Improved bounds on the multicolor
Ramsey numbers of paths and even cycles

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Abstract

We study the multicolor Ramsey numbers for paths and even cycles, $R_k(P_n)$ and $R_k(C_n)$, which are the smallest integers N such that every coloring of the complete graph K_N has a monochromatic copy of P_n or C_n respectively. For a long time, $R_k(P_n)$ has only been known to lie between $(k - 1 + o(1))n$ and $(k + o(1))n$. A recent breakthrough by Sárközy and later improvement by Davies, Jenssen and Roberts give an upper bound of $(k - \frac{1}{4} + o(1))n$. We improve the upper bound to $(k - \frac{1}{2} + o(1))n$. Our approach uses structural insights in connected graphs without a large matching.

Mathematics Subject Classifications: 05C15, 05C38, 05C55, 05C75

1   Introduction

A classical theorem by Ramsey from 1930 [18] proves the existence of (finite) Ramsey numbers. The multicolor Ramsey number $R(H_1, H_2, \ldots, H_k)$ is defined as the smallest positive integer N such that for every coloring of the edges of the complete graph K_N with k colors there is a monochromatic subgraph H_i in some color i. For convenience, we write $R_k(H)$ if $H_i = H$, for all $i \in [k]$.

The study of Ramsey numbers started with complete graphs but has been extended to general graphs and has found the interest of many researchers. Several surveys on the topic can be found in [6, 12, 17]. Ramsey numbers for complete graphs are known to grow exponentially and are hard to analyze. The upper and lower bounds for these still differ by an exponential factor, the most recent bound was by Conlon [5].

Besides the complete graphs, other natural families of graphs to study are paths P_n and cycles C_n. The two-color Ramsey numbers for paths are known since 1967 [11]. Already for three colors the problem is more difficult and solving it took almost 40 years,

the invention of Szemerédi's regularity lemma [20] and the idea of *connected matchings*. A matching M is connected in G if all edges of M are in the same component of G. This idea was suggested by Łuczak [14]. Using these tools, in 2007, Gyárfás, Ruszinkó, Sárközy and Szemerédi [13] showed the exact bounds for three colors and for large n

$$R_3(P_n) = \begin{cases} 2n - 1, & n \text{ odd,} \\ 2n - 2, & n \text{ even.} \end{cases}$$

Figaj and Łuczak [9] applied them to prove that in the case of Ramsey numbers, connected matchings, paths and even cycles are closely related (see Lemma 4 below). Connected matchings have since been used in a series of papers (see e.g. [2, 9, 13, 15]).

For k colors, an easy application of the Erdős-Gallai extremal theorem [Lemma 5] on each color class gives a bound of $R_k(P_n) \leqslant kn$ for even n. This does not match the current best known lower bound by Yongqi, Yuansheng, Feng and Bingxi [21], who showed that for all $k \geqslant 3$ we have $R_k(C_n) \geqslant (k - 1 + o(1)) n$. Note that the bound is given for cycles but easily extends to paths. The upper bound was best known until a recent breakthrough by Sárközy [19], who further improved this to $(k - \frac{k}{16k^3+1})n$. Davies, Jenssen and Roberts [7] refined his ideas to get

$$R_k(P_n) \leqslant \left(k - \frac{1}{4} + \frac{1}{2k} \right) n.$$

Contribution. In this paper, we analyze the structural properties of graphs not containing large connected matchings as a subgraph. Our improvement on the k-color Ramsey numbers for paths and even cycles comes from the fact that graphs without connected matching are conceptually easier to analyze than graphs without paths. We have a tool similar to Lemma 7 in [10] which states that in these graphs the vertices can be very clearly partitioned into categories of 'low', 'intermediate' and 'high' degree vertices. In particular, if a connected component is large compared to n, then only a small number of vertices have high degree. We introduce a tool for counting edges in such graphs.

While the arguments in the two most recent papers on $R_k(P_n)$ revolve around the overlap of dense connected components, we add the analysis on large connected components. This is important because the construction for the lower bound [21] also contains these large components. We believe this is an essential step towards achieving exact bounds for the multicolor Ramsey numbers for paths and cycles.

For this, we consider the following variant of finding monochromatic connected matchings in k-colored dense graphs. This will allow us to generalize our results to paths and cycles using the regularity lemma. The size of a connected matching corresponds to the number of edges in the matching.

Theorem 1. *Let $k \geqslant 4$ be a positive integer and ε, δ constants, such that $0 < \varepsilon \leqslant 1/2$ and $0 \leqslant \delta < \frac{\varepsilon^3}{3k^2}$. Then for every even integer $n \geqslant 4$, every k-colored graph G with $v(G) > (k - 1/2 + \varepsilon)n$ and $e(G) \geqslant (1 - \delta)\binom{v(G)}{2}$ has a monochromatic connected matching of size $n/2$.*

We will then use the mentioned relationship between avoiding even cycles and connected matching to derive the following theorem.

Theorem 2. *For every integer $k \geqslant 4$ and an even integer n*

$$R_k(C_n) \leqslant \left(k - \frac{1}{2} + o(1) \right) n.$$

As as immediate consequence of the above result we obtain an upper bound on the k-color Ramsey numbers for paths which follows from the fact that $P_n \subseteq C_n$.

