On the Turán density of $\{1,3\}$-Hypergraphs

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Abstract

In this paper, we consider the Turán problems on $\{1,3\}$-hypergraphs. We prove that a $\{1,3\}$-hypergraph is degenerate if and only if it's $H_5^{\{1,3\}}$-colorable, where $H_5^{\{1,3\}}$ is a hypergraph with vertex set $V=[5]$ and edge set $E=\{\{2\},\{3\},\{1,2,4\},\{1,3,5\},\{1,4,5\}\}$. Using this result, we further prove that for any finite set R of distinct positive integers, except the case $R=\{1,2\}$, there always exist non-trivial degenerate R-graphs. We also compute the Turán densities of some small $\{1,3\}$-hypergraphs.

Mathematics Subject Classifications: 5D05, 05C65, 05D40

1 Background

Turán theory is an important and active area in the extremal combinatorics. In 1941, Turán [10] determined the graph with maximum number of edges among all simple graphs on n vertices that doesn't contain the complete graph K_ℓ as a sub-graph. The Turán density $\pi(H)$ of a graph H is the least number α so that any large graph with edge density $(\alpha+\epsilon)$ will always contain a sub-graph isomorphic to H, for any $\epsilon>0$. Erdős-Simonovits-Stone theorem [1, 2] determined the Turán densities of all non-bipartite graphs. (The Turán density of any bipartite graph is always 0. Those are called the *degenerate* graphs.)

Turán problems on uniform hypergraphs have been actively studied for many decades. However, on non-uniform hypergraphs, these problems are rarely considered. Johnston and Lu [4] established the framework of the Turán theory for non-uniform hypergraphs. A hypergraph $H=(V,E)$ consists of a vertex set V and an edge set $E\subseteq 2^V$. Here the edges of E could have different cardinalities. The set of all the cardinalities of edges in

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H is denoted by $R(H)$, the set of edge types. In this paper, we will fix a finite set R of positive integers and consider all simple hypergraphs H with $R(H) \subseteq R$, which are called R-hypergraphs (or R-graphs, for short). We say a hypergraph is simple if there is at most one edge connecting any collection of vertices. A general hypergraph allows every edge to be a multi-set of vertices.

For example, $\{2\}$-graphs are just graphs and $\{r\}$-graphs are just r-uniform hypergraphs. An R-graph H on n vertices is denoted as H_n^R. We denote H^r as the rth *level hypergraph* of H which consists of all edges of cardinality r of H. We denote K_n^R as the complete hypergraph on n vertices with edge set $\cup_{i \in R} \binom{[n]}{i}$. We say H' is a *sub-graph* of H, denoted by $H' \subseteq H$, if there exists a 1-1 map $f : V(H') \to V(H)$ so that $f(e) \in E(H)$ for any $e \in E(H')$. A necessary condition for $H' \subseteq H$ is $R(H') \subseteq R(H)$. A chain C^R is a special R-graph containing exactly one edge of each size such that any pair of these edges are comparable under inclusion relation.

To measure the edge density of a non-uniform hypergraph, we use the Lubell function, which is the expected number of edges in the hypergraph hit by a random full chain [4]. For a non-uniform hypergraph G on n vertices, the Lubell function of G is defined by

$$h_n(G) := \sum_{e \in E(G)} \frac{1}{\binom{n}{|e|}} = \sum_{r \in R(G)} \frac{|E(G^r)|}{\binom{n}{r}}.$$

Given a family of hypergraphs $\mathcal{H}$ with common set of edge types R, we say G is *$\mathcal{H}$-free* if G doesn't contain any member of $\mathcal{H}$ as a sub-graph. Let $\pi_n(\mathcal{H})$ be the maximum edge density of any $\mathcal{H}$-free R-graph on n vertices. The Turán density of $\mathcal{H}$ is defined to be:

$$\begin{aligned} \pi(\mathcal{H}) &= \lim_{n \to \infty} \pi_n(\mathcal{H}) \\ &= \lim_{n \to \infty} \max \left\{ h_n(G) : |v(G)| = n, G \subseteq K_n^R, \text{and } G \text{ is } \mathcal{H}\text{-free} \right\}. \end{aligned}$$

A hypergraph $G := G_n^R$ is *extremal* with respect to the family $\mathcal{H}$ if G is *$\mathcal{H}$-free* and $h_n(G)$ is maximized.

Lu and Johnston [4] proved that this limit always exists by a simple average argument of Katona-Nemetz-Simonovits theorem [6]. They completely classified the Turán densities of $\{1,2\}$-graphs.

Theorem 1 (Lu and Johnston [4])**.** *For any hypergraph H with $R(H) = \{1,2\}$, we have*

$$\pi(H) = \begin{cases} 2 - \frac{1}{\mathcal{X}(H^2)-1} & \text{if } H^2 \text{ is not bipartite;} \\ \frac{5}{4} & \text{if } H^2 \text{ is bipartite and } \min\{k : \overline{P}_{2k} \subseteq H\} = 1; \\ \frac{9}{8} & \text{if } H^2 \text{ is bipartite and } \min\{k : \overline{P}_{2k} \subseteq H\} \geqslant 2; \\ 1 & \text{if } H^2 \text{ is bipartite and } \overline{P}_{2k} \not\subseteq H \text{ for any } k \geqslant 1. \end{cases}$$

where $H^2 \in H$ is the graph with all edges of cardinality 2. $\overline{P}_{2k}$ is a closed path of length $2k$, and $\mathcal{X}(H^2)$ is the chromatic number of H^2.

It is trivial that $\pi(H) \leqslant |R(H)|$ and it is easy to see that $\pi(H) \geqslant |R(H)| - 1$, since we can take an $(|R(H)| - 1)$-complete hypergraph $K_n^{|R(H)|-1}$ without the appearance of H. We are interested in these R-graphs with the smallest Turán density.

Definition 1 (Degenerate hypergraphs). A hypergraph H is called *degenerate* if $\pi(H) = |R(H)| - 1$.

What do the degenerate R-graphs look like? For the special case $R = \{r\}$, Erdős [3] showed that an r-uniform hypergraph H is degenerate if and only if it is r-partite, that is, a sub-graph of a blow-up of a single edge of cardinality r. As a natural extension of a single edge, the chain C^R for any set R is degenerate. Thus every sub-graph of a blow-up of a chain is also degenerate. We say a degenerate R-graph is *trivial* if it is a sub-graph of a blow-up of the chain C^R. For $R = \{1, 2\}$, by Theorem 1 all degenerate $\{1, 2\}$-graphs are trivial. However, a nontrivial degenerate $\{2, 3\}$-graph is found in [4]. It indicates that this question is more intrigue for other R-graphs.

In this paper, we will give a necessary and sufficient condition for the degenerate $\{1, 3\}$-graphs. Given two graphs G and H, we say G is H-colorable if and only if there exists a hypergraph homomorphism f from G to H. (see Definition 3). And we have

Theorem 2. *A $\{1, 3\}$-hypergraph is degenerate if and only if it's $H_5^{\{1,3\}}$-colorable, where $H_5^{\{1,3\}}$ is a hypergraph with vertex set $V = [5]$ and edge set*

$$E = \{\{2\}, \{3\}, \{1, 2, 4\}, \{1, 3, 5\}, \{1, 4, 5\}\}.$$

Using this non-trivial degenerate $\{1, 3\}$-graph, we prove the following result.

