Perfect matching covers of cubic graphs of oddness 2

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Submitted: Jul 25, 2017; Accepted: Feb 27, 2019; Published: Mar 8, 2019

Abstract

A perfect matching cover of a graph G is a set of perfect matchings of G such that each edge of G is contained in at least one member of it. Berge conjectured that every bridgeless cubic graph has a perfect matching cover of order at most 5. The Berge Conjecture is largely open and it is even unknown whether a constant integer c does exist such that every bridgeless cubic graph has a perfect matching cover of order at most c. In this paper, we show that a bridgeless cubic graph G has a perfect matching cover of order at most 11 if G has a 2-factor in which the number of odd circuits is 2.

Mathematics Subject Classifications: 05C70

1 Introduction

Only finite and simple graphs are considered in this paper. A *k-factor* of a graph G is a spanning k-regular subgraph of G. The set of edges in a 1-factor of a graph G is called a *perfect matching* of G. A *matching* of a graph G is a set of edges in a 1-regular subgraph of G. A *near-perfect matching* of a graph G is a matching of G which misses exactly one vertex of G. A *perfect matching cover* of a graph G is a set of perfect matchings of G such that each edge of G is contained in at least one member of it. The *order* of a perfect matching cover is the number of perfect matchings in it.

One of the first theorems in graph theory, Petersen's Theorem from 1891 [17], states that every bridgeless cubic graph has a perfect matching. By Tutte's 1-factor theorem from 1947 [20], we can obtain that every edge in a bridgeless cubic graph G is contained

[*]Supported by Shandong Provincial Natural Science Foundation of China grant ZR2017PA012 and by NSFC grant 11701332.
[†]Supported by NSFC grant 11501282

in a perfect matching of G. This implies that every bridgeless cubic graph has a perfect matching cover. What is the minimum number k such that every bridgeless cubic graph has a perfect matching cover of order k? Berge conjectured this number is 5 (unpublished, see e.g. [6, 7]).

Conjecture 1 (Berge Conjecture)**.** Every bridgeless cubic graph has a perfect matching cover of order at most 5.

The following stronger conjecture is attributed to Berge in [18], and was first published in a paper by Fulkerson [3].

Conjecture 2 (Fulkerson Conjecture)**.** Every bridgeless cubic graph has six perfect matchings such that each edge belongs to exactly two of them.

Conjectures 1 and 2, which are proved to be equivalent by Mazzuoccolo [7], are largely open in graph theory. The equivalence of these two conjectures does not imply that Conjecture 2 holds for a given bridgeless cubic graph satisfying Conjecture 1. It is still an open question whether this holds.

Clearly, Conjectures 1 and 2 hold true for 3-edge-colorable cubic graphs. Conjecture 2 has also been verified for some non-3-edge-colorable cubic graphs, such as flower snarks, Goldberg snarks and Loupekine snarks [5, 12, 14]. Besides the above snarks, Conjecture 1 has been verified for some special classes of cubic graphs. Steffen [16] showed that Conjecture 1 holds for bridgeless cubic graphs which have no nontrivial 3-edge-cuts and have 3 perfect matchings which miss at most 4 edges. It is proved by Hou et al. [10] that every almost Kotzig graph (defined in [10]) has a perfect matching cover of order at most 5. In [19], one author of this paper showed that a cubic graph G with n vertices has a perfect matching cover of order at most 5 if G has a circuit of length $n-1$ or has a 2-factor with exactly two circuits.

Esperet and Mazzuoccolo [2] and Abreu et al. [1] showed respectively that there are infinite cubic graphs of which every perfect matching cover has order at least 5. Mazzuoccolo [8] showed that every bridgeless cubic graph G with m edges has 5 perfect matchings which cover at least $\lceil \frac{215}{231} m \rceil$ edges of G. Esperet and Mazzuoccolo [2] also proved that the problem that deciding whether a bridgeless cubic graph has a perfect matching cover of order at most 4 is NP-complete.

As Conjecture 1 is a problem with large challenges, the following weaker problem (suggested by Berge) maybe should be considered firstly.

Problem 3. Is there a constant integer c such that every bridgeless cubic graph has a perfect matching cover of order at most c?

Unluckily, this weaker problem is still open. If we don't restrict such integer c in Problem 3 to be constant, the best upper bound of c, which was given by Mazzuoccolo [8], is logarithmic in the number of vertices.

In this paper, we consider the perfect matching covers of bridgeless cubic graphs of oddness 2. Recall that the *oddness* of a bridgeless cubic graph G is the minimum number of odd circuits in a 2-factor of G. Since oddness is always an even integer and oddness zero means that the graph is 3-edge-colorable, cubic graphs of oddness 2 constitute a natural class of graphs with structure close to that of 3-edge-colorable graphs. The famous Cycle Double Cover Conjecture have been verified for bridgeless cubic graphs of oddness at most 4 [9, 11]. Recently, Máčajová and Škoviera [15] proved that Fan and Raspaud Conjecture [4], which states that every bridgeless cubic graph has three perfect matchings with empty intersection, holds for bridgeless cubic graphs of oddness 2.

In this paper, we show that such constant integer c in Problem 3 exists under the assumption that the oddness of a bridgeless cubic graph is 2. Our result is as follows.

Theorem 4. *If G is a connected bridgeless cubic graph of oddness 2, then G has a perfect matching cover of order at most 11.*

2 Notations and two technical lemmas

Some notations and notions will be used in this paper. Let G be a graph with vertex-set $V(G)$ and edge-set $E(G)$. The *order* of G is the size of $V(G)$. The *degree* of a vertex u of G is the number of edges incident to u in G. For $X \subseteq V(G)$, we denote by $G - X$ the subgraph of G which is obtained from G by deleting all vertices in X and all edges incident to a vertex in X. If $X = \{u\}$, then $G - X$ is usually written to $G - u$ in short. For two graphs H_1 and H_2, we denote by $H_1 \cup H_2$ the graph with vertex-set $V(H_1) \cup V(H_2)$ and edge-set $E(H_1) \cup E(H_2)$. For a positive integer n, we denote by $[n]$ the set $\{1, 2, \dots, n\}$. For two set Y_1 and Y_2, we denote by $Y_1 \triangle Y_2$ the set $(Y_1 \backslash Y_2) \cup (Y_2 \backslash Y_1)$.

For a path P in G, we denote by $V_0(P)$ the set consisting of the two ends of P and denote by $E_0(P)$ the set of the edges in P which have an end in $V_0(P)$. For $X_1, X_2 \subseteq V(G)$ with $X_1 \cap X_2 = \emptyset$, we say that a path P in G is *from* X_1 *to* X_2 if $V_0(P) \cap X_i \neq \emptyset$ for $i = 1, 2$ and $(V(P) \backslash V_0(P)) \cap (X_1 \cup X_2) = \emptyset$. For $F_1, F_2 \subseteq E(G)$, a path or a circuit in G is called F_1-F_2 *alternating* if its edges are alternating in F_1 and in F_2. An F_1-F_2 alternating path (or circuit) P is called F_1-*alternating* if $F_2 = E(G) \backslash F_1$. An F_1-F_2 alternating path P is called F_1-F_2-F_1 *alternating* if $E_0(P) \subseteq F_1$. An F_1-F_2-F_1 alternating path P is called an F_1-F_2-F_1 *ear* of a subgraph H of G if $V(P) \cap V(H) = V_0(P)$ and $E(P) \cap E(H) = \emptyset$.

Now we present two technical lemmas.

Lemma 5. *Let G be a graph of odd order which has a hamiltonian circuit C and let X be the set of vertices of degree 2 in G. Suppose $|X| \geqslant 3$ and that every vertex in G has degree at most 3. For a vertex u in X, let M_1 be the perfect matching of $C - u$. Then G has three near-perfect matchings M_2, M_3 and M_4 such that $E(C) \backslash M_1 \subseteq \bigcup_{i=2}^{4} M_i$, M_2 misses u and each of M_3 and M_4 misses a vertex in $X \backslash \{u\}$.*

Proof. Set $E_1 {:=} E(C) \backslash M_1$ and $E_2 {:=} E(G) \backslash E(C)$. Let H be the subgraph of G with vertex-set $V(G)$ and edge-set $E_1 \cup E_2$. Let D be the component of H which contains u. Noting that the degree of each vertex in G is 2 or 3, we know that D is a path or a circuit.

Assume that D is a path. We know $V_0(D) \subseteq X \backslash \{u\}$. Let P_1 and P_2 be the two edge-disjoint paths from u to $V_0(D)$ in D. We know that both P_1 and P_2 are E_1-E_2-E_1

alternating. For $i = 1, 2$, set $M_{i+1} := E(P_i) \triangle E_1$. We know that each of M_2 and M_3 misses an end of D and we have $E(C) \backslash M_1 \subseteq M_2 \cup M_3$. So M_1, M_2 and M_3 are 3 near-perfect matchings of G which meet the requirements.