Corollary 3. *For every integer $k \geqslant 4$ and integer n*

$$R_k(P_n) \leqslant \left(k - \frac{1}{2} + o(1) \right) n.$$

Outline. Our paper is organized as follows. In Section 2 we show the structural results for graphs avoiding connected matchings and two tools to apply these structures for showing Ramsey results. In Section 3 we prove the main result in two steps. First, we use our structural results from the preceding section to show that we can only have few high-degree vertices. Second, we remove these vertices and show that the resulting connected components are then all small (of size at most n). In Sections 4 and 5 we prove the lemmas we used in Section 2. Finally, in Section 6 we summarize our results and give some ideas for future work.

2 Methods

All graphs considered in this paper are simple, without loops or multiple edges. For a graph $G = (V, E)$ we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of the graph G and we set $v(G)$ and $e(G)$ to be the respective cardinalities.

As mentioned there is a relation between avoiding connected matchings and even cycles in form of the following lemma which is a variant of a lemma used in [9].

Lemma 4 (Lemma 8 in [15]). *Let a real number $c > 0$ be given. If for every $\varepsilon > 0$ there exists a $\delta > 0$ and an n_0 such that for every even $n > n_0$ and any graph G with $v(G) > (1 + \varepsilon)cn$ and $e(G) \geqslant (1 - \delta)\binom{v(G)}{2}$ and any k-edge-coloring of G has a monochromatic component containing a matching of $n/2$ edges then*

$$R_k(C_n) \leqslant (c + o(1)) n.$$

For $c \geqslant 1$ we surely have $(1 + \varepsilon)cn \geqslant (c + \varepsilon)n$ which is more convenient for later calculations. Observe that Theorem 2 follows from Theorem 1 by using Lemma 4 with $c = k - 1/2$.

2.1 Structure

The structure of graphs without large connected matchings play an important role in this paper. As a path on n vertices contains a connected matching on $\lfloor n/2 \rfloor$ edges, extremal results for paths directly give an upper bound for connected matchings.

Lemma 5 (Erdős-Gallai [8]). *Let H be a graph which does not contain an n-vertex path. Then*

$$e(H) \leqslant \frac{n-2}{2}v(H).$$

Corollary 6. *Let H be a graph which does not contain a connected matching of size $n/2$ for even n. Then*

$$e(H) \leqslant \frac{n-2}{2}v(H).$$

The extremal graph in both cases consists of disjoint cliques of size $n-1$. In [1] the extremal configurations of connected graphs without a long path are discussed. We provide a structure for connected graphs without large connected matchings which is very similar. We capture this in the following lemma which is similar to lemma 7 in [10].

Lemma 7. *For every connected graph $G = (V, E)$ without a matching of size $n/2$ there is a partition $S \cup Q \cup I$ of the vertex set such that*

(1) $|Q| + 2|S| = min\{v(G), n-1\}$,

(2) I is an independent set; additionally, if $v(G) \leqslant n-1$, then $I = \varnothing$,

(3) every vertex in Q has at most one neighbor in I,

(4) every vertex in I has degree less than $n/2$.

We only want to give an intuition on the structure here and defer its proof to Section 4. In Figure 1 an example of such a partition can be seen. Our structure has a series of properties that we use later. From $|Q| + 2|S| \leqslant n-1$ we can easily deduce that $|S| < n/2$ and $|Q| < n$. From $|Q| + 2|S| \leqslant v(G)$ we get $|S| \leqslant |I|$, with a strict inequality as soon as the number of vertices exceeds $n-1$. Note that the graph induced by Q and S can potentially be a clique, and we think of the vertices in S as high degree vertices because they can potentially have edges to every other vertex. Vertices in Q have at most $|S| + |Q| < n$ neighbors and if $v(G) < n$, then I is empty so $|S| = 0$ and $|Q| = v(G)$ (using (1)).

From the above properties it also follows that the number of edges is at most $\binom{|Q|+|S|}{2} + |I| \cdot |S| + |Q|$. After some careful consideration one can see that this bound is stronger than the Erdős-Gallai bound as soon as $v(G) > n+1$.
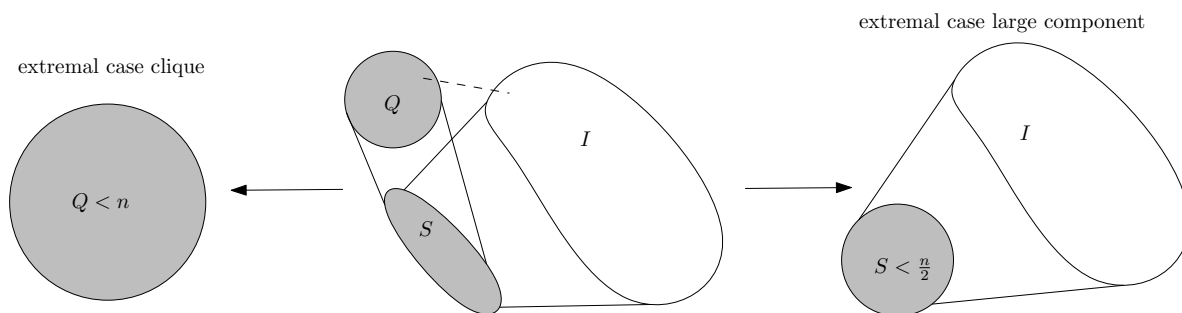
Figure 1: Visualization of the structure from Lemma 7 including extremal structures

2.2 Loss Function

In the following we define a function f which intuitively should capture the difference between the bound of Erdős-Gallai and the bound implied by Lemma 7. Remember Corollary 6 gave us $e(G) \leqslant \frac{n-2}{2}v(G)$ for general graphs. To simplify computations, we use the bound

$$e(G) \leqslant \frac{n-1}{2}v(G).$$

We denote by $\mathcal{G}$ the class of all graphs without a connected matching of size $n/2$. Then the function f maps $\mathcal{G}$ to a positive rational number.

