Permutations of type B with fixed number of descents and minus signs

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Abstract

We study three dimensional array of numbers $B(n,k,j)$, $0 \leqslant j,k \leqslant n$, where $B(n,k,j)$ is the number of type B permutations of order n with k descents and j minus signs. We prove in particular, that $b(n,k,j) := B(n,k,j)/\binom{n}{j}$ is an integer and provide two combinatorial interpretations for these numbers.

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Introduction

Let $B(n,k,j)$ denote the number of type B permutations $(0,\sigma_1,\ldots,\sigma_n)$ which have k descents and j minus signs. We study properties of the three-dimensional array $B(n,k,j)$, $0 \leqslant j,k \leqslant n$. Some of these properties appear in the work of Brenti [4]. In particular he computed the three-variable generating function and proved real rootedness of some linear combinations of the polynomials $P_{n,j}(x) := \sum_{k=0}^{n} B(n,k,j)x^k$ (Corollary 3.7 in [4], see also Corollary 6.9 in [2]). Here we will prove that the numbers $b(n,k,j) := B(n,k,j)/\binom{n}{j}$ are also integers. We provide two combinatorial interpretations of them.

For a subset $U \subseteq \{1,\ldots,n\}$ and $0 \leqslant k \leqslant n$ let $\mathcal{B}_{n,k,U}$ denote the family of all type B permutations $\sigma = (0,\sigma_1,\ldots,\sigma_n)$ that σ has k descents and satisfy: $\sigma_i < 0$ iff $|\sigma_i| \in U$. We will show (Theorem 9) that the cardinality of $\mathcal{B}_{n,k,U}$ is $b(n,k,|U|)$.

Conger [5, 6] defined the refined Eulerian number $\left\langle {n \atop k} \right\rangle_j$ as the cardinality of the set $\mathcal{A}_{n,k,j}$ of all type A permutations $\tau = (\tau_1,\ldots,\tau_n)$ such that $\tau_1 = j$ and τ has k descents. He proved many interesting properties of these numbers, like direct formula, asymptotic

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behavior, lexicographic unimodality, formula for the generating function and real rootedness of the corresponding polynomials. It turns out that for $0 \leqslant j, k \leqslant n$ we have $b(n,k,j) = \left\langle {n+1 \atop k} \right\rangle_{j+1}$. We will prove this equality providing a bijection $\mathcal{A}_{n+1,k,j+1} \to \mathcal{B}_{n,k,U}$, where $U = \{1, \ldots, j\}$ (Theorem 11). The array $b(n,k,1)$, $1 \leqslant k \leqslant n$, appears in OEIS [8] as A120434. It also counts permutations $\sigma \in \mathcal{A}_n$ which have $k-1$ big descents, i.e. such descents $\sigma_i > \sigma_{i+1}$ that $\sigma_i - \sigma_{i+1} \geqslant 2$.

Conger proved that the polynomials $p_{n,j}(x) := \sum_{k=0}^{n} b(n,k,j)x^k$ have only real roots (Theorem 5 in [5]). Brändén [3] showed something stronger: for every $n \geqslant 1$ the sequence of polynomials $\{p_{n,j}(x)\}_{j=0}^{n}$ is interlacing, in particular for every $c_0, c_1, \ldots, c_n \geqslant 0$ the polynomial $c_0 p_{n,0}(x) + c_1 p_{n,1}(x) + \ldots + c_n p_{n,n}(x)$ has only real roots. Here we remark, that $P_{n,j}(x) = \binom{n}{j} p_{n,j}(x)$, so the polynomials $P_{n,j}(x)$ admit the same property, which is a generalization of Corollary 3.7 in [4] and of Corollary 6.9 in [2].

1 Preliminaries

For a sequence $(a_0, \ldots, a_s)$, $a_i \in \mathbb{R}$, the *number of descents*, denoted $\operatorname{des}(a_0, \ldots, a_s)$, is defined as the cardinality of the set $\{i \in \{1, \ldots, s\} : a_{i-1} > a_i\}$. We will use the *Iverson bracket:* $[p] := 1$ if the statement p is true and $[p] := 0$ otherwise, see [7].

Denote by $\mathcal{A}_n$ the group of permutations of the set $\{1, \ldots, n\}$. We will identify $\sigma \in \mathcal{A}_n$ with the sequence $(\sigma_1, \ldots, \sigma_n)$ (we will usually write σ_k instead of $\sigma(k)$). For $0 \leqslant k \leqslant n$ we define $\mathcal{A}_{n,k}$ as the set of those $\sigma \in \mathcal{A}_n$ such that the sequence $(\sigma_1 \ldots, \sigma_n)$ has k descents. Then the classical *type A Eulerian number* $A(n,k)$ (see entry A123125 in OEIS) is defined as the cardinality of $\mathcal{A}_{n,k}$. We have the following recurrence relation:

$$A(n,k) = (n-k)A(n-1,k-1) + (k+1)A(n-1,k) \tag{1}$$

for $0 < k < n$, with the boundary conditions: $A(n,0) = 1$ for $n \geqslant 0$ and $A(n,n) = 0$ for $n \geqslant 1$. These numbers can be expressed as:

$$A(n,k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n+1}{k-i} (i+1)^n. \tag{2}$$

