Continuants, run lengths, and Barry's modified Pascal triangle

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Abstract

We show that the n'th diagonal sum of Barry's modified Pascal triangle can be described as the continuant of the run lengths of the binary representation of n. In other words, Stern's diatomic sequence appears in these diagonal sums. We also obtain an explicit description for the row sums.

Mathematics Subject Classifications: Primary: 05A10; Secondary: 05A19, 11A55, 11A63

1 Introduction

In 2006 in the *On-Line Encyclopedia of Integer Sequences* (OEIS) [9], sequence `A119326`, Paul Barry introduced a modified Pascal triangle, defined for integers k and n such that $0 \leqslant k \leqslant n$, as follows:

$$T(n,k) = \sum_{\substack{0 \leqslant j \leqslant n-k \\ 2 \mid j}} \binom{k}{j} \binom{n-k}{j}.$$

The first few rows of this triangle are as follows:

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$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 2 \quad 1 \quad 1 \\
1 \quad 1 \quad 4 \quad 4 \quad 1 \quad 1 \\
1 \quad 1 \quad 7 \quad 10 \quad 7 \quad 1 \quad 1 \\
1 \quad 1 \quad 11 \quad 19 \quad 19 \quad 11 \quad 1 \quad 1 \\
1 \quad 1 \quad 16 \quad 31 \quad 38 \quad 31 \quad 16 \quad 1 \quad 1 \\
1 \quad 1 \quad 22 \quad 46 \quad 66 \quad 66 \quad 46 \quad 22 \quad 1 \quad 1 \\
1 \quad 1 \quad 29 \quad 64 \quad 106 \quad 126 \quad 106 \quad 64 \quad 29 \quad 1 \quad 1
\end{array}$$

Similarly, one can consider $T(n, k) \bmod 2$, whose terms are given by sequence `A114213`:

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 0 \quad 1 \quad 1 \\
1 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \\
1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 1 \\
1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \\
1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 1 \quad 1
\end{array}$$

Sequences `A114212` and `A114214`, respectively, are the row sums and diagonal sums of this latter triangle. We denote them by $r(n)$ and $d(n)$, respectively:

$$r(n) = \sum_{k=0}^{n} (T(n, k) \bmod 2)$$

$$d(n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (T(n-k, k) \bmod 2)\,.$$

In May 2016, the first author observed, empirically, a connection between $d(n)$ and the binary representation of n. In this note we prove this connection, and also prove a formula for $r(n)$. The connection involves Stern's "diatomic sequence" $s(n)$, defined by $s(0) = 0$, $s(1) = 1$, $s(2n) = s(n)$, and $s(2n + 1) = s(n) + s(n + 1)$; see [10].

2 The diagonal sums

Let the binary representation of n be denoted by $\sum_{i=0}^{j} \varepsilon_i(n)2^i$. We show that the diagonal sum $d(n)$ can be expressed in terms of this representation. Given a string s of 0's and 1's, we consider its *run lengths*: the lengths of maximal blocks of consecutive identical elements. For example, if $s = 111000011111$, then the run lengths of s are $(3, 4, 5)$.

If m is a sequence of positive integers, we may associate an integer with it via the continued fraction expansion: if $m = (m_0, \ldots, m_k)$, we say that the *continuant* of m is the numerator of the continued fraction $[m_0; m_1, \ldots, m_k]$ (see [4, Ch. 34, §4]).

Theorem 1. *Let $n \geqslant 0$ be an integer and let m be the sequence of run lengths of the binary representation of n. Then $d(n)$ equals the continuant of m.*

We will use Lucas' famous congruence for binomial coefficients [8, p. 230]: if p is a prime number and $n = (n_\nu \cdots n_0)_p$ and $k = (k_\nu \cdots k_0)_p$, then

$$\binom{n}{k} \equiv \binom{n_\nu}{k_\nu} \cdots \binom{n_0}{k_0} \pmod{p}.$$

This implies that $\binom{n}{k}$ is not divisible by p if and only if $k_i \leqslant n_i$ for all i.

We prove the following statement, which reduces the problem to divisibility by 2 of binomial coefficients. We will derive Theorem 1 from it in a moment.

Proposition 2. *Let n and k be nonnegative integers such that $k \leqslant n$. If $2 \mid n + k$, then $T(n, k) \equiv \binom{n}{k} \pmod 2$. Otherwise, $T(n, k) \equiv \binom{n-1}{k} \pmod 2$.*

Proof. We prove the first statement. By replacing n with $n + k$ we get the equivalent assertion that if $2 \mid n$ or $2 \mid k$, then

$$\sum_{\substack{0 \leqslant j \leqslant n \\ 2 \mid j}} \binom{n}{j}\binom{k}{j} \equiv \binom{n+k}{k} \pmod 2. \tag{1}$$

By Lucas' congruence the left-hand side is congruent to

$$\sum_{j=0}^{n} \binom{n}{j}\binom{k}{j} \equiv \sum_{j=0}^{n} \binom{n \wedge k}{j} \equiv 2^{s_2(n \wedge k)} \pmod 2,$$

where $n \wedge k$ is the integer whose binary digits satisfy $\varepsilon_i(n \wedge k) = \min(\varepsilon_i(n), \varepsilon_i(k))$. This expression is odd if and only if $s_2(n \wedge k) = 0$, which is the case if and only if the binary representations of n and k are disjoint. To handle the right-hand side of Eq. (1), we note that $\binom{n+k}{k}$ is odd if $n \wedge k = 1$. On the other hand, if the binary representations of n and k are not disjoint, then the condition $\varepsilon_i(k) \leqslant \varepsilon_i(n + k)$ is violated for $i = \min\{j : \varepsilon_j(n) = 1, \varepsilon_j(k) = 1\}$; therefore $\binom{n+k}{k}$ is even. This proves the first assertion.