It is important to note that whenever we have connected components we can partition them as in Lemma 7. This partition may not be unique but we can fix an arbitrary one to make the definitions consistent.

Definition 8. Let $\mathcal{G}$ be the class of all graphs without a connected matching of size $n/2$. We define $f \colon \mathcal{G} \to \mathbb{Q}_{\geqslant 0}$ as the difference between $\frac{n-1}{2}v(G)$ and the number of edges in G, i.e.

$$f(G) = \frac{n-1}{2}v(G) - e(G).$$

Observe that Corollary 6 guarantees that $f(G) \geqslant 0$ for any graph without a connected matching of size $n/2$. One way to think about $f(G)$ is as the number of edges G loses in comparison to $\frac{n-1}{2}v(G)$. In the following we also refer to this value as the loss of G. Intuitively, this loss can happen for two reasons:

(1) G can lose edges because of large connected components. This part can be captured by the difference between $\frac{n-1}{2}v(G)$ and the bound of edges for connected graphs obtained from Lemma 7.

(2) Edges missing because G is not saturated. These are the edges that are not present in G but could be added without creating a connected matching of size $n/2$.

Throughout this paper we only look at the number of lost edges, it will not be important which exact edges are lost or what the cause was. We introduce a function distributing this loss among the vertices of G.

Definition 9. Let G be a graph without a connected matching of size $n/2$. We denote by $f\colon V(G) \to \mathbb{Q}_{\geqslant 0}$ the loss of edges in G compared to $\frac{n-1}{2}$ caused by a single vertex $v \in V(G)$ defined as follows. Let C be the connected component of G including v and let $S \cup Q \cup I$ be a partition of $V(C)$ as described in Lemma 7. We set

$$f(v) = \begin{cases} \frac{n-1}{4}, & \text{if } v \in S, \\ \frac{n-1}{2} - \frac{\deg(v)}{2}, & \text{if } v \in Q, \\ 0, & \text{if } v \in I. \end{cases}$$

From Lemma 7 we have that $\deg(v) \leqslant n - 1$ for vertices in Q and so the function $f(v)$ is non-negative and well defined. Since we want to distribute the loss of the graph over the different vertices we need that the sum of the losses over all vertices is at most the loss of G. We claim that this holds for f defined as above. We defer the proof of the following lemma to Section 5.

Lemma 10. *Let G be a graph without a connected matching of size $n/2$. Then*

$$\sum_{v \in V(G)} f(v) \leqslant f(G).$$

Let G be a graph with a k coloring of its edges. We introduce three different categories for the vertices of G. Let $v \in V(G)$ be some vertex of our graph and let $G_1, \ldots, G_k$ be the monochromatic subgraphs in each color. Then for every color i the vertex v is in exactly one component of G_i. We denote this component by C_i^v. Then we have $C_1^v, \ldots, C_k^v$ which are k monochromatic components in different colors, each including the vertex v. We partition the vertices into three classes depending on their role in the k components.

Definition 11. For a graph G with a k-edge-coloring not containing a monochromatic connected matching of size $n/2$ and a vertex $v \in V(G)$, let $C_1^v, \ldots, C_k^v$ be the k components containing v in $G_1, \ldots, G_k$. Consider the partition $C_i^v = S_i^v \cup Q_i^v \cup I_i^v$ given by Lemma 7 for every color i. We call the vertex v

(1) **strong** if it is in S_i^v for some color i,

(2) **Q-saturated** if it is in Q_i^v for every color i,

(3) **small** if it is in I_i^v for some color i and in Q_j^v or I_j^v for all other colors j.

Generalizing Definition 8, we define $F(G)$ to be the total loss of edges over all colors in a graph G without a monochromatic connected matching of size $n/2$. We consider G to be the union of k monochromatic graphs, each of which avoids a connected matching of size $n/2$. Applying Corollary 6, again with a slight weakening of the bound, for each of the color classes, gives

$$e(G) \leqslant k \cdot \frac{n-1}{2} v(G).$$

This is the bound we use for comparison throughout the paper.

Definition 12. Let $\mathcal{G}_c$ be the class of all k-edge-colored graphs which do not contain a monochromatic connected matching of size $n/2$ and let $F\colon \mathcal{G}_c \to \mathbb{Q}_{\geqslant 0}$ be defined as the difference between $k \cdot \frac{n-1}{2} v(G)$ and the number of edges in G, i.e.

$$F(G) = k \cdot \frac{n-1}{2} v(G) - e(G).$$

Observe that $F(G) = k \cdot \frac{n-1}{2} v(G) - e(G) = \sum_{i=1}^{k} f(G_i)$. The above equality holds because every edge of G is colored in exactly one color and thus counted in exactly one G_i. This also implies that $F(G)$ is non-negative by the non-negativity of $f(G_i)$.

We again distribute this loss over the vertices in G.