Assume that D is a circuit. Set $M_4 := E(D) \triangle E_1$. We know $E_1 \backslash M_4 = E_1 \cap E(D)$ and that M_4 is a near-perfect matching of G which misses u. As $|X| \geqslant 3$, G has an E_1-E_2-E_1 alternating path P_3 with two ends in $X \backslash \{u\}$. Let P_4 and P_5 be the two edge-disjoint paths from u to $V(P_3)$ in C. For $i = 4, 5$, let P_i' be the path from $V_0(P_i) \backslash \{u\}$ to $V_0(P_3)$ in P_3 such that $P_i \cup P_i'$ is E_1-alternating. For $i = 4, 5$, set $M_{i+1} := E(P_i \cup P_i') \triangle E_1$. We know that each of M_5 and M_6 misses an end of P_3. Noting $E_1 \backslash M_4 = E_1 \cap E(D)$ and $E(P_4) \cap E(P_5) = \emptyset$, we have $E(C) \backslash M_1 \subseteq M_4 \cup M_5 \cup M_6$. So M_4, M_5 and M_6 are 3 near-perfect matchings of G which meet the requirements. □

Lemma 6. *Let M be a matching of a cubic graph G and let C be an M-alternating circuit of G. Suppose that P_1 and P_2 are two vertex-disjoint M-alternating paths of G such that $\emptyset \neq E(P_i) \cap E(C) \subseteq M$ for $i = 1, 2$, $E_0(P_1) \subseteq E(C)$ and $V_0(P_2) \cap V(C) = \emptyset$. Then G has two M-alternating paths P_3 and P_4 from $V_0(P_1)$ to $V_0(P_2)$ in $C \cup P_1 \cup P_2$ such that $V_0(P_3) \cup V_0(P_4) = V_0(P_1) \cup V_0(P_2)$, $E_0(P_3) \cup E_0(P_4) = E_0(P_1) \cup E_0(P_2)$ and $E(P_3) \cap E(P_4) \subseteq E(P_1) \cup E(P_2)$.*

Proof. Noting $E(P_1) \cap E(C) \subseteq M$ and $E_0(P_1) \subseteq E(C)$, we have $E_0(P_1) \subseteq M$. Add two new vertices $u_{1,1}$ and $u_{1,2}$ such that $u_{1,1}$ is incident with an end of P_1 and $u_{1,2}$ is incident with the other end of P_1. Extend P_1 to a new path P_1' such that $V_0(P_1') = \{u_{1,1}, u_{1,2}\}$ and $V(P') \backslash V_0(P_1') = V(P_1)$. It suffice to show that there are two M-alternating paths P_3 and P_4 from $V_0(P_1')$ to $V_0(P_2)$ in $C \cup P_1' \cup P_2$ such that $V_0(P_3) \cup V_0(P_4) = V_0(P_1') \cup V_0(P_2)$, $E_0(P_3) \cup E_0(P_4) = E_0(P_1') \cup E_0(P_2)$ and $E(P_3) \cap E(P_4) \subseteq E(P_1') \cup E(P_2)$. This is because $P_3 - V_0(P_1')$ and $P_4 - V_0(P_1')$ are two paths that we need find if such P_3 and P_4 exist.

Let J_1, J_2, …, J_r be the (inclusionwise) maximal M-$(E(C) \backslash M)$-M alternating paths in C which have both ends in $V(P_1')$ and contain no edges in $E(P_2) \cap E(C)$. Let J_1', J_2', …, J_r' be the (inclusionwise) maximal paths in C which have both ends in $V(P_2)$ and contain no edges in $E(P_1') \cap E(C)$. We know that the paths in $\{J_1, J_2, \dots, J_r\}$ and the paths in $\{J_1', J_2', \dots, J_r'\}$ appear alternately in C. (See Fig. 1(a) for an example of J_1, J_2, …, J_r and J_1', J_2', …, J_r'.)

For each $i \in [r]$, we construct two paths $J_{i,1}$ and $J_{i,2}$ as follows. For each $i \in [r]$, we know $E_0(J_i) \subseteq E(P_1') \cap E(C) \subseteq M$ and let P_i'' be the path between the two ends of J_i in P_1'. If the length of P_i'' is odd, then for $j = 1, 2$, we denote by $J_{i,j}$ the path from $u_{1,j}$ to $V_0(J_i)$ in P_1' such that $E_0(J_{i,j}) \cap E_0(J_i) \neq \emptyset$. Next we assume that the length of P_i'' is even. We know that P_1' has an end $u_{1,\alpha}$ such that the path from $u_{1,\alpha}$ to $V_0(J_i)$ in P_1' contains no edges in $E_0(J_i)$. Let $P_{i,1}''$ be the path from $u_{1,\alpha}$ to $V(J_i)$ in P_1' and $P_{i,2}''$ be the M-$(E(C) \backslash M)$-M alternating path from $V_0(P_{i,1}'') \cap V(J_i)$ to $V_0(J_i)$ in J_i. Set $J_{i,\alpha} := P_{i,1}'' \cup P_{i,2}''$. Let β be the number in $\{1, 2\} \backslash \{\alpha\}$. Let $J_{i,\beta}$ be the path from $u_{1,\beta}$ to $V_0(J_i) \backslash V_0(J_{1,\alpha})$ in P_1'.

We can see that for each $i \in [r]$, $J_{i,1}$ and $J_{i,2}$ are two M-alternating path from $V_0(P_1')$ to $V_0(J_i)$ in $P_1' \cup J_i$ such that $u_{1,j} \in V_0(J_{i,j})$ for $i = 1, 2$, $(V_0(J_{i,1}) \cup V_0(J_{i,2})) \backslash V_0(P_1') = V_0(J_i)$, $E_0(J_{i,1}) \cup E_0(J_{i,2}) = E_0(P_1') \cup E_0(J_i)$ and $E(J_{i,1}) \cap E(J_{i,2}) \subseteq E(P_1')$.

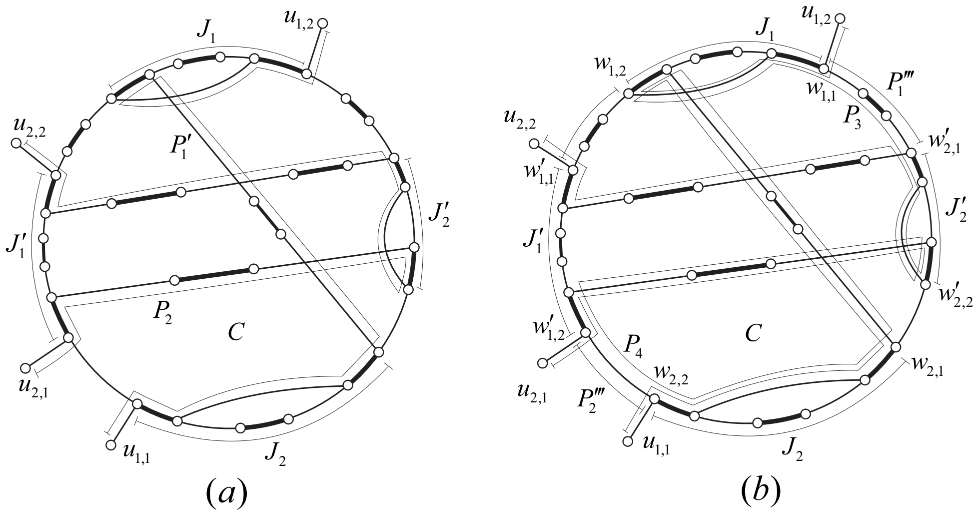

Figure 1: An example for the proof of Lemma 6. In this example, the edges in M are drawn as thick lines. In (b), $P_3 = J_{1,1} \cup P_1''' \cup J'_{2,1}$ and $P_4 = J_{2,2} \cup P_2''' \cup J'_{1,2}$.

Let $u_{2,1}$ and $u_{2,2}$ be the two ends of P_2. Similarly as the above statement, we can know that for each $i \in [r]$, G has two M-alternating paths $J'_{i,1}$ and $J'_{i,2}$ from $V_0(P_2)$ to $V_0(J'_i)$ in $P_2 \cup J'_i$ such that $u_{2,j} \in V_0(J'_{i,j})$ for $j = 1, 2$, $(V_0(J'_{i,1}) \cup V_0(J'_{i,2})) \backslash V_0(P_2) = V_0(J'_i)$, $E_0(J'_{i,1}) \cup E_0(J'_{i,2}) = E_0(P_2) \cup E_0(J'_i)$ and $E(J'_{i,1}) \cap E(J'_{i,2}) \subseteq E(P_2)$.

For each $i \in [r]$ and each $j \in \{1, 2\}$, let $w_{i,j}$ be the end of $J_{i,j}$ in C and $w'_{i,j}$ be the end of $J'_{i,j}$ in C. We know $V_0(J_i) = \{w_{i,1}, w_{i,2}\}$ and $V_0(J'_i) = \{w'_{i,1}, w'_{i,2}\}$ for each $i \in [r]$.

let P_1''' be the path from $w_{1,1}$ to $V(P_2)$ in C such that $E(P_1''') \cap E(P'_1) = \emptyset$. We know $V_0(P_1''') \backslash \{w_{1,1}\} \subseteq V_0(J'_{p_1})$ for some $p_1 \in [r]$. Let w'_{p_1,q_1} be the end of P_1''' in $V_0(J'_{p_1})$ and let q_2 be the number in $\{1, 2\} \backslash \{q_1\}$. Noting that the paths in $\{J_1, J_2, \ldots, J_r\}$ and the paths in $\{J'_1, J'_2, \ldots, J'_r\}$ appear alternately in C, we have that G has a path P_2''' in C which contains no edges in $E(P'_1) \cup E(P_2)$ and has an end in $\{w_{1,2}, w_{2,2}, \ldots, w_{r,2}\}$ and the other end in $\{w'_{1,q_2}, w'_{2,q_2}, \ldots, w'_{r,q_2}\}$. (See Fig. 1$(b)$ for an example of P_2'''.) Let $w_{p_2,2}$ and w'_{p_3,q_2} be the ends of P_2'''.

