Decomposing graphs into a spanning tree, an even graph, and a star forest

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Abstract

We prove that every connected graph can be edge-decomposed into a spanning tree, an even graph, and a star forest.

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1 Introduction

All graphs in this paper are simple and finite. A decomposition of a graph G is a collection of edge-disjoint subgraphs whose union is G. A graph is called *even* if every vertex has even degree. It is easy to see that every graph can be decomposed into a forest and an even graph. In 1979, Malkevitch [9] studied cubic graphs which admit such a decomposition where the forest is a spanning tree. In this case it is equivalent to the existence of a spanning tree containing no vertices of degree 2. Such a spanning tree is called *homeomorphically irreducible*, or a *HIST*.

For general graphs, the existence of a HIST is much less restrictive than the existence of a decomposition into a spanning tree and an even graph. However, even the existence of a HIST is not guaranteed by large connectivity or regularity, as was shown by Albertson et al. [3]. Douglas [5] showed that it is NP-complete to decide whether a planar subcubic graph contains a HIST.

In cubic graphs, the removal of the edges of a spanning tree results in a collection of cycles and paths. Hoffmann-Ostenhof [4] (see also [6]) conjectured that the spanning tree can be chosen such that the collection of paths is a matching.

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Conjecture 1 (3-Decomposition Conjecture)**.** Every connected cubic graph can be decomposed into a spanning tree, a collection of cycles, and a matching.

Akbari, Jensen, and Siggers [2] showed that any cubic graph has a decomposition into a spanning forest, a collection of cycles, and a matching. Abdolhosseini et al. [1] verified the 3-Decomposition Conjecture for traceable cubic graphs and Ozeki and Ye [10] verified it for 3-connected planar cubic graphs. The latter was extended by Hoffmann-Ostenhof, Kaiser, and Ozeki [7] to all planar cubic graphs. The following theorem is the main result of our paper and is in some sense a generalization of the 3-Decomposition Conjecture to the class of all connected graphs.

Theorem 2. *Every connected graph can be decomposed into a spanning tree, an even subgraph, and a star forest.*

As a special case our result implies that every cubic graph has a decomposition into a spanning tree, a collection of cycles, and a collection of paths of length at most 2. This was previously shown by Li and Cui [8].

One might be tempted to think that every connected graph admits a decomposition into a spanning tree, an even graph, and a matching. However, this is easily seen to be false since the complete bipartite graph $K_{2,n}$ has no such decomposition. Such a decomposition is also not guaranteed if we restrict our attention to regular graphs.

Theorem 3. *For each $r \geqslant 4$, there exists an r-regular connected graph which has no decomposition into a spanning tree, an even graph, and a matching.*

Proof. Let $r \geqslant 4$ be given and let G be the graph obtained from K_{r+1} by subdividing each edge once. Let G' be a graph obtained from K_{r+1} by subdividing $r-2$ edges once and adding an edge between each pair of vertices of degree 2. For each vertex v of degree 2 in G, let G_v denote a copy of G'. Now let G'' be obtained from the disjoint union of G and all the graphs G_v by adding edges between v and the vertices of degree $r-1$ in G_v, for each vertex v of degree 2 in G. Note that G'' is r-regular and any decomposition of G'' into a spanning tree, an even graph, and a matching also induces such a decomposition of G. Clearly, the even graph cannot contain any edges of G, therefore this corresponds to a decomposition of G into a spanning tree and a matching. The graph G has $r(r+1)$ edges, and every spanning tree of G has $r + \frac{r(r+1)}{2}$ edges, thus the matching has to contain at least $\frac{r(r-1)}{2} \geqslant r+2$ edges. However, the size of a maximal matching in G is $r+1$, so G cannot be decomposed into a spanning tree and a matching. □

The construction in the proof above shows that for r-regular graphs the size of the stars in the forest in Theorem 2 grows at least linearly in r.

2 Proof of Theorem 2

Before we begin the proof of our main theorem, we introduce a few definitions.

