The cd-indices of intervals
in the uncrossing partial order on matchings

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Abstract

We study flag enumeration in intervals in the uncrossing partial order on matchings. We produce a recursion for the cd-indices of intervals in the uncrossing poset P_n. We explicitly describe the matchings by constructing an order-reversing bijection. We obtain a recursion for the ab-indices of intervals in the poset $\hat{P}_n$, the poset P_n with a unique minimum $\hat{0}$ adjoined.

Mathematics Subject Classifications: 06A07, 06A11

1 Introduction

A graded poset P with a unique maximum and a unique minimum is *Eulerian* if, for every non-trivial interval $[x, y]$ in P, the number of elements of odd rank in $[x, y]$ is equal to the number of elements of even rank in $[x, y]$. One nice property of Eulerian posets is their very simple Möbius functions $\mu_P(s, t) = (-1)^{\ell(s,t)}$. Another nice property of Eulerian posets is that they have a cd-index. A cd-index is a non-commutative generating function, which is an efficient way to encode the flag enumeration of Eulerian posets. The cd-index arose from the work of Bayer-Billera [2] on flag f-vectors and flag h-vectors of Eulerian posets. Bayer-Klapper [3] and Stanley [15] proved the existence of the cd-indices of Eulerian posets. Ehrenborg-Readdy [7] showed the way to obtain the cd-index of some operations, for example, the cd-index of the *pyramid* over P, the product of a poset P with a chain of length one. Reading [13] presented a recursive formula for the cd-indices of intervals in the Bruhat order on a Coxeter group.

The uncrossing partially ordered set P_n on matchings is defined on the set of matchings on $2n$ points on a circle represented with wires, with an order relation: $\tau' \leqslant \tau$ in P_n if and only if τ' is obtained by resolving a crossing of τ. For example, if $\tau = \{(1, 3), (2, 4)\}, \tau' =$

$\{(1, 2), (3, 4)\}$ and $\tau'' = \{(1, 4), (2, 3)\}$ in P_2 then $\tau' \leqslant \tau$ and $\tau'' \leqslant \tau$. This partial order has been studied by Alman-Lian-Tran [1], Huang-Wen-Xie [8], Kenyon [9], and Lam [12], [11]. The uncrossing posets emerged from studies of circular planar electrical networks. Circular planar networks are finite weighted undirected graphs embedded into a disk, with boundary vertices and interior vertices. By Curtis-Ingerman-Morrow [6] and de Verdière-Gitler-Vertigan [5], the electrical networks can be encoded with response matrices. By Lam [12] the space of response matrices for electrical networks has a cell structure, and this cell structure can be described by the uncrossing partial orders (see [12], Proposition 4.10). Let $\hat{P}_n$ denote P_n with a unique minimum element $\hat{0}$ adjoined. The poset $\hat{P}_n$ was conjectured to be Eulerian by Alman-Lian-Tran [1], Huang-Wen-Xie [8], and Kim-Lee [10]. This conjecture was proved by Lam [11]. Thus it is natural for us to pay attention to the cd-index of the posets $\hat{P}_n$ and their intervals.

The main result in this work is a recursive formula in Theorem 18 for the cd-indices of intervals in the uncrossing poset P_n. To find this, we construct an order-reversing bijection in Theorem 10 between P_n and $\mathcal{MP}_n$, an induced subposet of the group $\tilde{S}_{2n}$ of affine permutations. We explicitly describe the elements in the induced subposet. Lam [12] introduced a representation of matchings in P_n: to a matching $\tau \in P_n$, associate $g_\tau : [2n] \to \mathbb{Z}$ by

$$g_\tau(i) \;=\; \begin{cases} \tau(i) & \text{if } i < \tau(i) \\ \tau(i) + 2n & \text{if } i > \tau(i). \end{cases} \tag{1}$$

Lam showed that this map $\tau \mapsto g_\tau$ identifies P_n with an induced subposet of dual Bruhat order of affine permutations of type $(n, 2n)$ (see [12], Theorem 4.16). However, we need a map to the group $\tilde{S}_{2n}$ of affine permutations in order to apply a technique in Reading [13] for finding recursions for the cd-indices of intervals in Bruhat order on Coxeter groups. We slightly modify Lam's map $\tau \mapsto g_\tau$ to adapt Reading's ideas to our situation. Finally, using the cd-indices of intervals in P_n, we present a recursion in Theorem 30 for the ab-indices of intervals in the poset $\hat{P}_n$.

The paper is structured as follows. In Section 2, we give basic definitions needed to understand this paper: posets, the cd-index, affine permutations, and Bruhat order. In Section 3, the definition of uncrossing posets is presented. In Section 4, we define an induced subposet $\mathcal{MP}_n$ of Bruhat order on affine permutations $\tilde{S}_{2n}$. We prove that there is an order-reversing bijection between P_n and $\mathcal{MP}_n$. In Section 5, we recall the recursive formula for the cd-indices of intervals in Bruhat order of a Coxeter group [13]. In Section 6, we prove that there is a recursive formula for the cd-indices of intervals in P_n. Moreover, we present a recursive formula for the ab-index of intervals in the poset $\hat{P}_n$.

2 Preliminaries

In this section we give background information on partially ordered sets, the cd-index, affine permutations, and Bruhat order. The definitions and notations in this section are mainly obtained from Stanley [16] and Björner-Brenti [4].

2.1 Posets

Let P be a poset and $x, y \in P$. If x covers y we write $x \gtrdot y$. By an *induced subposet*, or *subposet* for short, of P, we mean a subset Q of P and a partial ordering on Q such that for $x, y \in Q$ we have $x \leqslant y$ in Q if and only if $x \leqslant y$ in P. For $x \leqslant y$ in P, a (closed) *interval* $[x, y] = \{z \in P : x \leqslant z \leqslant y\}$ is a subposet of P. Given two posets P and Q, form their *product* $P \times Q$ on the set $\{(x, y) : x \in P, y \in Q\}$ such that $(x_1, y_1) \leqslant (x_2, y_2)$ in $P \times Q$ if $x_1 \leqslant x_2$ in P and $y_1 \leqslant y_2$ in Q. A graded poset P with a unique maximum and a unique minimum is *Eulerian* if, for every interval $[x, y]$ in P where $x < y$, the number of elements of odd rank in $[x, y]$ is equal to the number of elements of even rank in $[x, y]$.

2.2 The cd-index

The cd-index arose from the work of Bayer-Billera [2] on flag f-vectors of Eulerian posets. They extended the f-vector and the h-vector of a polytope Δ to the flag f-vector and the flag h-vector of the order complex $\Delta(P)$ of a finite graded poset P.

Let Δ be a finite $(d-1)$-dimensional simplicial complex with f_i i-dimensional faces. The vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is called the *f-vector* of Δ. The vector $h(\Delta) = (h_0, h_1, \ldots, h_d)$, called the *h-vector*, is defined by the relation

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}$$

where $f_{-1} = 1$ unless $\Delta = \emptyset$ [2].

Let P be a graded poset, rank n, with a $\hat{0}$ and a $\hat{1}$. The *order complex* $\Delta(P)$ of P is defined as follows: the vertices of $\Delta(P)$ are the elements of $P - \{\hat{0}, \hat{1}\}$, and the faces of $\Delta(P)$ are the chains of $P - \{\hat{0}, \hat{1}\}$. The order complex $\Delta(P)$ is a simplicial complex [2]. Stanley [14] enumerated faces of this order complex $\Delta(P)$ as follows. For a chain C in $P - \{\hat{0}, \hat{1}\}$, define $\rho(C) = \{\rho(x) : x \in C\}$. Let $[n] := \{1, 2, \ldots, n\}$ and let $2^{[n]}$ be the set of all subsets of $[n]$. For any $S \subseteq [n-1]$, define

$$\alpha_P(S) = |\{C \subseteq P : C \text{ is a chain such that } \rho(C) = S\}| .$$

We call the function $\alpha_P : 2^{[n-1]} \to \mathbb{N}$ the *flag f-vector*. We define

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha_P(T).$$

The function $\beta_P : 2^{[n-1]} \to \mathbb{N}$ is called the *flag h-vector* of P. Then we can check for the order complex $\Delta = \Delta(P)$

$$f_i(\Delta) = \sum_{|S|=i+1} \alpha_P(S)$$

$$h_i(\Delta) = \sum_{|S|=i} \beta_P(S).$$

Thus the flag f-vector α_P of the order complex $\Delta(P)$ counts flags (or chains) of $P - \{\hat{0}, \hat{1}\}$ by length, which are faces of $\Delta(P)$ by dimension, as the f-vector counts faces of a polytope by dimension. The flag h-vector β_P plays a role as the h-vector of a polytope.

Now we define ab-index of a poset and cd-index of an Eulerian poset. Define the *characteristic monomial* u_S of $S \subseteq [n-1]$ by $u_S = e_1 e_2 \ldots e_{n-1}$, where

$$
e_i = \begin{cases} b & \text{if } i \in S \\ a & \text{if } i \notin S. \end{cases}
$$

Define a noncommutative polynomial $\Psi_P(a, b)$, called the *ab-index* of P, by

$$
\Psi_P(a, b) = \sum_{S \subseteq [n-1]} \beta_P(S) u_S.
$$

Thus $\Psi_P(a, b)$ is a noncommutative generating function for the flag h-vector β_P. J. Fine (unpublished) conjectured and Bayer and Klapper [3] proved that if P is an Eulerian poset of rank n, then there exists a polynomial $\Phi_P(c, d)$ in the noncommutative variables c and d such that $\Psi_P(a, b) = \Phi_P(a + b, ab + ba)$. The polynomial $\Phi_P(c, d)$ is called the *cd-index* of P.

