

Dual graded graphs and Bratteli diagrams of towers of groups

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Abstract

An r -dual tower of groups is a nested sequence of finite groups, like the symmetric groups, whose Bratteli diagram forms an r -dual graded graph. Miller and Reiner introduced a special case of these towers in order to study the Smith forms of the up and down maps in a differential poset. Agarwal and the author have also used these towers to compute critical groups of representations of groups appearing in the tower. In this paper the author proves that when $r = 1$ or r is prime, wreath products of a fixed group with the symmetric groups are the only r -dual tower of groups, and conjecture that this is the case for general values of r . This implies that these wreath products are the only groups for which one can define an analog of the Robinson-Schensted bijection in terms of a growth rule in a dual graded graph.

Mathematics Subject Classifications: 05E10, 05C25, 06A07, 06A11, 20C30

1 Introduction

1.1 Differential posets and dual graded graphs

Differential posets are a class of partially ordered sets introduced by Stanley [13] which generalize many of the enumerative and combinatorial properties of Young's lattice Y , the poset of integer partitions ordered by inclusion of Young diagrams. The reader should see [14] for basic definitions and conventions for posets in what follows. Dual graded graphs are a generalization of differential posets developed independently by Fomin [4, 5].

A *graded graph* is an undirected multigraph P together with a rank function $\rho : P \rightarrow \mathbb{Z}_{\geq 0}$ such that P has a unique element $\hat{0}$ of rank zero, all ranks $P_n = \rho^{-1}(\{n\})$ are

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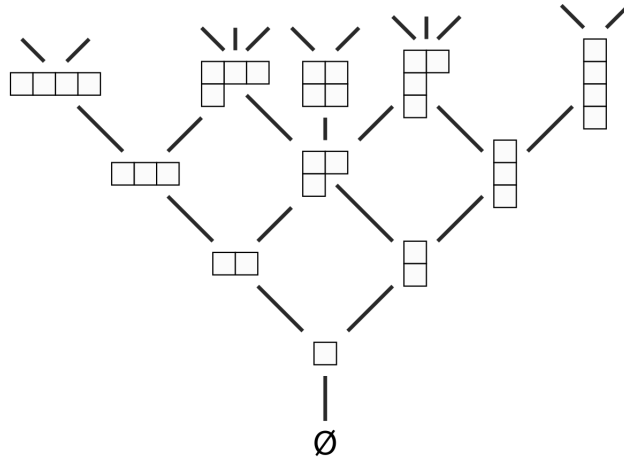


Figure 1: Young's lattice Y is a 1-dual graded graph and a 1-differential poset.

finite, and such that all edges are between consecutive ranks: if (x, y) is an edge, then $|\rho(x) - \rho(y)| = 1$; we denote the multiplicity of this edge by $m(x, y)$. In analogy with Hasse diagrams of partially ordered sets, we write $x \leq y$ if $\rho(x) \leq \rho(y)$ and there is a path from x to y in P taking only upward steps. If $\rho(y) = \rho(x) + 1$ and $x < y$, we write $x \prec y$ and say that y covers x . For $a, b \in \mathbb{Z}_{\geq 0}$, we let $P_{[a,b]}$ denote the induced subgraph on the elements of ranks $a, a + 1, \dots, b$.

Definition 1. Let r be a positive integer, let P be a graded graph, and let $\mathbb{C}P$ denote the complex vector space with basis P ; then P is an r -dual graded graph if $DU - UD = rI$ where the linear operators $U, D : \mathbb{C}P \rightarrow \mathbb{C}P$ are defined by

$$Ux = \sum_{x \prec y} m(x, y)y$$

$$Dy = \sum_{x \prec y} m(x, y)x.$$

If in addition all edges multiplicities $m(x, y)$ are 0 or 1, then P is (strictly speaking, the Hasse diagram of) an r -differential poset.

Remark 2. What we have defined here are usually called *self-dual* graded graphs. In the context of towers of groups, self-duality is implied by Frobenius reciprocity, so we do not address the more general definition here.

Many of the combinatorial properties of Young's lattice Y are shared by all dual graded graphs. First, define a pairing $\langle, \rangle : \mathbb{C}P \times \mathbb{C}P \rightarrow \mathbb{C}$ by requiring that $\langle x, y \rangle = \delta_{xy}$ for $x, y \in P$ and extending by bilinearity; let $e(x) = \langle U^n \widehat{0}, x \rangle$ where $x \in P_n$. It is easy to see that $e(x)$ counts the number of paths from $\widehat{0}$ to x in P , allowing only upward steps. Then we have

Proposition 3 ([4]).

$$\sum_{x \in P_n} e(x)^2 = r^n n!$$

When $P = Y$, it is easy to see that $e(\lambda)$ is the number f_λ of standard Young tableaux of shape λ , and so Proposition 3 reduces to the well known fact that $\sum_{\lambda \in Y_n} f_\lambda^2 = n!$, which follows from the Robinson-Schensted correspondence.