Set $P_3 {:=} J_{1,1} \cup P_1''' \cup J'_{p_1,q_1}$ and $P_4 {:=} J_{p_2,2} \cup P_2''' \cup J'_{p_3,q_2}$. (See Fig. 1$(b)$ for an example of P_3 and P_4.) We know that P_3 and P_4 are two M-alternating paths from $V_0(P'_1)$ to $V_0(P_2)$ in $C \cup P'_1 \cup P_2$ such that $V_0(P_3) \cup V_0(P_4) = V_0(P'_1) \cup V_0(P_2)$ and $E_0(P_3) \cup E_0(P_4) = E_0(P'_1) \cup E_0(P_2)$. If $p_2 \neq 1$, we know $E(J_{1,1}) \cap E(J_{p_2,2}) \subseteq E(P'_1)$, noting that $E(J_{1,j}) \subseteq E(P'_1 \cup J_j)$ for $j = 1, p_2$ and $E(J_1) \cap E(J_{p_2}) = \emptyset$. Noting also $E(J_{1,1}) \cap E(J_{1,2}) \subseteq E(P'_1)$, we have $E(J_{1,1}) \cap E(J_{p_2,2}) \subseteq E(P'_1)$. Similarly, we can obtain $E(J'_{p_1,q_1}) \cap E(J'_{p_3,q_2}) \subseteq E(P_2)$. Noting also $E(P_1''') \cap E(P_2''') = \emptyset$, we have $E(P_3) \cap E(P_4) \subseteq E(P'_1) \cup E(P_2)$. So P_3 and P_4 are two paths meeting the requirements. $\square$

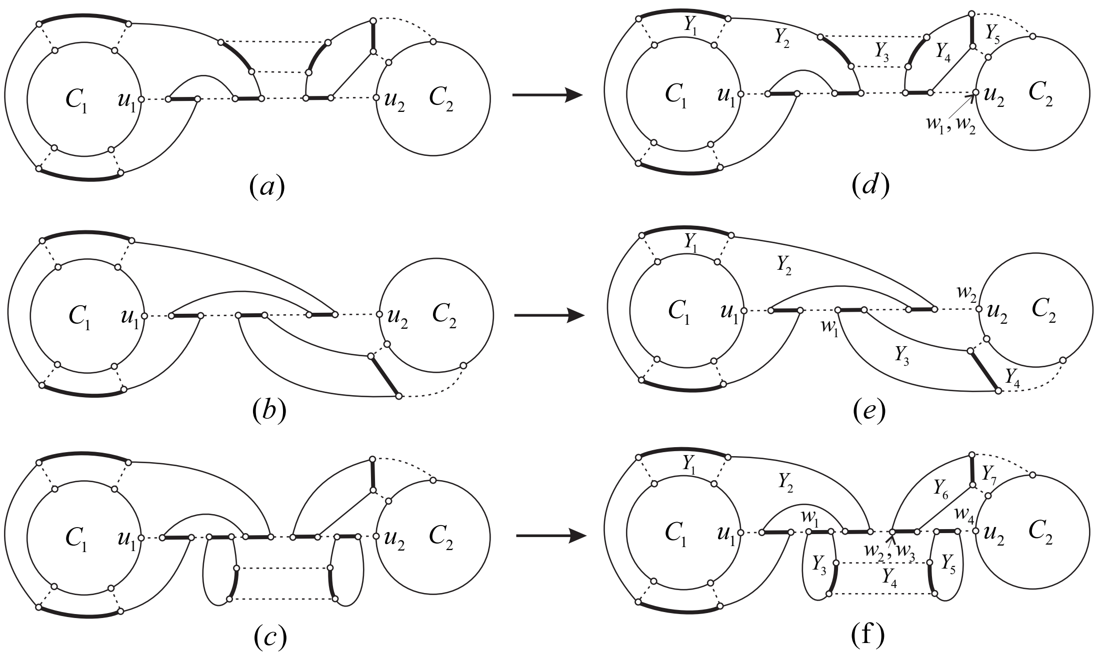

Figure 2: Examples of H_s and $Y_1Y_2\cdots Y_n$. In these examples in Fig. 2 and Fig. 3, the edges in M_0, M_1 and M_2 are drawn as dotted lines, thick lines and thin lines, respectively. In (a) and (b), the value of r is 0 when H_s is constructed by Algorithm 1. In (c), the value of r is 1 when H_s is constructed by Algorithm 1. In (d), G_2 has no directed edge $Y_i\rightarrow Y_j$ in $Y_1Y_2\cdots Y_n$ such that both Y_i and Y_j are M_1-M_2 alternating circuits. In (e) and (f), G_2 has an directed edge $Y_i\rightarrow Y_j$ in $Y_1Y_2\cdots Y_n$ such that both Y_i and Y_j are M_1-M_2 alternating circuits.

3 Proof of Theorem 4

In this section, we present the proof of Theorem 4.

Suppose that G is a connected bridgeless cubic graph of oddness 2. We know that G has a 2-factor K such that there are exactly two odd circuits in K. Let C_1 and C_2 be the two circuits in K. We know that $E(K)\backslash(E(C_1)\cup E(C_2))$ can be decomposed into two matchings M_1 and M_2 of G. Set M_0:=$E(G)\backslash K$. Next we will show that the edges in $E(C_1)\cup M_1$ can be covered by at most 5 perfect matchings of G.

Let $\mathscr{P}_1$ be the set of M_0-M_1-M_0 ears of $C_1\cup C_2$. Set H_1:=$C_1\cup C_2\cup(\bigcup_{P\in\mathscr{P}_1}P)$. Set $\mathscr{P}'_1$:=$\{P\in\mathscr{P}_1 : V_0(P)\subseteq V(C_1)\}$. For every $P\in\mathscr{P}'_1$, let f_P be an edge which has the same ends as P and does not belong to $E(G)$. Set F:=$\{f_P : P\in\mathscr{P}'_1\}$.

Assume that H_1 is bridgeless. Let G_1 be a new graph with vertex-set $V(C_1)$ and edge-set $E(C_1)\cup F$. Let X be the set of vertices of degree 2 in G_1. Noting that H_1 is bridgeless and C_1 is an odd circuit, we have $|X|\geqslant 3$. Choose a vertex v in X. Let N_1 be the perfect matching of C_1-v. By Lemma 5, G_1 has three near-perfect matchings N_2, N_3 and N_4 such that $E(C_1)\backslash N_1\subseteq\bigcup_{i=2}^4 N_i$, N_2 misses v and each of N_3 and N_4 misses a vertex in

$X\backslash\{v\}$. For each $i \in [4]$, let P_i be the path in $\mathscr{P}_1$ which is from the vertex missed by N_i in $V(C_1)$ to $V(C_2)$ and let N_i' be the perfect matching of $C_2 - (V_0(P_i) \cap V(C_2))$. For each $i \in [4]$, set $\mathscr{Q}_i{:=}\{P \in \mathscr{P}_1' : f_P \in F \cap N_i\}$. For each $i \in [4]$, set

$$M_{i+2} := (N_i \cap E(C_1)) \cup (E((\bigcup_{P \in \mathscr{Q}_i} P) \cup P_i) \triangle M_1) \cup N_i'.$$

Noting $E(C_1)\backslash N_1 \subseteq \bigcup_{i=2}^{4} N_i$ and $E(P_1) \cap E(P_3) = \emptyset$, we can obtain that M_3, M_4, M_5 and M_6 are 4 perfect matchings of G which cover all the edges in $E(C_1) \cup M_1$.

Algorithm 1 Constructing a connected bridgeless subgraph H_s of G.

set $s{:=}1$, $t{:=}2$, $r{:=}0$, $H_1'{:=}H_1 - V(P_0)$ and $U_1{:=}V(H_1')$;
while H_s has a bridge **do**
  if $r = 0$ **then**
    if H_s' has an M_t-M_1-M_t ear with both ends in U_s **then**
      let $\mathscr{P}_{s+1}$ be the set of M_t-M_1-M_t ears of H_s' which have both ends in U_s;
      set $H_{s+1}'{:=}H_s' \cup (\bigcup_{P \in \mathscr{P}_{s+1}} P)$ and $U_{s+1}{:=}V(\bigcup_{P \in \mathscr{P}_{s+1}} P)\backslash V(P_0)$;
    else
      set $r{:=}1$ and $t{:=}2$;
    end if
  end if
  if $r{=}1$ **then**
    let $\mathscr{P}_{s+1}$ be the set of M_t-M_1-M_t ears of H_s;
  end if
  set $H_{s+1}{:=}H_s \cup (\bigcup_{P \in \mathscr{P}_{s+1}} P)$ and $s{:=}s + 1$;
  let t' be the number in $\{0, 2\}\backslash\{t\}$ and set $t{:=}t'$;
end while
return s, r and H_s;

Next we assume that H_1 has a bridge. We know that G has a unique M_0-M_1-M_0 alternating path P_0 from $V(C_1)$ to $V(C_2)$. Let u_1 and u_2 be the ends of P_0 such that $u_1 \in V(C_1)$ and $u_2 \in V(C_2)$. We need construct a bridgeless subgraph H_s of G such that $H_1 \subseteq H_s$ and H_s has 5 M_1-alternating paths from $V(C_1)$ to $V(C_2)$ with no common edges, which can be used to constructed 5 perfect matchings of G which cover $E(C_1) \cup M_1$. We construct the subgraph H_s of G by Algorithm 1.

