also a canonical meetand of $m_1 \wedge m_2$. The dual argument establishes the desired isomorphism. □

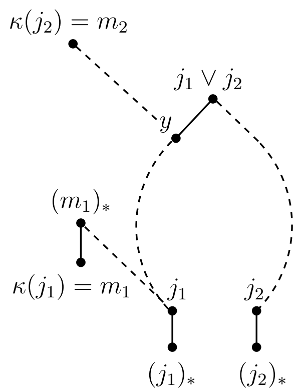

Figure 9: An illustration of the argument for the proof of Corollary 5. Dashed lines represent relations in L while solid black lines represent cover relations.

We close this section by relating Corollary 5 to Example 10 and Example 13, from Section 2.

Remark 32. Suppose that F is a face of the canonical join complex of a finite semidistributive lattice L. Corollary 5 says that $\bigwedge \kappa(F)$ is a canonical meet representation. By taking the canonical join representation of $\bigwedge \kappa(F)$, we can view the map κ as an operation on the canonical join complex. Similarly, we can view κ_* as an operation on the canonical meet complex. In fact, *rowmotion*—an operation on the set of antichains in a finite poset

$\mathcal{P}$—may be viewed as an instance of the operation of κ_* as it acts on the canonical meet complex of $J(\mathcal{P})$. For each antichain A in $\mathcal{P}$, we define $\text{Row}(A)$ to be the set of $x \in \mathcal{P}$ such that x is minimal among elements not in I_A. (Our definition is based on [29]. See also [2, 6, 7, 12, 20, 26].) Observe that $\text{Row}(A)$ is an antichain, and, in the notation from Example 10, $I_A = I^{\text{Row}(A)}$. It follows immediately from the definition of κ_* that $\kappa_*(I^y) \mapsto I_y$. We obtain the following result.

Proposition 33. *Suppose that $\mathcal{P}$ is a finite poset, and A is an antichain in $\mathcal{P}$. Then the map κ_*, acting on faces of the canonical meet complex of $J(\mathcal{P})$, sends the order ideal I^A to the order ideal $I^{\text{Row}(A)}$.*

3.3 Crosscut-simplicial lattices

In this section, we prove Theorem 2. Recall that one direction of the proof was given as [19, Theorem 3.1]. Because it is easy, we give an alternative argument in the next paragraph.

Write A for the set of atoms in L. When L is a finite semidistributive lattice every join of two atoms is a canonical join representation. In particular, Theorem 1 implies that each distinct subset of atoms gives rise to a distinct element in L. Thus the crosscut complex for L is either the boundary of the simplex on A or equal to the simplex on A, depending on whether $\bigvee A = \hat{1}$ or $\bigvee A < \hat{1}$. Since each interval in L inherits semidistributivity, it follows that L is crosscut-simplicial.

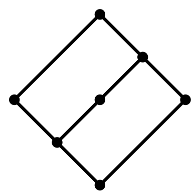

Figure 10: A finite crosscut-simplicial lattice failing both $SD_\vee$ and $SD_\wedge$.

Before we proceed with the proof of the converse, we point out that the hypothesis of join-semidistributivity in Corollary 2 is crucial. For example, consider the crosscut-simplicial lattice shown in Figure 10. This lattice fails both $SD_\vee$ and $SD_\wedge$. Join-semidistributivity gives us a powerful restriction: A finite join-semidistributive lattice L fails $SD_\wedge$ if and only if L contains the lattice shown in Figure 1 as a sublattice ([10, Theorem 5.56]).

We now begin our proof. The following lemmas will be useful. The first lemma is a local version of Theorem 16, and appears as [22, Lemma 9-2.5].

Lemma 34. *Suppose that L is a finite lattice satisfying the following property:*

> *If x, y, and z are elements of L with $x \wedge y = x \wedge z$ and also, y and z cover a common element, then $x \wedge (y \vee z) = x \wedge y$.*

Then L is meet-semidistributive.

