Decomposing graphs into a spanning tree, an even graph, and a star forest

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Submitted: Jun 29, 2018; Accepted: Feb 21, 2019; Published: Mar 8, 2019
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Abstract

We prove that every connected graph can be edge-decomposed into a spanning
tree, an even graph, and a star forest.

Mathematics Subject Classifications: 05C05, 05C38

1 Introduction

All graphs in this paper are simple and finite. A decomposition of a graph \( G \) is a collection
of edge-disjoint subgraphs whose union is \( G \). A graph is called even if every vertex
has even degree. It is easy to see that every graph can be decomposed into a forest
and an even graph. In 1979, Malkevitch [9] studied cubic graphs which admit such a
decomposition where the forest is a spanning tree. In this case it is equivalent to the
existence of a spanning tree containing no vertices of degree 2. Such a spanning tree is
called homeomorphically irreducible, or a HIST.

For general graphs, the existence of a HIST is much less restrictive than the existence
of a decomposition into a spanning tree and an even graph. However, even the existence of
a HIST is not guaranteed by large connectivity or regularity, as was shown by Albertson
et al. [3]. Douglas [5] showed that it is NP-complete to decide whether a planar subcubic
graph contains a HIST.

In cubic graphs, the removal of the edges of a spanning tree results in a collection of
cycles and paths. Hoffmann-Ostenhof [4] (see also [6]) conjectured that the spanning tree
can be chosen such that the collection of paths is a matching.

*Supported by the Danish Council for Independent Research, Natural Sciences, grant DFF-1323-00178,
AlgoDisc.
**Conjecture 1** (3-Decomposition Conjecture). Every connected cubic graph can be decomposed into a spanning tree, a collection of cycles, and a matching.

Akbari, Jensen, and Siggers [2] showed that any cubic graph has a decomposition into a spanning forest, a collection of cycles, and a matching. Abdolhosseini et al. [1] verified the 3-Decomposition Conjecture for traceable cubic graphs and Ozeki and Ye [10] verified it for 3-connected planar cubic graphs. The latter was extended by Hoffmann-Ostenhof, Kaiser, and Ozeki [7] to all planar cubic graphs. The following theorem is the main result of our paper and is in some sense a generalization of the 3-Decomposition Conjecture to the class of all connected graphs.

**Theorem 2.** Every connected graph can be decomposed into a spanning tree, an even subgraph, and a star forest.

As a special case our result implies that every cubic graph has a decomposition into a spanning tree, a collection of cycles, and a collection of paths of length at most 2. This was previously shown by Li and Cui [8].

One might be tempted to think that every connected graph admits a decomposition into a spanning tree, an even graph, and a matching. However, this is easily seen to be false since the complete bipartite graph $K_{2,n}$ has no such decomposition. Such a decomposition is also not guaranteed if we restrict our attention to regular graphs.

**Theorem 3.** For each $r \geq 4$, there exists an $r$-regular connected graph which has no decomposition into a spanning tree, an even graph, and a matching.

*Proof.* Let $r \geq 4$ be given and let $G$ be the graph obtained from $K_{r+1}$ by subdividing each edge once. Let $G'$ be a graph obtained from $K_{r+1}$ by subdividing $r - 2$ edges once and adding an edge between each pair of vertices of degree 2. For each vertex $v$ of degree 2 in $G$, let $G_v$ denote a copy of $G'$. Now let $G''$ be obtained from the disjoint union of $G$ and all the graphs $G_v$ by adding edges between $v$ and the vertices of degree $r - 1$ in $G_v$, for each vertex $v$ of degree 2 in $G$. Note that $G''$ is $r$-regular and any decomposition of $G''$ into a spanning tree, an even graph, and a matching also induces such a decomposition of $G$. Clearly, the even graph cannot contain any edges of $G$, therefore this corresponds to a decomposition of $G$ into a spanning tree and a matching. The graph $G$ has $r(r+1)$ edges, and every spanning tree of $G$ has $r + \frac{r(r+1)}{2}$ edges, thus the matching has to contain at least $\frac{r(r-1)}{2} \geq r + 2$ edges. However, the size of a maximal matching in $G$ is $r + 1$, so $G$ cannot be decomposed into a spanning tree and a matching. \qed

The construction in the proof above shows that for $r$-regular graphs the size of the stars in the forest in Theorem 2 grows at least linearly in $r$.

## 2 Proof of Theorem 2

Before we begin the proof of our main theorem, we introduce a few definitions.
Definition 4 (separating cycle). A cycle $C$ in a connected graph $G$ is called \textbf{separating} if $G - E(C)$ is disconnected.