For the Eulerian polynomials

$$P_n^{\mathrm{A}}(t) := \sum_{k=0}^{n} A(n,k) t^k$$

the exponential generating function is equal to

$$f^{\mathrm{A}}(t,z) := \sum_{n=0}^{\infty} \frac{P_n^{\mathrm{A}}(t)}{n!} z^n = \frac{(1-t)e^{(1-t)z}}{1-te^{(1-t)z}}. \tag{3}$$

By $\mathcal{B}_n$ we will denote the group of such permutations σ of the set

$$\{-n, \ldots, -1, 0, 1, \ldots, n\}$$

such that σ is odd, i.e. $\sigma(-k) = -\sigma(k)$ for every $-n \leqslant k \leqslant n$. Then $|\mathcal{B}_n| = 2^n n!$. We will identify $\sigma \in \mathcal{B}_n$ with the sequence $(0, \sigma_1, \ldots, \sigma_n)$. For $\sigma \in \mathcal{B}_n$ we define $\operatorname{des}(\sigma)$ (resp. $\operatorname{neg}(\sigma)$) as the number of descents (resp. of negative numbers) in the sequence $(0, \sigma_1, \ldots, \sigma_n)$. For $0 \leqslant k, j \leqslant n$ we define sets

$$\mathcal{B}_{n,k} := \{\sigma \in \mathcal{B}_n : \operatorname{des}(\sigma) = k\},$$
$$\mathcal{B}_{n,k,j} := \{\sigma \in \mathcal{B}_n : \operatorname{des}(\sigma) = k, \operatorname{neg}(\sigma) = j\},$$

and the numbers $B(n,k) := |\mathcal{B}_{n,k}|$ (*type B Eulerian numbers*, see entry A060187 in OEIS), $B(n,k,j) := |\mathcal{B}_{n,k,j}|$. The numbers $B(n,k)$ satisfy the following recurrence relation:

$$B(n,k) = (2n - 2k + 1)B(n-1, k-1) + (2k+1)B(n-1,k), \tag{4}$$

$0 < k < n$, with the boundary conditions $B(n,0) = B(n,n) = 1$, and can be expressed as

$$B(n,k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n+1}{k-i} (2i+1)^n. \tag{5}$$

The type B Eulerian polynomials are defined by

$$P_n^{\mathrm{B}}(t) := \sum_{k=0}^{n} B(n,k) t^k,$$

and the corresponding exponential generating function is equal to

$$f^{\mathrm{B}}(t,z) := \sum_{n=0}^{\infty} \frac{P_n^{\mathrm{B}}(t)}{n!} z^n = \frac{(1-t)e^{(1-t)z}}{1 - te^{2(1-t)z}}. \tag{6}$$

2 Descents and signs in type B permutations

This section is devoted to the numbers $B(n,k,j) := |\mathcal{B}_{n,k,j}|$. First we observe the following symmetry.

Proposition 1. *For $0 \leqslant j, k \leqslant n$ we have*

$$B(n,k,j) = B(n, n-k, n-j). \tag{7}$$

Proof. It is sufficient to note that the map

$$(0, \sigma_1, \ldots, \sigma_n) \mapsto (0, -\sigma_1, \ldots, -\sigma_n)$$

is a bijection of $\mathcal{B}_{n,k,j}$ onto $\mathcal{B}_{n,n-k,n-j}$. □

Now we provide two summation formulas.

Proposition 2.

$$\sum_{j=0}^{n} B(n,k,j) = B(n,k), \tag{8}$$

$$\sum_{k=0}^{n} B(n,k,j) = \binom{n}{j} n!. \tag{9}$$

Proof. The former sum counts all $\sigma \in \mathcal{B}_n$ which have k descents, while the latter counts all $\sigma \in \mathcal{B}_n$ which have j minus signs in the sequence $(\sigma_1, \ldots, \sigma_n)$. □

From Corollary 4.4 in [1] we have also

$$\sum_{\substack{j=0 \\ j \text{ even}}}^{n} B(n,k,j) = \frac{1}{2} B(n,k) + \frac{(-1)^k}{2} \binom{n}{k}, \tag{10}$$

$$\sum_{\substack{j=0 \\ j \text{ odd}}}^{n} B(n,k,j) = \frac{1}{2} B(n,k) - \frac{(-1)^k}{2} \binom{n}{k}, \tag{11}$$

see A262226 and A262227 in OEIS.

Now we present the basic recurrence relations for the numbers $B(n,k,j)$.

Theorem 3. *The numbers $B(n,k,j)$ admit the following recurrence:*

$$\begin{aligned} B(n,k,j) &= (k+1)B(n-1,k,j) + (n-k)B(n-1,k-1,j) \\ &+ kB(n-1,k,j-1) + (n-k+1)B(n-1,k-1,j-1) \end{aligned} \tag{12}$$

for $0 < k, j < n$, with boundary conditions:

$$B(n,0,j) = [j=0], \qquad B(n,n,j) = [j=n], \tag{13}$$

$$B(n,k,0) = A(n,k), \qquad B(n,k,n) = A(n,n-k) \tag{14}$$

for $0 \leqslant k, j \leqslant n$.