For the second assertion, we use Lucas' congruence again: for $2 \mid j$ and $2 \mid m$ we have $\binom{m}{j} \equiv \binom{m+1}{j} \pmod 2$. Since $2 \nmid n - k$, we obtain $\binom{n-k}{j} \equiv \binom{n-1-k}{j} \pmod 2$. Moreover, by $2 \nmid n - k$ the ranges of summation in $T(n, k)$ and $T(n - 1, k)$ are the same. □

From this proposition we obtain in particular the identity

$$d(2n) = d(2n+1). \tag{2}$$

Carlitz [3] proved that Stern's diatomic sequence $s(n)$ satisfies $s(n+1) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\binom{n-k}{k} \bmod 2\right)$. By Proposition 2 and Eq. (2) we therefore have

$$d(2n) = d(2n+1) = s(2n+1). \tag{3}$$

Of course, by the recurrence for s this can be written as

$$d(n) = s(\lfloor n/2 \rfloor) + s(\lfloor n/2 \rfloor + 1).$$

It is well-known [6, 7] that if $m = (m_0, \ldots, m_k)$ is the sequence of run-lengths of the binary representation of n and n is odd, then $s(n)$ is the continuant of m. Therefore $d(n)$ is the continuant of m. In order to complete the proof of the conjecture, we have to show that the same is true for even n. By Eq. (3) it is sufficient to prove the following lemma.

Lemma 3. *If n is even, then the continuant of the sequence of run-lengths of the binary representation of n is equal to the continuant corresponding to $n+1$.*

Proof. Let $n = \mathtt{1}^{m_0}\mathtt{0}^{m_1} \cdots \mathtt{1}^{m_{k-1}}\mathtt{0}^{m_k}$. We distinguish between two cases. If $m_k = 1$, then $n+1 = (\mathtt{1}^{m_0}\mathtt{0}^{m_1} \cdots \mathtt{0}^{m_{k-2}}\mathtt{1}^{m_{k-1}+1})$ and the statement follows from the identity $[m_0; m_1, \ldots, m_{k-1}, 1] = [m_0; m_1, \ldots, m_{k-1} + 1]$.
If $m_k \geqslant 2$, then $n+1 = (\mathtt{1}^{m_0}\mathtt{0}^{m_1} \cdots \mathtt{0}^{m_{k-2}}\mathtt{1}^{m_{k-1}}\mathtt{0}^{m_k - 1}\mathtt{1})$ and the statement follows from $[m_0; m_1, \ldots, m_k] = [m_0; m_1, \ldots, m_{k-1}, m_k - 1, 1]$. □

Remark. The sequence $(d(n))_{n \geqslant 0}$ is a 2-regular sequence [1], as it satisfies the equalities

$$\begin{aligned} d(2n+1) &= d(2n) \\ d(4n+2) &= 3d(2n) - d(4n) \\ d(8n) &= -d(2n) + 2d(4n) \\ d(8n+4) &= 4d(2n) - d(4n). \end{aligned}$$

3 The row sums

In order to state our result, we define the *sum-of-digits function* s_2 in base 2: this function returns the number of 1s in the binary expansion of a nonnegative integer n. By Lucas' congruence, it follows that the number of odd binomial coefficients $\binom{n}{k}$ equals $2^{s_2(n)}$ [5].

We will prove an analogous result for $r(n)$.

Theorem 4.

$$r(n) = \begin{cases} 2^{s_2(n)}, & \text{if } n \text{ odd;} \\ 2^{s_2(n)} + 2^{s_2(n-2)}, & \text{if } n \text{ even.} \end{cases}$$

Again, this expression can be written in a unified way:

$$r(n) = 2^{s_2(\lfloor n/2 \rfloor)} + 2^{s_2(\lfloor (n-1)/2 \rfloor)}.$$

A similar characterization was stated, without proof or attribution, in the notes to `A114212` of the OEIS.

Proof. From Proposition 2 we get, for integers $n \geqslant k \geqslant 0$, that

$$T(2n, 2k) \equiv T(2n+1, 2k) \equiv T(2n+1, 2k+1) \equiv \binom{n}{k} \pmod 2;$$
$$T(2n, 2k+1) \equiv \binom{n-1}{k} \pmod 2.$$

Then

$$\begin{aligned}
r(2m) &= \sum_{k=0}^{2m} (T(2m, k) \bmod 2) \\
&= \sum_{k=0}^{m} (T(2m, 2k) \bmod 2) \quad + \quad \sum_{k=0}^{m-1} (T(2m, 2k+1) \bmod 2) \\
&= \sum_{k=0}^{m} \left(\binom{m}{k} \bmod 2 \right) \quad + \quad \sum_{k=0}^{m-1} \left(\binom{m-1}{k} \bmod 2 \right) \\
&= 2^{s_2(m)} + 2^{s_2(m-1)} \\
&= 2^{s_2(2m)} + 2^{s_2(2m-2)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
r(2m+1) &= \sum_{k=0}^{2m+1} (T(2m+1, k) \bmod 2) \\
&= \sum_{k=0}^{m} (T(2m+1, 2k) \bmod 2) \quad + \quad \sum_{k=0}^{m} (T(2m+1, 2k+1) \bmod 2) \\
&= \sum_{k=0}^{m} \left(\binom{m}{k} \bmod 2 \right) \quad + \quad \sum_{k=0}^{m} \left(\binom{m}{k} \bmod 2 \right) \\
&= 2^{s_2(m)} + 2^{s_2(m)} \\
&= 2^{s_2(2m+1)}.
\end{aligned}$$

□

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