1.2 Towers of groups

A *tower of groups* \mathfrak{G} is an infinite nested sequence of finite groups $\mathfrak{G} : \{e\} = G_0 \subset G_1 \subset G_2 \subset \dots$ (see Section 1.3 where this notion is distinguished from the more restrictive sense in which Bergeron, Lam, and Li [2] use the term *tower of algebras*).

For a finite group G , we let $R(G)$ denote the representation ring of complex linear combinations of complex G -representations subject to the relations $[V \oplus W] = [V] + [W]$ and $[V \otimes W] = [V] \cdot [W]$. As a vector space, $R(G)$ has a distinguished basis consisting of the classes of irreducible representations $[V_\lambda]$ for $V_\lambda \in \text{Irr}(G)$. For \mathfrak{G} a tower of groups, we let

$$R(\mathfrak{G}) = \bigoplus_{i=0}^{\infty} R(G_i).$$

Although each $R(G_i)$ has a ring structure, we regard $R(\mathfrak{G})$ only as a complex vector space (see Section 1.3), together with an inner product $\langle V_\lambda, V_\mu \rangle = \delta_{\lambda\mu}$ defined so that the distinguished basis of irreducibles is orthonormal; this inner product coincides with the usual inner product of characters when restricted to a subspace $R(G_i)$.

The space $R(\mathfrak{G})$ has a natural pair of adjoint linear operators $\text{Ind}, \text{Res} : R(\mathfrak{G}) \rightarrow R(\mathfrak{G})$ defined by

$$\begin{aligned} \text{Ind}([V_\lambda]) &= [\text{Ind}_{G_i}^{G_{i+1}} V_\lambda] \\ \text{Res}([V_\lambda]) &= [\text{Res}_{G_{i-1}}^{G_i} V_\lambda] \end{aligned}$$

where $V_\lambda \in \text{Irr}(G_i)$, and extended by linearity. We define $\text{Res}([V]) = 0$ for $[V] \in R(G_0) \cong \mathbb{C}$. We now define the main object of study.

Definition 4 ([9]). A tower of groups \mathfrak{G} is an *r-dual tower of groups* if the linear operators $\text{Ind}, \text{Res} : R(\mathfrak{G}) \rightarrow R(\mathfrak{G})$ satisfy the relation

$$\text{Res Ind} - \text{Ind Res} = rI. \tag{1}$$

That is, \mathfrak{G} is an *r-dual tower of groups* if and only if the Bratteli diagram of \mathfrak{G} is an *r-dual graded graph* P . In this case the operators $\text{Ind}, \text{Res} : R(\mathfrak{G}) \rightarrow R(\mathfrak{G})$ correspond to $U, D : \mathbb{C}P \rightarrow \mathbb{C}P$ and the inner products on $\mathbb{C}P$ and $R(\mathfrak{G})$ coincide; we write $P = P(\mathfrak{G})$. We also abuse terminology by saying V_λ covers V_μ if λ covers μ in P .

If in addition the branching rules between G_i and G_{i+1} are multiplicity-free for all i then we say that \mathfrak{G} is an *r-differential tower of groups*. In this case the Bratteli diagram forms an *r-differential poset* $P = P(\mathfrak{G})$.

Miller and Reiner introduced differential towers of groups in order to study the Smith forms of the up and down maps in the associated differential poset [9]. Agarwal and the author have also used these towers to compute critical groups of representations of groups appearing in the tower [1, 6].

Example 5. It is well known (see for example [8]) that irreducible representations of the symmetric group S_n are indexed by partitions λ of n , and that

$$\text{Res}_{S_{n-1}}^{S_n} V_\lambda = \bigoplus_{\mu} V_\mu$$

where the direct sum is over all partitions μ of $n - 1$ whose Young diagrams are obtained from that of λ by removing a single box. That is, $\mathfrak{S} : \{e\} \subset S_1 \subset S_2 \subset \dots$ is a 1-differential tower of groups, with $P(\mathfrak{S}) = Y$.

