Cubic graphs with small independence ratio

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Submitted: Aug 30, 2017; Accepted: Feb 11, 2019; Published: Mar 22, 2019

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Abstract

Let $i(r, g)$ denote the infimum of the ratio $\frac{\alpha(G)}{|V(G)|}$ over the r-regular graphs of girth at least g, where $\alpha(G)$ is the independence number of G, and let $i(r, \infty) := \lim\limits_{g\to\infty} i(r, g)$. Recently, several new lower bounds of $i(3, \infty)$ were obtained. In particular, Hoppen and Wormald showed in 2015 that $i(3, \infty) \geqslant 0.4375$, and Csóka improved it to $i(3, \infty) \geqslant 0.44533$ in 2016. Bollobás proved the upper bound $i(3, \infty) < \frac{6}{13}$ in 1981, and McKay improved it to $i(3, \infty) < 0.45537$in 1987. There were no improvements since then. In this paper, we improve the upper bound to $i(3, \infty) \leqslant 0.454$.

Mathematics Subject Classifications: 05C15, 05C35

[*]Research of this author is partially supported by NSF Grants DMS-1500121, DMS-1764123, Arnold O. Beckman Research Award (UIUC) Campus Research Board 18132 and the Langan Scholar Fund (UIUC).

[†]Research of this author is supported in part by NSF grant DMS-1600592, by Award RB17164 of the UIUC Campus Research Board, and by grants 18-01-00353 and 19-01-00682 of the Russian Foundation for Basic Research.

[‡]Research of this author is supported in part by Award RB17164 of the Research Board of the University of Illinois at Urbana-Champaign.

1 Introduction

A set S of vertices in a graph G is *independent* if no two vertices of S are joined by an edge. The *independence number*, $\alpha(G)$, is the maximum size of an independent set in G. The *independence ratio*, $i(G)$, of a graph G is the ratio $\frac{\alpha(G)}{|V(G)|}$. For positive integers r and g, $i(r,g)$ denotes the infimum of $i(G)$ over the r-regular graphs of girth at least g, and $i(r,\infty)$ denotes $\lim_{g\to\infty} i(r,g)$. The first interesting upper bounds on $i(r,\infty)$ were obtained by Bollobás [2] in 1981. In particular, he proved $i(3,\infty) < \frac{6}{13}$. Refining the method, McKay [12] in 1987 showed

Theorem 1 (McKay [12])**.**

$$i(3,\infty) < 0.45537. \tag{1}$$

In the next 30 years, there were no improvements of Theorem 1, but recently some interesting lower bounds on $i(r,\infty)$ and in particular on $i(3,\infty)$ were proved. Hoppen [7] showed $i(3,\infty) \geqslant 0.4328$. Then Kardoš, Král and Volec [10] improved the bound to 0.4352. Csóka, Gerencsér, Harangi, and Virág [5] pushed the bound to 0.4361 and Hoppen and Wormald [8] — to 0.4375. Moreover, Csóka et al [5] claimed a computer assisted lower bound $i(3,\infty) \geqslant 0.438$, and Csóka [4] later improved the bound to 0.44533. The lower bound of $i(3,\infty)$ was also studied in [6], [11] and [13]. Our result is an improvement of (1) to $i(3,\infty) \leqslant 0.454$. The improvement is small, but it decreases the gap between the upper and lower bounds on $i(3,\infty)$ by approximately 14%.

Theorem 2. $i(3,\infty) \leqslant 0.454$.

The proof uses the language of configurations introduced by Bollobás [3], and shows that "many" 3-regular configurations have "small" independence ratio. The proof of our improvement is based on analyzing the presence not of largest independent sets, but of larger structures, so called MAI-sets (defined in Section 3) that contain largest independent sets.

2 Preliminaries

2.1 Notation

We mostly use standard notation. The complete n-vertex graph is denoted by K_n. If G is a multigraph and $v,u \in V(G)$, then $E_G(v,u)$ denotes the set of all edges in G connecting v and u, $e_G(v,u) := |E_G(v,u)|$, and $\deg_G(v) := \sum_{u\in V(G)\setminus\{v\}} e_G(v,u)$. By $\Delta(G)$ we denote the maximum degree of G, and by $g(G)$ — the *girth* (the length of a shortest cycle) of G. For $A \subseteq V(G)$, $G[A]$ denotes the submultigraph of G induced by A. For $k \in \mathbb{Z}_{>0}$, $[k]$ denotes the set $\{1,\ldots,k\}$.

2.2 The Configuration Model

The configuration model in different versions is due to Bender and Canfield [1] and Bollobás [3]. Our work is based on the version of Bollobás. Let n be an even positive integer and $V_n = [n]$. Consider the Cartesian product $W_n = V_n \times [3]$. A *configuration/pairing* (of order n and degree 3) is a perfect matching on the vertex set W_n. There are $(3n-1)\cdot(3n-3)\cdot\ldots\cdot 1 = (3n-1)!!$ such matchings.