Theorem 3. *Let R be a set of distinct positive integers with $|R| \geqslant 2$ and $R \neq \{1, 2\}$. Then a non-trivial degenerate R-graph always exists.*

We then continue to study the non-degenerate $\{1, 3\}$-graphs. Let $K_3^{\bullet\bullet}$ be a $\{1, 3\}$-graph with edges $\{\{1\}, \{2\}, \{1, 2, 3\}\}$, we have

Theorem 4. *For any $K_3^{\bullet\bullet}$-colorable $\{1, 3\}$-graph H, we have*

1. *if $K_3^{\bullet\bullet} \nsubseteq H$, H must be $H_5^{\{1,3\}}$-colorable, then $\pi(H) = 1$;*
2. *if $K_3^{\bullet\bullet} \subseteq H$, then $\pi(H) = \pi(K_3^{\bullet\bullet}) = 1 + \frac{\sqrt{3}}{18}$.*

A result following Theorem 4 indicates a break for the Turán density of $\{1, 3\}$-graphs:

Corollary 5. *Let α be a real value in $[1, \frac{4}{9} + \frac{\sqrt{3}}{3})$. For any $\{1, 3\}$-graph H with $\pi(H) \leqslant \alpha$, it must be the case that $\pi(H) = 1$.*

We also obtain the Turán densities of some 3-partite $\{1, 3\}$-graphs, the results are shown in Section 4.

The paper is organized as follows. In section 2 we introduce some notations and lemmas for non-uniform hypergraphs. In section 3 we will prove the Theorem 2, Theorem 4 and Corollary 5. In section 4, we determine Turán densities of some 3-partite $\{1, 3\}$-graphs. In section 5, we prove Theorem 3.

2 Notation and lemmas

In this section, we introduce some notations and lemmas for R-graphs and then for the $\{1,3\}$-graphs. We call an edge of cardinality i as an i-edge, for each $i \in R$. For convenience, we call a vertex that forms a 1-edge as "black vertex", otherwise, "white vertex". We use notations of form $H_n^{\bullet}$ to represent a hypergraph on n vertices that contains only one "black vertex", similarly, $H_n^{\bullet\bullet}$ represents a hypergraph on n vertices that contains two "black vertices", and so on. To simplify our notations for $\{1,3\}$-graphs, we use form of abc to denote the edge $\{a,b,c\}$.

For a fixed set $R = \{k_1, k_2, \ldots, k_r\}$, with $(k_1 < k_2 < \ldots < k_r)$, *R-flag* is an R-graph containing exactly one edge of each size. The *chain* C^R is the special R-flag with the edge set $E(C^R) = \{[k_1], [k_2], \ldots, [k_r]\}$, where $[k_i]$ is the set of all positive integers from 1 to k_i for each $i \in [r]$. For example, the chain $C^{\{1,3\}} = \{1, 123\}$. For any R-flag L, we have $\pi(L) = |R| - 1$ (see [4]). Thus the chain $C^{\{1,3\}}$ is a degenerate $\{1,3\}$-graph.

The following definitions and lemmas on non-uniform hypergraphs are generalized from uniform hypergraphs.

Definition 2 (Blow-up hypergraphs). [4] For any hypergraph H on n vertices and positive integers $s_1, s_2, \ldots, s_n$, the *blow-up* of H is a new hypergraph (V, E), denoted by $H_n(s_1, s_2, \ldots, s_n)$, satisfying

- $V := \bigsqcup_{i=1}^{n} V_i$, where $|V_i| = s_i$,
- $E := \bigcup_{F \in E(H)} \prod_{i \in F} V_i$.

When $s_1 = s_2 = \ldots = s_n = s$, we simply write it as $H(s)$.

The blow-up operation does not change the Turán density.

Theorem 6 (Blow-up Families). *[4] Let $\mathcal{H}$ be a finite family of hypergraphs and let $s \geqslant 2$. Then $\pi(\mathcal{H}(s)) = \pi(\mathcal{H})$.*

A direct corollary of Theorem 6 is the following result.

Theorem 7 (Squeeze Theorem). *[4] Let H be any hypergraph. If there exists a hypergraph H' and an integer $s \geqslant 2$ such that $H' \subseteq H \subseteq H'(s)$, then $\pi(H) = \pi(H')$.*

It is easy to generalize the concepts of homomorphisms and H-coloring to general R-graphs.

Definition 3. Given two R-graphs G and H, a hypergraph *homomorphism* is a vertex map $f : V(G) \to V(H)$ such that, if $\{v_1, \ldots, v_r\} \in E(G)$ then $\{f(v_1), \ldots, f(v_r)\} \in E(H)$, for all $r \in R$.

Definition 4. A hypergraph G is called *H-colorable* if and only if there exists a homomorphism from G to H.

Note that, if there exists a homomorphism from G to H, then G is isomorphic to a sub-graph of a blow-up of H. Thus we have:

Lemma 8. *If G is H-colorable, then $\pi(G) \leqslant \pi(H)$.*

2.1 R-graphs with loops, blow-up, and Lagrangian

A loop edge is a multiset of vertices. Sometimes we need to enlarge the concept of R-graphs to *R-graphs with loops*. For example, consider a $\{1,3\}$-graph H_1 with the edge set $\{x, xyy, yyy\}$. Here xyy is a loop edge with vertex x occurring once and vertex y twice. In general, a loop edge $e = x_1^{m_1} \cdots x_l^{m_l}$ consists of m_1 copies of vertex x_1, m_2 copies of vertex x_2, and so on. For a loop edge $e = x_1^{m_1} \cdots x_l^{m_l}$, the *cardinality* of e is $|e| = \sum_i m_i$. We also define a multinomial coefficient c_e to be

$$c_e := \binom{|e|}{m_1, m_2, \ldots, m_l} = \frac{|e|!}{m_1! m_2! \cdots m_l!}.$$

Definition 5. The *polynomial form* of an R-graph H with loops on n vertices, denoted by $\lambda(H, \vec{x})$ with $\vec{x} = (x_1, x_2, \ldots, x_n)$ is defined as

$$\lambda(H, \vec{x}) := \sum_{e \in E(H)} c_e \prod_{i \in e} x_i.$$

The *Lagrangian* of H, denoted by $\lambda(H)$, is the maximum value of the polynomial $\lambda(H, \vec{x})$ over the simplex $S_n = \{(x_1, x_2, \ldots, x_n) \in [0,1]^n : \sum_{i=1}^n x_i = 1\}$.

For any R-graph H (with possible loops), one can construct the family of H-colorable R-graph by blowing up H in a certain way. The Lagrangian of H is the maximum edge density of the H-colorable R-graphs that one can get in this way. This definition of Lagrangian is the same as the one in [5]; but differs from the classical Lagrangian for r-uniform hypergraphs such as in [8] by a constant multiplicative factor. This is not essential. This is a special case of more general Lagrangian of non-uniform hypergraphs introduced by Peng-Wu-Yao [9].