Definition 4 (separating cycle)**.** A cycle C in a connected graph G is called **separating** if $G - E(C)$ is disconnected.

Note that in the literature a cycle is called separating in a graph if the removal of its vertex-set results in a disconnected graph, while for us the removal of its edge-set is relevant. In particular, every cycle containing a vertex of degree 2 is separating.

Definition 5 (fragile)**.** A graph G is called **fragile**, if G is connected and every cycle of G is separating.

Fragile graphs have also been investigated in the context of planar graphs by Hoffmann-Ostenhof et al. [7]. Their 2-Decomposition Conjecture, which is equivalent to Conjecture 1, states that every subcubic fragile graph can be decomposed into a spanning tree and a matching.
Note that if we remove an even subgraph H such that $G - E(H)$ is connected and if we choose such a subgraph of maximal size, then $G - E(H)$ is fragile. In particular, every connected graph decomposes into an even graph and a fragile graph. Therefore it is sufficient to prove that every fragile graph has a decomposition into a spanning tree and a star forest. For brevity, we introduce the following notation.

Definition 6 (starlit)**.** A spanning tree T of a graph G is called **starlit** if $G - E(T)$ is a star forest.

All we need to show is that every fragile graph contains a starlit spanning tree. We prove an even stronger result, where we prescribe that all edges at a specified vertex belong to the spanning tree in the decomposition.

Definition 7 (***v***-full)**.** A spanning tree T of a graph G is called ***v*-full** for some vertex v in G, if all edges incident with v in G are also in T.

We can now state the theorem we are going to prove.

Theorem 8. *If v is a vertex in a fragile graph G, then G has a starlit v-full spanning tree.*

As already discussed, Theorem 2 follows immediately from Theorem 8. We finish this section by proving Theorem 8.

Proof of Theorem 8. Let G be a counterexample of minimal size.

Claim 1: G is 2-connected.

Proof. Suppose the claim is false and u is a cutvertex in G. Let K be a component of $G - u$, let G_1 denote the subgraph of G induced by $V(K) \cup \{u\}$, and let G_2 denote the graph induced by the edges in $G - E(G_1)$. We can assume that $v \in V(G_1)$. Clearly G_1 and G_2 are fragile and contain fewer edges than G, so G_1 contains a starlit v-full spanning tree T_1, and G_2 contains a starlit u-full spanning tree T_2. Now the union of T_1 and T_2 is a starlit v-full spanning tree in G. □

Note that Claim 1 implies that the minimum degree of G is at least 2.

Claim 2: There are no adjacent vertices of degree 2 in G.

Proof. Suppose x and y are two adjacent vertices of degree 2 and let z denote the neighbour of y different from x. We may assume without loss of generality that $v \neq y$. The graph $G' = G - xy$ is fragile, so by minimality of G there exists a starlit v-full spanning tree T' of G'. If $v \neq x$, then $T = T'$ is also a starlit v-full spanning tree of G. If $v = x$, then we choose instead a starlit z-full spanning tree T'' of G'. Now $T'' + xy - yz$ is a starlit v-full spanning tree of G. □

Let H be the subgraph of G induced by the vertices of degree at least 3.

Claim 3: H contains no isolated vertices and no cycles of length 3.

Proof. Suppose u is an isolated vertex in H. That is, u is a vertex of degree at least 3 in G all of whose neighbours have degree 2.
First, suppose $u = v$. Let x be a neighbour of u, and y the neighbour of x different from u. By Claim 2, y has degree at least 3 and is therefore not adjacent to u. Let G' be the graph obtained from G by removing x and adding the edge uy. Since u has only one neighbour of degree greater than 2 in G', every cycle through u is still separating. Thus, G' is fragile and contains a starlit u-full spanning tree T'. Now $T = T' - uy + ux + xy$ is a starlit u-full spanning tree of G.
Thus we can assume $u \neq v$. The graph $G' = G - u$ is connected by Claim 1. Clearly G' is fragile and therefore contains a v-full starlit spanning tree T'. If v is a neighbour of u, then $T = T' + uv$ is a starlit v-full spanning tree of G. If v is not a neighbour of u, then adding an arbitrary edge incident with u to T' results in a starlit v-full spanning tree of G. This contradiction shows that the minimum degree of H is at least 1.
Finally, suppose H contains a cycle C of length 3. Since every vertex of C has degree at least 3, and since G is 2-connected, it is easy to see that C is not separating, which contradicts G being fragile. □