2.3 Affine permutations and Bruhat order

Let $\tilde{S}_n, n \geqslant 2$ be the group of all bijections w of $\mathbb{Z}$ in itself such that

1. $w(x + n) = w(x) + n$ for all $x \in \mathbb{Z}$ and

2. $\sum_{x=1}^{n} w(x) = (1 + 2 + \cdots + n)$,

with composition as group operation. This is the group of affine permutations. The *window notation* of $w \in \tilde{S}_n$ is $w = [a_1, \ldots, a_n]$ if $w(i) = a_i$ for $i \in [n]$. As a set of generators for $\tilde{S}_n$ we take the set of periodic adjacent transpositions $S = \{s_0, s_1, \ldots, s_{n-1}\}$ where

$$
\begin{aligned}
s_1 &:= [2, 1, 3, \ldots, n] \\
s_i &:= [1, 2, \ldots, i-1, i+1, i, i+2, \ldots, n] \text{ for } 2 \leqslant i \leqslant n-2, \\
s_{n-1} &:= [1, 2, 3, \ldots, n-2, n, n-1] \\
s_0 &:= [0, 2, 3, \ldots, n-1, n+1].
\end{aligned}
$$

Then, the pair $(\tilde{S}_n, S)$ is a Coxeter system and $\tilde{S}_n$ is the Coxeter group of *affine permutations* of the integers.

There are several partial orders defined on $\tilde{S}_n$. We need Bruhat order and from now on, $\tilde{S}_n$ denotes the affine permutations, together with this order. For $w \in \tilde{S}_n$, a decomposition $w = s_{i_1} s_{i_2} \ldots s_{i_k}$ with letters in S is called a reduced decomposition for w if k is minimal. The word $i_1 i_2 \ldots i_k$ is called a reduced word for w. We say that k is the length of w and denote it by $\ell(w)$. Fix a reduced decomposition for $w = t_1 t_2 \ldots t_k$ where $t_i \in S$ for all $1 \leqslant i \leqslant k$. The Bruhat order is defined by $v \leqslant_B w$ if and only if there is a reduced subword $t_{i_1} t_{i_2} \ldots t_{i_j}$ of $t_1 t_2 \ldots t_k$ for v such that $1 \leqslant i_1 < i_2 < \cdots < i_j \leqslant k$. We will write $v \leqslant w$ for $v \leqslant_B w$ if there is no possibility of confusion.

3   The Uncrossing Posets

A (complete) matching on $[2n]$ is a partition of $[2n]$ of type $(2, 2, \ldots, 2)$. Let P_n be the set of matchings on $[2n]$. For a matching $\{(i_1, j_1), \ldots, (i_n, j_n)\}$ where $i_r < j_r$ for all r, two blocks (i_r, j_r) and (i_s, j_s) form a crossing if $i_r < i_s < j_r < j_s$. Now resolve any crossing in τ in either of two ways:
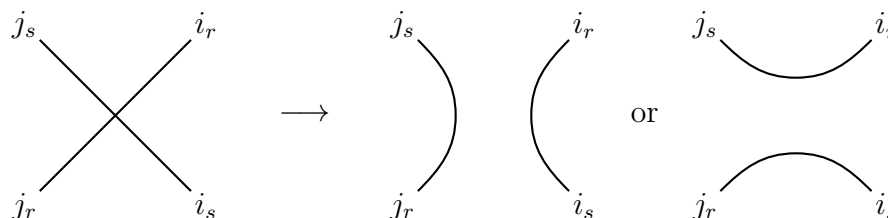
This gives a new matching τ' on $[2n]$. Then we define the partial order $\tau' \leqslant \tau$ on P_n if the matching τ' is obtained by resolving a crossing of the matching τ. Let $c(\tau)$ be the number of crossings of τ. The poset P_n is graded of rank $\binom{n}{2}$ with rank function given by $c(\tau)$. The poset P_n has a unique maximum element, namely $\{(1, n+1), (2, n+2), \ldots, (n, 2n)\}$, and Catalan number $\frac{1}{n+1}\binom{2n}{n}$ of minimal elements. Figure 1 is the Hasse diagram of P_3.

This partial order has been studied by Alman-Lian-Tran [1], Huang-Wen-Xie [8], Kenyon [9], and Lam [12], [11]. The uncrossing posets emerged from studies of circular planar electrical networks, finite weighted undirected graphs embedded into a disk, with boundary vertices and interior vertices. By Curtis-Ingerman-Morrow [6] and de Verdière-Gitler-Vertigan [5], the electrical networks can be encoded with response matrices. By Lam [12] the space of response matrices for electrical networks has a cell structure, and this cell structure can be described by the uncrossing partial orders (see [12], Proposition 4.10).

Let $\hat{P}_n$ denote P_n with a unique minimum element $\hat{0}$ adjoined, where we let $c(\hat{0}) = -1$. The poset $\hat{P}_n$ was conjectured to be Eulerian by Alman-Lian-Tran [1], Huang-Wen-Xie [8], and Kim-Lee [10], and was proved to be so by Lam [11].

Theorem 1 ([11], Theorem 1). *$\hat{P}_n$ is an Eulerian poset.*

Lam [11] used the map $\tau \mapsto g_\tau$ ([12], see (1) in Section 1) as the main tool for proving Theorem 1. Using this map, Lam showed that the number of odd elements equals the number of even rank elements in intervals in $\hat{P}_n$. To be specific, he first showed that the number of odd rank elements equals the number of even rank elements in any interval $[\tau, \eta] \subset P_n$ by descending induction on $c(\tau) + c(\eta)$. Then he showed the number of odd rank elements equals the number of even rank elements in any interval $[\hat{0}, \eta] \subset \hat{P}_n$ by establishing an involution $\sigma \mapsto s_i \cdot \sigma$ on the set $\{\sigma \in (\hat{0}, \eta] | s_i \cdot \sigma \neq \sigma\}$.

4   Modular Palindromic Permutations

Lam [12] mapped matchings $\tau \in P_n$ to affine permutations g_τ of type $(n, 2n)$. See (1) in Section 1. Lam showed that this map $\tau \mapsto g_\tau$ identifies P_n with an induced subposet of
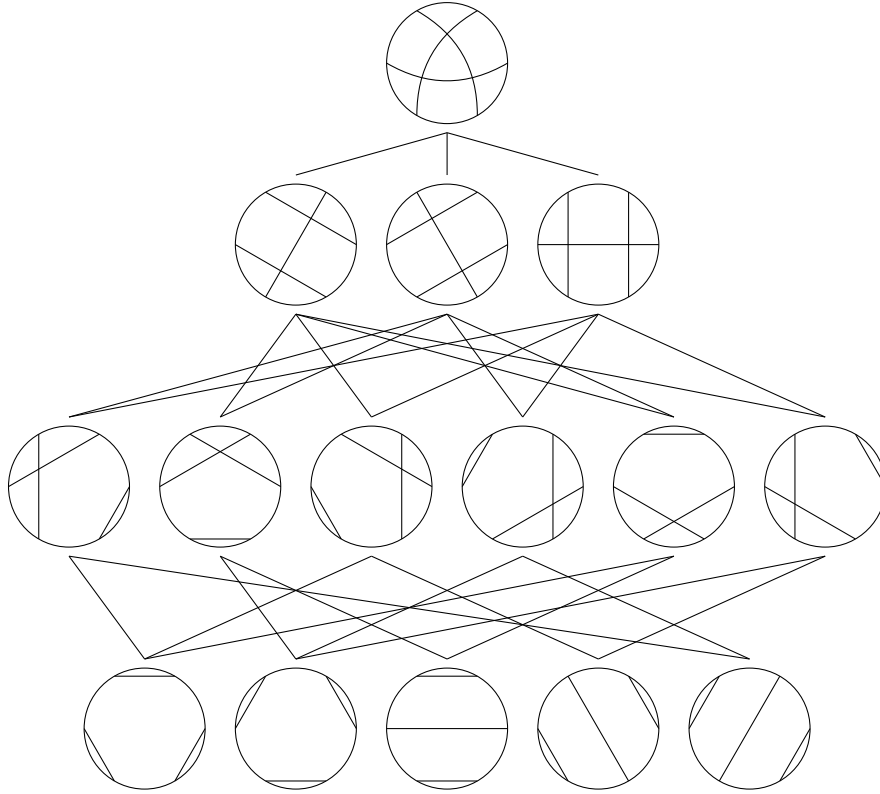
Figure 1: The Hasse diagram of P_3 (courtesy of Thomas Lam, used with permission)

dual Bruhat order of affine permutations of type $(n, 2n)$ (Theorem 4.16 in [12]), through Theorem 8.3.7 in Björner-Brenti [4] which characterizes affine Bruhat order in terms of a matrix which tracks inversions.

We notice that if we slightly modify Lam's map it is possible to show that P_n is identified with an induced subposet of dual Bruhat order of the group $\tilde{S}_{2n}$ of affine permutations without Theorem 8.3.7 in [4]. We believe that this identification is necessary to apply a technique in Reading [13] for finding recursions for the cd-indices of intervals in Bruhat order on Coxeter groups. Moreover, we are able to explicitly describe the elements in the induced subposet of affine permutations as modular palindromic permutations (see Theorem 10).