Equality (12) remains true for $0 \leqslant j, k \leqslant n$ under convention that $B(n,k,j) = 0$ whenever $j \in \{-1, n+1\}$ or $k \in \{-1, n+1\}$.

Proof. For $(\sigma_0, \ldots, \sigma_n) \in \mathcal{B}_n$, $n \geqslant 1$, we define

$$\Lambda\sigma := (\sigma_0, \ldots, \widehat{\sigma}_i, \ldots, \sigma_n) \in \mathcal{B}_{n-1},$$

where i is such that $\sigma_i = \pm n$, and the symbol "$\widehat{\sigma}_i$" means, that the element σ_i has been removed from the sequence.

For given $\sigma \in \mathcal{B}_{n,k,j}$, $0 < k, j < n$, we have four possibilities:

- $\sigma_i = n$ and either $i = n$ or $\sigma_{i-1} > \sigma_{i+1}$, $1 \leqslant i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k,j}$.

- $\sigma_i = n$ and $\sigma_{i-1} < \sigma_{i+1}$, $1 \leqslant i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k-1,j}$.
- $\sigma_i = -n$ and $\sigma_{i-1} > \sigma_{i+1}$, $1 \leqslant i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k,j-1}$.
- $\sigma_i = -n$ and either $i = n$ or $\sigma_{i-1} < \sigma_{i+1}$, $1 \leqslant i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k-1,j-1}$.

Now, suppose we are given a fixed $\tau = (\tau_0, \dots, \tau_{n-1})$ which belongs to one of the sets $\mathcal{B}_{n-1,k,j}$, $\mathcal{B}_{n-1,k-1,j}$, $\mathcal{B}_{n-1,k,j-1}$ or $\mathcal{B}_{n-1,k-1,j-1}$. We are going to count all $\sigma \in \mathcal{B}_{n,k,j}$ such that $\Lambda\sigma = \tau$.

If $\tau \in \mathcal{B}_{n-1,k,j}$ then we should either put n at the end of τ, or insert into a descent of τ, i.e. between τ_{i-1} and τ_i, where $1 \leqslant i \leqslant n-1$, $\tau_{i-1} > \tau_i$, therefore we have $k+1$ possibilities.

Similarly, if $\tau \in \mathcal{B}_{n-1,k-1,j}$ then we construct σ by inserting n between τ_{i-1} and τ_i, $1 \leqslant i \leqslant n-1$, where $\tau_{i-1} < \tau_i$. For this we have $n-k$ possibilities.

Now assume that $\tau \in \mathcal{B}_{n-1,k,j-1}$. Then we should insert $-n$ between τ_{i-1} and τ_i, $1 \leqslant i \leqslant n-1$, where $\tau_{i-1} > \tau_i$, for which we have k possibilities.

Finally, if $\tau \in \mathcal{B}_{n-1,k-1,j-1}$ then we put $-n$ either at the end of τ or between τ_{i-1} and τ_i, $1 \leqslant i \leqslant n-1$, where $\tau_{i-1} < \tau_i$, for which we have $n-k+1$ possibilities.

Therefore the number of $\sigma \in \mathcal{B}_{n,k,j}$ such that $\Lambda\sigma$ belongs to the set $\mathcal{B}_{n-1,k,j}$, $\mathcal{B}_{n-1,k-1,j}$, $\mathcal{B}_{n-1,k,j-1}$ or $\mathcal{B}_{n-1,k-1,j-1}$ is equal to $(k+1)B(n-1,k,j)$, $(n-k)B(n-1,k-1,j)$, $kB(n-1,k,j-1)$ or $(n-k+1)B(n-1,k-1,j-1)$ respectively. This proves (12).

For the boundary conditions it is clear that if $\operatorname{neg}(\sigma) > 0$ then $\operatorname{des}(\sigma) > 0$, which yields $B(n,0,j) = [j=0]$. We note that the map $(\sigma_0, \sigma_1, \dots, \sigma_n) \mapsto (\sigma_1, \dots, \sigma_n)$ is a bijection of $\mathcal{B}_{n,k,0}$ onto $\mathcal{A}_{n,k}$, consequently $B(n,k,0) = A(n,k)$. For the two others we refer to (7). $\square$

Below we present tables for the numbers $B(n,k,j)$ for $n = 0,1,2,3,4,5$:

$k \setminus j$	0
0	1

,

$k \setminus j$	0	1
0	1	0
1	0	1

,

$k \setminus j$	0	1	2
0	1	0	0
1	1	4	1
2	0	0	1

,

$k \setminus j$	0	1	2	3
0	1	0	0	0
1	4	12	6	1
2	1	6	12	4
3	0	0	0	1

,

$k \setminus j$	0	1	2	3	4
0	1	0	0	0	0
1	11	32	24	8	1
2	11	56	96	56	11
3	1	8	24	32	11
4	0	0	0	0	1

,

$k \setminus j$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	26	80	80	40	10	1
2	66	330	600	480	180	26
3	26	180	480	600	330	66
4	1	10	40	80	80	26
5	0	0	0	0	0	1

.