Definition 13. Let G be a k-edge-colored graph without a monochromatic connected matching of size $n/2$ and let $F\colon V(G) \to \mathbb{Q}_{\geqslant 0}$ denote the loss of edges in G compared to $k \cdot \frac{n-1}{2} v(G)$ caused by a single vertex $v \in V(G)$. We set

$$F(v) = \begin{cases} \frac{n-1}{4}, & \text{if } v \text{ is strong,} \\ k \cdot \frac{n-1}{2} - \frac{\deg(v)}{2}, & \text{if } v \text{ is } Q\text{-saturated,} \\ 0, & \text{if } v \text{ is small.} \end{cases}$$

This is again well defined as a Q-saturated vertex has degree at most $n-1$ in every color by Lemma 7. We conclude $\deg(v) \leqslant k \cdot (n-1)$ which directly implies that $F(v)$ is non-negative. We deduce the following corollary from Lemma 10 to again verify that we have a valid distribution of the loss. The proof is deferred to Section 5.

Corollary 14. *Let G be a k-edge-colored graph without a monochromatic matching of size $n/2$. Then*

$$\sum_{v \in V(G)} F(v) \leqslant F(G).$$

2.3 Small Components

Finally, we introduce a lemma to find an upper bound on the number of edges of graphs whose color classes have the special property of consisting only of components which are not too large.

Lemma 15. *For any two integers $k \geqslant 4$ and $n \geqslant 4$ let G be a k-edge-colored graph on $(k-1/2)n$ vertices with color classes $G_1, \ldots, G_k$ such that no G_i has a component of size larger than n. Then we have*

$$e(G) \leqslant \binom{v(G)}{2} - \frac{n^2}{32}.$$

Proof. For sake of contradiction we assume there exists a graph G on $(k - 1/2)n$ vertices with a k-coloring such that no color class contains a component of size larger than n and

$$e(G) > \binom{v(G)}{2} - \frac{n^2}{32}. \tag{1}$$

Additionally, let G be the graph such that the number of edges is maximal among all graphs satisfying the property above. Choose any color from the graph and without loss of generality we assume it is blue. Over all possible colorings satisfying that no color class has a component of size larger than n we choose the coloring which maximizes the number of blue edges, i.e. blue is the densest color. Let G_B be the subgraph of G induced by the blue edges. We take a closer look at the structure of the blue graph.

Claim 16. *The number of components in G_B is either k or $k + 1$.*

Proof. If we had only $k-1$ components, then we could cover at most $(k-1)n < (k-1/2)n$ vertices. Hence we can be sure we have at least k components. By edge maximality of G and the choice of the coloring we cannot have more than one component of size less than $n/2$ in the blue graph, as otherwise we could add or recolor any edge between the two components blue. But if we had $k + 2$ components, then by convexity and the fact that there is at most one component of size less than $n/2$, the number of edges is maximized when we have $k - 3$ cliques of size n, one clique of size $n - 1$, 3 cliques of size $n/2$ and one isolated vertex. Then

$$e(G_B) \leqslant (k - 2)\binom{n}{2} + 3\binom{n/2}{2} \leqslant (k - 2)\frac{n^2}{2} + 3\frac{n^2}{8} = \left(k - \frac{5}{4}\right)\frac{n^2}{2}.$$

As blue was the densest color, the total number of edges is at most $k \cdot e(G_B)$. We conclude

$$e(G) \leqslant k\left(k - \frac{5}{4}\right)\frac{n^2}{2}.$$

A simple algebraic transformation gives

$$e(G) \leqslant \left(k^2 - k + \frac{1}{4} - \frac{k - 1/2}{n}\right)\frac{n^2}{2} - \left(\frac{k}{4} + \frac{1}{4} - \frac{k - 1/2}{n}\right)\frac{n^2}{2}.$$

Using that the complete graph on $(k - 1/2)n$ vertices has

$$e(K_{v(G)}) = \binom{v(G)}{2} = \frac{n^2}{2}\left(k^2 - k + \frac{1}{4} - \frac{k - 1/2}{n}\right)$$

edges and assuming $n \geqslant 4$ we get

$$e(G) \leqslant \binom{v(G)}{2} - \frac{n^2}{16}.$$

Thus, G having at least $k + 2$ components contradicts our assumption (1) so we conclude that G_B has either k or $k + 1$ components. $\square$

By convexity and our previous observation that the blue graph has either k or $k + 1$ components, we obtain the number of blue edges is maximized when there are k components: $k - 1$ of size n and one of size $n/2$. In this case we have

$$e(G_B) \leqslant (k-1)\frac{n^2}{2} + \frac{n^2}{8} = \left(k - \frac{3}{4}\right)\frac{n^2}{2}.$$

Let us now see what this implies for the other components. Consider the color with the second most edges, say, red. Let G_R be the subgraph of G induced by the red edges. By the choice of G and the coloring, the blue components must be cliques. Thus for all components $C_B \subseteq G_B$ and $C_R \subseteq G_R$, the edges with both endpoints in $V(C_B) \cap V(C_R)$ are blue. Observe that as the blue color has at most $k + 1$ components, this implies that we can view the red graph as a union of $(k + 1)$-partite components, where the number of edges is maximized when all components are complete $(k + 1)$-partite where every part has the same size. We observe that by the same reasons as seen for the blue graph, the red graph cannot have fewer than k components. We conclude that the number of edges in the red graph is maximized, when there are again k components: $k - 1$ of size n and one of size $n/2$. As G_R is $(k + 1)$-partite we know that a component of size n has at most $\left(\left(1 - \frac{1}{k+1}\right)\frac{n^2}{2}\right)$ edges. We get

$$e(G_R) \leqslant \left(1 - \frac{1}{k+1}\right)\left((k-1)\frac{n^2}{2} + \frac{1}{4}\frac{n^2}{2}\right)$$
$$\leqslant \left(1 - \frac{1}{k+1}\right)\left(k - \frac{3}{4}\right)\frac{n^2}{2}.$$