For these reasons, we slightly modify the map $\tau \mapsto g_\tau$ to define an injective map $\phi : P_n \to \tilde{S}_{2n}$ as follows. For a matching $\tau \in P_n$, define $h_\tau : [2n] \to \mathbb{Z}$ by

$$h_\tau(i) = \begin{cases} \tau(i) - n & \text{if } i < \tau(i) \\ \tau(i) + n & \text{if } i > \tau(i). \end{cases}$$

Now define $\phi : P_n \to \tilde{S}_{2n}$ by $\phi(\tau) = [h_\tau(1), h_\tau(2), \ldots, h_\tau(2n)]$ in the window notation of the affine permutation group $\tilde{S}_{2n}$.

Example 2. Let $\tau = \{(1,4),(2,6),(3,5)\}$ and $\tau' = \{(1,3),(2,6),(4,5)\}$ in P_3. The images of τ and τ' under Lam's map are $g_\tau = [g_\tau(i)]_{i=1}^6 = [4,6,5,7,9,8]$ and $g_{\tau'} = [g_{\tau'}(i)]_{i=1}^6 = [3,6,7,5,10,8]$. On the other hands, the images under the map ϕ are $\phi(\tau) = [1,3,2,4,6,5] = s_2 s_5 \in \tilde{S}_6$ and $\phi(\tau') = [0,3,4,2,7,5] = s_2 s_0 s_3 s_5 \in \tilde{S}_6$.

With the map ϕ, we explicitly describe the subposet of $\tilde{S}_{2n}$ which is isomorphic to the dual of P_n. First, we define a *modular palindromic* permutation and a subset $\mathcal{MP}_n$ of the affine permutation group $\tilde{S}_{2n}$.

Definition 3. An affine permutation $w \in \tilde{S}_{2n}$ is *modular palindromic* if w has a reduced decomposition $w = s_{i_1} s_{i_2} \ldots s_{i_{2k}}$ with $|i_{2k+1-r} - i_r| = n$ for all $r \in [k]$.

Example 4. Let $\tau = \{(1,4),(2,6),(3,5)\}$ and $\tau' = \{(1,3),(2,6),(4,5)\}$ in P_3 as in Example 2. Both $\phi(\tau) = s_2 s_5$ and $\phi(\tau') = s_2 s_0 s_3 s_5$ are modular palindromic. We also observe that ϕ reverses the order, in other words, $\tau' \leqslant \tau$ and $\phi(\tau') \geqslant \phi(\tau)$.

$$\tau' \qquad\qquad \tau$$
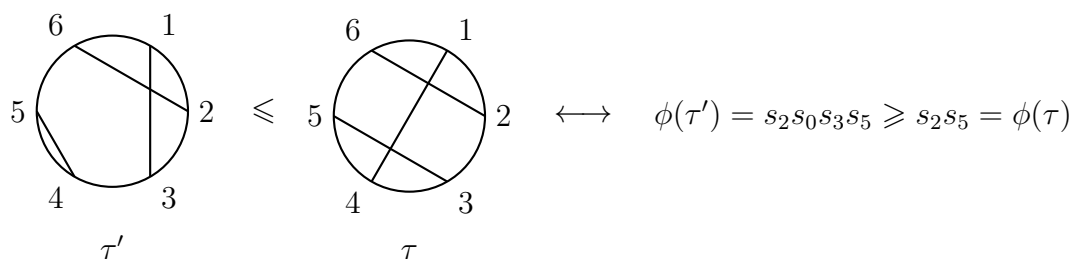
Example 5. Let $n = 3$. Let $w = s_0 s_5 s_1 s_4 s_2 s_3 \in \tilde{S}_6$. Notice that w is a modular palindromic permutation. The window notation of the permutation is $w = [2,3,7,0,4,5]$. Assume $w = \phi(\tau)$ for some $\tau \in P_3$. Since $w(3) = 7$, we must have $h_\tau(3) = 7$, which impossible because $1 \leqslant h_\tau(3) \leqslant 5$. Thus $w \notin \phi(P_3)$.

The previous example shows that not all modular palindromic permutations are images of matchings under the map ϕ. Which conditions are needed for modular palindromic permutations to be images of matchings under the map ϕ? We need the following definition.

Definition 6. The subset $\mathcal{MP}_n$ of $\tilde{S}_{2n}$ is the set of all modular palindromic permutations $w \in \tilde{S}_{2n}$ such that no reduced word of w contains a subword of n consecutive integers.

Notice that $i - n < p(i) < i + n$ for all $i \in [2n]$ since no reduced decomposition of p contains a subword of n consecutive integers.

Remark 7. The consecutive integers need not be in adjacent positions in the reduced word.

Now we use $\mathcal{MP}_n$ as a poset, equipped with Bruhat order.

Example 8. Let $n = 2$. We see that $\mathcal{MP}_2$ contains $e, s_0 s_2$ and $s_1 s_3$. Notice that all length four modular palindromic permutations, whose reduced words are 1023, 0132, 2130 and 3201 respectively, have a subword of 2-consecutive integers, and thus $\mathcal{MP}_2 = \{e, s_0 s_2, s_1 s_3\}$ and the followings are the Hasse diagrams of P_2 and the dual of $\mathcal{MP}_2$, respectively.

$$\cong$$

$$P_2 \qquad\qquad\qquad \mathcal{MP}_2^*$$
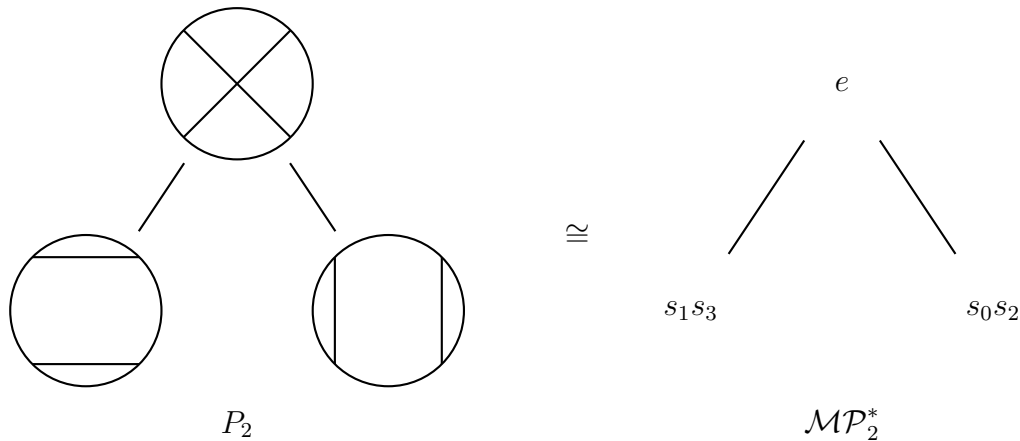
Example 9 (revisited)**.** Let $n = 3$. Let $w = s_0 s_5 s_1 s_4 s_2 s_3 \in \tilde{S}_6$ with a reduced word 051423. Notice that w is a modular palindromic permutation, which has a subword 543, a 3 consecutive integers. Thus $w \notin \mathcal{MP}_3$.

The following theorem is one of the main results in this paper.

Theorem 10. *The map ϕ is an order-reversing bijection between P_n and $\mathcal{MP}_n$.*

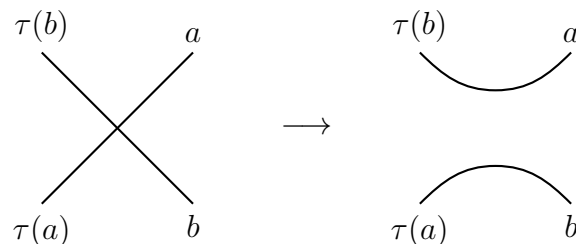To prove this theorem, we first state and prove a lemma.

Lemma 11. *The image of P_n under the map ϕ is contained in $\mathcal{MP}_n$. In other words, $\phi(P_n) \subseteq \mathcal{MP}_n$.*

Proof. First, we prove that $\phi(\tau)$ is modular palindromic using decreasing induction on the ranks of $\tau \in P_n$.

(i) Base case: the unique maximum element $\hat{1} = \{(1, n+1), (2, n+2), \ldots, (n, 2n)\}$. Notice that $\phi(\hat{1}) = e \in \mathcal{MP}_n$ as required.

(ii) Induction step: suppose $\tau \lessdot \hat{1}$ and $\phi(\tau) \in \mathcal{MP}_n$. Choose a crossing generated by a pair of wires $(a, \tau(a))$ and $(b, \tau(b))$ such that $a < b < \tau(a) < \tau(b)$. We can resolve the crossing in two ways.

Case 1: from $\{(a, \tau(a)), (b, \tau(b))\}$ to $\{(a, \tau(b)), (b, \tau(a))\}$. Let $\tau' \in \mathcal{MP}_n$ be the matching obtained by resolving the crossing this way.
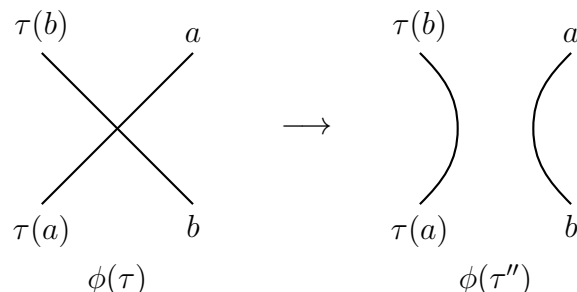

In window notation, we see that

$$\begin{array}{cccccccc} & \ldots, & a & ,\ldots, & b & ,\ldots, & \tau(a), \ldots, & \tau(b), \ldots \end{array}$$
$$\phi(\tau) = [\ldots, \tau(a) - n, \ldots, \tau(b) - n, \ldots, a + n, \ldots, b + n, \ldots]$$
$$\phi(\tau') = [\ldots, \tau(b) - n, \ldots, \tau(a) - n, \ldots, b + n, \ldots, a + n, \ldots].$$

Observe that $\phi(\tau')$ is obtained from $\phi(\tau)$ by swapping the numbers in the a-th spot and b-th spot and swapping the numbers $a+n$ and $b+n$. Thus, $\phi(\tau') = t_{a+n,b+n}\phi(\tau)t_{a,b}$ where $t_{a,b} = s_a s_{a+1} \ldots s_{b-2} s_{b-1} s_{b-2} \ldots s_{a+1} s_a$ is the periodic transposition of a and b. Therefore, $\phi(\tau')$ is modular palindromic.