For $u \in V(H)$, we write $d_H(u)$ to denote the degree of u in H and $d_G(u)$ for its degree in G.

Claim 4: If u is a vertex in H different from v, then $d_H(u) \geqslant 2$.

Proof. Suppose u is a vertex of degree 1 in H, $u \neq v$, and x is the neighbour of u in H. Let $G' = G - u$. First, suppose that v is not a degree 2 vertex adjacent to u in G. By Claim 1, the graph G' is fragile, so it has a starlit v-full spanning tree T' by minimality of G. Now $T = T' + ux$ is a v-full starlit spanning tree of G. Thus, we can assume that v has degree 2 and is a neighbour of u in G. Let T'' be a starlit x-full spanning tree of G'. Clearly $T = T'' + uv$ is a v-full spanning tree of G. Since T'' is x-full, the spanning tree T is also starlit, contradicting our choice of G. □

Claim 4 implies that there exists a cycle in H. The following claim shows that there are at most two vertices in H which have degree less than 3 in H.

Claim 5: If $u \in V(H)$ and $d_H(u) = 2$, then either $u = v$ or $d_G(v) = 2$ and $uv \in E(G)$.

Proof. Suppose u is a vertex of degree 2 in H, $u \neq v$, and $d_G(v) \geqslant 3$ or $uv \notin E(G)$. Let x and y denote the neighbours of u in H. Note that all other neighbours of u in G have degree 2. Let G' be the graph obtained from $G-u$ by adding the edge xy. Claim 1 implies that G is connected and Claim 3 implies that G' has no multiple edges. For a cycle C' in G' containing xy, the corresponding cycle C in G, which is obtained from C' by replacing xy with the path xuy, is separating if and only if C' is separating. Thus, G' is fragile and contains a starlit v-full spanning tree T'. If $xy \in E(T')$, then $T = T' - xy + ux + uy$ is a starlit v-full spanning tree in G. Thus, we can assume $xy \notin E(T')$. Since $G' - E(T')$ is a star forest, at least one of x and y has degree 1 in $G' - E(T')$, say x. Now $T = T' + uy$ is a starlit v-full spanning tree in G. □

Let C be a cycle in H for which the component of $G - E(C)$ containing v has maximal size. Note that C is induced. Let K denote the component of $G-E(C)$ containing v.

Claim 6: $H \subseteq K \cup C$.

Proof. Since C is chordless, it suffices to show $V(H) \subseteq V(K \cup C)$. Suppose u is a vertex in H which is not in K or C. Let L denote the component of $G - E(C)$ containing u. There exists no cycle in $L \cap H$ since that cycle would contradict the choice of C. Claim 4 now implies that L contains a path P joining two vertices a and b on C such that all intermediate vertices are in $V(H) \setminus V(C)$. Let P_1 and P_2 be the two edge-disjoint subpaths of C joining a and b. We may assume that P_2 contains a vertex of K. Now the cycle formed by the union of P and P_1 contradicts the choice of C. □

Since G is fragile, the graph $G - E(C)$ is disconnected so there is a vertex u on C which is not in K. Since C is induced, the vertex u has exactly two neighbours on C. Claim 6 implies that all neighbours of u not on C have degree 2. Now $d_H(u) = 2$ and $u \neq v$. By Claim 5, we have $d_G(v) = 2$ and $uv \in E(G)$, which implies that u is in K, contradicting our choice of u. □

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