From Algorithm 1, we know that the subgraph H_s of G is constructed by adding the paths in $\mathscr{P}_2$, $\mathscr{P}_3$, $\cdots$, $\mathscr{P}_s$ to H_1. If $r = 0$ when Algorithm 1 ends, we know that for all $i \in [s]\backslash\{1\}$, $\mathscr{P}_i$ is the set of M_0-M_1-M_0 or M_2-M_1-M_2 ears of H_{i-1}' with both ends in U_{i-1}. If $r = 1$ when Algorithm 1 ends, we know that there is some integer k such that $\mathscr{P}_i$ is the set of M_0-M_1-M_0 or M_2-M_1-M_2 ears of H_{i-1}' with both ends in U_{i-1} for all $i \in [k]\backslash\{1\}$ and $\mathscr{P}_i$ is the set of M_0-M_1-M_0 or M_2-M_1-M_2 ears of H_{i-1} for all $i \in [s]\backslash[k]$. If $r = 1$, we also know that $\mathscr{P}_{k+1}$ is the set of M_2-M_1-M_2 ears of H_k with both ends in $V(P_0)$. (See Fig. 2(a, b, c) for some examples of H_s.)

For each $i \in [s] \backslash \{1\}$, we know that the edge-induced subgraph $G[E(\bigcup_{P \in \mathscr{P}_i} P) \cup M_{1,i}]$ of G consists of vertex-disjoint circuits of G such that either all of them are M_0-M_1 alternating or all of them are M_1-M_2 alternating, where $M_{1,i}$ is the set of edges in $E(H_{i-1}) \cap M_1$ which are incident with an end of a path in $\mathscr{P}_i$. Let $\mathscr{C}$ be the set of circuits of H_s which are M_0-M_1 alternating or M_1-M_2 alternating.

For $i = 1, 2$ and a path or circuit Z in $(\mathscr{P}_1 \backslash \{P_0\}) \cup \mathscr{C}$, if $E(Z) \cap E(P_0) \neq \emptyset$, let P_Z^i be the path from u_i to a vertex of Z in P_0 such that $E(Z) \cap E(P_0) \subseteq E(P_Z^i)$; otherwise, let P_Z^i be the graph with no vertices and edges. For two vertices u and v in P_0, we denote by P_{uv} the path from u to v in P_0 if $u \neq v$ and denote by P_{uv} the graph with vertex-set $\{u\}$ and edge-set $\emptyset$ if $u = v$.

Let G_2 be a new directed graph with vertex-set $(\mathscr{P}_1 \backslash \{P_0\}) \cup \mathscr{C}$ such that for two distinct vertices S and T in G_2, G_2 has a directed edge from S to T, denoted by $S \rightarrow T$, if and only if $V(S) \cap V(T) \neq \emptyset$ or $V(P_S^1) \cap V(P_T^2) \neq \emptyset$.

Noting that H_s is bridgeless. It follows that G_2 has a directed path from $\mathscr{P}_1'$ to $\mathscr{P}_1 \backslash (\mathscr{P}_1' \cup \{P_0\})$. Let $Y_1 Y_2 \dots Y_n$ be a shortest directed path from $\mathscr{P}_1'$ to $\mathscr{P}_1 \backslash (\mathscr{P}_1' \cup \{P_0\})$ in G_2. By the minimality of $Y_1 Y_2 \dots Y_n$, we know that G_2 has no directed edge $Y_i \rightarrow Y_j$ if $j - i \geqslant 2$. That is, $E(Y_i) \cap E(Y_j) = \emptyset$ and $E(P_{Y_i}^1) \cap E(P_{Y_j}^2) = \emptyset$ for any $i, j \in [n]$ with $j - i \geqslant 2$. Let $Y_{y(1)}, Y_{y(2)}, \dots, Y_{y(m)}$ be the M_1-M_2 alternating circuits in $\{Y_1, Y_2, \dots, Y_n\}$, where $y(1) < y(2) < \cdots < y(m)$. We know $y(1) = 2$ and $y(m) = n - 1$.

For any two distinct M_0-M_1 alternating circuits S and T in $\mathscr{C}$, we know $V(S) \cap V(T) = \emptyset$ and $V(P_S^1) = \emptyset$. It implies that G_2 has no directed edge $Y_i \rightarrow Y_{i+1}$ such that both Y_i and Y_{i+1} are M_0-M_1 alternating circuits. Hence $y(i+1) = y(i)+2$ or $y(i+1) = y(i)+1$ for each $i \in [m-1]$. If $y(i+1) = y(i)+2$, we know that $Y_{y(i)+1}$ is an M_0-M_1 alternating circuit and we have $V(Y_{y(i)+1}) \cap V(Y_{y(i)}) \neq \emptyset$ and $V(Y_{y(i)+1}) \cap V(Y_{y(i+1)}) \neq \emptyset$. If $y(i+1) = y(i) + 1$, we have $V(P_{Y_{y(i)}}^1) \cap V(P_{Y_{y(i+1)}}^2) \neq \emptyset$

For each $i \in [m]$, if G_2 has a directed edge $Y_{y(j-1)} \rightarrow Y_{y(j)}$ for some j larger than i, we denote by w_i the vertex in $V_0(P_{Y_{y(k_i)}}^2) \backslash \{u_2\}$, where k_i is the minimum integer larger than i such that G_2 has a directed edge $Y_{y(k_i-1)} \rightarrow Y_{y(k_i)}$; otherwise, let w_i be the vertex u_2. Clearly, we know $w_m = u_2$. (See Fig. 2(d, e, f) for examples of Y_1, Y_2, $\dots$, Y_n and w_1, w_2, $\dots$, w_m.)

Claim 1. *It holds that* $V(P_{u_1 w_i}) \subseteq V(P_{u_1 w_j})$ *if* $i < j$.

Choose $i, j \in [m]$ with $i < j$. Clearly, Claim 1 holds if $w_j = u_2$ or $w_i = w_j$. So we assume $w_j \neq u_2$ and $w_i \neq w_j$. As $i < j$, we know $k_i < k_j$, $w_i \in V_0(P_{Y_{y(k_i)}}^2) \backslash \{u_2\}$ and $w_j \in V_0(P_{Y_{y(k_j)}}^2) \backslash \{u_2\}$. Noting that G_2 has a directed edge $Y_{y(k_i-1)} \rightarrow Y_{y(k_i)}$, we have $V(P_{Y_{y(k_i-1)}}^1) \cap V(P_{Y_{y(k_i)}}^2) \neq \emptyset$, which implies $w_i \in V(P_{Y_{y(k_i-1)}}^1)$. We also know $V(P_{Y_{y(k_i-1)}}^1) \cap V(P_{Y_{y(k_j)}}^2) = \emptyset$ by the minimality of $Y_1 Y_2 \dots Y_n$. So $V(P_{u_1 w_i}) \subseteq V(P_{u_1 w_j})$ and Claim 1 holds.

Claim 2. $E(P_{Y_j}^1) \subseteq E(P_{u_1 w_i})$ *for each* $i \in [m]$ *and each* $j \in [y(i) - 1]$.

Choose $i \in [m]$ and $j \in [y(i) - 1]$. Clearly, Claim 2 holds if $w_i = u_2$. So we assume $w_i \neq u_2$. We know $w_i \in V_0(P_{Y_{y(k_i)}}^2) \backslash \{u_2\}$. As $y(k_i) > y(i) > j$, we have $y(k_i) - j \geqslant 2$. Hence we know $V(P_{Y_j}^1) \cap V(P_{Y_{y(k_i)}}^2) = \emptyset$ by the minimality of $Y_1 Y_2 \dots Y_n$. So $E(P_{Y_j}^1) \subseteq$

$E(P_{u_1w_i})$ and Claim 2 holds.

Set $\mathscr{P}_1'':=\{P \in \mathscr{P}_1' : V(P) \cap V(Y_2) = \emptyset\}$. Let G_3 be a new graph with vertex-set $V(C_1)$ and edge-set $E(C) \cup \{f_P : P \in \mathscr{P}_1''\}$. Let X' be the set of vertices of degree 2 in G_3. We know $u_1 \in X'$. Noting also $Y_1 \in \mathscr{P}_1' \backslash \mathscr{P}_1''$, we have $|X'| \geqslant 3$. Let N_5 be the perfect matching of $C_1 - u_1$. By Lemma 5, G_3 has three near-perfect matchings N_6, N_7 and N_8 such that $E(C_1)\backslash N_5 \subseteq \bigcup_{i=6}^{8} N_i$, N_6 misses u_1 and each of N_7 and N_8 misses a vertex in $X'\backslash\{u_1\}$. Let u_3 and u_4 be respectively the vertex missed by N_7 and N_8 in $V(C_1)$. We know that for $i = 3, 4$, there is a path P_{i+2} in $\mathscr{P}_1' \backslash \mathscr{P}_1''$ which has u_i as an end. (It is possible that P_5 and P_6 are the same path.) Let u_5 be the vertex in $V_0(P_6)\backslash\{u_4\}$.

Now we will construct five M_1-alternating paths from $V(C_1)$ to $V(C_2)$ such that they have no edges in common. We construct these five paths, step by step, along the path $Y_1Y_2 \cdots Y_n$ of G_2 and the path P_0 of G. The construction of these five paths is as follows.

Let $J_{1,1}$ be the path from u_4 to $V(Y_{y(1)})$ in P_6 and $J_{1,2}$ be the path from u_5 to $V(Y_{y(1)})$ in P_6. Let $J_{1,3}$ be the path from u_3 to $V(Y_{y(1)})$ in P_5. Set $J_{1,j}:=P_{u_1w_1}$ for $j = 4, 5$. Next we will extend $J_{1,1}$, $J_{1,2}$, $J_{1,3}$, $J_{1,4}$ and $J_{1,5}$ to five M_1-alternating paths from $V(C_1)$ to $V(C_2)$ such that they have no edges in common.