Case 2: from $\{(a, \tau(a)), (b, \tau(b))\}$ to $\{(a, b), (\tau(a), \tau(b))\}$.

This case can be proved in exactly the same way as the previous case using transpositions $t_{b-n,\tau(a)-n}$ and $t_{b,\tau(a)}$.

Secondly, we prove that there is no reduced word for $\phi(\tau)$ which contains a subword of n consecutive integers. For the sake of contradiction, suppose there is a reduced word for $\phi(\tau)$ containing a subword of n consecutive integers. Without loss of generality, assume the consecutive integers are increasing. Take a maximal length subword $s_{a+1} s_{a+2} \cdots s_{a+m}$ of $\phi(\tau)$ where $m \geqslant n$ and the indices are taken modulo $2n$ if necessary. Then, observe that $\phi(\tau)(a + m + 1) = a + 1$, which contradicts that $i - n < \phi(\tau)(i) < i + n$ by the definition of ϕ for all i. $\qquad \square$

Corollary 12. *The map ϕ is order-reversing. In other words, $\tau' \leqslant \tau$ in P_n implies $\phi(\tau') \geqslant \phi(\tau)$ in $\mathcal{MP}_n$ or in $\tilde{S}_{2n}$.*

Proof. By the proof of the previous lemma, we see that if a matching τ' is obtained by resolving a crossing of a matching τ, then $\phi(\tau') = t_{a,b}\phi(\tau)t_{a+n,b+n}$ for some $a, b \in [2n]$. Since $\phi(\tau)$ is a subword of $\phi(\tau')$, we conclude that $\phi(\tau) \leqslant \phi(\tau')$ as required. $\qquad \square$

The following lemma will be used when we prove Theorem 10; in particular, we will use the lemma to show that the map ϕ is surjective.

Lemma 13. *Let $p, q \in \mathcal{MP}_n$ such that $q = s_a p s_{a+n}$ and $\ell(q) = \ell(p) + 2$. Let $x, y \in [2n]$ such that $p(x) \equiv a \pmod{2n}$ and $p(y) \equiv a + 1 \pmod{2n}$. Then,*

   (1) *The numbers $x, y, a+n$ and $a+n+1$ are distinct.*

   (2) *$q(i) = p(i)$ for all $i \in [2n] \backslash \{a+n, a+n+1, x, y\}$.*

   (3) *$q(a+n) = p(a+n+1)$, $q(a+n+1) = p(a+n)$, $q(x) \equiv a+1 \pmod{2n}$, and $q(y) \equiv a \pmod{2n}$.*

Proof. (1) It is clear that $x \neq y$ and $a + n \neq a + n + 1$. Since $i - n < p(i) < i + n$ for all $i \in [2n]$, we know $a < p(a + n) < a + 2n$ and $a + 1 < p(a + n + 1) < a + 2n + 1$, and $a + n \neq x$ and $a + n + 1 \neq y$. Asume $y = a + n$, then $p(a + n) \equiv a + 1 \pmod{2n}$. Then p should contain a subword of the form $s_{a+1}s_{a+2} \ldots s_{a+n-2}s_{a+n-1}$. Then $q = s_a p s_{a+n}$ should contain a subword of n consecutive integers, which contradicts $q \in \mathcal{MP}_n$. Assume $x = a + n + 1$, then $p(a + n + 1) \equiv a \pmod{2n}$. Then p should contain a subword of the form $s_{a+2n+1}s_{a+2n} \ldots s_{a+n+2}s_{a+n+1}$. Then $q = s_a p s_{a+n}$ should contain a subword of n consecutive integers, which contradicts $q \in \mathcal{MP}_n$.

(2) Let $i \in [2n] \backslash \{a + n, a + n + 1, x, y\}$. Then it is clear that $s_{a+n}(i) = i$. Note that $p(i) \notin \{a, a + 1\}$ since $i \notin \{x, y\}$, and thus $s_a(p(i)) = p(i)$. Then we have

$$q(i) = (s_a p s_{a+n})(i) = s_a(p(i)) = p(i).$$

(3) Since s_{a+n} acts on the right of p by swapping numbers in the $(a + n)$-th spot and $(a + n + 1)$-th spot and s_a acts on the left of p by swapping numbers a and $a + 1$, we have the third statement. $\qquad\square$

Proof of Theorem 10. By Lemma 11 and Corollay 12, we need to show that the map ϕ is surjective to complete the proof. Let $p \in \mathcal{MP}_n$. Since $i - n < p(i) < i + n$ for all $i \in [2n]$, we define τ_p by

$$\tau_p(i) \;=\; \begin{cases} p(i) + n & \text{if} \quad p(i) \leqslant n \\ p(i) - n & \text{if} \quad p(i) \geqslant n + 1 \end{cases}$$

for $i \in [2n]$. We claim that τ_p is a matching on $[2n]$. To show τ_p is a matching, we need to show that τ_p is an involution with no fixed point. In other words, we must show that $\tau_p(i) \neq i$ and $\tau_p^2(i) = i$ for all $i \in [2n]$. Since $i - n < p(i) < i + n$, either $i < \tau_p(i) < i + 2n$ or $i - 2n < \tau_p(i) < i$, and thus $\tau_p(i) \neq i$. Observe that

$$\tau_p^2(i) \;=\; \begin{cases} p(p(i) + n) + n & \text{if} \quad p(i) \leqslant n \text{ and } p(p(i) + n) \leqslant n \\ p(p(i) + n) - n & \text{if} \quad p(i) \leqslant n \text{ and } p(p(i) + n) \geqslant n + 1 \\ p(p(i) - n) + n & \text{if} \quad p(i) \geqslant n + 1 \text{ and } p(p(i) - n) \leqslant n \\ p(p(i) - n) - n & \text{if} \quad p(i) \geqslant n + 1 \text{ and } p(p(i) - n) \geqslant n + 1. \end{cases} \tag{2}$$

Since p is an affine permutation in $\tilde{S}_{2n}$, we see that

$$p(p(i) - n) + n = p(p(i) + n - 2n) + n = p(p(i) + n) - n$$

and

$$p(p(i) - n) - n = p(p(i) + n - 2n) - n = p(p(i) + n) - 3n.$$

By this observation, we have $\tau_p^2(i) \equiv p(p(i) + n) + n \pmod{2n}$.

Since $1 \leqslant \tau_p(i) \leqslant 2n$, we can simplify (2) as follows.

$$\tau_p^2(i) \equiv p(p(i) + n) + n \pmod{2n} \text{ and } 1 \leqslant \tau_p^2(i) \leqslant 2n. \tag{3}$$

Now we prove that τ_p is an involution by induction on $\ell(p)/2$.

Base case: $p = e$. Observe that

$$e(e(i) + n) + n = e(i + n) + n = (i + n) + n = i + 2n,$$

and $\tau_e^2(i) = i$ for all $i \in [2n]$ as desired.

Induction step: Suppose $\tau_p^2(i) = i$ for all $i \in [2n]$ for $p \in \mathcal{MP}_n$. Then $p(p(i)+n)+n \equiv i \pmod{2n}$, or equivalently, $p(p(i) + n) \equiv i + n \pmod{2n}$. Let $q = s_a p s_{a+n} \in \mathcal{MP}_n$ with $\ell(q) = \ell(p) + 2$. Let $x, y \in [2n]$ such that $p(x) \equiv a \pmod{2n}$ and $p(y) \equiv a + 1 \pmod{2n}$. Let $i \in [2n] \backslash \{a+n, a+n+1, x, y\}$. By Lemma 13, we see $p(i)+n \not\equiv a+n \pmod{2n}$ and $p(i) + n \not\equiv a + n + 1 \pmod{2n}$, and thus

$$
\begin{aligned}
q(q(i) + n) + n &\equiv q(p(i) + n) + n \\
&\equiv (s_a p s_{a+n})(p(i) + n) + n \\
&\equiv s_a(p(p(i) + n) + n \\
&\equiv s_a(i + n) + n \\
&\equiv i + 2n,
\end{aligned}
$$

where the last equality is due to $i \notin \{a + n, a + n + 1\}$, and hence $\tau_q^2(i) = i$ for $i \in [2n] \backslash \{a + n, a + n + 1, x, y\}$. We calculate $q(q(i) + n) + n$ for $i \in \{a + n, a + n + 1, x, y\}$ as follows.

$$
\begin{aligned}
q(q(a + n) + n) + n &\equiv a + 3n, \\
q(q(a + n + 1) + n) + n &\equiv a + 3n + 1, \\
q(q(x) + n) + n &\equiv x + 2n, \\
q(q(y) + n) + n &\equiv y + 2n,
\end{aligned}
$$

and thus $\tau_q^2(i) = i$ for $i \in \{a + n, a + n + 1, x, y\}$.