For example we have $B(n,1,0) = 2^n - n - 1$ and $B(n,1,j) = \binom{n}{j}2^{n-j}$ for $1 \leqslant j \leqslant n$ (cf. A038207 in OEIS). We will see that $B(n,k,j)/\binom{n}{j}$ is always an integer.

3 Generating functions

Now we define three families of polynomials corresponding to the numbers $B(n, k, j)$:

$$P_{n,j}(x) := \sum_{k=0}^{n} B(n, k, j)x^k, \tag{15}$$

$$Q_{n,k}(y) := \sum_{j=0}^{n} B(n, k, j)y^j, \tag{16}$$

$$R_n(x, y) := \sum_{j,k=0}^{n} B(n, k, j)x^k y^j. \tag{17}$$

The polynomials $R_n(x, y)$ were studied by Brenti [4], who called them "q-Eulerian polynomials of type B".

The symmetry (7) implies:

$$P_{n,j}(x) = x^n P_{n,n-j}(1/x), \tag{18}$$

$$Q_{n,k}(y) = y^n Q_{n,n-k}(1/y), \tag{19}$$

$$R_n(x, y) = x^n y^n R_n(1/x, 1/y). \tag{20}$$

Proposition 4. *The polynomials $P_{n,j}(x)$ satisfy the following recurrence:*

$$\begin{aligned} P_{n,j}(x) = (1 + nx - x)P_{n-1,j}(x) + (x - x^2)P'_{n-1,j}(x) \\ + nxP_{n-1,j-1}(x) + (x - x^2)P'_{n-1,j-1}(x), \end{aligned} \tag{21}$$

with the initial conditions: $P_{n,0}(x) = P_n^{\mathrm{A}}(x)$ for $n \geqslant 0$ and $P_{n,n}(x) = xP_n^{\mathrm{A}}(x)$ for $n \geqslant 1$.

Proof. It is easy to verify that

$$\sum_{k=0}^{n}(k + 1)B(n - 1, k, j)x^k = P_{n-1,j}(x) + xP'_{n-1,j}(x),$$

$$\sum_{k=0}^{n}(n - k)B(n - 1, k - 1, j)x^k = nxP_{n-1,j}(x) - xP_{n-1,j}(x) - x^2 P'_{n-1,j}(x),$$

$$\sum_{k=0}^{n} kB(n - 1, k, j - 1)x^k = xP'_{n-1,j-1}(x),$$

and

$$\sum_{k=0}^{n}(n - k + 1)B(n - 1, k - 1, j - 1)x^k = nxP_{n-1,j-1}(x) - x^2 P'_{n-1,j-1}(x).$$

Summing up and applying (12) we obtain (21). □

Brändén [2], Corollary 6.9, proved that for every nonempty subset $S \subseteq \{1, \ldots, n\}$ the polynomial $\sum_{j \in S} P_{n,j}(x)$ has only real and simple roots. Combining (47) with Example 7.8.8 in [3] we will note (Theorem 20) that in fact every linear combination $c_0 P_{n,0}(x) + c_1 P_{n,1}(x) + \ldots + c_n P_{n,n}(x)$, with $c_0, c_1, \ldots, c_n \geqslant 0$, has only real roots. The cases when S is the set of even or odd numbers in $\{1, \ldots, n\}$ were studied in [1]. The Newton's inequality implies that if $0 \leqslant j \leqslant n$ then the sequence $\{B(n,k,j)\}_{k=0}^{n}$ satisfies a stronger version of log-concavity, namely

$$B(n,k,j)^2 \geqslant B(n,k-1,j)B(n,k+1,j)\frac{(k+1)(n-k+1)}{k(n-k)} \tag{22}$$

for $0 < k < n$, in particular this sequence is unimodal.

For the polynomials $Q_{n,k}(y)$ we have the following, see (18) in [4]:

Proposition 5. *The polynomials $Q_{n,k}(y)$ satisfy the following recurrence:*

$$Q_{n,k}(y) = (k+1+ky)Q_{n-1,k}(y) + (n-k+(n-k+1)y)Q_{n-1,k-1}(y)$$

with the initial conditions: $Q_{n,0}(y) = 1$, $Q_{n,n}(y) = y^n$ for $n \geqslant 0$.

The polynomials $Q_{n,k}$ however do not have all roots real. They satisfy the following versions of Worpitzky identity:

$$\sum_{k=0}^{n} \binom{u+n-k}{n} Q_{n,k}(y) = (u+1+uy)^n, \tag{23}$$

$$\sum_{k=0}^{n} \binom{u+k}{n} Q_{n,k}(y) = (u+y+uy)^n. \tag{24}$$

The former is proved in [4], Theorem 3.4, the latter follows from the former and the symmetry (19).