Construction A: Consider a $\{1,3\}$-graph (with loops) H_A on two vertices $\{x, y\}$ with edges $\{x, xyy, yyy\}$. The polynomial form of H_A is

$$\lambda(H_A, \vec{x}) = x_1 + 3x_1x_2^2 + x_2^3.$$

It can be shown that $\lambda(H, \vec{x})$ reaches the maximum $1 + \frac{\sqrt{3}}{18}$ over the simplex $S_2 = \{(x_1, x_2) \in [0,1]^2, x_1 + x_2 = 1\}$ at $x_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$. Thus, we have

$$\lambda(H_A) = 1 + \frac{\sqrt{3}}{18} \approx 1.096225.$$

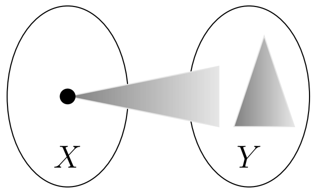
A $\{1,3\}$-graph G_A on n vertices is generated by blowing-up H_A as follows: set a vertex partition $V(G_A) = X \cup Y$ with $|X| \approx (\frac{1}{2} - \frac{\sqrt{3}}{6})n$ such that all 1-edges are in X (drawn by a black point), and all 3-edges are either formed by three vertices in Y or by one vertex in X plus two vertices in Y. In another words,

$$E(G_A) = \binom{X}{1} \cup \binom{X}{1} \times \binom{Y}{2} \cup \binom{Y}{3}.$$

We have

$$\begin{aligned} h_n(G_A) &= \frac{|X|}{n} + \frac{|X|\binom{|Y|}{2} + \binom{|Y|}{3}}{\binom{n}{3}} \\ &= \lambda(H_A, \vec{x}) + o_n(1). \\ &= \lambda(H_A) + o_n(1). \end{aligned}$$

Here $\vec{x} = (\frac{|X|}{n}, \frac{|Y|}{n}) = (\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6})$.



G_A: max $h_n(G_A) = 1 + \frac{\sqrt{3}}{18} + o_n(1)$, reached at $|X| = (\frac{1}{2} - \frac{\sqrt{3}}{6})n$.

Construction B: Let H_B be a general $\{1,3\}$-graph on three vertices $\{a, b, c\}$ with the edge set $\{a, b, abc\}$. We have

$$\lambda(H_B, \vec{x}) = x_1 + x_2 + 6x_1x_2x_3.$$

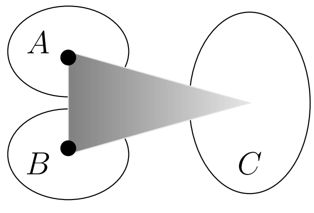
It is easy to check

$$\lambda(H_B) = \frac{4}{9} + \frac{\sqrt{3}}{3} \approx 1.021794714,$$

which is reached at $x_1 = x_2 = \frac{1+\sqrt{3}}{6}$, and $x_3 = \frac{2-\sqrt{3}}{3}$. A $\{1,3\}$-graph G_B on n vertices is generated by blowing-up H_B as follows: set a vertex partition $V(G_B) = A \cup B \cup C$. All 1-edges are in A and B(drawn by black points), all 3-edges are formed by exactly one vertex in each partition. We have

$$E(G_B) = \binom{A}{1} \cup \binom{B}{1} \cup \binom{A}{1} \times \binom{B}{1} \times \binom{C}{1}.$$

Note that G_B is H_B-colorable. Thus $h_n(G_B) = \lambda(H_B) + o_n(1)$.



G_B: max $h_n(G_B) = \frac{4}{9} + \frac{\sqrt{3}}{3} + o_n(1)$, reached at $|A| = |B| = \left(\frac{1+\sqrt{3}}{6}\right) n$.

2.2 Product of two R-graphs

Let's define the product of R-graphs (with loops):

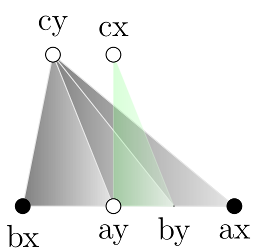
Definition 6. For any two general R-graphs H_1 and H_2 with vertices set V_1 and V_2 respectively, we define the product of H_1 and H_2, which is denoted by $H_1 \times H_2 = (V,\ E)$, where

$$V = V_1 \times V_2,\ E = \cup_{r \in R} E(H_1^r) \times E(H_2^r),$$

the $E(H_i^r)$ denotes the set of all edges of cardinality r in H_i for $i = 1, 2$. Here $E(H_1^r) \times E(H_2^r)$ consists of all products of $e \times_\sigma f$, where $\sigma = (\sigma(1), \ldots, \sigma(r))$ takes over all permutations of $[r]$. For example, given $e = \{v_1, \ldots, v_r\} \in E(H_1)$, $f = \{u_1, \ldots, u_r\} \in E(H_2)$, then $e \times_\sigma f = \{(v_1, u_{\sigma(1)}), (v_2, u_{\sigma(2)}), \ldots, (v_r, u_{\sigma(r)})\}$ is an edge in $E(H_1^r) \times E(H_2^r)$.

Example 1. The product of two $\{1, 3\}$-graphs H_A and H_B is given below. Let ax stand for (a, x), similar for other labels, then the vertex set is $V(H_A \times H_B) = \{ax, ay, bx, by, cx, cy\}$ and the edge set is

$$E(H_A \times H_B) = \{\{ax\}, \{bx\}, \{cy, bx, ay\}, \{cy, ay, by\}, \{cy, by, ax\}, \{cx, ay, by\}\}.$$



$H_A \times H_B$

Lemma 9. *For any two R-graphs H_1 and H_2, if hypergraph H is H_1 and H_2-colorable, then it's $(H_1 \times H_2)$-colorable.*

Proof. By definition, there exist two graph homomorphisms $f_1 : V(H) \mapsto V(H_1)$ and $f_2 : V(H) \mapsto V(H_2)$. Note that H could be an R-graph. Then for any $r \in R$, if edge $e = \{v_1, \ldots, v_r\} \in E(H)$, we have

$$f_1(e) = \{f_1(v_1), \ldots, f_1(v_r)\} \in E(H_1)$$

and

$$f_2(e) = \{f_2(v_1), \ldots, f_2(v_r)\} \in E(H_2).$$

Define a map $f := f_1 \times f_2$ from $V(H)$ to $V(H_1) \times V(H_2)$, such that $f(v) = (f_1(v), f_2(v))$. Then we have

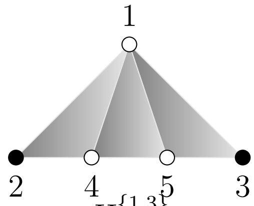
$$f(e) = \{(f_1(v_1), f_2(v_1)), \ldots, (f_1(v_r), f_2(v_r))\} \in f_1(e) \times f_2(e) \subseteq E(H_1 \times H_2).$$

Thus the map f takes edges in H to edges in $H_1 \times H_2$, it is a graph homomorphism. Therefore, H is $(H_1 \times H_2)$-colorable. □

3 Proof of Theorem 2 and Theorem 4

3.1 Proof of Theorem 2

For $\{1,3\}$-graph H, we have $1 \leqslant \pi(H) \leqslant 2$. Observe that the product of two general R-graphs could be an R-graph. This is very useful in determining the Turán density. A degenerate R-graph H must be G-colorable for any R-graph G with $\lambda(G) > 1$. By Lemma 9, it must be colorable by the product of these R-graphs. In this section, we will characterize the degenerate $\{1,3\}$-graph. Let's consider $H_5^{\{1,3\}}$, the 4-vertex $\{1,3\}$-graph with edge set $\{2, 3, 124, 135, 145\}$.