Hence by induction, τ_p is an involution and therefore τ_p is a matching on $[2n]$. By the construction of τ_p, observe that

$$\phi(\tau_p)(i) = h_{\tau_p}(i) = \begin{cases} \tau_p(i) - n = (p(i) + n) - n & \text{if} \quad i < \tau_p(i) \Leftrightarrow p(i) \leqslant n \\ \tau_p(i) + n = (p(i) - n) + n & \text{if} \quad i > \tau_p(i) \Leftrightarrow p(i) \geqslant n + 1 \end{cases}$$

and thus $\phi(\tau_p)(i) = p(i)$ for all $i \in [2n]$, which shows that the map ϕ is surjective. $\quad\square$

5   The cd-index of the posets P_n

In the previous section, we saw that the poset P_n, without $\hat{0}$ adjoined, is isomorphic to a subposet of the dual Bruhat order of affine permutations. Reading [13] provided a

recursive formula for the cd-indices of intervals in the Bruhat order on a Coxeter group, and it looks promising to examine the recursion to compute the cd-index of P_n.

The product of a poset P with a chain of length one is called the *pyramid* of P, and denoted by $Pyr(P)$. We will use the proposition from Ehrenborg-Readdy [7] which produces the cd-index of $Pyr(P)$ from the cd-index of P.

Proposition 14 ([7], Proposition 4.2). *Let P be an Eulerian poset. Then*

$$\Phi_{Pyr(P)} = \frac{1}{2}\left(\Phi_p \cdot c + c \cdot \Phi_P + \sum_{x:\hat{0}<x<\hat{1}} \Phi_{[\hat{0},x]} \cdot d \cdot \Phi_{[x,\hat{1}]}\right).$$

A *zipper* in a poset P is a triple of distinct elements $x, y, z \in P$ such that $\{w : w < x\} = \{w : w < y\}$ and $z = x \vee y$ covers x and y but covers no other element. Please see Figure 2.
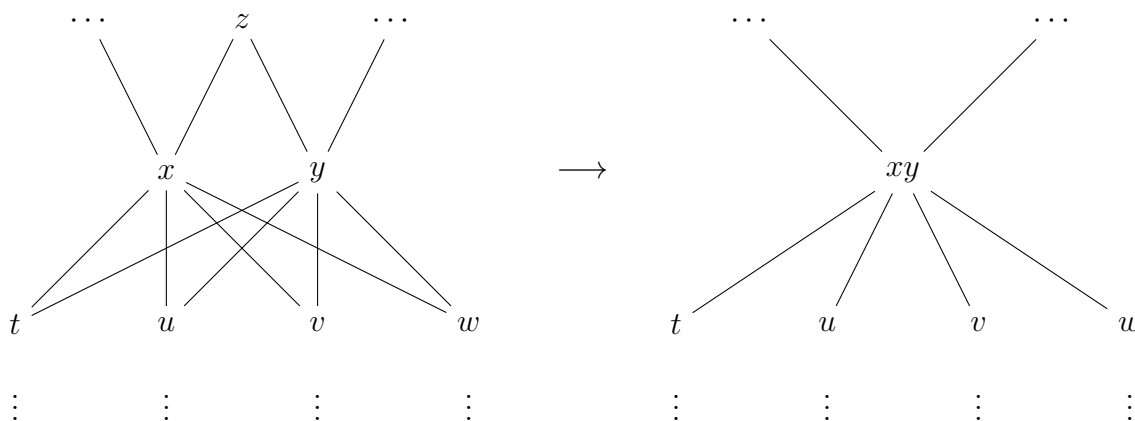

Figure 2: A zipper $\{x, y, z\}$ and zip of the zipper

The operation *zip* of a zipper $\{x, y, z\} \subset P$ is defined as follows. Let xy be a single new element not in P, and define $P' = (P - \{x, y, z\}) \cup \{xy\}$, with a binary relation $\preceq$ on P', given by:

$$
\begin{aligned}
a \preceq b \qquad & \text{if } a \leqslant b \text{ in } P - \{x, y, z\} \\
xy \preceq a \qquad & \text{if } x \leqslant a \text{ or if } y \leqslant a \text{ in } P - \{x, y, z\} \\
a \preceq xy \qquad & \text{if } a \leqslant x \text{ or (equivalently) if } a \leqslant y \text{ in } P - \{x, y, z\} \\
xy \preceq xy.
\end{aligned}
$$

We can think of the operation zip of a zipper $\{x, y, z\}$ as deleting z and identifying x with y. The zip operation of a zipper produces a new poset. The zip operation of an Eulerian poset is also Eulerian. Furthermore, the following theorem provides a formula for the cd-index of the resulting poset in terms of the cd-index of the initial Eulerian poset.

Theorem 15 ([13], Theorem 4.6). *P' is a poset under the partial order $\preceq$. If P is Eulerian then so is P'. Moreover, P has a cd-index Φ_P if and only if P' has a cd-index $\Phi_{P'}$, and*

$$\Phi_{P'} = \Phi_P - \Phi_{[\hat{0},x]} \cdot d \cdot \Phi_{[z,\hat{1}]}.$$

The following theorem is a structural recursion for Bruhat intervals.

Theorem 16 ([13], Theorem 5.5). *Let (W, S) be a Coxeter system. Let $w, u \in W$ and $s \in S$. Let $ws > w, us > u$ and $u \leqslant w$.*

(1) If $us \notin [u, w]$ then $[u, ws] \cong [u, w] \times [1, s]$ and $[us, ws] \cong [u, w]$.

(2) If $us \in [u, w]$, then $[u, ws]$ can be obtained from $[u, w] \times [1, s]$ by a sequence of zippings.

From these two theorems, Reading [13] produced recursions for the cd-indices of Bruhat intervals. For $v \in W$ and $s \in S$, define $\sigma_s(v) := \ell(vs) - \ell(v) \in \{-1, 1\}$.

Theorem 17 ([13], Theorem 6.1). *Let (W, S) be a Coxeter system. Let $w, u \in W$ and $s \in S$. Let $u < us, w < ws$ and $u \leqslant w$.*

(1) If $us \notin [u, w]$, then $\Phi_{[us,ws]} = \Phi_{[u,w]}$, and

$$
\begin{aligned}
\Phi_{[u,ws]} &= \Phi_{Pyr([u,w])} \\
&= \frac{1}{2}\left(\Phi_{[u,w]} \cdot c + c \cdot \Phi_{[u,w]} + \sum_{v:u<v<w} \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]} \right).
\end{aligned}
$$

(2) If $us \in [u, w]$, then

$$
\begin{aligned}
\Phi_{[u,ws]} &= \Phi_{Pyr([u,w])} - \sum_{\substack{v:u<v<w \\ vs<v}} \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]} \\
&= \frac{1}{2}\left(\Phi_{[u,w]} \cdot c + c \cdot \Phi_{[u,w]} + \sum_{v:u<v<w} \sigma_s(v)\Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]} \right).
\end{aligned}
$$

We claim that there are recurrence relations of the cd-indices of intervals in $\mathcal{MP}_n$, and also in P_n. For $v \in \mathcal{MP}_n$ and $s = s_i \in S$ and $s' = s_{i+n} \in S$, define $\sigma_s(v) := \frac{1}{2}[\ell(svs') - \ell(v)] \in \{-1, 1\}$.

Theorem 18. *Let $u, w \in \mathcal{MP}_n$ and let $s = s_i$ and $s' = s_{i+n}$ for some $0 \leqslant i < 2n$. Let $u < sus', w < sws'$ and $u \leqslant w$.*

(1) If $sus' \notin [u, w]$, then $\Phi_{[sus',sws']} = \Phi_{[u,w]}$, and

$$
\begin{aligned}
\Phi_{[u,sws']} &= \Phi_{Pyr([u,w])} \\
&= \frac{1}{2}\left(\Phi_{[u,w]} \cdot c + c \cdot \Phi_{[u,w]} + \sum_{\substack{v \in \mathcal{MP}_n \\ u<v<w}} \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]} \right).
\end{aligned}
$$

(2) If $sus' \in [u, w]$, then

$$
\begin{aligned}
\Phi_{[u,sws']} &= \Phi_{Pyr([u,w])} - \sum_{\substack{v \in \mathcal{MP}_n \\ u < v < w \\ svs' < v}} \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]} \\
&= \frac{1}{2} \left(\Phi_{[u,w]} \cdot c + c \cdot \Phi_{[u,w]} + \sum_{\substack{v \in \mathcal{MP}_n \\ u < v < w}} \sigma_s(v) \Phi_{[u,v]} \cdot d \cdot \Phi_{[v,w]} \right).
\end{aligned}
$$

Remark 19. We state but do not prove the recurrence relation of the cd-index of the interval $[sus', sws']$ if $sus' \in [u, w]$. In order to prove this, we need a poset operation Sh_v of vertex shaving (see Chapter 5 in [13] for details).

$$
\begin{aligned}
\Phi_{[sus',sws']} &= Sh_{sus'} \Phi_{[u,w]} - \sum_{\substack{v \in \mathcal{MP}_n \\ sus' < v < w \\ svs' < v}} \Phi_{[sus',v]} \cdot d \cdot \Phi_{[v,w]} \\
&= \Phi_{[u,w]} + \frac{1}{2} \left(\Phi_{[sus',w]} \cdot c - c \cdot \Phi_{[sus',w]} + \sum_{\substack{v \in \mathcal{MP}_n \\ sus' < v < w}} \sigma_s(v) \Phi_{[sus',v]} \cdot d \cdot \Phi_{[v,w]} \right).
\end{aligned}
$$

Example 20. Let $u = e$ and $w = s_1 s_2 s_5 s_4$ in $\mathcal{MP}_3$. The Hasse diagrams for intervals $[u, w]$, $[s_3 u s_0, s_3 w s_0]$, $[u, s_3 w s_0]$ and $[u, s_2 w s_5]$ are in Figure 3.