Now we recall the recurrence relation for $R_n(x,y)$ (see Theorem 3.4 in [4]):

Proposition 6. *The polynomials $R_n(x,y)$ admit the following recurrence:*

$$R_n(x,y) = (1+nxy+nx-x)R_{n-1}(x,y) + (x-x^2)(1+y)\frac{\partial}{\partial x}R_{n-1}(x,y),$$

$n \geqslant 1$, with initial condition $R_0(x,y) = 1$.

Brenti [4] also found the generating function for the numbers $B(n,k,j)$:

$$f(x,y,z) := \sum_{n=0}^{\infty} \frac{R_n(x,y)}{n!} z^n = \frac{(1-x)e^{(1-x)z}}{1-xe^{(1-x)(1+y)z}}. \tag{25}$$

Note that

$$f(x,y,z) = f^{\mathrm{A}}\big(x,(1+y)z\big)e^{(x-1)yz}. \tag{26}$$

4 Refined numbers

For $0 \leqslant k \leqslant n$ and a subset $U \subseteq \{1, 2, \ldots, n\}$ we define $\mathcal{B}_{n,k,U}$ as the set of those $\sigma \in \mathcal{B}_{n,k}$ which have minus sign at σ_i, $1 \leqslant i \leqslant n$, if and only if $|\sigma_i| \in U$. Therefore we have

$$\dot{\bigcup_{\substack{U \subseteq \{1,\ldots,n\} \\ |U|=j}}} \mathcal{B}_{n,k,U} = \mathcal{B}_{n,k,j}. \tag{27}$$

The cardinality of $\mathcal{B}_{n,k,U}$ will be denoted $b(n, k, U)$. By convention we put $b(n, -1, U) = b(n, n+1, U) := 0$. It is quite easy to observe boundary conditions.

Proposition 7. *For $n \geqslant 1$, $0 \leqslant k \leqslant n$, $U \subseteq \{1, \ldots, n\}$ we have*

$$b(n, 0, U) = [U = \emptyset], \qquad b(n, n, U) = [U = \{1, \ldots, n\}],$$
$$b(n, k, \emptyset) = A(n, k), \qquad b(n, k, \{1, \ldots, n\}) = A(n, n-k).$$

Now we provide a recurrence relation.

Proposition 8. *For $0 \leqslant k \leqslant n$, $U \subseteq \{1, 2, \ldots, n\}$ we have*

$$b(n, k, U) = (k+1) \cdot b(n-1, k, U) + (n-k) \cdot b(n-1, k-1, U) \tag{28}$$

if $n \notin U$ and

$$b(n, k, U) = k \cdot b(n-1, k, U') + (n-k+1) \cdot b(n-1, k-1, U') \tag{29}$$

if $n \in U$, where $U' := U \setminus \{n\}$.

Proof. Both formulas are true when $k = 0$ or $k = n$. Assume that $0 < k < n$. We will apply the same map $\Lambda : \mathcal{B}_n \to \mathcal{B}_{n-1}$ as in the proof of Theorem 2.1. Fix $\sigma \in \mathcal{B}_{n,k,U}$ and assume that i is such that $\sigma_i = n$ (when $n \notin U$) or $\sigma_i = -n$ (when $n \in U$), $1 \leqslant i \leqslant n$. We have now four possibilities:

- $n \notin U$ and either $i = n$ or $\sigma_{i-1} > \sigma_{i+1}$, $1 \leqslant i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k,U}$.
- $n \notin U$ and $\sigma_{i-1} < \sigma_{i+1}$, $1 \leqslant i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k-1,U}$.
- $n \in U$ and $\sigma_{i-1} > \sigma_{i+1}$, $1 \leqslant i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k,U\setminus\{n\}}$.
- $n \in U$ and either $i = n$ or $\sigma_{i-1} < \sigma_{i+1}$, $1 \leqslant i < n$. Then $\Lambda\sigma \in \mathcal{B}_{n-1,k-1,U\setminus\{n\}}$.

On the other hand, as in the proof of Theorem 3, we see that for a given τ in $\mathcal{B}_{n-1,k,U}$ (resp. in $\mathcal{B}_{n-1,k-1,U}$) there are $k+1$ (resp. $n-k$) such σ's in $\mathcal{B}_{n,k,U}$ that $\Lambda\sigma = \tau$. We simply insert n into a descent or at the end of τ (resp. into an ascent). Similarly, for a given τ in $\mathcal{B}_{n-1,k,V}$ (resp. in $\mathcal{B}_{n-1,k-1,V}$) there are k (resp. $n-k+1$) such σ's in $\mathcal{B}_{n,k,V\cup\{n\}}$ that $\Lambda\sigma = \tau$. □

Now we will see that $b(n, k, U)$ depends only on n, k and the cardinality of U.