As red was the second densest color, the number of edges in every other color is at most the number of red edges. We conclude for the total number of edges

$$e(G) \leqslant e(G_B) + (k-1) \cdot e(G_R)$$
$$\leqslant \left(k - \frac{3}{4}\right)\frac{n^2}{2} + (k-1)\left(1 - \frac{1}{k+1}\right)\left(k - \frac{3}{4}\right)\frac{n^2}{2}.$$

Simplifying gives

$$e(G) \leqslant \left(k^2 - \frac{7}{4}k + \frac{11}{4} - \frac{7}{2(k+1)}\right)\frac{n^2}{2}.$$

Using the fact that the graph G has $(k - \frac{1}{2})n$ vertices, $n \geqslant 4$ and $k \geqslant 4$, we conclude the proof of the lemma with

$$e(G) \leqslant \binom{v(G)}{2} - \frac{n^2}{32}. \qquad \square$$

3 Proof of the Main Theorem

The main part of this paper is the proof of Theorem 1 about finding monochromatic connected matchings in k-edge-colored dense graphs. We restate the theorem for convenience.

Theorem 1. *Let $k \geqslant 4$ be a positive integer and ε, δ constants, such that $0 < \varepsilon \leqslant 1/2$ and $0 \leqslant \delta < \frac{\varepsilon^3}{3k^2}$. Then for every even integer $n \geqslant 4$, every k-colored graph G with $v(G) > (k-1/2+\varepsilon)n$ and $e(G) \geqslant (1-\delta)\binom{v(G)}{2}$ has a monochromatic connected matching of size $n/2$.*

In the previous section we stated all the tools we need to prove Theorem 1, so all that remains is to put all the lemmas together.

Proof of Theorem 1. For $0 < \varepsilon \leqslant \frac{1}{2}$, let $\alpha = 1/2 - \varepsilon$ and $\delta < \frac{\varepsilon^3}{3k^2}$. We proceed to prove Theorem 1 by contradiction. Let G be an edge maximal graph on $(k-\alpha)n$ vertices with the property that there is a k-coloring of the edges of G that avoids a monochromatic connected matching of size $n/2$. Assume G has many edges,

$$e(G) \geqslant (1-\delta)\binom{v(G)}{2} > \binom{v(G)}{2} - \delta k^2 \frac{n^2}{2}. \tag{2}$$

We then derive a contradiction by showing that G cannot have enough edges.

Firstly, we show that the number of vertices of low degree in G is small. Let V_ℓ be all vertices $v \in V(G)$ with $\deg(v) < (k-1/2)n$.

Claim 17. $|V_\ell| \leqslant \frac{\delta k^2 n}{\varepsilon}$.

Proof. Observe that every vertex in V_ℓ misses at least εn incident edges. Thus we can find a lower bound on the number of edges missing in G compared to the complete graph $K_{v(G)}$ by $\frac{|V_\ell|\varepsilon n}{2}$. As we cannot miss more than $\delta k^2 \frac{n^2}{2}$ edges we conclude $|V_\ell| \leqslant \frac{\delta k^2 n}{\varepsilon}$. $\quad\square$

Observe next that every vertex in the set $V(G) \setminus V_\ell$ has degree at least $(k-1/2)n$. Recall that, by Lemma 7, we can partition every color class G_i into $S_i \cup Q_i \cup I_i$, where S_i, Q_i and I_i are the unions of the corresponding sets of Lemma 7 given for every connected component in color i respectively, such that all vertices with very large degree are in S_i and all vertices in I_i have degree less than $n/2$ and all vertices in Q_i have degree less than n. Let $v \in V(G) \setminus V_\ell$ be a vertex which is in the I_i for some color i and thus has degree less than $n/2$ in this color. If this vertex was not in S_j for any color j, then it has a maximum degree of $n-1$ in every other color, thus $\deg(v) < n/2 + (k-1)(n-1) < (k-1/2)n$, contradicting that $v \in V(G) \setminus V_\ell$. We conclude that no vertex in $V(G) \setminus V_\ell$ by Definition 11 is small and thus every vertex in $V(G) \setminus V_\ell$ is either strong or Q-saturated, meaning it is in S_i for some color i or it is in Q_i for all colors $1 \leqslant 1 \leqslant k$.

Secondly, we show that the number of strong vertices in G also has to be small. Let V_s be the set of all strong vertices in $V(G) \setminus V_\ell$ and let $\beta = \frac{|V_s|}{n}$.

Claim 18. $\beta \leqslant \frac{2\delta k^2}{\varepsilon^2}$.