(1) Let $s = s_3$ and $s' = s_0$. Observe that $u < sus'$, $w < sws'$, $u \leqslant w$ and $sus' \notin [u, w]$. We see that $\Phi_{[sus',sws']} = \Phi_{[s_3 s_0, s_3 s_1 s_2 s_5 s_4 s_0]} = c = \Phi_{[e, s_1 s_2 s_5 s_4]} = \Phi_{[u,w]}$. Notice that the intervals $[u, w]$ and $[s_3 u s_0, s_3 w s_0]$ are isomorphic to Boolean lattice B_2. The cd-index of B_2 is $\Phi_{B_2}(c, d) = c$. We also have

$$
\begin{aligned}
\Phi_{[u,sws']} &= \Phi_{[e, s_3 s_1 s_2 s_5 s_4 s_0]} \\
&= \frac{1}{2} \left(\Phi_{[e, s_1 s_2 s_5 s_4]} \cdot c + c \cdot \Phi_{[e, s_1 s_2 s_5 s_4]} + \sum_{e < v < s_1 s_2 s_5 s_4} \Phi_{[e,v]} \cdot d \cdot \Phi_{[v, s_1 s_2 s_5 s_4]} \right) \\
&= \frac{1}{2} \left(c \cdot c + c \cdot c + \Phi_{[e, s_1 s_4]} \cdot d \cdot \Phi_{[s_1 s_4, s_1 s_2 s_5 s_4]} + \Phi_{[e, s_2 s_5]} \cdot d \cdot \Phi_{[s_2 s_5, s_1 s_2 s_5 s_4]} \right) \\
&= \frac{1}{2} (c^2 + c^2 + d + d) = c^2 + d.
\end{aligned}
$$

Note that the interval $[u, s_3 w s_0]$ is isomorphic to Boolean lattice B_3. The cd-index of B_3 is $\Phi_{B_3}(c, d) = c^2 + d$.

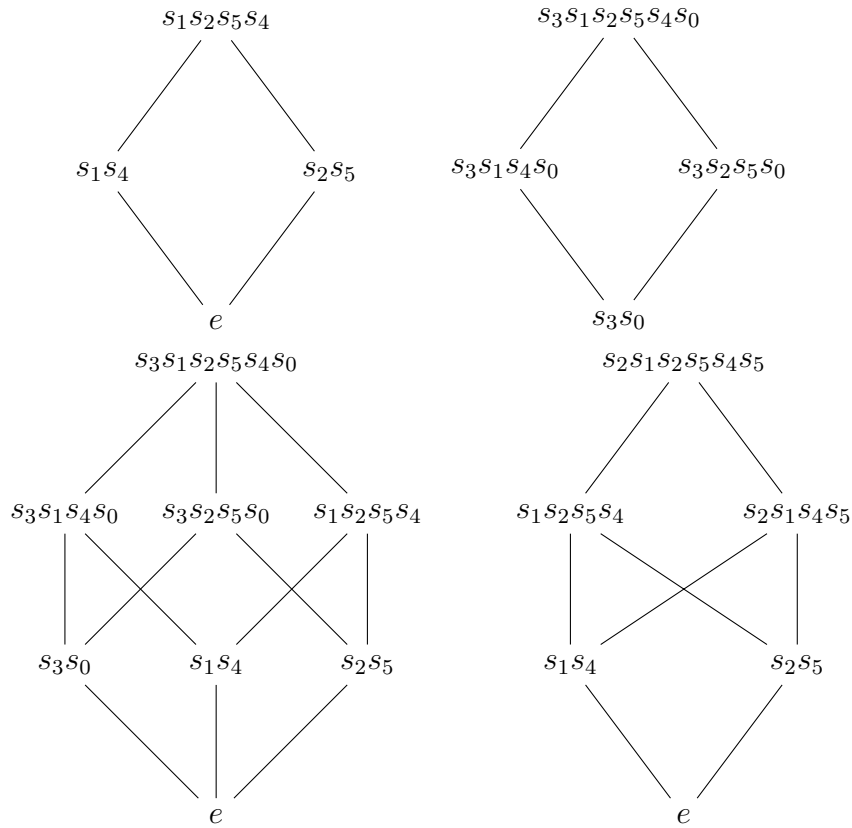(2) Let $s = s_2$ and $s' = s_5$. Observe that $u < sus'$, $w < sws'$, $u \leqslant w$ and $sus' \in [u, w]$.

Figure 3: The Hasse diagrams for some intervals in $\mathcal{MP}_3$

Note that $\sigma_{s_2}(s_1 s_4) = \frac{1}{2}(4 - 2) = 1$ and $\sigma_{s_2}(s_2 s_5) = \frac{1}{2}(0 - 2) = -1$. We see that

$$
\begin{aligned}
\Phi_{[u, sws']} &= \Phi_{[e, s_2 s_1 s_2 s_5 s_4 s_5]} \\
&= \frac{1}{2}\left(\Phi_{[e, s_1 s_2 s_5 s_4]} \cdot c + c \cdot \Phi_{[e, s_1 s_2 s_5 s_4]} + \sum_{e < v < w} \sigma_{s_2}(v) \Phi_{[e, v]} \cdot d \cdot \Phi_{[v, s_1 s_2 s_5 s_4]} \right) \\
&= \frac{1}{2}\left(c \cdot c + c \cdot c + \Phi_{[e, s_1 s_4]} \cdot d \cdot \Phi_{[s_1 s_4, s_1 s_2 s_5 s_4]} - \Phi_{[e, s_2 s_5]} \cdot d \cdot \Phi_{[s_2 s_5, s_1 s_2 s_5 s_4]} \right) \\
&= \frac{1}{2}(c^2 + c^2 + d - d) = c^2.
\end{aligned}
$$

Observe that the interval $[u, s_2 w s_5]$ is isomorphic to Bruhat order of symmetric group S_3. The cd-index of Bruhat order of S_3 is $\Phi_{S_3}(c, d) = c^2$.

6 Proof of Theorem 18

In this section we let $u, w \in \mathcal{MP}_n$ and let $s = s_i \in \tilde{S}$ and let $s' = s_{i+n} \in \tilde{S}$. Suppose $u < sus'$ and $w < sws'$. We slightly modify Reading's [13] map η to define a map

$\eta : [u, w] \times [e, ss'] \to [u, sws']$, as follows:

$$\eta(v, e) = v$$

$$\eta(v, ss') = \begin{cases} svs' & \text{if } svs' > v \\ v & \text{if } svs' < v \end{cases}$$

The following proposition is known as Lifting Property of Bruhat order.

Proposition 21 ([4], Proposition 1.2 (Lifting Property of Bruhat order)). *Let (W, S) be a Coxeter system. Let $u, w \in W$ and $s \in S$. If $w > ws$ and $us > u$, then the following are equivalent:*

(i) $w > u$

(ii) $ws > u$

(iii) $w > us$.

We will need the following proposition, which is the analogue of Proposition 21 for $\mathcal{MP}_n$.

Proposition 22 (Lifting Property of $\boldsymbol{\mathcal{MP}_n}$). *Let $u, w \in \mathcal{MP}_n$ and let $s = s_i \in \tilde{S}$. Let $s' = s_{i+n} \in \tilde{S}$. If $w > sws'$ and $sus' > u$, then the following are equivalent:*

(i) $w > u$

(ii) $sws' > u$

(iii) $w > sus'$.

Proof. Since $w > sws'$, by transitivity (ii) implies (i). Since $sus' > u$, by transitivity (iii) implies (i). So assume (i) $w > u$. Choose a reduced decomposition for $sws' = t_1 t_2 \ldots t_{2q}$ where $t_i \in \tilde{S}$ for all $i \in [2q]$ such that the reduced word is modular palindromic. Then $w = st_1 t_2 \ldots t_{2q} s'$ is also reduced and modular palindromic. There is a reduced decomposition for u

$$u = t_{i_1} t_{i_2} \ldots t_{i_{2r}}$$

which is a subword of $w = st_1 t_2 \ldots t_{2q} s'$. Since $sus' > u$, we have $t_{i_1} \neq s$ and $t_{i_{2r}} \neq s'$, and thus both (ii) $sws' > u$ and (iii) $w > sus'$ hold as desired. $\qquad\square$

We claim that the map η is well-defined. To show this, let $v \in [u, w]$. Because we have assumed $w < sws'$ we know that $v \in [u, w] \subset [u, sws']$, so we may assume that $\eta(v, ss') = svs'$. In this case, $v < svs'$, thus we see $u \leqslant v < svs' < sws'$ where the last inequality is due to the lifting property, and therefore $\eta(v, ss') \in [u, sws']$. The following proposition and Proposition 5.1 in [13] are the same statement on different posets: Bruhat order of Coxeter groups and the induced subposet $\mathcal{MP}_n$. Here, we show the order-preserving part of the statement on the induced subposet $\mathcal{MP}_n$. For the proof of the surjective part, see Proposition 5.1 in [13].

Proposition 23. *If $u < sus'$ and $w < sws'$, then $\eta : [u, w] \times [e, ss'] \to [u, sws']$ is an surjective order-preserving map.*

Proof. Suppose $(v_1, a_1) \leqslant (v_2, a_2)$ in $[u, w] \times [e, ss']$. Since $e \leqslant a_1 \leqslant a_2 \leqslant ss'$, so we break up into three cases.