Theorem 9. *If* $0 \leqslant k \leqslant n$, $U, V \subseteq \{1, \ldots, n\}$ *and* $|U| = |V|$ *then*

$$b(n, k, U) = b(n, k, V).$$

Proof. Fix $U, V \subseteq \{1, \ldots, n\}$, with $|U| = |V|$ and define $\tau \in \mathcal{A}_n$ as the unique permutation of $\{1, \ldots, n\}$ such that: $\tau(U) = V$, $\tau|_U$ preserves the order and $\tau|_{\{1,\ldots,n\}\setminus U}$ preserves the order. We extend τ to an element of $\mathcal{B}_n$ by putting $\tau(-i) = -\tau(i)$. Now let $\sigma \in \mathcal{B}_{n,k,U}$. Then, by definition, $\tau(\sigma(i)) < 0$ if and only if $\sigma(i) < 0$, $-n \leqslant i \leqslant n$. Moreover, if $1 \leqslant i \leqslant n$ then $\tau(\sigma(i-1)) < \tau(\sigma(i))$ if and only if $\sigma(i-1) < \sigma(i)$. This is clear when $\sigma(i-1)$ and $\sigma(i)$ have different signs. If they have the same sign then this is a consequence of the order preserving property of τ on U and on $\{1, \ldots, n\} \setminus U$. Consequently, the map $\sigma \mapsto \tau \circ \sigma$ is a bijection of $\mathcal{B}_{n,k,U}$ onto $\mathcal{B}_{n,k,V}$. ☐

The theorem justifies the following definition: for $0 \leqslant j, k \leqslant n$ we put

$$b(n, k, j) := b(n, k, U),$$

where U is an arbitrary subset of $\{1, \ldots, n\}$ with $|U| = j$. In addition, if $j < 0$ or $k < 0$ or $n < j$ or $n < k$ then we put $b(n, k, j) = 0$. From (27) we obtain

Corollary 10. *For* $0 \leqslant j, k \leqslant n$ *we have*

$$\binom{n}{j} b(n, k, j) = B(n, k, j). \tag{30}$$

5 Connections with permutations of type A

For given $n \geqslant 0$ we define a map $F_n : \mathcal{A}_{n+1} \to \mathcal{B}_n$ in the following way: $F_n(\sigma) = \widetilde{\sigma}$, where for $1 \leqslant i \leqslant n$ we put

$$\widetilde{\sigma}_i := \begin{cases} \sigma_{i+1} - \sigma_1 & \text{if } \sigma_{i+1} < \sigma_1, \\ \sigma_{i+1} - 1 & \text{if } \sigma_{i+1} > \sigma_1, \end{cases} \tag{31}$$

$\widetilde{\sigma}_{-i} := -\widetilde{\sigma}_i$ and $\widetilde{\sigma}_0 := 0$. Note that $\widetilde{\sigma}_{i-1} > \widetilde{\sigma}_i$ if and only if $\sigma_i > \sigma_{i+1}$ for $1 \leqslant i \leqslant n$, so the number of descents in $(0, \widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_n)$ is the same as in $(\sigma_1, \ldots, \sigma_{n+1})$. It is easy to see that F_n is one-to-one. Its image is the set of such $\tau \in \mathcal{B}_n$ which satisfy the following property: if $1 \leqslant i_1, i_2 \leqslant n$, $|\tau_{i_1}| < |\tau_{i_2}|$, $\tau_{i_2} < 0$ then $\tau_{i_1} < 0$. Denote

$$\mathcal{A}_{n,k,j} := \{\sigma \in \mathcal{A}_{n,k} : \sigma_1 = j\}.$$

The cardinalities of these sets were studied by Conger [5], who denoted $\left\langle {n \atop k} \right\rangle_j := |\mathcal{A}_{n,k,j}|$.

From our remarks we have

Theorem 11. *For* $0 \leqslant j, k \leqslant n$ *the function* F_n *maps* $\mathcal{A}_{n+1,k}$ *into* $\mathcal{B}_{n,k}$ *and is a bijection from* $\mathcal{A}_{n+1,k,j+1}$ *onto* $\mathcal{B}_{n,k,\{1,\ldots,j\}}$. *Consequently,*

$$b(n, k, j) = |\mathcal{A}_{n+1,k,j+1}|. \tag{32}$$

In the rest of this section we briefly collect some properties of the numbers $b(n,k,j) = \left\langle {n+1 \atop k} \right\rangle_{j+1}$, most of them are immediate consequences of the results of Conger [5, 6].

Proposition 12. *If* $0 \leqslant k, j \leqslant n$ *then*

$$b(n,0,j) = [j = 0], \tag{33}$$

$$b(n,n,j) = [j = n], \tag{34}$$

$$b(n,k,0) = A(n,k), \tag{35}$$

$$b(n,k,n) = A(n,n-k), \tag{36}$$

$$b(n,k,j) = (k+1)b(n-1,k,j) + (n-k)b(n-1,k-1,j), \quad j < n, \tag{37}$$

$$b(n,k,j) = kb(n-1,k,j-1) + (n-k+1)b(n-1,k-1,j-1), \quad j > 0, \tag{38}$$

$$b(n,k,j) = b(n,n-k,n-j). \tag{39}$$

Proof. These formulas are consequences of Proposition 7, Proposition 8, (7) and (30) (see formulas (3) and (8) in [5]). Note that (38) is absent in [5]. □