Let $J_{1,6}$ be the path from $V(J_{1,1})$ to $V(J_{1,2})$ in P_6. For an integer p with $2 \leqslant p \leqslant m+1$, we suppose that $J_{p-1,1}$, $J_{p-1,2}$, $J_{p-1,3}$, $J_{p-1,4}$, $J_{p-1,5}$ and $J_{p-1,6}$ have been constructed. The construction of $J_{p,1}$, $J_{p,2}$, $J_{p,3}$, $J_{p,4}$, $J_{p,5}$ and $J_{1,6}$ will be discussed in the following three cases.

Case 1. *$p \leqslant m$ and $y(p) = y(p-1) + 2$.*

We know that $Y_{y(p)-1}$ is an M_0-M_1 alternating circuit and we have $V(Y_{y(p)-1}) \cap V(Y_{y(p-1)}) \neq \emptyset$ and $V(Y_{y(p)-1}) \cap V(Y_{y(p)}) \neq \emptyset$. Let P_1' be a path from $V_0(J_{p-1,3})\backslash V(C_1)$ to $V(Y_{y(p)})$ in $Y_{y(p-1)} \cup Y_{y(p)-1}$ such that $J_{p-1,3} \cup P_1'$ is M_1-alternating. Let $J_{p,6}$ be the (inclusionwise) maximal M_1-M_0-M_1 alternating path in $Y_{y(p)-1}$ such that $V_0(P_1') \cap V(Y_{y(p)}) \subseteq V_0(J_{p,6})$, $E_0(J_{p,6}) \subseteq E(Y_{y(p)-1}) \cap E(Y_{y(p)})$ and $E(J_{p,6}) \cap E(Y_{y(p-1)}) = \emptyset$. Let $J_{p-1,7}$ be the (inclusionwise) maximal M_0-M_1-M_0 alternating path in $Y_{y(p)-1}$ such that $V_0(J_{p,6})\backslash V_0(P_1') \subseteq V_0(J_{p-1,7})$ and $E(J_{p-1,7}) \cap E(Y_{y(p)}) = \emptyset$. (See Fig. 3($a$) for an example of $J_{p,6}$ and $J_{p-1,7}$.) By the maximality of $J_{p,6}$ and $J_{p-1,7}$, we know $V_0(J_{p-1,7}) \subseteq V(Y_{y(p)})$ and $E(J_{p-1,7}) \cap E(Y_{p-1}) \neq \emptyset$.

By Lemma 6, G has two M_1-alternating paths P_2' and P_3' from $V_0(J_{p-1,6})$ to $V_0(J_{p-1,7})$ in $Y_{y(p-1)} \cup J_{p-1,6} \cup J_{p-1,7}$ such that $V_0(P_2') \cup V_0(P_3') = V_0(J_{p-1,6}) \cup V_0(J_{p-1,7})$, $E_0(P_2') \cup E_0(P_3') = E_0(J_{p-1,6}) \cup E_0(J_{p-1,7})$ and $E(P_2') \cap E(P_3') \subseteq E(J_{p-1,6}) \cup E(J_{p-1,7})$. Without loss of generality, we assume that P_2' has an end in $V_0(J_{p-1,7}) \cap V_0(J_{p,6})$ and P_3' has an end in $V_0(J_{p-1,7})\backslash V_0(J_{p,6})$. Let α_1 be the number in $\{1,2\}$ such that $V_0(J_{1,\alpha_1}) \cap V_0(P_2') \neq \emptyset$. Let α_2 be the number in $\{1, 2\}\backslash\{\alpha_1\}$.

Set $J_{p,1}:=J_{p-1,3} \cup P_1'$, $J_{p,2}:=J_{p-1,\alpha_1} \cup P_2'$, $J_{p,3}:=J_{p-1,\alpha_2} \cup P_3'$ and $J_{p,j}:=J_{p-1,j}$ for $j = 4, 5$. (See Fig. 3(a) for an example of $J_{p,1}$, $J_{p,2}$ and $J_{p,3}$.)

Case 2. *$p \leqslant M$ and $y(p) = y(p-1) + 1$.*

We know $V(P^1_{Y_{y(p-1)}}) \cap V(P^2_{Y_{y(p)}}) \neq \emptyset$ and $w_{p-1} \in V_0(P^2_{Y_{y(p)}})\backslash\{u_2\}$. Let P_1'' be the path from $V_0(J_{p-1,3}) \cap V(Y_{y(p-1)})$ to $V(P^2_{Y_{y(p)}})$ in $Y_{y(p-1)}$ such that $J_{p-1,3} \cup P_1''$ is an M_1-alternating path. Let v_1 be the end of P_1'' in $V(P^2_{Y_{y(p)}})$. We will construct $J_{p,1}$, $J_{p,2}$, $\ldots$,

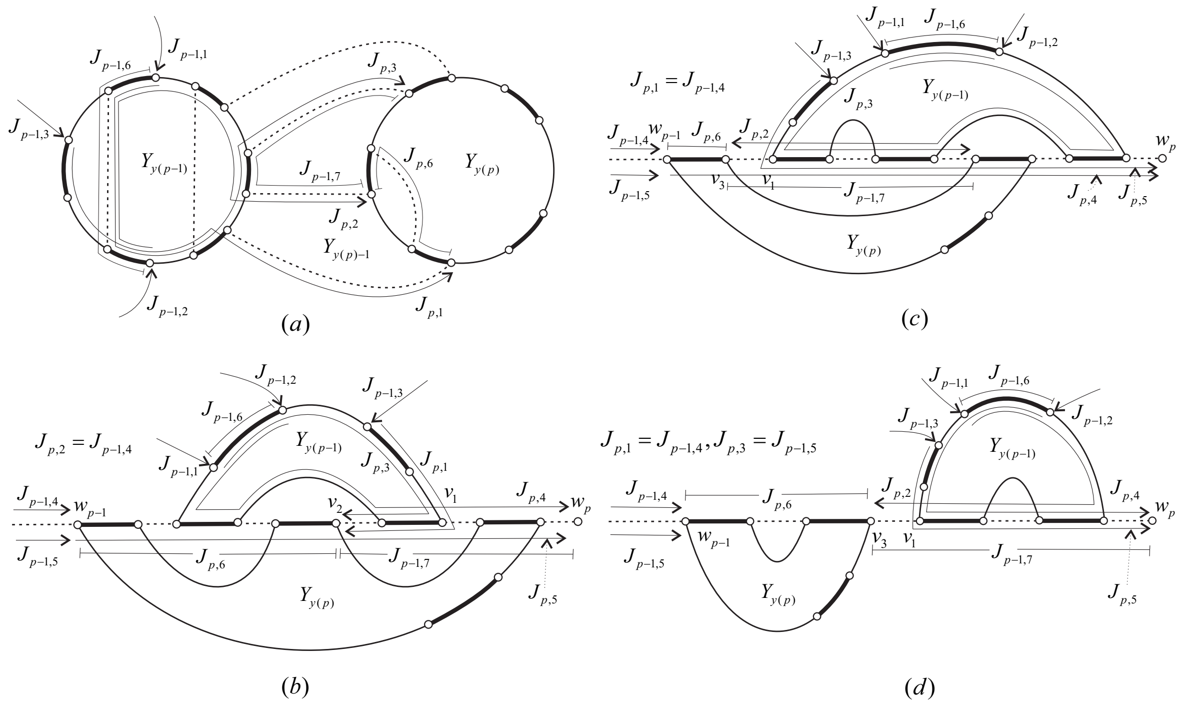

Figure 3: Examples for the proof of Theorem 4. These are examples for showing how to extend the paths $J_{p-1,1}$, $J_{p-1,2}$, $\dots$, $J_{p-1,5}$ to the paths $J_{p,1}$, $J_{p,2}$, $\dots$, $J_{p,5}$.

$J_{p,6}$ in the following two subcases.

Subcase 2.1. $P_1'' \cup P_{w_{p-1}v_1}$ is an M_1-alternating path.

Let P_2'' be the path from v_1 to $V(Y_{y(p)})$ in $P_{w_{p-1}v_1}$. Let v_2 be the vertex in $V_0(P_2'')\backslash\{v_1\}$. Let $J_{p-1,7}$ be the path from w_p to $V(Y_{y(p)}) \cap V(P^1_{Y_{y(p-1)}})$ in $P_{w_{p-1}w_p}$. By Claim 2, we have $E(P^1_{Y_{y(p-1)}}) \subseteq E(P_{u_1w_p})$. Noting also $V(P^1_{Y_{y(p-1)}}) \cap V(P^2_{Y_{y(p)}}) \neq \emptyset$, we know $E(J_{p-1,7}) \cap E(Y_{y(p-1)}) \neq \emptyset$. (See Fig. 3($b$) for an example of $J_{p-1,7}$, v_1 and v_2.)