Case 1: $a_1 = e$ and $a_2 = e$. Then $\eta(v_1, a_1) = v_1 \leqslant v_2 = \eta(v_2, a_2)$.

Case 2: $a_1 = e$ and $a_2 = ss'$. Then either $\eta(v_2, a_2) = v_2$ with $v_2 > sv_2s'$ or $\eta(v_2, a_2) = sv_2s'$ with $sv_2s' > v_2$. In either cases, we see that $\eta(v_2, a_2) \geqslant v_1$.

Case 3: $a_1 = ss'$ and $a_2 = ss'$. Then either $\eta(v_1, a_1) = v_1$ with $v_1 > sv_1s'$ or $\eta(v_1, a_1) = sv_1s'$ with $sv_1s' > v_1$. We also have either $\eta(v_2, a_2) = v_2$ with $v_2 > sv_2s'$ or $\eta(v_2, a_2) = sv_2s'$ with $sv_2s' > v_2$. We break up this case into four subcases.

Subcase 3-1: $\eta(v_1, a_1) = v_1$ and $\eta(v_2, a_2) = v_2$. Then, $\eta(v_1, a_1) \leqslant \eta(v_2, a_2)$.

Subcase 3-2: $\eta(v_1, a_1) = v_1$ and $\eta(v_2, a_2) = sv_2s'$. Then, $\eta(v_1, a_1) = v_1 \leqslant v_2 < sv_2s' = \eta(v_2, a_2)$.

Subcase 3-3: $\eta(v_1, a_1) = sv_1s'$ and $\eta(v_2, a_2) = v_2$. Then, $\eta(v_1, a_1) = sv_1s' \leqslant v_2 = \eta(v_2, a_2)$ by the lifting property.

Subcase 3-4: $\eta(v_1, a_1) = sv_1s'$ and $\eta(v_2, a_2) = sv_2s'$. Then, $\eta(v_1, a_1) = sv_1s' \leqslant sv_2s' = \eta(v_2, a_2)$ by the lifting property. $\qquad\square$

For every $v \in [u, w]$ with $svs' < v$, notice that the image of the elements (v, e), (svs', ss'), (v, ss') under the map η is the single element v. From this observation, we let $v_1, v_2, \ldots, v_k$ be a linear ordering of the elements of the set $Z = \{v : u < v < w, svs' < v\}$ such that the ranks (or lengths) of elements are weakly increasing, in other words, $\ell(v_1) \leqslant \ell(v_2) \leqslant \ldots \leqslant \ell(v_k)$. Define posets Q_i for $0 \leqslant i \leqslant k$ recursively as follows. Let $Q_0 = [u, w] \times [e, ss']$. Let Q_i to be the poset obtained by zipping $\{(v_i, e), (sv_is', ss'), (v_i, ss')\}$ in Q_{i-1}. In the following proposition, we show that this is indeed a proper zipping.

Proposition 24. *The triples $\{(v_i, e), (sv_is', ss'), (v_i, ss')\}$ in Q_{i-1} for $1 \leqslant i \leqslant k$ are zippers.*

Proof. First, we claim that $(v_i, e), (sv_is', ss')$ and (v_i, ss') are elements of Q_{i-1}. The element (v_i, ss') has not been deleted yet, and we have not identified (v_i, ss') with any element because it is at a rank higher than we have yet made identifications. The only elements ever deleted are of the form (x, ss') where $x > sxs'$, so (v_i, e) and (sv_is', ss') have not been deleted. The only identification one could make involving (v_i, e) and (sv_is, ss') is to identify them to each other, and that has not happened yet, and thus the claim is proved. Second, we check the conditions in the definition of a zipper. Suppose $(x, a) < (v_i, e)$, then $x < v_i$ and $a = e$. Then $x < sxs'$; otherwise the triple $\{(sxs', e), (x, ss'), (x, e)\}$ is a zipper in Q_{i-1} with $\ell(x) < \ell(v_i)$, which is impossible. Then we have $x \leqslant sv_is'$ by lifting property and thus $(x, a) < (sv_is', ss')$. Now suppose $(x, a) < (sv_is', ss')$; then either $a = e$ or $a = ss'$. If $a = e$ then $(x, a) < (v_i, e)$ since $x \leqslant sv_is' < v_i$. Assume $a = ss'$. Then $x < sv_is'$ which implies $\ell(x) < \ell(sv_is')$ so x is not v_r for any $r \leqslant i$, and then $x < sxs'$. Then the triple $\{(x, ss'), (sxs', e), (sxs', ss')\}$ is a zipper with $\ell(sxs') < \ell(v_i)$ in Q_{i-1}, which is impossible. Hence the triple satisfies the first condition $\{(x, a) : (x, a) < (v_i, e)\} = \{(x, a) : (x, a) < (sv_is', ss')\}$. The second condition is obvious
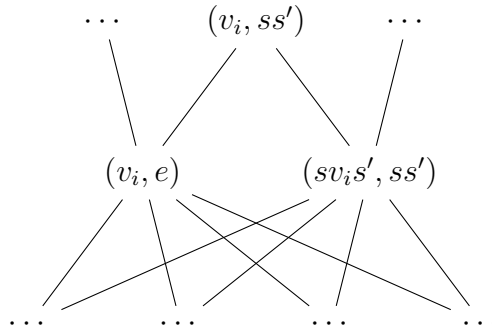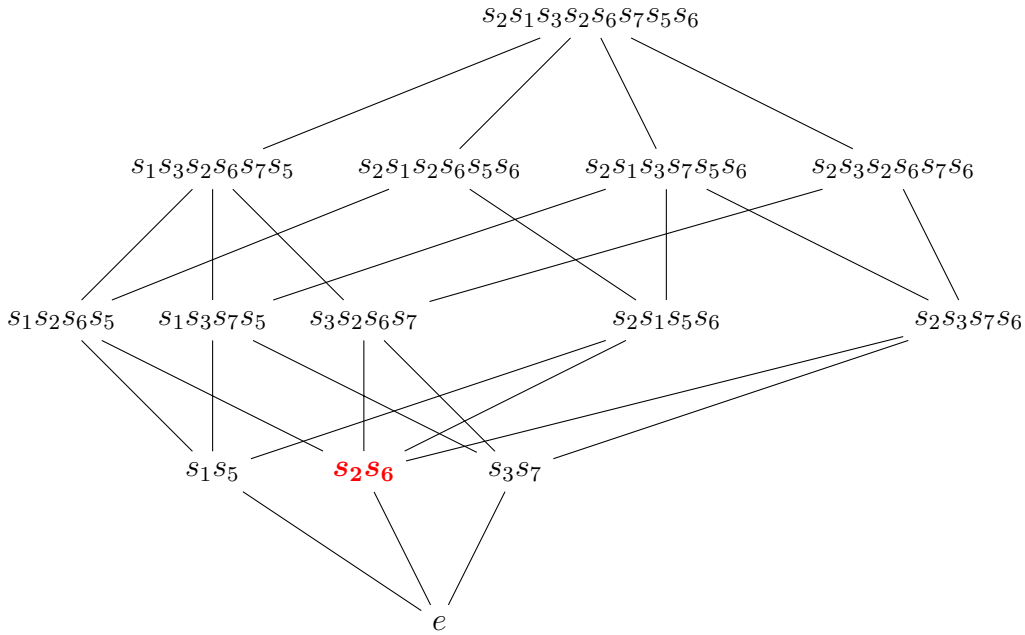
because $(v_i, e) \lessdot (v_i, ss')$ and $(sv_is', ss') \lessdot (v_i, ss')$. Assume (x, a) is covered by (v_i, ss'). If $a = e$ then $x = v_i$, so $(x, a) = (v_i, e)$. If $a = ss'$ then $x \lessdot v_i$. Since $sv_is' < v_i$ and $x < sxs'$, by the lifting property $x \leqslant sv_is'$ which implies $x = sv_is'$. Hence (v_i, ss') covers no other element than (v_i, e) and (sv_is', ss'), and thus the third condition holds. Therefore, the triple $\{(v_i, e), (sv_is', ss'), (v_i, ss')\}$ is a zipper in Q_{i-1}. $\qquad\square$

In terms of zipper and operation zip, we can think of the image of $[u, w] \times [e, ss']$ under the map η as a sequence of zipping operations of the triples $\{(v_i, e), (sv_is', ss'), (v_i, ss')\}$ for $v_i \in Z$.