Proof. Remember that we previously saw that we can use the function F from Definition 13 to capture the loss of edges in a graph caused by vertex $v \in V(G)$. By definition for Q-saturated vertices we have

$$F(v) = k \cdot \frac{n-1}{2} - \frac{\deg(v) - 1}{2}.$$

As $\deg(v) \leqslant v(G) - 1 = (k - \alpha)n - 1$ we conclude that for a Q-saturated vertex $v \in V(G)$

$$F(v) \geqslant k \cdot \frac{n-1}{2} - \frac{(k - \alpha)n - 1}{2} = \alpha \frac{n}{2} - \frac{k-1}{2}. \tag{3}$$

By definition of F we have $F(G) = k \cdot \frac{n-1}{2}v(G) - e(G)$ and Corollary 14 states that $\sum_{v \in V(G)} F(v) \leqslant F(G)$, so we can bound the number of edges in G by

$$e(G) \leqslant k \cdot \frac{n-1}{2}v(G) - \sum_{v \in V(G)} F(v).$$

Next, we use the loss function F of the vertices to find an upper bound on the number of edges in G. All small vertices in G have to be in V_ℓ and thus there cannot be too many of them, remember for strong vertices we have $F(v) = \frac{n-1}{4}$ by Definition 13. Using this, Equation (3) for Q-saturated vertices and the fact that there are only few small vertices allows us to find an upper bound on the number of strong vertices. We get

$$e(G) \leqslant k \cdot \frac{n-1}{2}(k - \alpha)n - \sum_{v \in V(G)} F(v)$$

$$\leqslant k \cdot \frac{n-1}{2}(k - \alpha)n - \overbrace{\left(k - \alpha - \beta - \frac{\delta k^2}{\varepsilon}\right)n\left(\alpha\frac{n}{2} - \frac{k-1}{2}\right)}^{\substack{\text{vertices in } V(G)\backslash V_\ell \\ \text{which are } Q\text{-saturated}}} - \overbrace{\beta n\left(\frac{n-1}{4}\right)}^{\substack{\text{vertices in } V(G)\backslash V_\ell \\ \text{which are strong}}}.$$

Now it follows from algebraic transformations and simple estimations that

$$e(G) \leqslant \frac{n^2}{2}\left(k^2 - 2\alpha k + \alpha^2 - \frac{k - \alpha}{n}\right) - \frac{n^2}{2}\left(\frac{\beta}{2} - \alpha\beta - \frac{\delta k^2}{\varepsilon}\alpha\right).$$

Observe that $\binom{v(G)}{2} = \frac{n^2}{2}\left(k^2 - 2k\alpha + \alpha^2 - \frac{k-\alpha}{n}\right)$. Together with $\alpha = 1/2 - \varepsilon$ this gives

$$e(G) \leqslant \binom{v(G)}{2} - \frac{n^2}{2}\left(\varepsilon\beta - \frac{\delta k^2}{\varepsilon}\alpha\right).$$

By the assumption $e(G) \geqslant \binom{v(G)}{2} - \frac{\delta k^2 n^2}{2}$ we conclude that $\varepsilon\beta - \frac{\delta k^2}{\varepsilon}\alpha \leqslant \delta k^2$. Using $\alpha \leqslant 1$ and $\varepsilon \leqslant 1$ we get a bound on the size of β.

$$\beta \leqslant \frac{\delta k^2}{\varepsilon} + \frac{\delta k^2}{\varepsilon^2}\alpha \leqslant \frac{2\delta k^2}{\varepsilon^2}.$$

This concludes the proof of Claim 2. $\qquad\square$

We look at the induced subgraph on the remaining vertices i.e. $V(G) \setminus \{V_\ell \cup V_s\}$. Observe that we get using that $\delta < \frac{\varepsilon^3}{3k^2}$, $\alpha = 1/2 - \varepsilon$ and Claims 17 and 18

$$|V(G) \setminus (V_\ell \cup V_s)| \geqslant v(G) - \frac{\delta k^2 n}{\varepsilon} - \frac{2\delta k^2 n}{\varepsilon^2}$$
$$\geqslant v(G) - \frac{3\delta k^2 n}{\varepsilon^2}$$
$$> v(G) - \varepsilon n = (k - 1/2)n.$$

We now remove εn vertices from the graph, including the vertices from V_s and V_ℓ. We are left with a graph G' on $(k - 1/2)n$ vertices such that every vertex was Q-saturated in G. This means we remove from all components in all colors i the vertices in I_i and S_i of the corresponding partition and we are thus left with only monochromatic components of size at most $n-1$. This allows us to apply Lemma 15 which gives us $e(G') \leqslant \binom{v(G')}{2} - \frac{n^2}{32} \leqslant \binom{v(G')}{2} - \delta k^2 \frac{n^2}{2}$. Which follows by the choice of ε and δ.

From here we conclude that there are at least $\delta k^2 \frac{n^2}{2}$ edges not present in G'. But then, even without considering the edges missing in the rest of G we can conclude that $e(G) \leqslant \binom{v(G)}{2} - \delta k^2 \frac{n^2}{2} < (1 - \delta)\binom{v(G)}{2}$ edges, yielding the desired contradiction. $\square$

4   Proof of Lemma 7

In the following we derive some properties of connected graphs without large matchings. For convenience we restate the lemma

Lemma 7. *For every connected graph $G = (V, E)$ without a matching of size $n/2$ there is a partition $S \cup Q \cup I$ of the vertex set such that*

(1) $|Q| + 2|S| = min\{v(G), n - 1\}$,

(2) I is an independent set; additionally, if $v(G) \leqslant n - 1$, then $I = \varnothing$,

(3) every vertex in Q has at most one neighbor in I,

(4) every vertex in I has degree less than $n/2$.