By Lemma 6, G has two M_1-alternating paths P_3'' and P_4'' from $V_0(J_{p-1,6})$ to $V_0(J_{p-1,7})$ in $Y_{y(p-1)} \cup J_{p-1,6} \cup J_{p-1,7}$ such that $V_0(P_3'') \cup V_0(P_4'') = V_0(J_{p-1,6}) \cup V_0(J_{p-1,7})$, $E_0(P_3'') \cup E_0(P_4'') = E_0(J_{p-1,6}) \cup E_0(J_{p-1,7})$ and $E(P_3'') \cap E(P_4'') \subseteq E(J_{p-1,6}) \cup E(J_{p-1,7})$. Without loss of generality, we assume that P_3'' has one end in $V(Y_p)$ and P_4'' has w_p as an end. Let β_1 be the number in $\{1,2\}$ such that $V_0(J_{p-1,\beta_1}) \cap V_0(P_3'') \neq \emptyset$. Let β_2 be the number in $\{1,2\}\backslash\{\beta_1\}$.

Set $J_{p,1}$:=$J_{p-1,3} \cup P_1'' \cup P_2''$, $J_{p,2}$:=$J_{p-1,4}$, $J_{p,3}$:=$J_{p-1,\beta_1} \cup P_3''$, $J_{p,4}$:=$J_{p-1,\beta_2} \cup P_4''$, $J_{p,5}$:=$J_{p-1,5} \cup P_{w_{p-1}w_p}$ and $J_{p,6}$:=$P_{w_{p-1}v_2}$. (See Fig. 3(b) for an example of $J_{p,1}$, $J_{p,2}$, $\dots$, $J_{p,6}$.)

Subcase 2.2. $P_1'' \cup P_{v_1w_p}$ is an M_1-alternating path.

Let $J_{p,6}$ be the (inclusionwise) maximal M_1-M_0-M_1 alternating path in $P_{w_{p-1}w_p}$ such that $w_{p-1} \in V_0(J_{p,6}) \subseteq V(Y_{y(p)})$ and $E(J_{p,6}) \cap E(Y_{y(p-1)}) = \emptyset$. Let v_3 be the vertex in $V_0(J_{p,6})\backslash\{w_{p-1}\}$. Let $J_{p-1,7}$ be the path from v_3 to $V(Y_{y(p)}) \cup \{w_p\}$ in $P_{v_3w_p}$ such that $E(J_{p-1,7}) \cap E(J_{p,6}) = \emptyset$. By the maximality of $J_{p,6}$, we know $E(J_{p-1,7}) \cap E(Y_{y(p-1)}) \neq \emptyset$.

((See Fig. 3(c, d) for two examples of $J_{p,6}$ and $J_{p-1,7}$.)

By Lemma 6, G has two M_1-alternating paths P_5'' and P_6'' from $V_0(J_{p-1,6})$ to $V_0(J_{p-1,7})$ in $Y_{y(p-1)} \cup J_{p-1,6} \cup J_{p-1,7}$ such that $V_0(P_5'') \cup V_0(P_6'') = V_0(J_{p-1,6}) \cup V_0(J_{p-1,7})$, $E_0(P_5'') \cup E_0(P_6'') = E_0(J_{p-1,6}) \cup E_0(J_{p-1,7})$ and $E(P_5'') \cap E(P_6'') \subseteq E(J_{p-1,6}) \cup E(J_{p-1,7})$. Without loss of generality, we assume that P_5'' has v_3 as one end and P_6'' has an end in $V_0(J_{p-1,7}) \backslash \{v_3\}$. Let γ_1 be the number in $\{1,2\}$ such that $V_0(J_{p-1,\gamma_1}) \cap V_0(P_5'') \neq \emptyset$. Let γ_2 be the number in $\{1,2\} \backslash \{\gamma_1\}$.

Set $J_{p,1}{:=}J_{p-1,4}$, $J_{p,2}{:=}J_{p-1,\gamma_1} \cup P_5''$ and $J_{p,5}{:=}J_{p-1,3} \cup P_1'' \cup P_{v_1 w_p}$. If the end of $J_{p-1,7}$ which is different from v_3 is in $V(Y_{y(p)})$, we set $J_{p,3}{:=}J_{p-1,\gamma_2} \cup P_6''$ and $J_{p,4}{:=}J_{p-1,5} \cup P_{w_{p-1} w_p}$. (See Fig 3$(c)$ for an example of $J_{p,1}$, $J_{p,2}$, $\ldots$, $J_{p,5}$ in this case.) If $w_p \in V_0(J_{p-1,7})$, we set $J_{p,3}{:=}J_{p-1,5}$ and $J_{p,4}{:=}J_{p-1,\gamma_2} \cup P_6''$. (See Fig. 3$(d)$ for an example of $J_{p,1}$, $J_{p,2}$, $\ldots$, $J_{p,5}$ in the case that $w_p \in V_0(J_{p-1,7})$.)

Case 3. $p = m + 1$.

Set $J_{p-1,7}{:=}Y_n$. Let P_1''' be a path from $V_0(J_{p-1,3}) \cap V(Y_{y(m)})$ to $V_0(Y_n)$ in $Y_{y(m)} \cup Y_n$ such that $J_{p-1,3} \cup P_1'''$ is M_1-alternating. By Lemma 6, G has two M_1-alternating paths P_2''' and P_3''' from $V_0(J_{p-1,6})$ to $V_0(J_{p-1,7})$ in $Y_{y(m)} \cup J_{p-1,6} \cup J_{p-1,7}$ such that $V_0(P_2''') \cup V_0(P_3''') = V_0(J_{p-1,6}) \cup V_0(J_{p-1,7})$, $E_0(P_2''') \cup E_0(P_3''') = E_0(J_{p-1,6}) \cup E_0(J_{p-1,7})$ and $E(P_2''') \cap E(P_3''') \subseteq E(J_{p-1,6}) \cup E(J_{p-1,7})$. Without loss of generality, we assume that P_2''' has an end in $V_0(J_{p-1,1})$ and P_3''' has an end in $V_0(J_{p-1,2})$.

Set $J_{p,1}{:=}J_{p-1,1} \cup P_2'''$, $J_{p,2}{:=}J_{p-1,2} \cup P_3'''$, $J_{p,3}{:=}J_{p-1,3} \cup P_1'''$ and set $J_{p,j}{=}J_{p-1,j}$ for $j = 4, 5$.

By the above construction, we can construct five M_1-alternating paths $J_{m+1,1}$, $J_{m+1,2}$, $J_{m+1,3}$, $J_{m+1,4}$ and $J_{m+1,5}$ from $V(C_1)$ to $V(C_2)$. Next we will show $\bigcap_{j=1}^{5} E(J_{m+1,j}) = \emptyset$.

For each $i \in [m]$, let π_i be a one-to-one correspondence from [5] to [5] such that $J_{i,j}$ is a subgraph of $J_{i+1,\pi_i(j)}$ for each $j \in [5]$. We know that for each $i \in [m]$ and each $j \in [5]$, $J_{i,j}$ is extended to $J_{i+1,\pi(j)}$.

Claim 3. *For each $i \in [m]$, $J_{i,1}$, $J_{i,2}$ and $J_{i,3}$ are three M_1-alternating paths from $V(C_1)$ to $V(Y_{y(i)})$ in G and $J_{i,4}$, $J_{i,5}$ are two M_1-alternating paths from $V(C_1)$ to w_i in G, such that $\bigcup_{j=1}^{5} E(J_{i,j}) \subseteq E(P_5 \cup P_6) \cup E(Y_2 \cup Y_3 \cup \cdots \cup Y_{y(i)-1}) \cup E(P_{u_1 w_i})$.*

We prove Claim 3 by induction on i. Clearly, Claim 3 holds for $i = 1$. Choose a number $p \in \{2, 3, \ldots, m\}$. We suppose that Claim 3 holds for $i = p - 1$. At first, we can easily see, from the construction in the above three cases, that $J_{p,1}$, $J_{p,2}$ and $J_{p,3}$ are three M_1-alternating paths from $V(C_1)$ to $V(Y_{y(p)})$ in G and $J_{p,4}$, $J_{p,5}$ are two M_1-alternating paths from $V(C_1)$ to w_p in G.

Now we consider the set $\bigcup_{j=1}^{5} E(J_{p,j})$. From the construction of $J_{p,1}$, $J_{p,2}$, $J_{p,3}$, $J_{p,4}$ and $J_{p,5}$, we know $E(J_{p,\pi_{p-1}(j)}) \backslash E(J_{p-1,j}) \subseteq E(J_{p-1,6}) \cup E(Y_{y(p)-2}) \cup E(Y_{y(p)-1})$ for each $j \in [5]$ if $y(p) = y(p-1) + 2$ and we know $E(J_{p,\pi_{p-1}(j)}) \backslash E(J_{p-1,j}) \subseteq E(J_{p-1,6}) \cup E(Y_{y(p)-1}) \cup E(P^1_{Y_{y(p)-1}}) \cup E(P_{u_1 w_p})$ for each $j \in [5]$ if $y(p) = y(p-1) + 1$. From the construction of $J_{p-1,6}$, we know $E(J_{p-1,6}) \subseteq E(P_6)$ if $p = 2$, we know $E(J_{p-1,6}) \subseteq E(Y_{y(p-1)-1})$ if $p \geqslant 3$ and $y(p-1) = y(p-2)+2$, and we know $E(J_{p-1,6}) \subseteq E(P_{w_{p-2} w_{p-1}})$ if $p \geqslant 3$ and $y(p-1) = y(p-2) + 1$. Noting also $E(P_{u_1 w_{p-1}}) \subseteq E(P_{u_1 w_p})$ by Claim 1 and $E(P^1_{Y_{y(p)-1}}) \subseteq E(P_{u_1 w_p})$ by Claim 2, we have $E(J_{p,j}) \subseteq E(P_5 \cup P_6) \cup E(Y_2 \cup Y_3 \cup \cdots \cup Y_{y(p)-1}) \cup E(P_{u_1 w_p})$ for each

$j \in [5]$ and hence Claim 3 holds for $i = p$. Claim 3 is proved.