Applying (37), with $j-1$ instead of j, and (38) we obtain (see (10) in [5])

Corollary 13. *For* $1 \leqslant j, k \leqslant n$

$$b(n,k,j-1) - b(n,k,j) = b(n-1,k,j-1) - b(n-1,k-1,j-1). \tag{40}$$

Below we present tables for the numbers $b(n,k,j)$ for $n = 0,1,2,3,4,5,6$ (they also appear in Appendix A of [6]):

$k \setminus j$	0
0	1

,

$k \setminus j$	0	1
0	1	0
1	0	1

,

$k \setminus j$	0	1	2
0	1	0	0
1	1	2	1
2	0	0	1

,

$k \setminus j$	0	1	2	3
0	1	0	0	0
1	4	4	2	1
2	1	2	4	4
3	0	0	0	1

,

$k \setminus j$	0	1	2	3	4
0	1	0	0	0	0
1	11	8	4	2	1
2	11	14	16	14	11
3	1	2	4	8	11
4	0	0	0	0	1

,

$k \setminus j$	0	1	2	3	4	5
0	1	0	0	0	0	0
1	26	16	8	4	2	1
2	66	66	60	48	36	26
3	26	36	48	60	66	66
4	1	2	4	8	16	26
5	0	0	0	0	0	1

,

$k \setminus j$	0	1	2	3	4	5	6
0	1	0	0	0	0	0	0
1	57	32	16	8	4	2	1
2	302	262	212	160	116	82	57
3	302	342	372	384	372	342	302
4	57	82	116	160	212	262	302
5	1	2	4	8	16	32	57
6	0	0	0	0	0	0	1

.

From (30), (37) and (38) we can provide new recurrence formulas for the numbers $B(n,k,j)$:

Corollary 14. *For $0 \leqslant j,k \leqslant n$ we have*

$$B(n,k,j) = \frac{(k+1)n}{n-j} B(n-1,k,j) + \frac{(n-k)n}{n-j} B(n-1,k-1,j),$$

if $0 \leqslant j < n$ and

$$B(n,k,j) = \frac{kn}{j} B(n-1,k,j-1) + \frac{(n-k+1)n}{j} B(n-1,k-1,j-1),$$

if $0 < j \leqslant n$.

Now we introduce the following lexicographic order on the set $\{0,1,\dots,n\}^2$: $(k_1,j_1) \preceq (k_2,j_2)$ if and only if either $k_1 < k_2$ or $k_1 = k_2$, $j_1 \geqslant j_2$. This is a linear order, in which the successor of $(k,0)$, with $0 \leqslant k < n$, is $(k+1,n)$, and for $1 \leqslant j \leqslant n$ the successor of (k,j) is $(k,j-1)$. It turns out that for every $n \geqslant 1$ the array $\big(b(n,k,j)\big)_{k,j=0}^{n}$ is lexicographically unimodal, cf. Theorem 7 in [5].

Proposition 15. *For every $n \geqslant 1$ we have the following:*

a) If either $0 \leqslant k < n/2$, $1 \leqslant j \leqslant n$ or $k = n/2$, $n/2 < j \leqslant n$ then

$$b(n,k,j-1) \geqslant b(n,k,j).$$

This inequality is sharp unless either $k = 0$, $2 \leqslant j \leqslant n$ or n is odd, $k = (n-1)/2$, $j = 1$.

b) If either $1 \leqslant k \leqslant n/2$, $0 \leqslant j \leqslant n$ or n is odd, $k = (n+1)/2$, $(n+1)/2 \leqslant j \leqslant n$ then

$$b(n,k-1,j) \leqslant b(n,k,j)$$

and this inequality is sharp unless n is even, $k = n/2$, $j = 0$.

c) The array of numbers $b(n,k,j)$, $0 \leqslant j,k \leqslant n$, is unimodal with respect to the order "$\preceq$", with the maximal value $b(n,n/2,n/2)$ if n is even and

$$b(n,(n-1)/2,n) = b(n,(n+1)/2,0)$$

if n is odd.

Proof. First we note that (a) implies (c) as a consequence of the symmetry (39) and the equality

$$b(n,k-1,0) = A(n,k-1) = A(n,n-k) = b(n,k,n).$$

Similarly we get (b).

Now assume that the statement holds for $n-1$. If either $k < n/2$ or $k = n/2$, $n/2 < j$ then, due to (3), the right hand side of (40) is nonnegative which proves (a), (b) and consequently (c) for n. Moreover, it is positive unless $j = 1$, $n-1 = 2k$, as $A(2k,k-1) = A(2k,k)$. □

Now we note two summation formulas (see (4) and (5) in [5]).

Proposition 16. *For* $0 \leqslant j, k \leqslant n$ *we have*

$$\sum_{j=0}^{n} b(n,k,j) = A(n+1,k), \tag{41}$$

$$\sum_{k=0}^{n} b(n,k,j) = n!. \tag{42}$$

Proof. For (41) we apply (32) to the following decomposition:

$$\mathcal{A}_{n+1,k,1} \dot{\cup} \mathcal{A}_{n+1,k,2} \dot{\cup} \ldots \dot{\cup} \mathcal{A}_{n+1,k,n+1} = \mathcal{A}_{n+1,k}.$$

The latter identity is a consequence of (9) and (30). □

It turns out that (2) can be generalized to a formula which expresses the numbers $b(n,k,j)$, see Theorem 1 in [5].