$H_5^{\{1,3\}}$

We first prove the following lemma.

Lemma 10. *Any degenerate* $\{1,3\}$*-graph is* $H_5^{\{1,3\}}$*-colorable.*

Proof. Observe that a degenerate $\{1,3\}$-graph H must be contained in G_A and G_B. Equivalently, H is both H_A and H_B-colorable, then it must be colorable by the product $H_A \times H_B$. We define a map $f : V(H_A \times H_B) \to [5]$ such that: $f(cx) = f(cy) = 1$, $f(bx) = 2$, $f(ax) = 3$, $f(ay) = 4$, $f(by) = 5$. Obviously, f is a graph homomorphism from $H_A \times H_B$ to $H_5^{\{1,3\}}$. The result follows. □

Let $K_3^{\bullet\bullet}$ be a $\{1,3\}$-graph on 3 vertices with edges $\{1, 2, 123\}$, and $G_4^{\bullet}$ be a $\{1,3\}$-graph on 4 vertices with edges $\{1, 123, 134, 234\}$.

Remark 11. $K_3^{\bullet\bullet}$ is not contained in G_A whose edge density reaches $1 + \frac{\sqrt{3}}{18}$, and $G_4^{\bullet}$ is not contained in G_B whose edge density reaches $\frac{4}{9} + \frac{\sqrt{3}}{3}$. Thus both $K_3^{\bullet\bullet}$ and $G_4^{\bullet}$ are non-degenerate $\{1,3\}$-graphs.

Lemma 12. $\pi(\{K_3^{\bullet\bullet}, G_4^{\bullet}\}) = 1$.

Proof. For any positive integer n, let G be a $\{K_3^{\bullet\bullet}, G_4^{\bullet}\}$-free $\{1,3\}$-graph on n vertices. Denote S as the set of all 1-edges of G, i.e. $S = \{v \in V(G) : \{v\} \in E(G)\}$, and let $|S| = xn$ for some $x \in (0,1)$. Let $\overline{S}$ be the complement of S, i.e. $\overline{S} = V(G) \setminus S$, then $|\overline{S}| = (1-x)n$.

Denote $E(G^3)$ as the set of all 3-edges of G. To forbidden $K_3^{\bullet\bullet}$, there is at most one black vertex in any 3-edges of G, thus we have

$$E(G^3) \subseteq \binom{S}{1} \times \binom{\overline{S}}{2} \cup \binom{\overline{S}}{3}.$$

We consider the 3-edges of G in edge set $\binom{S}{1} \times \binom{\overline{S}}{2}$. Define y as the average edge density of such 3-edges in G. Thus

$$y = \frac{|E(G^3) \cap (S \times \binom{\overline{S}}{2})|}{|S| \times \binom{|\overline{S}|}{2}}.$$

Note that there exists one vertex $s_0 \in S$ such that $|C(s_0)| \geqslant y \times \binom{|\overline{S}|}{2}$, where $C(s_0)$ is the set of 3-edges that contain the black vertex s_0. For any vertex $u \in \overline{S}$, define

$$W_u := \{v \in \overline{S} | s_0 uv \in E(G)\}.$$

We then have

$$\sum_{u \in \overline{S}} |W_u| \leqslant |\overline{S}| \times (|\overline{S}| - 1)$$

and

$$|C(s_0)| = \frac{1}{2} \sum_{u \in \overline{S}} |W_u|,$$

which implies

$$\sum_{u \in \overline{S}} |W_u| \geqslant 2y \times \binom{|\overline{S}|}{2}.$$

To forbidden $G_4^{\bullet}$, if $s_0uv, s_0uk \in E(G)$, then $uvk \notin E(G)$. Since for each $u \in \overline{S}$, there are $\binom{|W_u|}{2}$ pair of vertices each can form a 3-edge with u, we need to remove these edges in $\binom{\overline{S}}{3}$. Let N be the number of 3-edges in $\binom{\bar{S}}{3}$ but not in G, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} N &\geqslant \frac{1}{3} \sum_{u \in \overline{S}} \binom{|W_u|}{2} \\ &\geqslant \frac{1}{6} \frac{1}{|\overline{S}|} (\sum_{u \in \overline{S}} |W_u|)^2 - \frac{1}{6} \sum_{u \in \overline{S}} |W_u| \\ &\geqslant \frac{1}{6} y^2 |\overline{S}|^3 - \frac{1}{6} |\overline{S}| \times (|\overline{S}| - 1). \end{aligned} \tag{1}$$

Thus we have

$$h_n(G) \leqslant x + y \times 3x \times (1-x)^2 + (1-x)^3 - y^2(1-x)^3 + o_n(1).$$

When $x \leqslant \frac{2}{5}$, the above expression reaches the maximum value when $y = \frac{3}{2} \frac{x}{1-x} \leqslant 1$. When $x \geqslant \frac{2}{5}$, the above expression reaches the maximum value when $y = 1$. Thus we obtain

$$h_n(G) \leqslant \begin{cases} x + (1-x)^3 + \frac{9}{4} x^2 (1-x) & \text{for } x \leqslant \frac{2}{5}; \\ x + 3x(1-x)^2 & \text{for } x \geqslant \frac{2}{5}. \end{cases}$$

When $x \leqslant \frac{2}{5}$, by solving $f(x) = x + (1-x)^3 + \frac{9}{4}x^2(1-x) \leqslant 1$, we get $x \leqslant \frac{8}{13}$. This always holds since $x \leqslant \frac{2}{5} \leqslant \frac{8}{13}$. When $x \geqslant \frac{2}{5}$, $g(x) = x + 3x(1-x)^2 \leqslant 1$ is equivalent to $3x(1-x) \leqslant 1$ which is always true. Thus in both cases, we have $h_n(G) \leqslant 1 + o_n(1)$, implies that $\pi(\{K_3^{\bullet\bullet}, G_4^{\bullet}\}) = \lim_{n\to\infty} h_n(G) = 1$. The proof is complete. $\square$

Proof of Theorem 2. Note that $H_5^{\{1,3\}}$ is $K_3^{\bullet\bullet}$ and $G_4^{\bullet}$-colorable respectively, we have

$$\pi(H_5^{\{1,3\}}) \leqslant \pi(\{K_3^{\bullet\bullet}, G_4^{\bullet}\}).$$

By Lemma 10 and Lemma 12, the result follows. $\square$

3.2 Proof of Theorem 4

In this subsection, we will consider the non-degenerate $\{1,3\}$-graphs. In particularly, we consider the non-degenerate 3-partite $\{1,3\}$-graphs. A hypergraph is called 3-*partite* if its vertex set V can be partitioned into 3 different classes V_1, V_2, V_3 such that every edge intersects each class in exactly one vertex. A 3-partite $\{1,3\}$-graph is $K_3^{\bullet\bullet\bullet}$-colorable, where $K_3^{\bullet\bullet\bullet}$ is a $\{1,3\}$-graph on 3 vertices with edge set $\{1,2,3,123\}$.