Claim 4. $\bigcap_{j=3}^{6} E(J_{p,j}) = \emptyset$ and $(\bigcap_{j=4}^{6} E(J_{p,j})) \cap E(Y_{y(p)}) = \emptyset$ for each $p \in [m]$.

Noting that $E(J_{1,6}) \subseteq E(P_6)$, $E(J_{1,4}) \subseteq E(P_0)$ and $E(P_6) \cap E(P_0) = \emptyset$, we have $\bigcap_{j=3}^{6} E(J_{1,j}) = \emptyset$ and $E(J_{1,4}) \cap E(J_{1,5}) \cap E(J_{1,6}) \cap E(Y_{y(1)}) = \emptyset$. So Claim 4 holds for $p = 1$. Now we consider the case that $2 \leqslant p \leqslant m$.

Assume $y(p) = y(p-1) + 2$. We know from the construction of $J_{p,4}$ and $J_{p,6}$ in Case 1 that $J_{p,4} = J_{p-1,4}$ and $E(J_{p,6}) \subseteq E(Y_{y(p)-1})$. By Claim 3, we know $E(J_{p-1,4}) \subseteq E(P_5 \cup P_6) \cup E(Y_2 \cup Y_3 \cup \cdots \cup Y_{y(p-1)-1}) \cup E(P_{u_1 w_{p-1}})$. Noting that $Y_{y(p)-1}$ is an M_0-M_1 alternating circuit in $\mathscr{C}$ and P_0, P_5 and P_6 are M_0-M_1-M_0 alternating paths in $\mathscr{P}_1$, we have $E(Y_{y(p)-1}) \cap E(P_5 \cup P_6 \cup P_{u_1 w_{p-1}}) = \emptyset$. By the minimality of $Y_1 Y_2 \cdots Y_n$, we know $V(Y_j) \cap V(Y_{y(p)-1}) = \emptyset$ for every integer j with $2 \leqslant j \leqslant y(p-1)-1$. So $E(J_{p,6}) \cap E(J_{p,4}) = \emptyset$. Hence $\bigcap_{j=3}^{6} E(J_{p,j}) = \emptyset$ and $E(J_{p,4}) \cap E(J_{p,5}) \cap E(J_{p,6}) \cap E(Y_{y(p)}) = \emptyset$.

Assume $y(p) = y(p-1) + 1$ and that $J_{p,3}$, $J_{p,4}$, $J_{p,5}$ and $J_{p,6}$ are constructed in Subcase 2.1. We know $J_{p,3}{:=}J_{p-1,\beta_1} \cup P''_3$, $J_{p,4}{:=}J_{p-1,\beta_2} \cup P''_4$, $E(J_{p,6}) \subseteq E(P_{w_{p-1}w_p})$ and $E(P_{p,6}) \cap E(J_{p-1,7}) = \emptyset$. We firstly show $\bigcap_{j=3}^{6} E(J_{p,j}) = \emptyset$. From the construction of $J_{p-1,6}$, we know that either $E(J_{p-1,6}) \subseteq E(P_{u_1 w_{p-1}})$ or $J_{p-1,6}$ is a subgraph of P_6 or a subgraph of an M_0-M_1 alternating circuit in $\mathscr{C}$. So $E(J_{p,6}) \cap E(J_{p-1,6}) = \emptyset$. By Claims 1, 2 and 3, we can obtain $E(J_{p,6}) \cap (E(J_{p-1,1}) \cup E(J_{p-1,2})) = \emptyset$. Noting also $E(P''_3) \cap E(P''_4) \subseteq E(J_{p-1,6}) \cup E(J_{p-1,7})$ and $E(P_{p,6}) \cap E(J_{p-1,7}) = \emptyset$, we have

$$\begin{aligned}\bigcap_{j=3}^{6} E(J_{p,j}) &\subseteq E(J_{p,6}) \cap E(J_{p,3}) \cap E(J_{p,4}) \\ &= E(J_{p,6}) \cap E(P''_3) \cap E(P''_4) \\ &\subseteq E(J_{p,6}) \cap (E(J_{p-1,6}) \cup E(J_{p-1,7})) \\ &= (E(J_{p,6}) \cap E(J_{p-1,6})) \cup (E(J_{p,6}) \cap E(J_{p-1,7})) \\ &= \emptyset\end{aligned}$$

Now we show $E(J_{p,4}) \cap E(J_{p,5}) \cap E(J_{p,6}) \cap E(Y_{y(p)}) = \emptyset$. From the construction of $J_{p,4}$, we know $E(J_{p,4}) \subseteq E(J_{p-1,\beta_2}) \cup E(J_{p-1,6}) \cup E(Y_{y(p-1)}) \cup E(J_{p-1,7})$. We already know $E(J_{p,6}) \cap (E(J_{p-1,1}) \cup E(J_{p-1,2})) = \emptyset$, $E(J_{p,6}) \cap E(J_{p-1,6}) = \emptyset$ and $E(P_{p,6}) \cap E(J_{p-1,7}) = \emptyset$. So $E(J_{p,6}) \cap E(J_{p,4}) \subseteq E(Y_{y(p-1)})$. Noting that both $Y_{y(p-1)}$ and $Y_{y(p)}$ are distinct M_1-M_2 alternating circuits in $\mathscr{C}$, we have $E(Y_{y(p-1)}) \cap E(Y_{y(p)}) = \emptyset$. So $E(J_{p,6}) \cap E(J_{p,4}) \cap E(Y_{y(p)}) = \emptyset$. Hence $E(J_{p,4}) \cap E(J_{p,5}) \cap E(J_{p,6}) \cap E(Y_{y(p)}) = \emptyset$.

Assume $y(p) = y(p-1)+1$ and that $J_{p,3}$, $J_{p,4}$, $J_{p,5}$ and $J_{p,6}$ are constructed in Subcase 2.2. We know from the construction that $E(J_{p,6}) \subseteq E(P_{w_{p-1}w_p})$, $J_{p,5}{=}J_{p-1,3} \cup P''_1 \cup P_{v_1 w_p}$ and $E(J_{p,6}) \cap E(P''_1 \cup P_{v_1 w_p}) = \emptyset$. By Claims 1, 2 and 3, we can obtain $E(J_{p,6}) \cap E(J_{p-1,3}) = \emptyset$. So $E(J_{p,6}) \cap E(J_{p,5}) = \emptyset$. Thus $\bigcap_{j=3}^{6} E(J_{p,j}) = \emptyset$ and $E(J_{p,4}) \cap E(J_{p,5}) \cap E(J_{p,6}) \cap E(Y_{y(p)}) = \emptyset$.

Claim 4 is proved.

Claim 5. $E(J_{p-1,6}) \cap (\bigcap_{j=3}^{5} E(J_{p,\pi_{p-1}(j)})) = \emptyset$ for each $p \in \{2, 3, \ldots, m+1\}$.

Choose a number $p \in \{2, 3, \ldots, m+1\}$. From the construction of $J_{p,\pi_{p-1}(3)}$, we know $E(J_{p,\pi_{p-1}(3)}) \subseteq E(J_{p-1,3}) \cup E(Y_{y(p-1)}) \cup E(Y_{y(p)-1}) \cup E(P_{w_{p-1}w_p})$ if $p \leqslant m$ and we know

$E(J_{p,\pi_{p-1}(3)}) \subseteq E(J_{p-1,3}) \cup E(Y_{y(p-1)}) \cup E(Y_n)$ if $p = m+1$. From the construction of $J_{p-1,6}$, we know $E(J_{p-1,6}) \subseteq E(P_6)$, $E(J_{p-1,6}) \subseteq E(P_{u_1 w_{p-1}})$, or that $J_{p-1,6}$ is a subgraph of the M_0-M_1 alternating circuit $Y_{y(p-1)-1}$ when $y(p-1) = y(p-2)+2$ holds. Hence we have $E(J_{p-1,6}) \cap E(P_{w_{p-1}w_p}) = \emptyset$ and $E(J_{p-1,6}) \cap E(Y_n) = \emptyset$. If $p \leqslant m$ and $y(p) = y(p-1)+2$, we know that $Y_{y(p)-1}$ is an M_0-M_1 alternating circuit and we have $E(J_{p-1,6}) \cap E(Y_{y(p)-1}) = \emptyset$. So $E(J_{p-1,6}) \cap E(J_{p,\pi_{p-1}(3)}) = (E(J_{p-1,6}) \cap E(J_{p-1,3})) \cup (E(J_{p-1,6}) \cap E(Y_{y(p-1)}))$.