Theorem 17. *For any* $0 \leqslant j, k \leqslant n$ *we have*

$$b(n,k,j) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n+1}{k-i} i^j (i+1)^{n-j}, \tag{43}$$

under convention that $0^0 = 1$.

Proof. It can be proved by induction by applying (2), (36) and (38). □

From (43) and (30) we can derive a formula for the numbers $B(n,k,j)$.

Corollary 18. *For any* $0 \leqslant j, k \leqslant n$ *we have*

$$B(n,k,j) = \binom{n}{j} \sum_{i=0}^{k} (-1)^{k-i} \binom{n+1}{k-i} i^j (i+1)^{n-j}, \tag{44}$$

under convention that $0^0 = 1$.

Now we can prove Worpitzky type formula:

Proposition 19. *For* $0 \leqslant j \leqslant n$ *we have*

$$\sum_{k=0}^{n} b(n,k,j) \binom{x+n-k}{n} = x^j (1+x)^{n-j}. \tag{45}$$

Proof. If $x \in \{0,1,\ldots,n\}$ then

$$\sum_{k=0}^{n} (-1)^{k-i} \binom{n+1}{k-i} \binom{x+n-k}{n} = [x=i]$$

(see (5.25) in [7]). Applying (43) we see that (45) holds for $x \in \{0,1,\ldots,n\}$ (see formula (4.18) in [6]). Since the left hand side is a polynomial of degree at most n, this implies that (45) is true for all $x \in \mathbb{R}$. □

6 Real rootedness

For $0 \leqslant j \leqslant n$ denote

$$p_{n,j}(x) := \sum_{k=0}^{n} b(n,k,j)x^k \tag{46}$$

so that

$$P_{n,j}(x) = \binom{n}{j} p_{n,j}(x). \tag{47}$$

By Proposition 4 we have the following recurrence:

$$\begin{aligned} p_{n,j}(x) = \frac{n-j}{n}(1+xn-x)p_{n-1,j}(x) + \frac{n-j}{n}(x-x^2)p'_{n-1,j}(x) \\ + jxp_{n-1,j-1}(x) + \frac{j}{n}(x-x^2)p'_{n-1,j-1}(x), \end{aligned} \tag{48}$$

with the initial conditions: $p_{n,0}(x) = P_n^{\mathrm{A}}(x)$ for $n \geqslant 0$ and $p_{n,n}(x) = xP_n^{\mathrm{A}}(x)$ for $n \geqslant 1$. By (32) the polynomial $p_{n,j}(x)$ coincides with $A_{n+1,j+1}(x)$ considered by Brändén [3], Example 7.8.8. He noted that

$$p_{n,j}(x) = \sum_{i=0}^{j-1} xp_{n-1,i}(x) + \sum_{i=j}^{n-1} p_{n-1,i}(x), \tag{49}$$

which is equivalent to

$$b(n,k,j) = \sum_{i=0}^{j-1} b(n-1,k-1,i) + \sum_{i=j}^{n-1} b(n-1,k,i) \tag{50}$$

(see (9) in [5]). Note that if $0 \leqslant j < n$ then $\deg p_{n,j}(x) = n-1$ and $\deg p_{n,n}(x) = n$. In fact, $p_{n,n}(x) = xp_{n,0}(x)$. Conger [5], Theorem 5, proved that all $p_{n,j}(x)$ have only real roots. It turns out that they admit a much stronger property.

Let $f, g \in \mathbb{R}[x]$ be real-rooted polynomials with positive leading coefficients. We say that f is an *interleaver* of g, which we denote $f \ll g$, if

$$\ldots \leqslant \alpha_2 \leqslant \beta_2 \leqslant \alpha_1 \leqslant \beta_1,$$

where $\{\alpha_i\}_{i=1}^m$, $\{\beta_i\}_{i=1}^n$ are the roots of f and g respectively. A sequence $\{f_i\}_{i=0}^n$ of real-rooted polynomials is called *interlacing* if $f_i \ll f_j$ whenever $0 \leqslant i < j \leqslant n$.

From [3], Example 7.8.8 and (47) we have the following property of the polynomials $p_{n,j}(x)$ and $P_{n,j}(x)$:

Theorem 20. *For every $n \geqslant 1$ the sequence $\{p_{n,j}(x)\}_{j=0}^n$ is interlacing. Consequently, for any $c_0, c_1, \ldots, c_n \geqslant 0$ the polynomial*

$$c_0p_{n,0}(x) + c_1p_{n,1}(x) + \ldots + c_np_{n,n}(x)$$

has only real roots.

The same statement holds for the polynomials $P_{n,j}(x)$.

Note that Theorem 20 generalizes Corollary 3.7 in [4] and Corollary 6.9 in [2].

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