Example 25. Let $u = e, w = s_1s_3s_2s_6s_7s_5$ in $\mathcal{MP}_4$ and let $s = s_2$ and $s' = s_6$. Then, $u < sus', w < sws'$ and $u < w$. The Hasse diagram of $[u, w] \times [e, ss']$ is
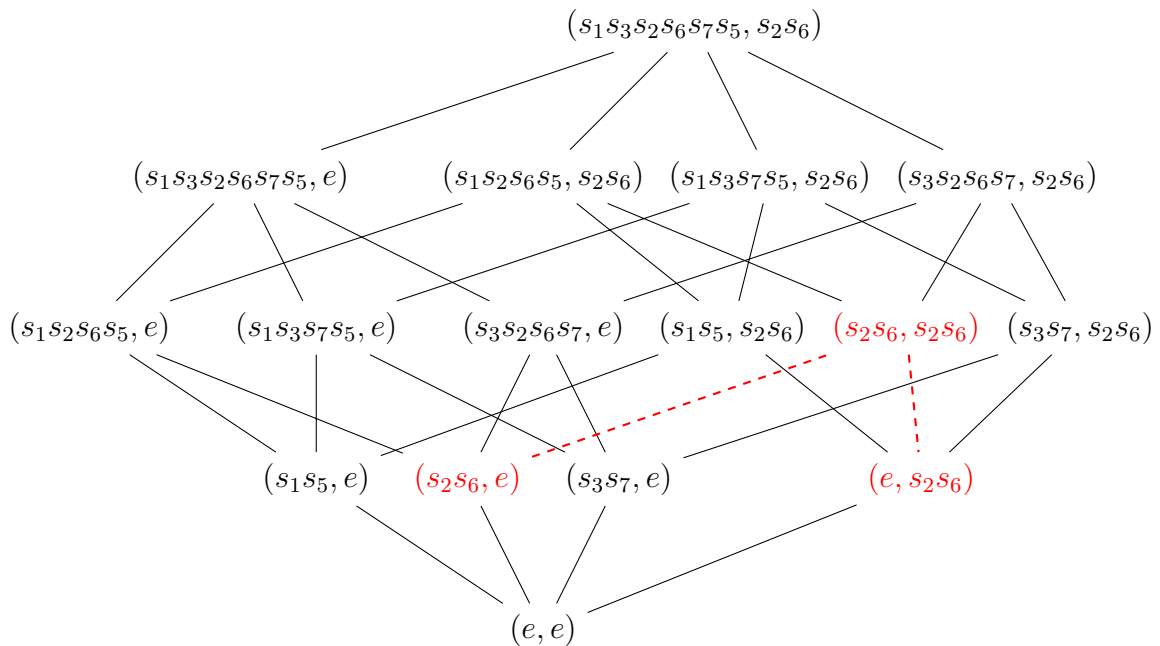
We consider the map $\eta : [u, w] \times [e, ss'] \to [u, sws']$. After zipping the zipper $\{(s_2s_6, e), (e, s_2s_6), (s_2s_6, s_2s_6)\}$, we get

$$s_2s_1s_3s_2s_6s_7s_5s_6$$

$$s_1s_3s_2s_6s_7s_5 \qquad s_2s_1s_2s_6s_5s_6 \qquad s_2s_1s_3s_7s_5s_6 \qquad s_2s_3s_2s_6s_7s_6$$

$$s_1s_2s_6s_5 \quad s_1s_3s_7s_5 \quad s_3s_2s_6s_7 \qquad\qquad s_2s_1s_5s_6 \qquad\qquad s_2s_3s_7s_6$$

$$s_1s_5 \qquad \mathbf{\color{red}{s_2s_6}} \qquad s_3s_7$$

$$e$$

As the following proposition is an analogue of Proposition 5.2 in [13], we omit the proof.

Proposition 26. *Let $u < sus'$ and $w < sws'$ and $sus' \not\leqslant w$. Then $svs' > v$ for all $v \in [u, w]$, and η is an order-preserving bijection.*

Proposition 26 directly implies the following corollary.

Corollary 27. *Let $u < sus'$ and $w < sws'$ and $sus' \not\leqslant w$. Then the map $\theta : [u, w] \to [sus', sws']$ with $\theta(v) = svs'$ is an order-preserving bijection.*

In other words, if the condition in Corollary 27 holds, then the map θ makes a copy $[sus', sws']$ of the interval $[u, w]$.

We have proven the following theorem which is an analogue of Theorem 5.5 in [13].

Theorem 28. *Let $w < sws'$ and $u < sus'$ and $u \leqslant w$. If $sus' \notin [u, w]$ then $[u, sws'] \cong [u, w] \times [1, ss']$ and $[sus', sws'] \cong [u, w]$. If $sus' \in [u, w]$, then $[u, sws']$ can be obtained from $[u, w] \times [1, ss']$ by a sequence of zippings.*

Now we are ready to prove Theorem 18. This proof is essentially the same as the proof of Theorem 6.1 in [13]. Here we reproduce this proof for completeness.

Proof of Theorem 18. Part (1) follows from Proposition 26 and Corollary 27. So we focus on part (2), in the case of $sus' \in [u, w]$. Define the posets Q_i as in Proposition 24. By Theorem 15,

$$\Phi_{Q_{i-1}} - \Phi_{Q_i} = \Phi_{[(u,e),(v_i,e)]} \cdot d \cdot \Phi_{[(v_i,ss'),(w,ss')]}.$$

where intervals in the right hand side are in Q_{i-1}. Since $Q_k = [u, sws']$, sum from $i = 1$ to $i = k$ to obtain

$$\Phi_{[u,sws']} = \Phi_{Q_0} - \sum_{i=1}^{k} \Phi_{[(u,e),(v_i,e)]} \cdot d \cdot \Phi_{[(v_i,ss'),(w,ss')]}.$$

where intervals in right hand side are in Q_{i-1}. Notice that the interval $[(u,e),(v_j,e)]$ in Q_{j-1} is isomorphic to the interval $[(u,e),(v_j,e)]$ in Q_0 which is also isomorphic to $[u, v_j]$. Similarly, the interval $[(v_j, ss'),(w, ss')]$ in Q_{j-1} is isomorphic to the interval $[(v_j, ss'),(w, ss')]$ in Q_0 which is isomorphic to $[v_j, w]$, and the second part of the theorem is proved. $\qquad\square$

Therefore, we have a recurrence relation for the cd-indices $\Phi(c, d)$ of intervals in P_n, or equivalently, $\mathcal{MP}_n$. Then we also have a recurrence relation for the ab-indices $\Psi(a, b)$ of the intervals by the relation $\Psi(a, b) = \Phi(a + b, ab + ba)$. Recall that poset $\hat{P}_n$ is P_n with a unique minimum element $\hat{0}$ adjoined. We prove the following proposition which helps us compute ab-indices of intervals $[\hat{0}, \tau]$ in $\hat{P}_n$ recursively.

Proposition 29. *Let P be a graded poset of rank n with a unique maximum element $\hat{1}$ and multiple minimal elements such that every interval in P is Eulerian. Let $\hat{P}$ be the poset P with a unique minimum element $\hat{0}$ adjoined, where $\rho(\hat{0}) = -1$. If $\hat{P}$ is Eulerian, then the ab-index of $\hat{P}$ is given by*

$$\Psi_{\hat{P}}(a, b) = (a - b)^n + \sum_{i=0}^{n-1} (a - b)^i b \sum_{\rho(x)=i} \Psi_{[x,\hat{1}]}(a, b).$$

Proof. By the definition of ab-index and the definition of flag f-vector and h-vector,

$$
\begin{aligned}
\Psi_{\hat{P}}(a + b, b) &= \sum_{S \subseteq [n-1] \cup \{0\}} \alpha_P(S) u_S \\
&= \sum_{\hat{0} < t_1 < \cdots < t_{k-1} < \hat{1}} a^{\rho(\hat{0}, t_1)-1} b a^{\rho(t_1, t_2)-1} b \ldots b a^{\rho(t_{k-1}, \hat{1})-1},
\end{aligned}
$$

and thus

$$\Psi_{\hat{P}}(a, b) = \sum_{\hat{0} < t_1 < \cdots < t_{k-1} < \hat{1}} (a - b)^{\rho(\hat{0}, t_1)-1} b (a - b)^{\rho(t_1, t_2)-1} b \ldots b (a - b)^{\rho(t_{k-1}, \hat{1})-1}. \qquad (4)$$

We rearrange this summation in terms of t_1, the lowest element in the chain except $\hat{0}$. If there is no t_1 in the chain, then the summand will be $(a - b)^n$. If $\rho(t_1) = i \in \{0, 1, \ldots, n-1\}$, then the summand will be $(a-b)^i b[(a-b)^{\rho(t_1,t_2)-1} b \ldots b(a-b)^{\rho(t_{k-1},\hat{1})-1}]$. By this observation, we can write (4) as

$$
\begin{aligned}
\Psi_{\hat{P}}(a, b) &= (a - b)^n + \sum_{i=0}^{n-1} (a - b)^i b \sum_{t_1 < \cdots < t_{k-1} < \hat{1}} (a - b)^{\rho(t_1, t_2)-1} b \ldots b(a - b)^{\rho(t_{k-1}, \hat{1})-1} \\
&= (a - b)^n + \sum_{i=0}^{n-1} (a - b)^i b \sum_{\rho(t_1)=i} \Psi_{[t_1, \hat{1}]}(a, b)
\end{aligned}
$$

as desired. $\square$

The previous proposition directly implies the following theorem. We have recursions for the ab-indices of the poset $\hat{P}_n$ and its intervals $[\hat{0}, \tau]$.

Theorem 30. *The ab-index of $\hat{P}_n$ is recursively given by*

$$\Psi_{\hat{P}_n}(a,b) = (a-b)^{\binom{n}{2}} + \sum_{i=0}^{\binom{n}{2}-1} (a-b)^i b \sum_{\ell(x)=2\binom{n}{2}-2i} \Psi_{[e,x]}(a,b)$$

where $x \in \mathcal{MP}_n$. Let $\tau \in P_n$ such that $c(\tau) = k \leqslant \binom{n}{2}$ and $\phi(\tau) = w \in \mathcal{MP}_n$. The ab-index of the interval $[\hat{0}, \tau] \subset \hat{P}_n$ is recursively given by

$$\Psi_{[\hat{0},\tau]}(a,b) = (a-b)^k + \sum_{i=0}^{k-1} (a-b)^i b \sum_{\substack{x:x>w \\ \ell(x)=2\binom{n}{2}-2i}} \Psi_{[w,x]}(a,b).$$

Acknowledgements

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