Note that in the literature a cycle is called separating in a graph if the removal of its vertex-set results in a disconnected graph, while for us the removal of its edge-set is relevant. In particular, every cycle containing a vertex of degree 2 is separating.

Definition 5 (fragile). A graph $G$ is called \textbf{fragile}, if $G$ is connected and every cycle of $G$ is separating.

Fragile graphs have also been investigated in the context of planar graphs by Hoffmann-Ostenhof et al. [7]. Their 2-Decomposition Conjecture, which is equivalent to Conjecture 1, states that every subcubic fragile graph can be decomposed into a spanning tree and a matching. Note that if we remove an even subgraph $H$ such that $G - E(H)$ is connected and if we choose such a subgraph of maximal size, then $G - E(H)$ is fragile. In particular, every connected graph decomposes into an even graph and a fragile graph. Therefore it is sufficient to prove that every fragile graph has a decomposition into a spanning tree and a star forest. For brevity, we introduce the following notation.

Definition 6 (starlit). A spanning tree $T$ of a graph $G$ is called \textbf{starlit} if $G - E(T)$ is a star forest.

All we need to show is that every fragile graph contains a starlit spanning tree. We prove an even stronger result, where we prescribe that all edges at a specified vertex belong to the spanning tree in the decomposition.

Definition 7 ($v$-full). A spanning tree $T$ of a graph $G$ is called \textbf{$v$-full} for some vertex $v$ in $G$, if all edges incident with $v$ in $G$ are also in $T$.

We can now state the theorem we are going to prove.

\textbf{Theorem 8.} If $v$ is a vertex in a fragile graph $G$, then $G$ has a starlit $v$-full spanning tree.

As already discussed, Theorem 2 follows immediately from Theorem 8. We finish this section by proving Theorem 8.

\textbf{Proof of Theorem 8.} Let $G$ be a counterexample of minimal size.

\textbf{Claim 1:} $G$ is 2-connected.

\textit{Proof.} Suppose the claim is false and $u$ is a cutvertex in $G$. Let $K$ be a component of $G - u$, let $G_1$ denote the subgraph of $G$ induced by $V(K) \cup \{u\}$, and let $G_2$ denote the graph induced by the edges in $G - E(G_1)$. We can assume that $v \in V(G_1)$. Clearly $G_1$ and $G_2$ are fragile and contain fewer edges than $G$, so $G_1$ contains a starlit $v$-full spanning tree $T_1$, and $G_2$ contains a starlit $u$-full spanning tree $T_2$. Now the union of $T_1$ and $T_2$ is a starlit $v$-full spanning tree in $G$. \hfill $\square$
Note that Claim 1 implies that the minimum degree of $G$ is at least 2.

**Claim 2:** There are no adjacent vertices of degree 2 in $G$.

*Proof.* Suppose $x$ and $y$ are two adjacent vertices of degree 2 and let $z$ denote the neighbour of $y$ different from $x$. We may assume without loss of generality that $v \neq y$. The graph $G' = G - xy$ is fragile, so by minimality of $G$ there exists a starlit $v$-full spanning tree $T'$ of $G'$. If $v \neq x$, then $T = T'$ is also a starlit $v$-full spanning tree of $G$. If $v = x$, then we choose instead a starlit $z$-full spanning tree $T''$ of $G'$. Now $T'' + xy - yz$ is a starlit $v$-full spanning tree of $G$. \hfill $\Box$

Let $H$ be the subgraph of $G$ induced by the vertices of degree at least 3.

**Claim 3:** $H$ contains no isolated vertices and no cycles of length 3.

*Proof.* Suppose $u$ is an isolated vertex in $H$. That is, $u$ is a vertex of degree at least 3 in $G$ all of whose neighbours have degree 2.

First, suppose $u = v$. Let $x$ be a neighbour of $u$, and $y$ the neighbour of $x$ different from $u$. By Claim 2, $y$ has degree at least 3 and is therefore not adjacent to $u$. Let $G'$ be the graph obtained from $G$ by removing $x$ and adding the edge $uy$. Since $u$ has only one neighbour of degree greater than 2 in $G'$, every cycle through $u$ is still separating. Thus, $G'$ is fragile and contains a starlit $u$-full spanning tree $T'$. Now $T = T' - uy + ux + xy$ is a starlit $u$-full spanning tree of $G$.

Thus we can assume $u \neq v$. The graph $G' = G - u$ is connected by Claim 1. Clearly $G'$ is fragile and therefore contains a $v$-full starlit spanning tree $T'$. If $v$ is a neighbour of $u$, then $T = T' + uv$ is a starlit $v$-full spanning tree of $G$. If $v$ is not a neighbour of $u$, then adding an arbitrary edge incident with $u$ to $T'$ results in a starlit $v$-full spanning tree of $G$. This contradiction shows that the minimum degree of $H$ is at least 1.