This correspondence can be extended to show that for any abelian group A of order r the tower

$$A \wr \mathfrak{S} : \{e\} \subset A \subset A \wr S_2 \subset A \wr S_3 \subset \dots$$

is an r -differential tower of groups with $P(A \wr \mathfrak{S}) \cong Y^r$ [10]. When A is cyclic, $A \wr \mathfrak{S}$ is a complex reflection group [12]; the trivial representation corresponds to the tuple $((n), \emptyset, \dots, \emptyset)$ of partitions, and the reflection representations are of the form $((n - 1), \emptyset, \dots, \emptyset, (1), \emptyset, \dots, \emptyset)$ when $r > 1$ or $((n - 1, 1))$ when $r = 1$. If $\lambda = (\lambda^1, \dots, \lambda^r)$ is a tuple of partitions with total size n , then the dimension of the corresponding representation V_λ of $A \wr S_n$ is:

$$\binom{n}{|\lambda^1|, \dots, |\lambda^r|} \prod_{i=1}^r f_{\lambda^i}$$

where f_μ is the dimension of the irreducible S_n representation indexed by μ .

More generally, if H is any finite group of order r with k -conjugacy classes and irreducible representations of dimensions d_1, \dots, d_k , then

$$H \wr \mathfrak{S} : \{e\} \subset H \subset H \wr S_2 \subset H \wr S_3 \subset \dots$$

is an r -dual tower of groups with $P(H \wr S_n) \cong (d_1 Y) \times (d_2 Y) \times \dots \times (d_k Y)$ where dY denotes the d^2 -dual graded graph whose underlying simple graph is Y , but where all edge multiplicities are multiplied by d (see [10], Theorem 4.1).

Theorem 6 below shows that this is in fact the *only* r -dual tower of groups when $r = 1$ or r is prime. In the rest of the paper, we use the convention that $A \wr S_0 = \{e\}$ while $A \wr S_1 = A$.

In [10], Okada also gives an explicit bijection which generalizes the Robinson-Schensted correspondence to a bijection between elements of the groups $H \wr S_n$ and pairs of paths in Y^r . Fomin's theory of dual graded graphs [4, 5] later showed that this bijection was a special case of one which holds for any dual graded graph, defined in terms of growth rules. The main results of this paper, Theorem 6 and Conjecture 7 below, together assert that the groups $H \wr S_n$ are the *only* groups for which one can define an analog of the Robinson-Schensted correspondence via growth rules.

Theorem 6. Let $\mathfrak{G} : \{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots$ be an r -dual tower of groups where r is one or prime. Then $P(\mathfrak{G}) \cong Y^r$ and $G_n \cong (\mathbb{Z}/r\mathbb{Z}) \wr S_n$ for all $n \geq 0$.

Conjecture 7. Let $\mathfrak{G} : \{e\} = G_0 \subset G_1 \subset G_2 \subset \cdots$ be a tower of groups and let $r \in \mathbb{Z}_{>0}$.

- (a) If \mathfrak{G} is r -differential, then $P(\mathfrak{G}) \cong Y^r$ and there exists an abelian group A of order r , not depending on n , such that $G_n \cong A \wr S_n$ for all $n \geq 0$.
- (b) If \mathfrak{G} is an r -dual tower of groups then $P(\mathfrak{G}) \cong (d_1 Y) \times \cdots \times (d_k Y)$ for some d_1, \dots, d_k and there exists a group H of order r , not depending on n , such that $G_n \cong H \wr S_n$ for all $n \geq 0$.

Clearly part (b) of the conjecture implies part (a).

Remark 8. The general construction outlined in [7] shows that any graded graph can be realized as the Bratteli diagram of some sequence $A_0 \subset A_1 \subset A_2 \subset \cdots$ of complex semisimple algebras. In particular, for all r each of the (uncountably many) r -differential posets and r -self-dual graded graphs can be realized in this way; thus the implication in Theorem 6 that Y^r is the only differential poset or dual graded graph which can arise from a sequence of *group algebras* may initially be surprising.

1.3 Relation to Bergeron, Lam, and Li's towers of algebras

In [2], Bergeron, Lam, and Li study what they call towers of algebras; I will call their notion a *strong tower of algebras* in order to avoid confusion. Strong towers of algebras are certain algebras $A = \bigoplus_{i=0}^n A_i$ whose summands A_i are algebras in their own right which are subject to several additional conditions; the corresponding branching rules satisfy the same differential relation as in (1). A pair of Grothendieck groups $(G(A), K(A))$ bears a similar relationship to A as the vector space $R(\mathfrak{G})$ does to a tower of groups \mathfrak{G} . The additional conditions required of a strong tower of algebras are restrictive enough to guarantee that $G(A)$ and $K(A)$ in fact have the structure of dual combinatorial Hopf algebras. If the A_i are complex semisimple algebras one obtains that $G(A) = K(A)$ is a positive self-dual Hopf algebra. By Zelevinsky's classification of such Hopf algebras [15], this forces $G(A)$ to be isomorphic to a tensor product of copies of the Hopf algebra Λ of symmetric functions, and therefore forces the corresponding Bratteli diagram for A to be isomorphic to a product of copies of Young's lattice.