Let $\mathcal{F}_3(n)$ denote the collection of all $(3n-1)!!$ possible pairings on W_n. We project each pairing $F \in \mathcal{F}_3(n)$ to a multigraph $\pi(F)$ on the vertex set V_n by ignoring the second coordinate. Then $\pi(F)$ is a 3-regular multigraph (which may or may not contain loops and/or multiple edges). Let $\pi(\mathcal{F}_3(n)) = \{\pi(F) : F \in \mathcal{F}_3(n)\}$ be the set of 3-regular multigraphs on V_n. By definition,

$$\textit{each simple graph } G \in \pi(\mathcal{F}_3(n)) \textit{ corresponds to } (3!)^n \textit{ distinct pairings in } \mathcal{F}_3(n). \quad (2)$$

We will call the elements of V_n - *vertices*, and of W_n - *points*.

Definition 3. Let $\mathcal{G}_g(n)$ be the set of all cubic graphs with vertex set $V_n = [n]$ and girth at least g and $\mathcal{G}'_g(n) = \{F \in \mathcal{F}_3(n) : \pi(F) \in \mathcal{G}_g(n)\}$.

We will heavily use the following result:

Theorem 4 (Wormald [14], Bollobás [3]). *For each fixed $g \geqslant 3$,*

$$\lim_{n\to\infty} \frac{|\mathcal{G}'_g(n)|}{|\mathcal{F}_3(n)|} = \exp\left\{-\sum_{k=1}^{g-1} \frac{2^{k-1}}{k}\right\}. \quad (3)$$

Remark. When we say that *a pairing F has a multigraph property $\mathcal{A}$*, we mean that $\pi(F)$ has property $\mathcal{A}$.

Since dealing with pairings is simpler than working with labeled simple regular graphs, we need the following well-known consequence of Theorem 4.

Corollary 5 ([12](Corollary 1.1), [9](Theorem 9.5)). *For fixed $g \geqslant 3$, any property that holds for $\pi(F)$ for almost all pairings $F \in \mathcal{F}_3(n)$ also holds for almost all graphs in $\mathcal{G}_g(n)$.*

Definition 6. For a graph G, let $I(G)$ denote the total number of all independent sets in G, including the empty set. For integers $r \geqslant 0$, $g \geqslant 3$, we define

$$I(r,g) = \inf I(G)^{1/|V(G)|},$$

where the infimum is over all graphs G of maximum degree at most r and girth at least g.

Recall that the *Fibonacci numbers* F_n are defined by $F_1 = F_2 = 1$, and $F_i = F_{i-1}+F_{i-2}$, for $i \geqslant 3$. The exact formula for F_i is

$$F_i = \frac{\varphi^i - \psi^i}{\sqrt{5}},$$

where $i \geqslant 0$, $\varphi = \frac{1+\sqrt{5}}{2}$, and $\psi = \frac{1-\sqrt{5}}{2}$.

Lemma 7 (McKay [12]). *For any $g \geqslant 4$, $I(2,g) = (F_{s-1} + F_{s+1})^{\frac{1}{s}}$, where $s = 2\lfloor g/2 \rfloor + 1$.*

Remark 8. The numbers $s-1$ and $s+1$ in Lemma 7 are even. Therefore,

$$I(2,g) = (F_{s-1}+F_{s+1})^{1/s} = \left(\frac{\varphi^{s-1}+\varphi^{s+1}-\varphi^{1-s}-\varphi^{-s-1}}{\sqrt{5}}\right)^{1/s}$$

$$= \varphi\cdot\left((1-\varphi^{-2s})\frac{\varphi^{-1}+\varphi}{\sqrt{5}}\right)^{1/s} = \varphi(1-\varphi^{-2s})^{1/s}.$$

Since the function $(1-\varphi^{-2s})^{1/s}$ monotonically increases for $s \geqslant 1$, and $\varphi(1-\varphi^{-18})^{1/9} \geqslant 1.618002$, we conclude that for each graph H with maximum degree at most 2 and girth at least 8,

$$1.618 \leqslant I(2,8) \leqslant I(H)^{1/|V(H)|}. \tag{4}$$

3 MAI sets in cubic graphs

Definition 9. A vertex set A in a graph G is *an AI set* (an almost independent set), if every component of $G[A]$ is an edge or an isolated vertex. In other words, A is *an AI set* if $\Delta(G[A]) \leqslant 1$.

Definition 10. A vertex set A is a *maximum almost independent set (MAI set)* in a graph G if all of the following hold:

M1. A is an AI set;

M2. A contains an independent set A' of size $\alpha(G)$;

M3. A is largest among all sets satisfying M1 and M2.

Let $G \in \mathcal{G}_{16}(n)$ and A be a MAI set. Denote $B = V(G) - A$.

Lemma 11. *B is an AI set.*

Proof. Let $b \in B$. We prove that $d_{G[B]}(b) \leqslant 1$. Let A' be a maximum independent set in A. If $d_{G[B]}(b) = 3$, then there is no edge from b to A, and $A' \cup \{b\}$ is an independent set in G with size $|A'|+1 = \alpha(G)+1$, contradicting the definition of $\alpha(G)$.