Lemma 35. *Suppose that L is a finite join-semidistributive lattice that fails $SD_\wedge$. Then there exists x, y, and z such that $y \vee z > x$ and x, y, and z cover a common element.*

Proof. We prove the proposition by induction on the size of L. As mentioned above, L contains the lattice shown in Figure 1 as sublattice, and this proves the base case. By Lemma 34, we can assume that there exist x, y, and z in L such that $x \wedge y = x \wedge z$, $x \wedge (y \vee z) \neq x \wedge y$, and $\mathrm{cov}_\downarrow(y) \cap \mathrm{cov}_\downarrow(z)$ is not empty. Among all such triples, we choose $\{x, y, z\}$ minimal in join-refinement. Write a for the element in $\mathrm{cov}_\downarrow(y) \cap \mathrm{cov}_\downarrow(z)$ (if there is more than one element in $\mathrm{cov}_\downarrow(y) \cap \mathrm{cov}_\downarrow(z)$, then $y \wedge z$ does not exist). If x also covers a, then we are done (because if $x \gtrdot a$ and $y \vee z \not> x$, then $(y \vee z) \wedge x = a$, and that contradicts our assumption that $\{x, y, z\}$ fail $SD_\wedge$). So we assume that x does not cover a.

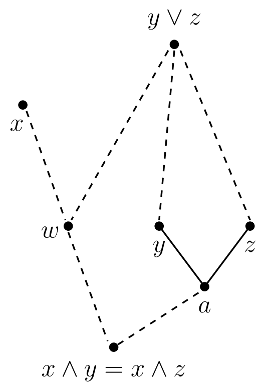

Figure 11: Dashed lines represent relations in L and solid lines represent cover relations.

We first prove that $y \vee z > x$. We write w for $x \wedge (y \vee z)$. See Figure 11. Observe that $x \wedge y < w$ (because x, y and z fail $SD_\wedge$, the inequality is strict). On the one hand $w \wedge (x \wedge y) = x \wedge y$. On the other hand, $x \geqslant w$, so $(x \wedge w) \wedge y = w \wedge y$. By symmetry, $w \wedge z = x \wedge z$. Therefore, $w \wedge y = w \wedge z$. Note that $w \neq y \wedge w$ (otherwise $w \leqslant y \wedge x$, and that is absurd). Finally, we observe that $w \wedge (y \vee z) = w$. Thus, $\{w, y, z\}$ fails $SD_\wedge$. The minimality of $\{x, y, z\}$ in join-refinement implies that $w = x$. We have proved the claim that $y \vee z > x$. By induction, we may assume $y \vee z = \hat{1}$.

Next, we claim that $x \vee y$ and $x \vee z$ are incomparable. By way of contradiction assume that $x \vee z \geqslant x \vee y$, so we have $x \vee z \geqslant x, y, z$. Therefore, $z \vee x = z \vee y$, as shown on the left in Figure 12. Observe that $z \vee (x \wedge y) = z$. Since L is join-semidistributive, we have $z = \hat{1}$. This contradicts the fact that $x \wedge z \neq x \wedge (y \vee z)$. We have proved the claim that $x \vee y$ and $x \vee z$ are incomparable.

Finally, we claim that there is some $w' \in \mathrm{cov}^\uparrow(a) \setminus \{y, z\}$. Suppose that $\{y, z\} = \mathrm{cov}^\uparrow(a)$, and consider the righthand of Figure 12. Either $y \leqslant a \vee x$ or $z \leqslant a \vee x$, but not both. Indeed, if $x \vee a \geqslant y, z$ then $x \vee a = \hat{1}$, so $x \vee a = x \vee y = x \vee z$. This contradicts the fact that $x \vee y$ and $x \vee z$ are incomparable. By symmetry, we assume that $y \leqslant x \vee a$.

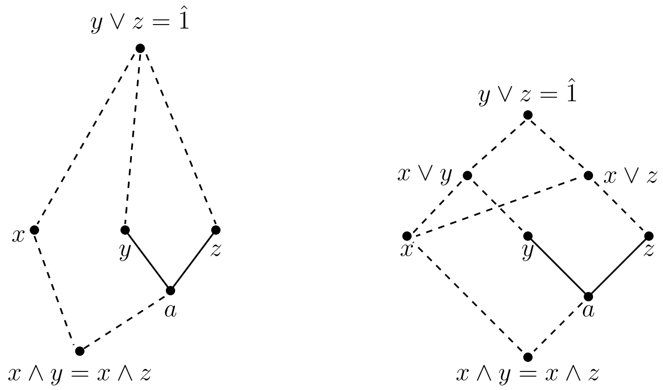

Figure 12: Dashed lines represent relations in L while solid lines represent cover relations.