For a graph G and $A \subseteq V(G)$ we define $G \setminus A$ as the subgraph of G where all vertices from A and their incident edges are removed. For a matching $M \subseteq E$ we call all vertices which are not incident to any edge in M the unmatched vertices in M. Furthermore, we denote by $q(G \setminus S)$ the number of odd components in $G \setminus S$. We use a generalization of Tutte's Theorem by Berge [3] in our proof.

Theorem 19 (Berge [3]). *Let $G = (V, E)$ be a graph. For any set $S \subseteq V$ and any matching M, the number of unmatched vertices in M is at least $q(G \setminus S) - |S|$. Moreover, there exists a set $S \subset V$ such that every maximum matching of G misses exactly $q(G \setminus S) - |S|$ vertices.*

Proof of Lemma 7. We distinguish two cases in the proof.

Case 1 ($v(G) \leqslant n - 1$). In this case we set $Q = V$ and $S = I = \varnothing$. It can be easily verified that all four conditions are satisfied in this case.

Case 2 ($v(G) > n - 1$). Let M be a maximum matching in G and let V_M be all vertices covered by the matching. As G has no matching covering n vertices and n is even, $2|M| \leqslant n - 2$. Then we know by Theorem 19 that there exists a subset of $S \subset V$ such that $|V \setminus V_M| = q(G \setminus S) - |S|$.

Let $\mathcal{Q} = \{Q_1, \ldots, Q_{q(G \setminus S)}\}$ be the set of all odd components in $G \setminus S$ where we assume without loss of generality that $|Q_1| \geqslant |Q_2| \geqslant \ldots \geqslant |Q_{q(G \setminus S)}|$. Let I be be an arbitrary set of vertices from the odd components such that $|I \cap Q_1| = 0$ and $|I \cap Q_i| = 1$ for all $2 \leqslant i \leqslant q(G \setminus S)$. Then clearly we have that I is an independent set (proving (2)). Note that we have

$$|I| = q(G \setminus S) - 1.$$

Let $Q = V \setminus (I \cup S)$ be the set of all remaining vertices. Then $S \cup Q \cup I$ is clearly a partition of V and thus $|V| = |S| + |Q| + |I|$. By Theorem 19 we know

$$
\begin{aligned}
n - 2 \geqslant 2|M| &= |V| - (q(G \setminus S) - |S|) \\
&= |S| + |Q| + |I| - q(G \setminus S) + |S| \\
&= 2|S| + |Q| + 1.
\end{aligned}
$$

We conclude $n - 1 \geqslant 2|S| + |Q|$. In case $n - 1 > 2|S| + |Q|$ we move vertices from I to Q until the above holds with equality. As we are in the case $v(G) > n - 1$ this proves (1).

To see that (3) holds, recall that I contains at most one vertex from each odd component. Every vertex in Q can thus be adjacent to at most one of the vertices in I, the one which was in the same component in $G \setminus S$. As we by construction did not remove any vertex from the largest odd component, no vertex in I can have more that $\frac{|Q|}{2}$ neighbors in Q. Together with the vertices from S we conclude for $v \in I$ that

$$\deg(v) \leqslant |S| + \frac{|Q|}{2} = \frac{n-1}{2},$$

proving (4). $\qquad\square$

5 Proof of Lemma 10 and Corollary 14

Next we proof Lemma 10, recall that

$$f(G) = \frac{n-1}{2} v(G) - e(G)$$

and for every connected component C and every vertex $v \in V(C) = S \cup Q \cup I$ we have

$$
f(v) = \begin{cases}
\frac{n-1}{4}, & \text{if } v \in S, \\
\frac{n-1}{2} - \frac{\deg(v)}{2}, & \text{if } v \in Q, \\
0, & \text{if } v \in I.
\end{cases}
$$

Lemma 10. *Let G be a graph without a connected matching of size $n/2$. Then*

$$\sum_{v \in V(G)} f(v) \leqslant f(G).$$

Proof of Lemma 10. Let G be a graph, not necessarily connected, and let $C_1, \ldots, C_m$ be the connected components of G. Every vertex and every edge is in exactly one component thus we get

$$
\begin{aligned}
f(G) &= \frac{n-1}{2} v(G) - e(G) \\
&= \sum_{i=1}^{m} \frac{n-1}{2} v(C_i) - e(C_i) \\
&= \sum_{i=1}^{m} f(C_i).
\end{aligned}
$$

If we know that for every component we have $\sum_{v \in v(C_i)} f(v) \leqslant f(C_i)$, then

$$\sum_{v \in V(G)} f(v) = \sum_{i=1}^{m} \sum_{v \in v(C_i)} f(v) \leqslant \sum_{i=1}^{m} f(C_i) = f(G).$$

We can thus without loss of generality assume G is connected. This allows us to partition G with Lemma 7. We distinguish two cases.