In the case where $A_i = \mathbb{C}[G_i]$ are group algebras for $i = 0, 1, \dots$, the additional conditions required of a strong tower of algebras imply in particular that for all $n \geq k \geq 0$ the group $G_k \times G_{n-k}$ is isomorphic to a subgroup of G_n . This condition is clearly not true for general towers of groups, and it is therefore a notable feature of Theorem 6 and Conjecture 7 that this condition, and the resulting Hopf structure on $R(\mathfrak{G})$, emerge from the local condition (1) which only relates the branching rules of consecutive pairs of groups in the tower.

2 Proof of Theorem 6

Throughout Section 2, let $\mathfrak{G} : \{e\} = G_0 \subset G_1 \subset G_2 \subset \dots$ be an r -dual tower of groups where r is one or prime. The proof of Theorem 6 will proceed by showing inductively that the groups G_n are complex reflection groups and applying the known classification [12] of these groups. We first prove the following useful fact:

Proposition 9. *For all $n \geq 0$, the group G_n has order $r^n n!$.*

Proof. First, note that $e(\lambda) = \dim(V_\lambda)$, since both satisfy the same recurrence (as the sum of the values for $\mu < \lambda$, with multiplicity) and the initial condition $e(\widehat{0}) = \dim(\mathbb{1}_{G_0}) = 1$. Thus by Proposition 3 and the standard fact that the sum of the squares of the dimensions of the irreducibles is the order of the group we have:

$$|G_n| = \sum_{V_\lambda \in \text{Irr}(G_n)} \dim(V_\lambda)^2 = \sum_{\lambda \in P_n} e(\lambda)^2 = r^n n!. \quad \square$$

Before proceeding with the proof, we state Clifford's Theorem, which will be used several times in what follows.

Theorem 10 (Clifford's Theorem). *Let G be a finite group, N a normal subgroup and $\varphi : G \rightarrow GL(V)$ an irreducible representation of G . Then the irreducible factors appearing in $\text{Res}_N^G V$ are in a single orbit under the action of G given by $\psi^{(g)}(n) = \psi(gng^{-1})$.*

In particular, all irreducible factors of $\text{Res}_N^G V$ are of the same dimension, and if $\mathbb{1}_N$ appears as a factor, then $\text{Res}_N^G V$ is a direct sum of trivial representations.

2.1 Base case

By assumption we have that $G_0 = \{e\}$ is trivial, and since r is 1 or prime, we know by Proposition 9 that $G_1 \cong \mathbb{Z}/r\mathbb{Z}$. In this section we will show:

Proposition 11. *When r is one or prime we must have $G_2 \cong (\mathbb{Z}/r\mathbb{Z}) \wr S_2$ and $P_{[0,2]} \cong (Y^r)_{[0,2]}$.*

Proof. When $r = 1$ the claim is clear, since $|G_2| = 2$, so assume r is prime. If $r \neq 2$, then the Sylow r -subgroup of G_2 has order r^2 and so must be abelian; since it has index two, it is also normal in G_2 . It is a standard fact (see [11]) that dimensions of irreducible representations must divide the index of an abelian normal subgroup. Thus all irreducible representations of G_2 have dimension 1 or 2. In order to satisfy the relation (1), there must be $\binom{r}{2}$ 2-dimensional irreducibles of G_2 which when restricted to G_1 give each of the possible pairs of characters, and there must be $2r$ linear characters, two of which restrict to each character of G_1 . It is a straightforward exercise (see, for example, [3] p. 185) to check that there are three nonabelian groups G of order $2r^2$, all of which are semidirect products of $H = \langle h \rangle \cong \mathbb{Z}/2\mathbb{Z}$ and the r -Sylow subgroup N . If N the cyclic group C_{r^2} , then h acts on N by inversion; if N is $C_r \times C_r$, then h acts either by inversion of a single factor, or inversion of both. In the first and last cases one calculates that the commutator

subgroup $[G, G]$ is of order r^2 , thus G has only two linear characters and cannot be G_2 ; the only remaining possibility is $G_2 = (\mathbb{Z}/r\mathbb{Z}) \wr S_2$.