Finally, suppose $H$ contains a cycle $C$ of length 3. Since every vertex of $C$ has degree at least 3, and since $G$ is 2-connected, it is easy to see that $C$ is not separating, which contradicts $G$ being fragile. \hfill $\Box$

For $u \in V(H)$, we write $d_H(u)$ to denote the degree of $u$ in $H$ and $d_G(u)$ for its degree in $G$.

**Claim 4:** If $u$ is a vertex in $H$ different from $v$, then $d_H(u) \geq 2$.

*Proof.* Suppose $u$ is a vertex of degree 1 in $H$, $u \neq v$, and $x$ is the neighbour of $u$ in $H$. Let $G' = G - u$. First, suppose that $v$ is not a degree 2 vertex adjacent to $u$ in $G$. By Claim 1, the graph $G'$ is fragile, so it has a starlit $v$-full spanning tree $T'$ by minimality of $G$. Now $T = T' + ux$ is a $v$-full starlit spanning tree of $G$. Thus, we can assume that $v$ has degree 2 and is a neighbour of $u$ in $G$. Let $T''$ be a starlit $x$-full spanning tree of $G'$. Clearly $T = T'' + uv$ is a $v$-full spanning tree of $G$. Since $T''$ is $x$-full, the spanning tree $T$ is also starlit, contradicting our choice of $G$. \hfill $\Box$
Claim 4 implies that there exists a cycle in $H$. The following claim shows that there are at most two vertices in $H$ which have degree less than 3 in $H$.

**Claim 5:** If $u \in V(H)$ and $d_H(u) = 2$, then either $u = v$ or $d_G(v) = 2$ and $uv \in E(G)$.

*Proof.* Suppose $u$ is a vertex of degree 2 in $H$, $u \neq v$, and $d_G(v) \geq 3$ or $uv \notin E(G)$. Let $x$ and $y$ denote the neighbours of $u$ in $H$. Note that all other neighbours of $u$ in $G$ have degree 2. Let $G'$ be the graph obtained from $G - u$ by adding the edge $xy$. Claim 1 implies that $G$ is connected and Claim 3 implies that $G'$ has no multiple edges. For a cycle $C'$ in $G'$ containing $xy$, the corresponding cycle $C$ in $G$, which is obtained from $C'$ by replacing $xy$ with the path $xuy$, is separating if and only if $C'$ is separating. Thus, $G'$ is fragile and contains a starlit $v$-full spanning tree $T'$. If $xy \in E(T')$, then $T = T' - xy + ux + uy$ is a starlit $v$-full spanning tree in $G$. Thus, we can assume $xy \notin E(T')$. Since $G' - E(T')$ is a star forest, at least one of $x$ and $y$ has degree 1 in $G' - E(T')$, say $x$. Now $T = T' + uy$ is a starlit $v$-full spanning tree in $G$. \hfill $\Box$

Let $C$ be a cycle in $H$ for which the component of $G - E(C)$ containing $v$ has maximal size. Note that $C$ is induced. Let $K$ denote the component of $G - E(C)$ containing $v$.

**Claim 6:** $H \subseteq \overline{K} \cup C$.

*Proof.* Since $C$ is chordless, it suffices to show $V(H) \subseteq V(\overline{K} \cup C)$. Suppose $u$ is a vertex in $H$ which is not in $K$ or $C$. Let $L$ denote the component of $G - E(C)$ containing $u$. There exists no cycle in $L \cap H$ since that cycle would contradict the choice of $C$. Claim 4 now implies that $L$ contains a path $P$ joining two vertices $a$ and $b$ on $C$ such that all intermediate vertices are in $V(H) \setminus V(C)$. Let $P_1$ and $P_2$ be the two edge-disjoint subpaths of $C$ joining $a$ and $b$. We may assume that $P_2$ contains a vertex of $K$. Now the cycle formed by the union of $P$ and $P_1$ contradicts the choice of $C$. \hfill $\Box$

Since $G$ is fragile, the graph $G - E(C)$ is disconnected so there is a vertex $u$ on $C$ which is not in $K$. Since $C$ is induced, the vertex $u$ has exactly two neighbours on $C$. Claim 6 implies that all neighbours of $u$ not on $C$ have degree 2. Now $d_H(u) = 2$ and $u \neq v$. By Claim 5, we have $d_G(v) = 2$ and $uv \in E(G)$, which implies that $u$ is in $K$, contradicting our choice of $u$. \hfill $\Box$

**References**