Then $y \leqslant x \vee a \leqslant x \vee z$. Thus we have $x \vee y \leqslant x \vee z$, also contradicting the fact that $x \vee y$ and $x \vee z$ are incomparable. We conclude that there exists some $w' \in \mathrm{cov}^{\uparrow}(a) \setminus \{y, z\}$ (in particular, $w' \leqslant a \vee x$), and we have proved the second claim. The triple $\{w', y, z\}$ satisfies the statement of the proposition. □

Proof of Theorem 2. We prove that if L is join-semidistributive and crosscut-simplicial then it is semidistributive. Assume that L is fails $SD_{\wedge}$. Lemma 35 says that there exists x, y and z covering a common element $a \in L$ such that $y \vee z > x$. In particular, $[a, y \vee z]$ is not crosscut-simplicial because $\{y, z\}$ is not a face in the crosscut complex. That is a contradiction. Therefore, L is a finite semidistributive lattice. □

4 Lattice-theoretic constructions

In the following propositions, we construct new semidistributive lattices from old ones, and give the corresponding construction for the canonical join complex.

4.1 Products, Quotients, and Sublattices

Recall that the ***join*** of the simplicial complexes Δ and Δ' is the complex $\Delta * \Delta' = \{F \cup F' : F \in \Delta \text{ and } F' \in \Delta'\}$.

Proposition 36. *Suppose that L_1 and L_2 are finite, join-semidistributive lattices. Then the canonical complex for $L_1 \times L_2$ is the join $\Gamma(L_1) * \Gamma(L_2)$.*

The ***ordinal sum*** of lattices L_1 and L_2 written $L_1 \oplus L_2$ is the lattice whose set of elements is the disjoint union $L_1 \uplus L_2$, ordered as follows: $x \leqslant y$ if and only if $x \leqslant y$ in L_i, for $i = 1, 2$, or $x \in L_1$ and $y \in L_2$.

Proposition 37. *Suppose that L_1 and L_2 are finite, join-semidistributive lattices. Then $\Gamma(L_1 \oplus L_2)$ is equal to the disjoint union $\Gamma(L_1) \uplus \Gamma(L_2) \uplus \{v\}$, in which the vertex v corresponds to the minimal element of L_2.*

We define the ***wedge sum*** $L_1 \curlyvee L_2$ to be the lattice quotient of the ordinal sum $L_1 \oplus L_2$ in which the minimal element of L_2 is identified with the maximal element of L_1. (Our nonstandard terminology is inspired by the wedge sum of topological spaces.)

Proposition 38. *Suppose that L_1 and L_2 are finite, join-semidistributive lattices. Then $\Gamma(L_1 \curlyvee L_2)$ is equal to the disjoint union $\Gamma(L_1) \uplus \Gamma(L_2)$.*

A map $\phi : L \to L'$ between lattices L and L' is a ***lattice homomorphism*** if ϕ respects the meet and join operations. The image of ϕ is a ***sublattice*** of L' and a ***lattice quotient*** of L. It is immediate that each sublattice of a semidistributive lattice is also semidistributive. When L is finite, the image $\phi(L)$ also inherits semidistributivity (see [22, Proposition 1-5.24]). Outside of the finite case, it is not generally true that if L is semidistributive, then $\phi(L)$ is semidistributive. (Similar statements hold for meet and join-semidistributivity.) We obtain the following result as an immediate corollary of Theorem 1.

Corollary 39. *Suppose that L is a finite join-semidistributive lattice whose canonical join complex is flag. Then, the canonical join complex of each sublattice and quotient lattice of L is also flag.*

An equivalence relation Θ on L is a ***lattice congruence*** if Θ satisfies the following: if $x \equiv_\Theta y$, then $x \vee t \equiv_\Theta y \vee t$ and $x \wedge t \equiv_\Theta y \wedge t$ for each x, y, and t in L (see [14, Lemma 8]). It is immediate that the fibers of the lattice homomorphism ϕ constitute a lattice congruence of L. Conversely, each lattice congruence also gives rise to a lattice quotient (see [14, Theorem 11]).