So far we know that the chain $C^{\{1,3\}} = \{1,123\}$ is 3-partite and it is degenerate, while a slightly larger 3-partite $\{1,3\}$-graph $K_3^{\bullet\bullet} = \{1,2,123\}$ is not degenerate. We have $\pi(K_3^{\bullet\bullet}) \geqslant 1 + \frac{\sqrt{3}}{18}$ since it's not contained in the G_A. Now we are ready to prove Theorem 4 and Corollary 5.

Proof of Theorem 4. For Item 1: For graph H in the assumption, there exists a vertex partition $V(H) = V_1 \cup V_2 \cup V_3$ so that H is 3-partite and the level-graph H^1 only appears in at most two vertex partitions (say V_2 and V_3). Since H does not contain $K_3^{\bullet\bullet}$ as subgraph, then each edge of the level-graph H^3 can only intersect one vertex in V_1, plus one white (black) vertex in V_2 and one black (white) vertex in V_3, or intersect one vertex in V_1 plus two white vertices in V_2 and V_3. Let f be a map such that $f(v) = 1$ if $v \in V_1$, $f(v) = 2$ if v is a black vertex in V_2, $f(v) = 5$ if v is a white vertex in V_2, and $f(v) = 3$ if v is a black vertex in V_3, $f(v) = 4$ if v is a white vertex in V_3. One can check that f is a hypergraph homomorphism from H to $H_5^{\{1,3\}}$. Thus H is $H_5^{\{1,3\}}$-colorable, we have $\pi(H) = 1$.

For Item 2: Since $K_3^{\bullet\bullet} \subseteq H \subseteq K_3^{\bullet\bullet}(s)$ for $s \geqslant 2$, by Theorem 7, we have $\pi(H) = \pi(K_3^{\bullet\bullet})$. For any $K_3^{\bullet\bullet}$-free $\{1,3\}$-graph G on n vertices, let X be the set of all 1-edges in G, and $Y \subseteq V(G)$ be the complement of X. On one hand, since G is $K_3^{\bullet\bullet}$-free, there is no 3-edge of form $\binom{X}{3}$ or form $\binom{X}{2} \times \binom{Y}{1}$. Thus G is H_A-colorable. Therefore $\lim_{n\to\infty} h_n(G) \leqslant \lambda(H_A) = 1 + \frac{\sqrt{3}}{18}$. On the other hand, the construction G_A is $K_3^{\bullet\bullet}$-free. We have $\pi(K_3^{\bullet\bullet}) \geqslant \lambda(H_A) = 1 + \frac{\sqrt{3}}{18}$. Thus, $\pi(H) = \pi(K_3^{\bullet\bullet}) = 1 + \frac{\sqrt{3}}{18}$. $\square$

Proof of Corollary 5: Let H be any $\{1,3\}$-graph with $\pi(H) < \lambda(H_B) = \frac{4}{9} + \frac{\sqrt{3}}{3}$. Then H must be H_B-colorable, hence $K_3^{\bullet\bullet}$-colorable. By Item 2 of Theorem 4, $\pi(H)$ is either 1 or $1 + \frac{\sqrt{3}}{18}$, thus we must have $\pi(H) = 1$. $\square$

4 The 3-partite $\{1,3\}$-graphs

In previous section, all $\{1,3\}$-graphs we studied are 3-partite. In this section, we continue to study the Turán densities of 3-partite $\{1,3\}$-graphs.

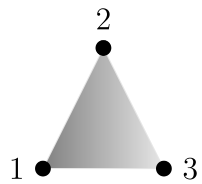
Lemma 13. *Let H be a 3-partite $\{1,3\}$-graph such that $K_3^{\bullet\bullet\bullet} \subseteq H$, where $K_3^{\bullet\bullet\bullet} = \{1,2,3,123\}$. Then $\pi(H) = 1 + \frac{2\sqrt{3}}{9}$.*

Proof. Since any 3-partite $\{1,3\}$-graph is $K_3^{\bullet\bullet\bullet}$-colorable, then $\pi(H) = \pi(K_3^{\bullet\bullet\bullet})$. We will prove $\pi(K_3^{\bullet\bullet\bullet}) = 1 + \frac{2\sqrt{3}}{9}$.

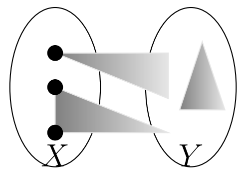
On one hand, consider an extremal $K_3^{\bullet\bullet\bullet}$-free $\{1,3\}$-graph G_n. Let X be the set of vertices of 1-edges in G_n. Projecting all the vertices in X into a single vertex x and all the vertices not in X into a single vertex y, we get an $\{1,3\}$-graph (with loops) H_C: where $E(H_C) = \{x, xxy, xyy, yyy\}$. This projection is a hypergraph homomorphism from G_n to H_C since G_n is $K_3^{\bullet\bullet\bullet}$-free. Thus G_n is H_C-colorable. In particular, we have

$$\pi(K_3^{\bullet\bullet\bullet}) = \lim_{n\to\infty} h_n(G_n) \leqslant \lambda(H_C) = 1 + \frac{2\sqrt{3}}{9}.$$

On the other hand, any blow-up of H_C does not contain the sub-graph $K_3^{\bullet\bullet\bullet}$. The following blow-up graph G_C has the maximal edge density $1 + \frac{2\sqrt{3}}{9}$:



$K_3^{\bullet\bullet\bullet}$



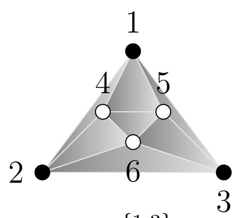
G_C: $h_n(G_C) = 1 + \frac{2\sqrt{3}}{9} + o_n(1)$, at $|X| = \frac{\sqrt{3}}{3}n$.

Hence, $\pi(H) = \pi(K_3^{\bullet\bullet\bullet}) = \lambda(H_C) = 1 + \frac{2\sqrt{3}}{9}$. □

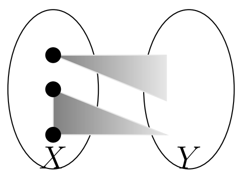
Let us restrict H to be 3-partite but containing no $K_3^{\bullet\bullet\bullet}$ as sub-graph. One can check such H must be $H_6^{\{1,3\}}$-colorable, where

$$H_6^{\{1,3\}} = \{1,2,3,124,145,135,236,246,356,456\}.$$

$H_6^{\{1,3\}}$ is not contained in G_D with maximal edge density $\frac{4}{3}$.