When $r = 2$, there is only one 2-dual graded graph up to rank 2, namely $(Y^2)_{[0,2]}$. These branching rules imply that G_1 is not normal in G_2 , since the 2-dimensional irreducible restricts to the sum of the trivial representation and a non-trivial summand, violating Clifford's theorem. The only nonabelian groups of order 8 are the dihedral group $D_4 \cong (\mathbb{Z}/2\mathbb{Z}) \wr S_2$ and the quaternion group Q_8 . However Q_8 is well-known to be a Hamiltonian group, contradicting the fact that G_1 is not normal. Thus we must have $G_2 \cong (\mathbb{Z}/2\mathbb{Z}) \wr S_2$ as desired. \square

2.2 Facts about dual graded graphs

Lemma 12. *Let P be an r -dual graded graph such that $P_{[0,m]}$ has no multiple edges and let $x \neq y \in P_m$. Then*

- (a) *If $z, z' \in P_{m+1}$ both cover both x and y , then $z = z'$.*
- (b) *If $z \in P_{m+1}$ covers both x and y , then there is some $w \in P_{m-1}$ which is covered by both x and y and $m(x, z) = m(y, z) = m(w, x) = m(w, y) = 1$.*

Proof.

- (a) Suppose $z \neq z'$, then the coefficient of y in DUx is at least 2. Therefore the coefficient of y in UDx must also be at least 2. Since $P_{[0,m]}$ has no multiple edges, there must be some $w \neq w' \in P_{m-1}$ such that x, y both cover w and w' . Repeating this argument, we arrive at a contradiction, since P_0 has a single element. Thus $z = z'$.
- (b) Since z covers x and y , we know y appears in DUx , and so it must also appear in UDx ; that is, there must be some $w \in P_{m-1}$ covered by x and y , and by part (a) this w is unique. There are no multiple edges in $P_{[0,m]}$, so y appears with coefficient 1 in UDx , thus it must also have coefficient 1 in DUx which forces $m(x, z) = m(y, z) = m(w, x) = m(w, y) = 1$. \square

2.3 Inductive step

By a *partial r -dual tower of groups* we mean a finite sequence of finite groups $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_{m+1}$ such that for $i = 1, \dots, m$ we have

$$\text{Res}_{G_i}^{G_{i+1}} \text{Ind}_{G_i}^{G_{i+1}} - \text{Ind}_{G_{i-1}}^{G_i} \text{Res}_{G_{i-1}}^{G_i} = rI.$$

and we similarly define a partial r -dual graded graph.

Proposition 13. *For $m \geq 2$, let $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_{m+1}$ be a partial r -dual tower of groups with corresponding partial dual graded graph P . Suppose that for $n \leq m$ we have $G_n \cong (\mathbb{Z}/r\mathbb{Z}) \wr S_n$ and that $P_{[0,m]} \cong (Y^r)_{[0,m]}$. Then $G_{m+1} \cong (\mathbb{Z}/r\mathbb{Z}) \wr S_{m+1}$ and $P_{[0,m+1]} \cong (Y^r)_{[0,m+1]}$.*

Proof. The goal is to show that G_{m+1} must be an irreducible complex reflection group of rank $m + 1$ by identifying a faithful irreducible representation W of dimension $m + 1$ and a generating set for G_{m+1} such that all elements of the generating set act in W as complex reflections.

Since $P_{[0,m]} \cong (Y^r)_{[0,m]}$ we know that $\text{Ind}_{G_{m-1}}^{G_m} \mathbb{1}_{G_{m-1}}$ decomposes into distinct irreducibles as $\mathbb{1}_{G_m} \oplus \bigoplus_{i=1}^r U_i$ where U_1 has dimension $m - 1$ and U_i has dimension m for $i \geq 2$. Define distinct irreducible representations W_k by $\text{Ind}_{G_m}^{G_{m+1}} \mathbb{1}_{G_m} \cong \mathbb{1}_{G_{m+1}} \oplus \bigoplus_k W_k^{\oplus c_k}$ with $c_k > 0$. Let N be the smallest normal subgroup of G_{m+1} which contains G_m . By Lemma 12, there must be some W_k whose restriction to G_m contains both $\mathbb{1}_{G_m}$ and U_1 , since both of these representations cover $\mathbb{1}_{G_{m-1}}$. Then by Clifford's Theorem, G_m is not normal in G_{m+1} since $\dim(\mathbb{1}_{G_m}) \neq \dim(U_1)$. Thus N properly contains G_m . Let the V_j be the distinct nontrivial irreducible representations of N appearing in $\text{Res}_N^{G_{m+1}} W_k$ for some k .

We now collect some facts about the representations U_i, V_j and W_k . In order to avoid naming homomorphisms $\varphi : G \rightarrow GL(V)$ corresponding to all representations V , we abuse notation by writing $\ker(V)$ for $\ker(\varphi)$.

Lemma 14.

