

Continuants, run lengths, and Barry's modified Pascal triangle

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Abstract

We show that the n 'th diagonal sum of Barry's modified Pascal triangle can be described as the continuant of the run lengths of the binary representation of n . In other words, Stern's diatomic sequence appears in these diagonal sums. We also obtain an explicit description for the row sums.

Mathematics Subject Classifications: Primary: 05A10; Secondary: 05A19, 11A55, 11A63

1 Introduction

In 2006 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [9], sequence A119326, Paul Barry introduced a modified Pascal triangle, defined for integers k and n such that $0 \leq k \leq n$, as follows:

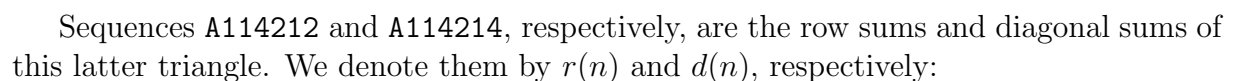
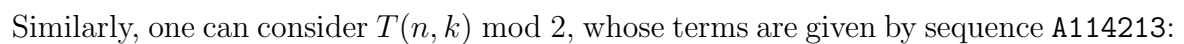
$$T(n, k) = \sum_{\substack{0 \leq j \leq n-k \\ 2|j}} \binom{k}{j} \binom{n-k}{j}.$$

The first few rows of this triangle are as follows:

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In May 2016, the first author observed, empirically, a connection between $d(n)$ and the binary representation of n . In this note we prove this connection, and also prove a formula for $r(n)$. The connection involves Stern’s “diatomic sequence” $s(n)$, defined by $s(0) = 0$, $s(1) = 1$, $s(2n) = s(n)$, and $s(2n + 1) = s(n) + s(n + 1)$; see [10].

2 The diagonal sums

Let the binary representation of n be denoted by $\sum_{i=0}^j \varepsilon_i(n)2^i$. We show that the diagonal sum $d(n)$ can be expressed in terms of this representation. Given a string s of 0's and 1's, we consider its *run lengths*: the lengths of maximal blocks of consecutive identical elements. For example, if $s = 111000011111$, then the run lengths of s are $(3, 4, 5)$.

If m is a sequence of positive integers, we may associate an integer with it via the continued fraction expansion: if $m = (m_0, \dots, m_k)$, we say that the *continuant* of m is the numerator of the continued fraction $[m_0; m_1, \dots, m_k]$ (see [4, Ch. 34, §4]).

Theorem 1. *Let $n \geq 0$ be an integer and let m be the sequence of run lengths of the binary representation of n . Then $d(n)$ equals the continuant of m .*

We will use Lucas' famous congruence for binomial coefficients [8, p. 230]: if p is a prime number and $n = (n_\nu \cdots n_0)_p$ and $k = (k_\nu \cdots k_0)_p$, then

$$\binom{n}{k} \equiv \binom{n_\nu}{k_\nu} \cdots \binom{n_0}{k_0} \pmod{p}.$$

This implies that $\binom{n}{k}$ is not divisible by p if and only if $k_i \leq n_i$ for all i .

We prove the following statement, which reduces the problem to divisibility by 2 of binomial coefficients. We will derive Theorem 1 from it in a moment.

Proposition 2. *Let n and k be nonnegative integers such that $k \leq n$. If $2 \mid n + k$, then $T(n, k) \equiv \binom{n}{k} \pmod{2}$. Otherwise, $T(n, k) \equiv \binom{n-1}{k} \pmod{2}$.*

Proof. We prove the first statement. By replacing n with $n + k$ we get the equivalent assertion that if $2 \mid n$ or $2 \mid k$, then

$$\sum_{\substack{0 \leq j \leq n \\ 2 \mid j}} \binom{n}{j} \binom{k}{j} \equiv \binom{n+k}{k} \pmod{2}. \quad (1)$$

By Lucas' congruence the left-hand side is congruent to

$$\sum_{j=0}^n \binom{n}{j} \binom{k}{j} \equiv \sum_{j=0}^n \binom{n \wedge k}{j} \equiv 2^{s_2(n \wedge k)} \pmod{2},$$

where $n \wedge k$ is the integer whose binary digits satisfy $\varepsilon_i(n \wedge k) = \min(\varepsilon_i(n), \varepsilon_i(k))$. This expression is odd if and only if $s_2(n \wedge k) = 0$, which is the case if and only if the binary representations of n and k are disjoint. To handle the right-hand side of Eq. (1), we note that $\binom{n+k}{k}$ is odd if $n \wedge k = 1$. On the other hand, if the binary representations of n and k are not disjoint, then the condition $\varepsilon_i(k) \leq \varepsilon_i(n+k)$ is violated for $i = \min\{j : \varepsilon_j(n) = 1, \varepsilon_j(k) = 1\}$; therefore $\binom{n+k}{k}$ is even. This proves the first assertion.