When L is finite, Θ is lattice congruence if and only if it satisfies the following: Each class is an interval; the map $\pi_\downarrow^\Theta$ sending $x \in L$ to the smallest element in its Θ-class is order preserving; the map $\pi^\uparrow_\Theta$ sending $x \in L$ to the largest element in its Θ-class is order preserving. Both $\pi_\downarrow^\Theta$ and $\pi^\uparrow_\Theta$ are lattice homomorphisms onto their images, and $\pi_\downarrow^\Theta(L)$ and $\pi^\uparrow_\Theta(L)$ are isomorphic lattice quotients of L. As lattice quotients, both $\pi_\downarrow^\Theta(L)$ and $\pi^\uparrow_\Theta(L)$ are endowed with their own join and meet operations. So, for example, when we write $\bigvee A$ or $\bigwedge A$ for some subset $A \subset \pi_\downarrow^\Theta(L)$, we must indicate whether the join or meet is taken in L or in its lattice quotient. It turns out that $\pi_\downarrow^\Theta(L)$ is also a sub-join-semilattice of L, meaning that the join operation in $\pi_\downarrow^\Theta(L)$ coincides with the join operation in L. However, $\pi_\downarrow^\Theta(L)$ is generally *not* a sublattice of L. Similar statements hold for $\pi^\uparrow_\Theta(L)$. Below we quote [24, Proposition 6.3]. In the proposition, a join-irreducible element $j \in L$ is ***contracted*** by the congruence Θ if j is congruent to the unique element that it covers.

Proposition 40. *Suppose that L is a finite join-semidistributive lattice and Θ is a lattice congruence on L with associated projection map $\pi_\downarrow^\Theta$. Then, the element w belongs to $\pi_\downarrow^\Theta(L)$ if and only if no canonical joinand of w is contracted by Θ.*

Suppose that $w \in \pi_\downarrow^\Theta(L)$. Since $\pi_\downarrow^\Theta(L)$ is a sub-join-semilattice of L, the canonical join representation of w taken in L is equal to the canonical join representation taken in the lattice quotient $\pi_\downarrow^\Theta(L)$. We obtain the follow result.

Corollary 41. *Suppose that L is a finite join-semidistributive lattice and Θ is a lattice congruence on L. Then, the canonical join complex of $\pi_{\downarrow}^{\Theta}(L)$ is the induced subcomplex of $\Gamma(L)$ supported on the set of join-irreducible elements not contracted by Θ.*

Example 42. Recall from Example 13 that the Tamari lattice T_n may be realized as the subposet of the weak order on S_n induced by the set of 312-avoiding permutations. It is well-known that the map which sends a permutation w to the largest 312-avoiding permutation below it (in the weak order) is a lattice surjection. Thus, T_n is lattice quotient of the weak order on S_n. (The corresponding lattice congruence on S_n is called a c-Cambrian congruence. See [21, Theorem 5.1].) It follows immediately from Corollary 41 that $\Gamma(T_n)$ is isomorphic to the subcomplex of compatible arcs induced by the set of right arcs.

Remark 43. In general, not every induced subcomplex of Δ is the canonical join complex of a lattice quotient of L. Each lattice congruence is determined by the set of join-irreducible elements that it contracts. But, a given collection of join-irreducible elements may not correspond to a lattice congruence. For j and j' in $\mathrm{Irr}(L)$, we say that j ***forces*** j' if every congruence that contracts j also contracts j'. In N_5 pictured in Figure 13 both a and b force c. So, for example, there is no quotient of N_5 whose canonical join complex is the subcomplex induced by $\{b, c\}$.

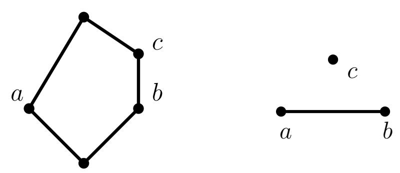

Figure 13: The pentagon lattice N_5 and its canonical join complex.

The canonical join complex of a sublattice L' of L is not as well-behaved as a quotient lattice. In general, $\Gamma(L')$ need not be an induced subcomplex of $\Gamma(L)$. In fact, the sets $\mathrm{Irr}(L')$ and $\mathrm{Irr}(L)$ may be disjoint. (For example, consider the canonical join complex of the sublattice $\{\hat{0}, \hat{1}\}$ in the boolean lattice B_n, where $n > 1$.) However, in certain special cases, the canonical join complex of L' may be obtained by "folding" a certain subcomplex of L, as we explain below.