- (a) *Either $\text{Res}_N^{G_{m+1}} W_k = \mathbb{1}_N^{\oplus c}$ or $\text{Res}_{G_m}^{G_{m+1}} W_k$ is a multiplicity-free direct sum of $\mathbb{1}_{G_m}$ and some of the U_i . In the latter case, $\text{Res}_N^{G_{m+1}} W_k$ contains a unique V_j such that $\text{Res}_{G_m}^N V_j$ contains $\mathbb{1}_{G_m}$, and V_j does so with multiplicity one.*
- (b) *$\text{Res}_{G_m}^N V_j$ must contain some U_i .*
- (c) *No distinct $W_k, W_{k'}$ may contain the same V_j in their restriction to N .*
- (d) *No distinct $V_j, V_{j'}$ may contain the same U_i in their restriction to G_m .*

Proof of lemma 14.

- (a) Since N is normal, by Clifford's Theorem either all irreducible factors in $\text{Res}_N^{G_{m+1}} W_k$ are trivial, or none are; in the first case we are done, so assume we are in the second case. By definition, $\text{Res}_{G_m}^{G_{m+1}} W_k$ must contain $\mathbb{1}_{G_m}$; however it must also contain some nontrivial factor, otherwise $G_m \subseteq \ker(W_k)$ but $N \not\subseteq \ker(W_k)$, contradicting the fact that N is the smallest normal subgroup containing G_m . Thus $\text{Res}_{G_m}^{G_{m+1}} W_k$ contains $\mathbb{1}_{G_m}$ and some nontrivial irreducible U . By Lemma 12, U must cover $\mathbb{1}_{G_{m-1}}$, so U is one of the U_i and both $\mathbb{1}_{G_m}$ and U_i appear with multiplicity one in $\text{Res}_{G_m}^{G_{m+1}} W_k$. The last statement now follows immediately.
- (b) Suppose V_j does not contain any U_i in its restriction; pick W_k which contains V_j in its restriction. Then, to satisfy the multiplicity freeness condition in part (a) we must have $\text{Res}_{G_m}^N V_j = \mathbb{1}_{G_m}$, so $\dim(V_j) = 1$. By Clifford's theorem W_k only contains 1-dimensional representations in its restriction to N . But any other possible 1-dimensional representation appearing in this restriction itself restricts to $\mathbb{1}_{G_m}$, so in order to satisfy multiplicity freeness, it must be that $\dim(W_k) = 1$ and $\text{Res}_N^{G_{m+1}} W_k =$

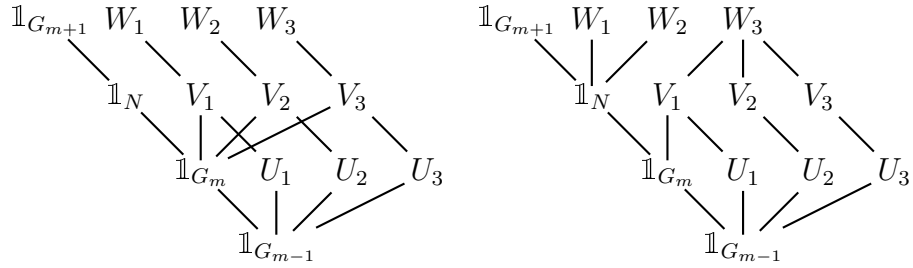


Figure 2: The two possibilities for the branching rules between the U_i, V_j, W_k when $r = 3$. On the left is the case $r' = 1$ and on the right the case $r' = r = 3$.

V_j . But then $\ker(W_k)$ is a normal subgroup of G_{m+1} which contains G_m but which does not contain N , contradicting the minimality of N .

- (c) Suppose $\text{Res}_N^{G_{m+1}} W_k$ and $\text{Res}_N^{G_{m+1}} W_{k'}$ both contain V_j . Since, by part (b), $\text{Res}_{G_m}^N V_j$ must contain some U_i we see that $W_k, W_{k'}$ both cover both $\mathbb{1}_{G_m}$ and U_i , thus by Lemma 12, $W_k = W_{k'}$.
- (d) Suppose $\text{Res}_{G_m}^N V_j$ and $\text{Res}_{G_m}^N V_{j'}$ both contain U_i . If $V_j, V_{j'}$ are both contained in the restriction of the same W_k , then this violates multiplicity freeness from part (a). Otherwise, $V_j, V_{j'}$ are contained in the restrictions of $W_k \neq W_{k'}$ respectively and then both $W_k, W_{k'}$ cover both $\mathbb{1}_{G_m}$ and U_i , contradicting Lemma 12. \square