For the second assertion, we use Lucas' congruence again: for $2 \mid j$ and $2 \mid m$ we have $\binom{m}{j} \equiv \binom{m+1}{j} \pmod{2}$. Since $2 \nmid n - k$, we obtain $\binom{n-k}{j} \equiv \binom{n-1-k}{j} \pmod{2}$. Moreover, by $2 \nmid n - k$ the ranges of summation in $T(n, k)$ and $T(n-1, k)$ are the same. \square

From this proposition we obtain in particular the identity

$$d(2n) = d(2n + 1). \quad (2)$$

Carlitz [3] proved that Stern's diatomic sequence $s(n)$ satisfies $s(n + 1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} \pmod{2}$. By Proposition 2 and Eq. (2) we therefore have

$$d(2n) = d(2n + 1) = s(2n + 1). \quad (3)$$

Of course, by the recurrence for s this can be written as

$$d(n) = s(\lfloor n/2 \rfloor) + s(\lfloor n/2 \rfloor + 1).$$

It is well-known [6, 7] that if $m = (m_0, \dots, m_k)$ is the sequence of run-lengths of the binary representation of n and n is odd, then $s(n)$ is the continuant of m . Therefore $d(n)$ is the continuant of m . In order to complete the proof of the conjecture, we have to show that the same is true for even n . By Eq. (3) it is sufficient to prove the following lemma.

Lemma 3. *If n is even, then the continuant of the sequence of run-lengths of the binary representation of n is equal to the continuant corresponding to $n + 1$.*

Proof. Let $n = 1^{m_0}0^{m_1} \dots 1^{m_{k-1}}0^{m_k}$. We distinguish between two cases. If $m_k = 1$, then $n + 1 = (1^{m_0}0^{m_1} \dots 0^{m_{k-2}}1^{m_{k-1}+1})$ and the statement follows from the identity $[m_0; m_1, \dots, m_{k-1}, 1] = [m_0; m_1, \dots, m_{k-1} + 1]$. If $m_k \geq 2$, then $n + 1 = (1^{m_0}0^{m_1} \dots 0^{m_{k-2}}1^{m_{k-1}}0^{m_k-1}1)$ and the statement follows from $[m_0; m_1, \dots, m_k] = [m_0; m_1, \dots, m_{k-1}, m_k - 1, 1]$. \square

Remark. The sequence $(d(n))_{n \geq 0}$ is a 2-regular sequence [1], as it satisfies the equalities

$$\begin{aligned} d(2n + 1) &= d(2n) \\ d(4n + 2) &= 3d(2n) - d(4n) \\ d(8n) &= -d(2n) + 2d(4n) \\ d(8n + 4) &= 4d(2n) - d(4n). \end{aligned}$$

3 The row sums

In order to state our result, we define the *sum-of-digits function* s_2 in base 2: this function returns the number of 1s in the binary expansion of a nonnegative integer n . By Lucas' congruence, it follows that the number of odd binomial coefficients $\binom{n}{k}$ equals $2^{s_2(n)}$ [5].

We will prove an analogous result for $r(n)$.

Theorem 4.

$$r(n) = \begin{cases} 2^{s_2(n)}, & \text{if } n \text{ odd;} \\ 2^{s_2(n)} + 2^{s_2(n-2)}, & \text{if } n \text{ even.} \end{cases}$$

Again, this expression can be written in a unified way:

$$r(n) = 2^{s_2(\lfloor n/2 \rfloor)} + 2^{s_2(\lfloor (n-1)/2 \rfloor)}.$$

A similar characterization was stated, without proof or attribution, in the notes to A114212 of the OEIS.

Proof. From Proposition 2 we get, for integers $n \geq k \geq 0$, that

$$\begin{aligned} T(2n, 2k) &\equiv T(2n+1, 2k) \equiv T(2n+1, 2k+1) \equiv \binom{n}{k} \pmod{2}; \\ T(2n, 2k+1) &\equiv \binom{n-1}{k} \pmod{2}. \end{aligned}$$

Then

$$\begin{aligned} r(2m) &= \sum_{k=0}^{2m} (T(2m, k) \pmod{2}) \\ &= \sum_{k=0}^m (T(2m, 2k) \pmod{2}) + \sum_{k=0}^{m-1} (T(2m, 2k+1) \pmod{2}) \\ &= \sum_{k=0}^m \left(\binom{m}{k} \pmod{2} \right) + \sum_{k=0}^{m-1} \left(\binom{m-1}{k} \pmod{2} \right) \\ &= 2^{s_2(m)} + 2^{s_2(m-1)} \\ &= 2^{s_2(2m)} + 2^{s_2(2m-2)}. \end{aligned}$$

Similarly,

$$\begin{aligned} r(2m+1) &= \sum_{k=0}^{2m+1} (T(2m+1, k) \pmod{2}) \\ &= \sum_{k=0}^m (T(2m+1, 2k) \pmod{2}) + \sum_{k=0}^m (T(2m+1, 2k+1) \pmod{2}) \\ &= \sum_{k=0}^m \left(\binom{m}{k} \pmod{2} \right) + \sum_{k=0}^m \left(\binom{m}{k} \pmod{2} \right) \\ &= 2^{s_2(m)} + 2^{s_2(m)} \\ &= 2^{s_2(2m+1)}. \end{aligned}$$

□

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