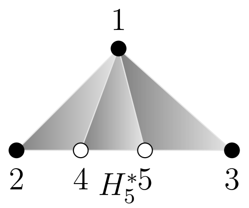


$H_6^{\{1,3\}}$

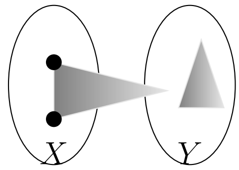


G_D: $h_n(G_D) = \frac{4}{3} + o_n(1)$, at $|X| = \frac{2}{3}n$.

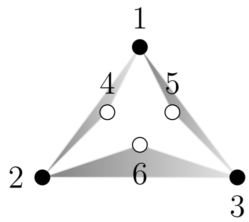
So far we couldn't determine the upper bound of $\pi(H_6^{\{1,3\}})$. We leave this open. Let's turn our attention to the sub-graphs of $H_6^{\{1,3\}}$ and we aim to determine their Turán densities. We first consider two sub-graphs of $H_6^{\{1,3\}}$: $H_5^* = \{1, 2, 3, 124, 145, 135\}$ and $H_6^* = \{1, 2, 3, 124, 135, 236\}$. For both of them, the Turán density is greater than $1 + \frac{\sqrt{3}}{9}$, since they are not contained in G_E and G_F respectively($\lim_{n\to\infty} h_n(G_E) = \lim_{n\to\infty} h_n(G_F)$).



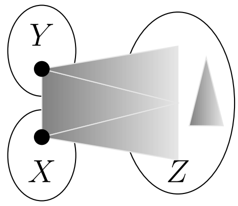
H_5^*



G_E: $h_n(G_E) = \frac{9+\sqrt{3}}{9} + o_n(1)$, at $|X| = (\frac{3+\sqrt{3}}{6})n$.



H_6^*



G_F: $h_n(G_F) = \frac{9+\sqrt{3}}{9} + o_n(1)$, at $|X| = |Y| = (\frac{3-\sqrt{3}}{6})n$.

To calculate the upper bounds of $\pi(H_5^*)$, we need the following lemma.

Lemma 14. *Let* $H_4^{\bullet\bullet} = \{1, 2, 123, 124, 134\}$*, then* $\pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet}\}) = 1 + \frac{\sqrt{3}}{9}$.

Proof. To see the lower bound, observe that both $H_4^{\bullet\bullet}$ and $K_3^{\bullet\bullet\bullet}$ are not contained in G_E.

To see the upper bound, let G represent a $\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet}\}$-free graph on n vertices, let $X \subseteq V(G)$ be the set of all 1-edges of G, $|X| = xn$ for some real $x \in (0, 1)$, and let $Y = V(G) \setminus X$, then $|Y| = (1 - x)n$. To forbidden $K_3^{\bullet\bullet\bullet}$, there is no 3-edge of form $\binom{X}{3}$. Let y be the density of 3-edges in G among all edges of form $\binom{X}{2} \times \binom{Y}{1}$. For any pair of vertices (i, j) in X, denote d_{ij} as the number of vertices $k \in Y$ so that $\{ijk\} \in E(G)$. Then

$$y = \frac{\sum_{(i,j)\in\binom{X}{2}} d_{ij}}{\binom{|X|}{2} \times \binom{|Y|}{1}}.$$

To forbidden $H_4^{\bullet\bullet}$, for each pair of vertices $(i, j) \in \binom{X}{2}$, if ijk and ijl are in $E(G)$, neither kli nor klj can be contained in $E(G)$. Thus for every pair of $\{i, j\}$, the number of 3-edges not shown in G is at least $2\binom{d_{ij}}{2}$. Let M be the total number of 3-edges of form $\binom{X}{1} \times \binom{Y}{2}$ not shown in G, then by Cauchy-Schwarz inequality, we have

$$M \geqslant \frac{\sum_{i,j\in\binom{X}{2}} 2\binom{d_{ij}}{2}}{|X|}$$

$$\geqslant \frac{\left(\sum_{i,j\in X} d_{ij}\right)^2}{\binom{xn}{2}(xn)} - \frac{\sum_{i,j\in X} d_{ij}}{xn}$$
$$\geqslant \frac{1}{2}y^2x(1-x)^2n^3 - \frac{1}{2}yx(1-x)n^2.$$

Thus

$$h_n(G) \leqslant x + (1-x)^3 + 3x^2(1-x)y + 3x(1-x)^2 - 3x(1-x)^2y^2 + o_n(1).$$

A simple calculation can show that $h_n(G)$ achieves maximum value at $y = 1$, which implies that for any positive integer n, any extremal $\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet}\}$-free $\{1,3\}$-graph is H_E-colorable where $E(H_E) = \{x, xxy, yyy\}$. Therefore, $\pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet}\}) = \lim_{n\to\infty} h_n(G_n) \leqslant \lambda(H_E) = 1 + \frac{\sqrt{3}}{9}$. The result follows. □

Lemma 15. $\pi(H_5^*) = 1 + \frac{\sqrt{3}}{9}$.

Proof. On one hand, H_5^* is not contained in G_E, then $\pi(H_5^*) \geqslant 1 + \frac{\sqrt{3}}{9}$. On the other hand, H_5^* is $K_3^{\bullet\bullet\bullet}$ and $H_4^{\bullet\bullet}$-colorable, thus $\pi(H_5^*) \leqslant \pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet}\}) \leqslant 1 + \frac{\sqrt{3}}{9}$. The result follows. □

Corollary 16. *The proper sub-graphs of H_5^* can be classified to two different sets: either the sub-graph contains $K_3^{\bullet\bullet}$ and is $K_3^{\bullet\bullet}$-colorable, in this case the Turán density is $1 + \frac{\sqrt{3}}{18}$; or the sub-graph does not contain $K_3^{\bullet\bullet}$, then it is $H_5^{\{1,3\}}$-colorable, in this case the Turán density is 1.*

To calculate the upper bounds of $\pi(H_6^*)$, we need the following lemma.

Lemma 17. *Let $H_4^{\bullet\bullet\bullet} = \{1, 2, 3, 124, 134, 234\}$, then $\pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet\bullet}\}) = 1 + \frac{\sqrt{3}}{9}$.*

Proof. To see the lower bound, observe that both $K_3^{\bullet\bullet\bullet}$ and $H_4^{\bullet\bullet\bullet}$ are not contained in G_F. To see the upper bound, let G represent a $\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet\bullet}\}$-free graph on n vertices, let $X \subseteq V(G)$ be the set of all 1-edges of G, $|X| = xn$ for some real $x \in (0,1)$, let $Y = V(G) \setminus X$, then $|Y| = (1-x)n$. To forbidden $K_3^{\bullet\bullet\bullet}$, there is no 3-edge of form $\binom{X}{3}$.