We now return to the proof of Proposition 13. Define $r' = [G_{m+1} : N]$. Since representations of G_{m+1}/N are in dimension-preserving bijection with representations of G_{m+1} which restrict to a multiple of the trivial representation on N , we see that $r' = 1 + \sum a_k^2$ where a_k is the multiplicity of $\mathbb{1}_N$ in $\text{Res}_N^{G_{m+1}} W_k$. Let $b_k > 0$ denote the multiplicity of $\mathbb{1}_{G_m}$ in $\text{Res}_{G_m}^{G_{m+1}} W_k$; clearly we have $a_k \leq b_k$, and $a_k = b_k$ if $a_k > 0$. By the definition of an r -dual tower of groups, we know $\sum_k b_k^2 = r$. Furthermore, at least one $W_{k'}$ must contain some U_i in its restriction, since $\mathbb{1}_{G_m}$ and U_i must have an upper bound in rank $m + 1$, and thus $a_{k'} = 0$ by Lemma 14. Therefore $1 \leq r' \leq r$.

Now, we have

$$\dim \text{Ind}_{G_m}^N \mathbb{1}_{G_m} = [N : G_m] = [G_{m+1} : G_m] / [G_{m+1} : N] = \frac{r(m+1)}{r'}.$$

On the other hand, if $a_k = 0$, then $b_k = 1$ by Lemma 14, thus there are $r + 1 - r'$ representations W_k which contain some V_j . By the same lemma there must be $r + 1 - r'$ of the V_j whose restriction contains $\mathbb{1}_{G_m}$ and each of these V_j 's must contain some U_i , while only one of them can contain U_1 . This implies that

$$\dim \text{Ind}_{G_m}^N \mathbb{1}_{G_m} \geq 1 + m + (r - r')(m + 1).$$

Thus we have $\frac{r(m+1)}{r'} \geq 1 + m + (r - r')(m + 1)$, and so

$$rm + r \geq (r' + rr' - (r')^2)m + (r' + rr' - (r')^2)$$

This forces $r \geq r' + rr' - (r')^2$, that is:

$$(r' - 1)(r - r') \leq 0.$$

Therefore r' must be equal to 1 or r .

When $r = r'$, we have $[N : G_m] = (m + 1)$, this forces there to be some V_j which restricts exactly to $\mathbb{1}_{G_m} \oplus U_1$. Then by Clifford's Theorem, each of the V_j must have dimension m , so the branching rules are as depicted in Figure 2.3 for $r = 3$, and extend in the obvious way to larger r .

Lemma 15. $r' = 1$.

Proof of Lemma 15. Suppose $r > 1$ and $r' = r$. Let V_1 be the unique V_j whose restriction contains U_1 . The representation U_1 is the $(m - 1)$ -dimensional representation of $G_m = (\mathbb{Z}/r\mathbb{Z}) \wr S_m$ obtained by projecting onto the symmetric group S_m and applying the usual irreducible reflection representation of S_m . Thus $\ker(U_1) \cong (\mathbb{Z}/r\mathbb{Z})^m$ is the *base group* (see [8], Chapter 4) of the wreath product. In particular $|\ker(U_1)| = r^m$, and since $\text{Res } V_1 = \mathbb{1}_{G_m} \oplus U_1$, we see that $|\ker(V_1)| \geq r^m$. By Clifford's Theorem, the representation V_2 is conjugate to V_1 ; that is, if $\varphi_1, \varphi_2 : N \rightarrow GL_m$ are the corresponding maps, then for some $g \in G_{m+1}$ we have $\varphi_2(x) = \varphi_1(gxg^{-1})$ for all x . This implies that $\ker(V_2) = g^{-1}(\ker(V_1))g$, so $|\ker(V_2)| \geq r^m$. However $\text{Res } V_2 = U_2$ is faithful, so $\ker(V_2)$ intersects G_m trivially. This means that the product group $K = (\ker(V_2)) \cdot G_m$ has order at least $r^m \cdot r^m m!$. But if $r > 1$ this is greater than $|N| = r^m(m + 1)!$, a contradiction. \square

Lemma 16. *We have $N = G_{m+1}$ (and so we can identify $V_j = W_j$ for all j , relabeling if necessary). Let V_1 be the unique V_j whose restriction contains U_1 , then $\dim(V_1) = m$ and $\dim(V_j) = m + 1$ for $j \neq 1$. Furthermore, V_j is a faithful reflection representation of G_{m+1} for $j \neq 1$ (if $r = 1$ then V_1 is a faithful reflection representation).*

Proof of Lemma 16. By definition $r' = [G_{m+1} : N]$, since $r' = 1$ by Lemma 15 we have $G_{m+1} = N$. The V_j , which were defined by restricting the W_k to N are thus the same as the W_k and we can relabel to let $V_j = W_j$. The representation U_1 has dimension $m - 1$ and each other U_i has dimension m . Letting U_i be the unique nontrivial representation appearing in $\text{Res } V_i$ for all i we have $\text{Res}_{G_m}^{G_{m+1}} V_i = U_i \oplus \mathbb{1}_{G_m}$, so $\dim(V_1) = m$ and $\dim(V_j) = m + 1$ for $j \neq 1$.