From the construction of $J_{p,\pi_{p-1}(4)}$ and $J_{p,\pi_{p-1}(5)}$, we know $J_{p,\pi_{p-1}(4)} = J_{p-1,4}$ and either $J_{p,\pi_{p-1}(5)} = J_{p-1,5}$ or $J_{p,\pi_{p-1}(5)} = J_{p-1,5} \cup P_{w_{p-1}w_p}$. We already know $E(J_{p-1,6}) \cap E(P_{w_{p-1}w_p}) = \emptyset$. Now we have

$$
\begin{aligned}
&E(J_{p-1,6}) \cap E(J_{p,\pi_{p-1}(3)}) \cap E(J_{p,\pi_{p-1}(4)}) \cap E(J_{p,\pi_{p-1}(5)}) \\
&\quad = \big((E(J_{p-1,6}) \cap E(J_{p-1,3})) \cup (E(J_{p-1,6}) \cap E(Y_{y(p-1)}))\big) \cap E(J_{p-1,4}) \cap E(J_{p-1,5}) \\
&\quad = \big(\bigcap_{j=3}^{6} E(J_{p-1,j})\big) \cup \big((\bigcap_{j=4}^{6} E(J_{p-1,j})) \cap E(Y_{y(p-1)})\big).
\end{aligned}
$$

By Claim 4, we know $\bigcap_{j=3}^{6} E(J_{p-1,j}) = \emptyset$ and $(\bigcap_{j=4}^{6} E(J_{p-1,j})) \cap E(Y_{y(p-1)}) = \emptyset$. So $E(J_{p-1,6}) \cap E(J_{p,\pi_{p-1}(3)}) \cap E(J_{p,\pi_{p-1}(4)}) \cap E(J_{p,\pi_{p-1}(5)}) = \emptyset$ and Claim 5 holds.

Claim 6. *$E(J_{p-1,7}) \cap E(J_{p,\pi_{p-1}(4)}) = \emptyset$ for each $p \in \{2, 3, \ldots, m+1\}$.*

Choose a number $p \in \{2, 3, \ldots, m+1\}$. We know $J_{p,\pi_{p-1}(4)} = J_{p-1,4}$ from the construction of $J_{p,\pi_{p-1}(4)}$. From the construction of $J_{p-1,7}$, we know $E(J_{p-1,7}) \subseteq E(Y_n)$ if $p = m+1$, we know $E(J_{p-1,7}) \subseteq E(Y_{y(p)-1})$ if $p \leqslant m$ and $y(p) = y(p-1)+2$, and we know $E(J_{p-1,7}) \subseteq E(P_{w_{p-1}w_p})$ if $p \leqslant m$ and $y(p) = y(p-1)+1$. By Claims 1, 2 and 3, we can obtain $E(J_{p-1,4}) \cap E(P_{w_{p-1}w_p}) = \emptyset$. By Claim 3 and the minimality of $Y_1Y_2 \cdots Y_n$, we know $E(J_{p-1,4}) \cap E(Y_n) = \emptyset$ if $p = m+1$ and we also know $E(J_{p-1,4}) \cap E(Y_{y(p)-1}) = \emptyset$ if $p \leqslant m$ and $y(p) = y(p-1)+2$. So $E(J_{p-1,7}) \cap E(J_{p,\pi_{p-1}(4)}) = E(J_{p-1,7}) \cap E(J_{p-1,4}) = \emptyset$ and Claim 6 holds.

Claim 7. *$\bigcap_{j=1}^{5} E(J_{p,j}) = \emptyset$ for each $p \in [m+1]$.*

We proceed by induction on p. Clearly, Claim 7 holds for $i = 1$. For an integer p with $2 \leqslant p \leqslant m+1$, we suppose $\bigcap_{j=1}^{5} E(J_{p-1,j}) = \emptyset$.

Noting that $J_{p-1,1}$ and $J_{p-1,2}$ are two paths from $V(C_1)$ to $V(Y_{y(p-1)})$ in G by Claim 3, we have $E(J_{p-1,j}) \cap E(Y_{y(p-1)}) = \emptyset$ for $j = 1, 2$. Set $E_1 := E(J_{p,\pi_{p-1}(1)}) \backslash E(J_{p-1,1})$ and $E_2 := E(J_{p,\pi_{p-1}(2)}) \backslash E(J_{p-1,2})$. From above, we know $E_j \subseteq E(Y_{y(p-1)}) \cup E(J_{p-1,6}) \cup E(J_{p-1,7})$ for $j = 1, 2$ and $E_1 \cap E_2 \subseteq E(J_{p-1,6}) \cup E(J_{p-1,7})$. Now we have

$$
\begin{aligned}
&E(J_{p,\pi_{p-1}(1)}) \cap E(J_{p,\pi_{p-1}(2)}) \\
&\quad = (E(J_{p-1,1}) \cap E_1) \cup (E(J_{p-1,2}) \cap E_2) \\
&\quad = (E(J_{p-1,1}) \cap E(J_{p-1,2})) \cup (E(J_{p-1,1}) \cap E_2) \cup (E_1 \cap E(J_{p-1,2})) \cup (E_1 \cap E_2) \\
&\quad \subseteq (E(J_{p-1,1}) \cap E(J_{p-1,2})) \cup E(J_{p-1,6}) \cup E(J_{p-1,7}).
\end{aligned}
$$

From the construction of $J_{p,\pi_{p-1}(3)}$, $J_{p,\pi_{p-1}(4)}$ and $J_{p,\pi_{p-1}(5)}$, we know $E(J_{p,\pi_{p-1}(j)}) \subseteq E(J_{p-1,j}) \cup E(Y_{y(p-1)}) \cup E(Y_{y(p)-1}) \cup E(P_{w_{p-1}w_p})$ for $j = 3, 4, 5$ if $p \leqslant m$ and we know $E(J_{p,\pi_{p-1}(j)}) \subseteq E(J_{p-1,j}) \cup E(Y_{y(p-1)}) \cup E(Y_n)$ for $j = 3, 4, 5$ if $p = m+1$. By Claims 1-3 and the minimality of $Y_1Y_2\cdots Y_n$, we can know $E(J_{p-1,1})\cap(E(P_{w_{p-1}w_p})\cup E(Y_n))=\emptyset$. We also can know $E(J_{p-1,1})\cap E(Y_{y(p)-1})=\emptyset$ if $p\leqslant m$ and $y(p)=y(p-1)+2$. Noting also $E(J_{p-1,1})\cap E(Y_{y(p-1)})=\emptyset$, we have $E(J_{p-1,1})\cap E(J_{p,\pi_{p-1}(j)})=E(J_{p-1,1})\cap E(J_{p-1,j})$ for $j=3,4,5$. Hence we have

$$E(J_{p-1,1})\cap E(J_{p-1,2})\cap(\bigcap_{j=3}^{5}E(J_{p,\pi_{p-1}(j)}))=\bigcap_{j=1}^{5}E(J_{p-1,j})=\emptyset.$$

By Claim 5, we know $E(J_{p-1,6})\cap(\bigcap_{j=3}^{5}E(J_{p,\pi_{p-1}(j)}))=\emptyset$. By Claim 6, we know $E(J_{p-1,7})\cap E(J_{p,\pi_{p-1}(4)})=\emptyset$. Now we have

$$\begin{aligned}\bigcap_{j=1}^{5}E(J_{p,j})&=(E(J_{p,\pi_{p-1}(1)})\cap E(J_{p,\pi_{p-1}(2)}))\cap(\bigcap_{j=3}^{5}E(J_{p,\pi_{p-1}(j)}))\\&\subseteq((E(J_{p-1,1})\cap E(J_{p-1,2}))\cup E(J_{p-1,6})\cup E(J_{p-1,7}))\cap(\bigcap_{j=3}^{5}E(J_{p,\pi_{p-1}(j)}))\\&\subseteq(E(J_{p-1,1})\cap E(J_{p-1,2})\cap(\bigcap_{j=3}^{5}E(J_{p,\pi_{p-1}(j)})))\\&\qquad\cup(E(J_{p-1,6})\cap(\bigcap_{j=3}^{5}E(J_{p,\pi_{p-1}(j)})))\cup(E(J_{p-1,7})\cap E(J_{p,\pi_{p-1}(4)}))\\&=\emptyset.\end{aligned}$$

Claim 7 is proved.

By Claim 7, we know $\bigcap_{j=1}^{5}E(J_{m+1,j})=\emptyset$. At last, we will use these five paths $J_{m+1,1}$, $J_{m+1,2}$, $J_{m+1,3}$, $J_{m+1,4}$ and $J_{m+1,5}$ to constructed five perfect matchings of G which cover $E(C_1)\cup M_1$.

Let N_9 be the perfect matching of C_1-u_5. For each $j\in\{5,6,7,8,9\}$, let N'_j be the perfect matching of $C_2-V_0(J_{m+1,j-4})\cap V(C_2)$. Let δ be a one-to-one correspondence from $\{5,6,7,8,9\}$ to $\{5,6,7,8,9\}$ such that for each $j\in\{5,6,7,8,9\}$, the vertex in $V_0(J_{m+1,j-4})\cap V(C_1)$ is missed by $N_{\delta(j)}$. For each $j\in\{5,6,7,8,9\}$, set $\mathscr{Q}'_j:=\{P\in\mathscr{P}''_1: f_P\in F\cap N_{\delta(j)}\}$. Now for each $j\in\{5,6,7,8,9\}$, we set

$$M_{j+2}:=(N_{\delta(j)}\cap E(C_1))\cup(E((\bigcup_{P\in\mathscr{Q}'_j}P)\cup J_{m+1,j-4})\triangle M_1)\cup N'_j.$$

We know that M_7, M_8, M_9, M_{10} and M_{11} are 5 perfect matchings of G. Noting that $E(C_1)\backslash N_5\subseteq\bigcup_{j=6}^{8}N_j$ and $\bigcap_{j=1}^{5}E(J_{m+1,j})=\emptyset$, we have $E(C_1)\cup M_1\subseteq\bigcup_{j=7}^{11}M_j$.

By a similar argument as above, we know that the edges in $E(C_2)\cup M_2$ can also be covered by at most 5 perfect matchings of G. These at most 10 perfect matchings of G together with M_0 cover all edges of G.

The proof is complete.

Acknowledgements

The authors are grateful to the referees for their valuable suggestions in improving this paper.

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