Let y be the density of 3-edges in G among all edges of form $\binom{X}{2} \times \binom{Y}{1}$. For each $i \in Y$, let $D_i = \{\{j,k\} \in \binom{X}{2} |\ ijk \in E(G)\}$, denote $d_i = |D_i|$. Then

$$y = \frac{\sum_{i\in Y} d_i}{\binom{|X|}{2} \times \binom{|Y|}{1}}.$$

Suppose $y > \frac{1}{2}$, then there exists $i \in Y$, such that $d_i > \frac{1}{2}\binom{|X|}{2}$. By the fact that the Turán density of a triangle-free graph is $\frac{1}{2}$, there must exist a triple $\{j,k,l\} \in \binom{X}{3}$ such that $\{ijk, ijl, ikl\} \subseteq E(G)$, which is a copy of $H_4^{\bullet\bullet\bullet}$, a contradiction. Thus we have $y \leqslant \frac{1}{2}$. Note that the existence of 3-edge of form $\binom{Y}{3}$ or $\binom{X}{1} \times \binom{Y}{2}$ does not result in an occurrence of $H_4^{\bullet\bullet\bullet}$ or $K_3^{\bullet\bullet\bullet}$ in G. Thus we can take all such edges. then

$$h_n(G) \leqslant x + (1-x)^3 + \frac{3}{2}x^2(1-x) + 3x(1-x)^2 + o_n(1),$$

which achieves the maximum $1+\frac{\sqrt{3}}{9}$ at $x=1-\frac{\sqrt{3}}{3}$.

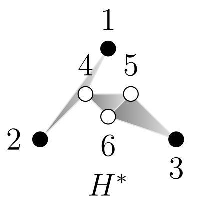
Hence, we have $\pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet\bullet}\}) = \lim_{n\to\infty} h_n(G_n) \leqslant 1+\frac{\sqrt{3}}{9}$. The result follows. □

Lemma 18. $\pi(H_6^*) = 1+\frac{\sqrt{3}}{9}$.

Proof. On one hand, H_6^* is not contained in G_F, then $\pi(H_6^*) \geqslant 1+\frac{\sqrt{3}}{9}$. On the other hand, H_6^* is $K_3^{\bullet\bullet\bullet}$ and $H_4^{\bullet\bullet\bullet}$-colorable, thus $\pi(H_6^*) \leqslant \pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet\bullet}\}) \leqslant 1+\frac{\sqrt{3}}{9}$. The result follows. □

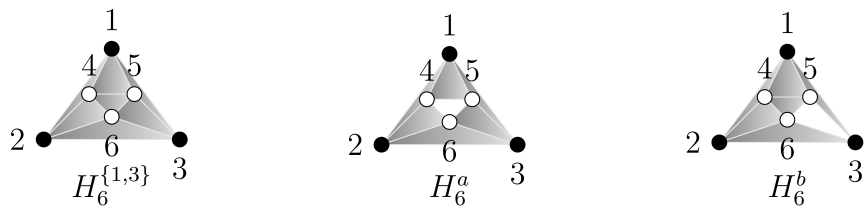
Since we are considering all sub-graphs of $H_6^{\{1,3\}}$, we start this by looking at the larger sub-graphs then the smaller sub-graphs. Using above two lemmas, we are able to determine the Turán density for a list of sub-graphs of $H_6^{\{1,3\}}$. Now we let H be a sub-graph of $H_6^{\{1,3\}}$, we have:

1. If H is $K_3^{\bullet\bullet}$-colorable, thus $\pi(H)=1$ or $\pi(H)=1+\frac{\sqrt{3}}{18}$.
2. If H is not above case, then H must contain all 1-edges: $1,2,3$, and none of them is isolated. Then we have several different cases:
   (a) Suppose H is obtained from $H_6^{\{1,3\}}$ by removing one 3-edge consisting of two black vertices and one white vertex (say 236 or equivalence), then one can check H is H_5^*-colorable. Note that $H_5^* \subseteq H$, by Lemmas 15, we have $\pi(H)=1+\frac{\sqrt{3}}{9}$. Similarly, for any sub-graph H' of H, if H' contains H_5^* or a H_5^*-colorable graph as sub-graph, then $\pi(H')=1+\frac{\sqrt{3}}{9}$. If H' is not above case, by trial and error, there is only one situation: H' contains the following sub-graph H^* (or its equivalence) which is not contained in G_E. Thus $\pi(H')=1+\frac{\sqrt{3}}{9}$:

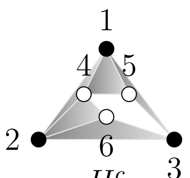


H^*

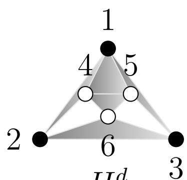
   (b) Let H be sub-graph of $H_6^{\{1,3\}}$ by removing edges 145, 246, 456 and 356, the resulting graph is H_6^*. By Lemma 18, $\pi(H)=\pi(H_6^*)=1+\frac{\sqrt{3}}{9}$.
3. The following graphs $H_6^{\{1,3\}}$, $H_6^a, H_6^b, , H_6^c, H_6^d$ and H_6^e are unsolved. We conjecture that the extremal configuration of $H_6^{\{1,3\}}$ and H_6^b is Construction G_D: thus we conjecture $\pi(H_6^{\{1,3\}}) = \pi(H_6^b) = \frac{4}{3}$.



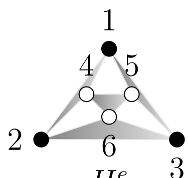
$H_6^{\{1,3\}}$ H_6^a H_6^b



H_6^c



H_6^d



H_6^e

5 Non-trivial degenerate $\boldsymbol{R}$-graphs

Recall that a degenerate R-graph H is trivial if it is contained in a blow-up of the chain C^R, otherwise, we say H is non-trivial. In this section, we will show that except the case $R = \{1, 2\}$, there always exist non-trivial degenerate R-graphs for any finite set R of at least two distinct positive integers. We will use the *suspension* operations on hypergraphs.

Definition 7. [4] The suspension of a hypergraph H, denoted by $S(H)$, is the hypergraph with $V = V(H) \cup \{v\}$ where $\{v\}$ is a new vertex not in $V(H)$, and the edge set $E = \{e \cup \{v\} : e \in E(H)\}$. We write $S^t(H)$ to denote the hypergraph obtained by iterating the suspension operation t-times, i.e. $S^2(H) = S(S(H))$ and $S^3(H) = S(S(S(H)))$, etc.

The relationship between $\pi(H)$ and $\pi(S(H))$ was investigated in [4].

Proposition 1. [4] For any family of hypergraphs $\mathcal{H}$ we have that $\pi(S(\mathcal{H})) \leqslant \pi(\mathcal{H})$.