The representations U_i for $i \neq 1$ can be realized as permutation matrices but with the 1's replaced by character values of one of the nontrivial representations of $\mathbb{Z}/r\mathbb{Z}$ (see [8]). Let r be prime, then these representations of $\mathbb{Z}/r\mathbb{Z}$ are faithful, and thus U_2, \dots, U_r are faithful as well. Let $\psi_2 : G_m \rightarrow GL(U_2)$ and $\varphi_2 : G_{m+1} \rightarrow GL(V_2)$ be the maps corresponding to the representations U_2, V_2 , where, since $\text{Res } V_2 = U_2 \oplus \mathbb{1}_{G_m}$, we identify $GL(U_2)$ as a subgroup of $GL(V_2)$ in the natural way. The representation U_2 is a reflection representation of G_m , so G_m is generated by elements $X_m \subset G_m$ which act in U_2 as complex reflections. Now, since $G_{m+1} = N$ is the smallest normal subgroup containing G_m , it is generated by the conjugacy classes in G_{m+1} of the elements in X_m . Since $\text{Res } V_2 = U_2 \oplus \mathbb{1}_{G_m}$, the elements of X_m also act as complex reflections in V_2 , and

thus so do their G_{m+1} conjugates. Therefore $\text{im}(\varphi_2)$ is generated as a group by complex reflections. It remains to check that φ_2 is faithful.

Let $H_{m+1} := \text{im}(\varphi_2)$ and let $G_m \cong H_m := \text{im}(\psi_2) \subset H_{m+1}$. We know that ψ_2 is faithful, so $|H_m| = r^m m!$, and that H_{m+1} is an irreducible complex reflection group of rank $m + 1$ whose order divides $|G_{m+1}| = r^{m+1}(m + 1)!$. By the classification of irreducible complex reflection groups [12], we must either have $|H_{m+1}| = r^m(m + 1)!$ or $r^{m+1}(m + 1)!$ (all of the exceptional reflection groups H have multiple prime factors dividing $|H|/\text{rank}(H)!$, thus H_{m+1} must belong to the infinite family). We have $[H_{m+1} : H_m] = \dim(\text{Ind}_{H_m}^{H_{m+1}} \mathbb{1}_{H_m})$. But $\text{Ind}_{H_m}^{H_{m+1}} \mathbb{1}_{H_m}$ contains summands $\mathbb{1}_{H_{m+1}}$ and V_2 , so $[H_{m+1} : H_m] \geq m + 2$. This forces $|H_{m+1}| = r^{m+1}(m + 1)!$, so φ_2 is faithful. Therefore, as it is an irreducible complex reflection group of order $r^{m+1}(m + 1)!$ and rank $m + 1$, we must have $G_{m+1} \cong (\mathbb{Z}/r\mathbb{Z}) \wr S_{m+1}$.

Finally, if $r = 1$, then U_1 is faithful, it is the usual reflection representation of S_m . We see that $\text{Ind} \mathbb{1}_{G_m} = V_1 \oplus \mathbb{1}_{G_{m+1}}$. Since U_1 is faithful and appears in the restriction of V_1 , we see that V_1 is faithful on any conjugacy class intersecting G_m . On any conjugacy class not intersecting G_m , the character value of $\text{Ind} \mathbb{1}_{G_m}$ is 0, and so $\chi_{V_1} = -1$ on these conjugacy classes. Thus V_1 is faithful and G_{m+1} is an irreducible complex reflection group of rank m and order $(m + 1)!$. By the classification, this forces $G_{m+1} \cong S_{m+1}$. \square

By [12], we can take H_{m+1} to be the standard realization of $(\mathbb{Z}/r\mathbb{Z}) \wr S_{m+1}$ in $GL(V_2)$ as monomial matrices. Since $\text{Res}_{G_m}^{G_{m+1}} V_2 \cong \mathbb{1}_{G_m} \oplus U_2$, we see that H_m is conjugate in H_{m+1} to the standard embedding of $m \times m$ monomial matrices into H_{m+1} . This shows that the embedding $G_m \subset G_{m+1}$ is conjugate to the usual one, and so we have the usual branching rules, thus $P_{[0, m+1]} \cong (Y^r)_{[0, m+1]}$. The proof of Proposition 13 is now complete. This result, together with the base case Proposition 11, implies Theorem 6. \square

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