Case 1 ($v(G) \leqslant n-1$). In this case all vertices are in Q as by Lemma 7 (2) we know $I = \varnothing$ and thus $|Q| + 2|S| = v(G)$ implies $S = \varnothing$. We clearly have

$$f(G) = \frac{n-1}{2} v(G) - e(G) = \sum_{v \in V(G)} \left(\frac{n-1}{2} - \frac{\deg(v)}{2} \right) = \sum_{v \in V(G)} f(v).$$

Case 2 ($v(G) > n-1$). As in the previous case we rewrite the function f

$$f(G) = \frac{n-1}{2} v(G) - e(G) = \sum_{v \in V(G)} \left(\frac{n-1}{2} - \frac{\deg(v)}{2} \right).$$

Let $V(G) = S \cup Q \cup I$. We now distinguish between vertices which are in Q, S or I. By definition

$$\sum_{v \in Q} \left(\frac{n-1}{2} - \frac{\deg(v)}{2} \right) = \sum_{v \in Q} f(v). \tag{4}$$

Then we know that $\deg(v)$ for $v \in S$ is at most $v(G) - 1$ and $\deg(v)$ for $v \in I$ is at most $|S|$ plus at most $|Q|$ edges in total from the set I to Q. Therefore, for vertices in S and I we have the inequalities

$$\sum_{v \in S} \left(\frac{n-1}{2} - \frac{\deg(v)}{2} \right) \geqslant \sum_{v \in S} \left(\frac{n-1}{2} - \frac{v(G)-1}{2} \right), \text{ and}$$

$$\sum_{v \in I} \left(\frac{n-1}{2} - \frac{\deg(v)}{2} \right) \geqslant \sum_{v \in I} \left(\frac{n-1}{2} - \frac{|S|}{2} \right) - \frac{|Q|}{2}.$$

Adding in the fact that $v(G) = |Q| + |I| + |S|$ and $n - 1 = |Q| + 2|S|$ we get

$$\frac{n-1}{2} - \frac{v(G)-1}{2} = \frac{|S| - |I| + 1}{2}, \text{ and}$$

$$\frac{n-1}{2} - \frac{|S|}{2} = \frac{|Q| + |S|}{2}.$$

Using the fact that $|I| \geqslant |S| + 1$ we obtain

$$\sum_{v \in S \cup I} \left(\frac{n-1}{2} - \frac{\deg(v)}{2} \right) \geqslant |S| \frac{|S| - |I| + 1}{2} + |I| \frac{|Q| + |S|}{2} - \frac{|Q|}{2}$$

$$\geqslant |S| \frac{|Q| + |S|}{2} \geqslant |S| \frac{n-1}{4}. \tag{5}$$

Putting inequalities (4) and (5) together finally gives

$$f(G) = \sum_{v \in V(G)} \left(\frac{n-1}{2} - \frac{\deg(v)}{2} \right)$$

$$\geqslant \sum_{v \in Q} \left(\frac{n-1}{2} - \frac{\deg(v)}{2} \right) + |S| \frac{n-1}{4} = \sum_{v \in V(G)} f(v).$$

This finishes the proof of Lemma 10. $\hfill\square$

We deduce Corollary 14 from Lemma 10. For this consider G to be a k-edge-colored graph. Recall that G_i is the monochromatic induced subgraph in color i. We denote the associated function from Definition 8 and 9 for the graph G_i in color i with f_i.

Proof of Corollary 14. By definition of $F(G)$ and $f_i(G)$ we have that $F(G) = \sum_{i=1}^k f(G_i)$. We conclude that if $F(v) \leqslant \sum_{i=1}^k f_i(v)$, then this together with Lemma 10 implies

$$\sum_{v \in V(G)} F(v) \leqslant \sum_{v \in V(G)} \sum_{i=1}^k f_i(v) = \sum_{i=1}^k \sum_{v \in V(G)} f_i(v) \leqslant \sum_{i=1}^k f_i(G_i) = F(G).$$

In the following we show that $F(v) \leqslant \sum_{i=1}^k f_i(v)$ indeed holds. For this we look at $F(v)$ depending on the class of the vertex v.

For every strong vertex v, we know that for at least one color i, $v \in S_i$, so we have $f_i(v) = \frac{n-1}{4}$. Hence then $F(v) = f_i(v) \leqslant \sum_{i=1}^k f_i(v)$ by the non-negativity of $f(v)$.

For every Q-saturated vertex by definition we have $f_i(v) = \frac{n-1}{2} - \frac{\deg_{G_i}(v)}{2}$. This means that for every Q-saturated vertex of G

$$\sum_{i=1}^k f_i(v) = \sum_{i=1}^k \left(\frac{n-1}{2} - \frac{\deg_{G_i}(v)}{2} \right) = k \cdot \frac{n-1}{2} - \frac{\deg_G(v)}{2} = F(v),$$

since the graphs G_i are edge-disjoint and their union is G. For every small vertex we have, by non-negativity of $f(v)$, that $F(v) = 0 \leqslant \sum_{i=1}^k f_i(v)$. $\hfill\square$

6 Conclusion

In this paper, we provided some insight into the behavior of graphs avoiding connected matchings. We introduced strong properties for large connected components without a matching of size $n/2$. Also these directly imply a better bound for the multicolor Ramsey numbers of paths and cycles.

The analysis of these large connected components is important because of the different extremal constructions that exist for the lower bound of $R_k(P_n)$. Even though there is a lower bound construction using finite affine planes (see [4]) where all colors have small connected components, the bound by Yongqi et al. [21] uses large connected components. We believe for tight bounds, both extremal constructions must be considered.

To prove the bound for $R_k(C_n)$, we first find large connected matchings in k-colored dense graphs (Theorem 1) and then conclude by applying Lemma 4, which itself applies the regularity lemma. Because the regularity lemma is such a strong tool, it should be possible to deduce bounds on Ramsey numbers for other structures than the path or even cycle, such as bounded degree trees, by adjusting Lemma 4 (see [16] for similar ideas with three colors).

Although we now have some consideration for large connected components and the overlap of small components, we do not look at the overlap of large components. Considering this might lead to better bounds. We would be interested to see these ideas used to prove an upper bound matching the lower bound.

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