Given a general set R and positive integer t, we denote $(R + t)$ as the set obtained from R by adding t to each element of R. Note that if the R-graph H is not contained in a blow-up of chain C^R, then $S^t(H)$ is not contained in a blow-up of the chain $C^{(R+t)}$. Thus we have the following fact:

Corollary 19. *Let H be a non-trivial degenerate R-graph, let t be any positive integer. Then the t-times suspension $S^t(H)$ is a non-trivial degenerate $(R + t)$-graph.*

Lemma 20. *Given a positive integer $t \geqslant 2$, and a $\{1, t\}$-graph H, let $T(H)$ be the $\{1, t + 1\}$-graph obtained from H by adding a new vertex $v \notin V(H)$ such that $V = V(H) \cup \{v\}$, $T(H)^1 = H^1$ and $T(H)^{1+t} = \{e \cup \{v\} : e \in E(H^t)\}$. Then we have $\pi(T(H)) \leqslant \pi(H)$.*

Proof. Let n be a positive integer and $G = (V, E)$ be an extremal $T(H)$-free $\{1, t + 1\}$-graph on n vertices. We have $\pi_n(T(H)) = h_n(G)$. Denote E_i as the set of i-edges of G, for $i = 1, t + 1$. For any vertex $v \in V(G)$, denote G_v as the hypergraph obtained from G with the vertex set $V(G_v) = V \setminus \{v\}$ and the edge sets $E(G_v) = E_{v,1} \cup E_{v,t}$, where $E_{v,1} = \{u \in V(G_v) : u \in E_1\}$ and $E_{v,t} = \{\{u_1, \ldots, u_t\} : \{v, u_1, \ldots, u_t\} \in E_{t+1}\}$. Observe that G_v is an H-free $\{1, t\}$-graph on $n - 1$ vertices. Thus $h_{n-1}(G_v) \leqslant \pi_{n-1}(H)$.

Since

$$|E_1| = \frac{1}{n-1} \sum_{v \in V(G)} |E_{v,1}| \quad \text{and} \quad |E_{t+1}| = \frac{1}{(t+1)} \sum_{v \in V(G)} |E_{v,t}|,$$

then

$$\begin{aligned}
h_n(G) &= \frac{|E_1|}{\binom{n}{1}} + \frac{|E_{1+t}|}{\binom{n}{1+t}} \\
&= \sum_{v \in V(G)} \frac{|E_{v,1}|}{(n-1)\binom{n}{1}} + \sum_{v \in V(G)} \frac{|E_{v,t}|}{(t+1)\binom{n}{1+t}} \\
&= \frac{1}{n} \sum_{v \in V(G)} \left(\frac{|E_{v,1}|}{\binom{n-1}{1}} + \frac{|E_{v,t}|}{\binom{n-1}{t}} \right) \\
&= \frac{1}{n} \sum_{v \in V(G)} h_{n-1}(G_v) \\
&\leqslant \pi_{n-1}(H).
\end{aligned}$$

Thus $\pi(T(H)) = \lim_{n\to\infty} \pi_n(T(H)) = \lim_{n\to\infty} h_n(G) \leqslant \pi(H)$. □

Lemma 21. *Let R be a set of two distinct positive integers, $R \neq \{1,2\}$. Then there exist non-trivial degenerate R-graphs.*

Proof. By Corollary 19, for every positive integer k, one can take the suspension of $H_5^{\{1,3\}}$ k-times, the resulting graph $S^k(H_5^{\{1,3\}})$ is a non-trivial degenerate $\{1+k, 3+k\}$-graph. Thus there are non-trivial degenerate hypergraphs of edge types: $\{1,3\}$, $\{2,4\}$, $\{3,5\}$, $\dots$, $\{k, k+2\}, \dots$.

In [4], the authors found a non-trivial degenerate $\{2,3\}$-graph: $H_4^{\{2,3\}} = \{12, 13, 234\}$. Similarly, by Corollary 19, there are non-trivial degenerate hypergraphs of edge types: $\{2,3\}, \{3,4\}, \{4,5\}, \dots, \{k, k+1\}, \dots$.

Using Lemma 20 on $H_5^{\{1,3\}}$, there are non-trivial degenerate hypergraphs of edge types: $\{1,4\}, \{1,5\}, \dots, \{1,t\}, \dots$, for integer $t \geqslant 4$. For each of these non-trivial degenerate $\{1,t\}$-graphs, applying Corollary 19, there are non-trivial degenerate hypergraphs of edge types: $\{2, 1+t\}, \{3, 2+t\}, \dots, \{k, k-1+t\}, \dots$.

To summarize, for each integer $k \geqslant 2$ and each integer $t \geqslant 3$, we have non-trivial degenerate hypergraphs of edge types $\{1,t\}, \{k,k+1\}, \{k,k+2\}, \{k,k+t\}$, which cover all sets of two distinct positive integers, except $\{1,2\}$. □

Lemma 22. *Let R be a set of distinct positive integers with $|R| \geqslant 2$ and $1 \notin R$. If there exist non-trivial degenerate R-graphs, then there exist non-trivial degenerate $\{1\} \cup R$-graphs.*

Proof. For each R stated in the lemma, let H be the non-trivial degenerate R-graph. Let H' be the disjoint union of H with a single 1-edge $v \notin H$. Clearly, H' is not contained in a blow-up of chain $C^{\{1\}\cup R}$. We will prove that H' is also degenerate.

Let n be a positive integer and $G = (V, E)$ be an extremal H'-free $\{1\} \cup R$-graph on n vertices. We have $\pi_n(H') = h_n(G)$. Denote E_i as the set of i-edges of G, for each $i \in \{1\} \cup R$. For any 1-edge $v \in E_1$, consider the sub-graph G_v of G by removing all

1-edges (keep the vertices of these 1-edges in G_v). Then the vertex set $V(G_v) = V$, set of i-edges $E_i(G_v) = E_i(G)$ for each $i \in R$. Then we have

$$|E_i(G)| = \frac{1}{|E_1|}\sum_{v\in E_1} |E_i(G_v)|, \ \forall i \in R.$$

Observe that G_v is an H-free R-graph on n vertices, so $\pi_n(H) \geqslant h_n(G_v)$. Then we have

$$\begin{aligned} h_n(G) &= \sum_{i\in\{1\}\cup R} \frac{|E_i|}{\binom{n}{i}} \\ &= \frac{|E_1|}{\binom{n}{1}} + \sum_{i\in R}\sum_{v\in E_1} \frac{|E_i(G_v)|}{|E_1|\binom{n}{i}} \\ &\leqslant 1 + \frac{1}{|E_1|}\sum_{v\in E_1} h_n(G_v) \\ &\leqslant 1 + \pi_n(H). \end{aligned}$$

Thus $\pi(H') = \lim_{n\to\infty} \pi_n(H') = \lim_{n\to\infty} h_n(G) \leqslant 1 + \pi(H) = |R|$, then $\pi(H') = |R|$. Therefore, H' is a non-trivial degenerate $\{1\} \cup R$-graph. □

Proof of Theorem 3. Using the non-trivial degenerate R-graph for R stated in Lemma 21, then apply Lemma 22, we obtain non-trivial degenerate R-graphs for $|R| = 3$ and $1 \in R$. Apply Corollary 19, we then obtain all other non-trivial degenerate R-graphs for $|R| = 3$. Repeatedly apply Lemma 22 and Corollary 19, we can obtain all R-graphs for $|R| \geqslant 4$, the result follows. □

We conjecture that for any set R, there exists an R-graph H^R such that if G^R is R-degenerate if and only if G^R is H^R-colorable. This conjecture is true for the case $R = \{r\}$ with $r \geqslant 2$ and $R = \{1, 2\}$ and is confirmed for $R = \{1, 3\}$ in this paper.

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