Let $\phi : L \to L$ be a lattice automorphism. Below, we use the following notation: L^{ϕ} is the sublattice of L that is fixed point-wise by ϕ; $\Gamma(L)^{\phi}$ is the set of faces A in $\Gamma(L)$ that are fixed (as sets) by ϕ; and $\mathcal{O}(j)$ is the orbit $\{\phi^k(j) : k \in \mathbb{Z}\}$.

Proposition 44. *The automorphism ϕ induces a bijection $\overline{\phi} : \Gamma(L)^{\phi} \to \Gamma(L^{\phi})$ that is defined by*

$$A \mapsto \{w = \bigvee \mathcal{O}(a) : a \in A\}$$

for each $A \in \Gamma(L)^{\phi}$.

Proof. First we check that $\overline{\phi}(A)$ is a canonical join representation in L^ϕ, for each face $A \in \Gamma(L)^\phi$. Let $a \in A$, and write x for the element $\bigvee \overline{\phi}(A)$. We make two easy observations: First $x \in L^\phi$; second $\bigvee A = \bigvee \overline{\phi}(A)$. Since $\bigvee \overline{\phi}(A) \setminus \{\overline{\phi}(a)\}$ is less than or equal to $\bigvee A \setminus \{a\}$, we conclude that $\bigvee \overline{\phi}(A)$ is irredundant. Suppose that $B \in L^\phi$ and $\bigvee B = x$. For each $a \in A$, there is element $b \in B$ such that $a \leqslant b$. The automorphism ϕ is, in particular, a poset isomorphism. So $\phi^k(a) \leqslant b$ for each $k \in \mathbb{Z}$. Thus $\bigvee \mathcal{O}(a) \leqslant b$. We conclude that $\overline{\phi}(A)$ join-refines B (in L^ϕ). Thus the map $\overline{\phi}$ is well-defined. Suppose that $A' \in \Gamma(L^\phi)$, and write x' for the element $\bigvee A'$. Let $\bigvee A$ be the canonical join representation of x' taken in L. The map $A' \mapsto A$ is the inverse of $\bar{\phi}$. □

Example 45. Let $S_{\pm n}$ denote the symmetric group on $\{-n, \ldots, -1, 1, \ldots, n\}$, and let w_0 be the permutation in $S_{\pm n}$ whose one-line notation is $n(n-1)\ldots -n$. It is well-known that conjugation by w_0 is an automorphism of the weak order on $S_{\pm n}$, and the sublattice of its fixed points is isomorphic to the weak order on B_n. Now consider a noncrossing arc diagram corresponding to $w \in S_{\pm n}$, with nodes labelled $-n, \ldots, -1, 1, \ldots n$ in increasing order from bottom to top. Conjugation by w_0 corresponds to a half-turn rotation through the center of the noncrossing arc diagram. We define a ***symmetric arc*** to be an arc or pair of arcs that are fixed by this central symmetry.

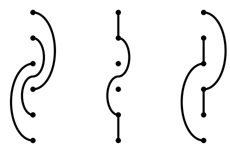

Figure 14: Each diagram contains two symmetric arcs.

Corollary 46. *The canonical join complex of the weak order on B_n is isomorphic to the complex of compatible symmetric arcs on the nodes $\{-n, \ldots, -1, 1, \ldots n\}$.*

5 Discussion and open problems

We close the paper be raising some questions about the canonical join graph of a finite semidistributive lattice.

Question 47. Which graphs can be realized as the canonical join graph of some finite semidistributive lattice?

Many examples of canonical join graphs were constructed in [3, Chapter 1] using Day's doubling construction [9], including certain chordal graphs and the cycle graphs C_n. One can use similar techniques to show that each finite threshold graph and each path graph can be realized as the canonical join graph of some finite semidistributive lattice.

Non-isomorphic semidistributive lattices may have isomorphic canonical join graphs. For example, consider the ordinal sum $B_0 \oplus B_2$, where B_n is the boolean lattice on an n-element set, and the pentagon lattice N_5 from Figure 13. The canonical join graph for both lattices consists of an edge and an isolated vertex.

Question 48. Suppose that G is the canonical join graph of a finite semidistributive lattice L. What data, in addition to G, is necessary in order to determine L up to isomorphism?

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