If $d_{G[B]}(b) = 2$, then there is only one edge e from b to A, say ba. If $d_{G[A]}(a) = 0$, then $G[A \cup \{b\}]$ is an AI set in G larger than A containing A'. This contradicts the fact that A is a MAI set. If $d_{G[A]}(a) = 1$, then without loss of generality, we may assume $a \in A - A'$. Then b has no neighbors in A', and $A' \cup \{b\}$ is an independent set in G with size $|A'|+1$, again contradicting the definition of $\alpha(G)$. □

Let A be a MAI set in $G \in G_{16}(n)$. Denote the set of vertices with degree 1 in $G[A]$ by Y, the set of vertices with degree 1 in $G[B]$ by Z. We introduce notation for the sizes of the sets: Let $x := |A'|$, $s := |Y|/2$, $t := |Z|/2$, and $i := \frac{n}{2} - |A|$. Then $|A| = \frac{n}{2} - i$ and $|B| = \frac{n}{2} + i$.

Lemma 12. *$i \geqslant 0$ and $t \geqslant s$.*

Proof. We count the number of edges with one end in A and one end in B in two ways. We have

$$2s \cdot 2 + \left(\frac{n}{2} - i - 2s\right) \cdot 3 = e[A,B] = 2t \cdot 2 + \left(\frac{n}{2} + i - 2t\right) \cdot 3, \tag{5}$$

i.e.,

$$t - s = 3i. \tag{6}$$

We also know that $x = \alpha(G)$, so

$$x = \frac{n}{2} - i - s \geqslant \frac{n}{2} + i - t,$$

i.e.,

$$2i \leqslant t - s = 3i,$$

which implies that

$$i \geqslant 0 \text{ and } t \geqslant s.$$

□

Lemma 13. *If $G \in \mathcal{G}_5(n)$, then*

(i) each vertex in Z has degree at most one to Y;

(ii) each vertex in Y has degree at most one to Z.

Proof. (i) Suppose $z \in Z$ and $N_G(z) = \{z', y_1, y_2\}$, where $z' \in Z$ and $y_1, y_2 \in Y$. Since $g(G) \geqslant 4$, $y_1 \neq y_2$, $y_1y_2 \notin E(G)$, and so $A - y_1 - y_2$ contains an independent set A' with $|A'| = \alpha(G)$. Thus the set $A' + z$ is an independent set of size $\alpha(G) + 1$ contradicting the definition of $\alpha(G)$.

(ii) Similarly, suppose $y \in Y$ and $N_G(y) = \{y', z_1, z_2\}$, where $y' \in Y$ and $z_1, z_2 \in Z$. Then $A - y$ contains an independent set A' with $|A'| = \alpha(G)$. For $i = 1, 2$, let $N_G(z_i) = \{z_i', y, a_i\}$, where $z_i' \in Z$. By Part (i), $a_1, a_2 \notin Y$. Since $g(G) \geqslant 5$, $a_2 \neq a_1$. Then $(A - y) \cup \{z_1, z_2\}$ is an AI set containing A' and is larger than A, a contradiction. □

Let $J = \{y_1z_1, \ldots, y_jz_j\}$ be the set of all edges connecting Y with Z in G. By Lemma 13, J is a matching in G. Define an auxiliary graph $H = H(A)$ as follows: $V(H) = J$, and $y_\ell z_\ell$ is adjacent to $y_{\ell'}z_{\ell'}$ if $y_\ell y_{\ell'} \in E(G)$ or $z_\ell z_{\ell'} \in E(G)$. By construction, the maximum degree of H is at most 2 and a cycle of length c in H corresponds to a cycle of length $2c$ in G.

Lemma 14. *The graph G contains at least $I(H)$ distinct MAIs.*

Proof. Let $J' = \{y_1z_1, \ldots, y_{j'}z_{j'}\}$ be an arbitrary independent set in H. Then the sets $Y_1 = \{y_1, \ldots, y_{j'}\}$ and $Z_1 = \{z_1, \ldots, z_{j'}\}$ are independent in G. By the definition of Y, $A - Y_1$ contains an independent set A' with $|A'| = \alpha(G)$. Let $A_1 = (A - Y_1) \cup Z_1$. By Lemma 13, the degree in $G[A_1]$ of every vertex in $(Y - Y_1) \cup Z_1$ is at most 1. If a vertex $a \in A - Y$ is adjacent to two vertices, say z_1, z_2 in Z_1, then the set $(A' - a) \cup \{z_1, z_2\}$ is independent and is larger than A', a contradiction. Thus, A_1 is an AI set. Since $|A_1| = |A|$, this proves the lemma. □

Remark 15. Recall that $|A| = \frac{n}{2} - i$, $|B| = \frac{n}{2} + i$, $|Y| = 2s = 2(\frac{n}{2} - i - x)$, and $|A - Y| = 2x - \frac{n}{2} + i$. By (6), we know that $t = 3i + s = \frac{n}{2} + 2i - x$. Therefore, $|Z| = 2t = 2(\frac{n}{2} + 2i - x)$ and $|B - Z| = 2x - \frac{n}{2} - 3i$. By (5), $e[A,B] = 2x + \frac{n}{2} - i$.

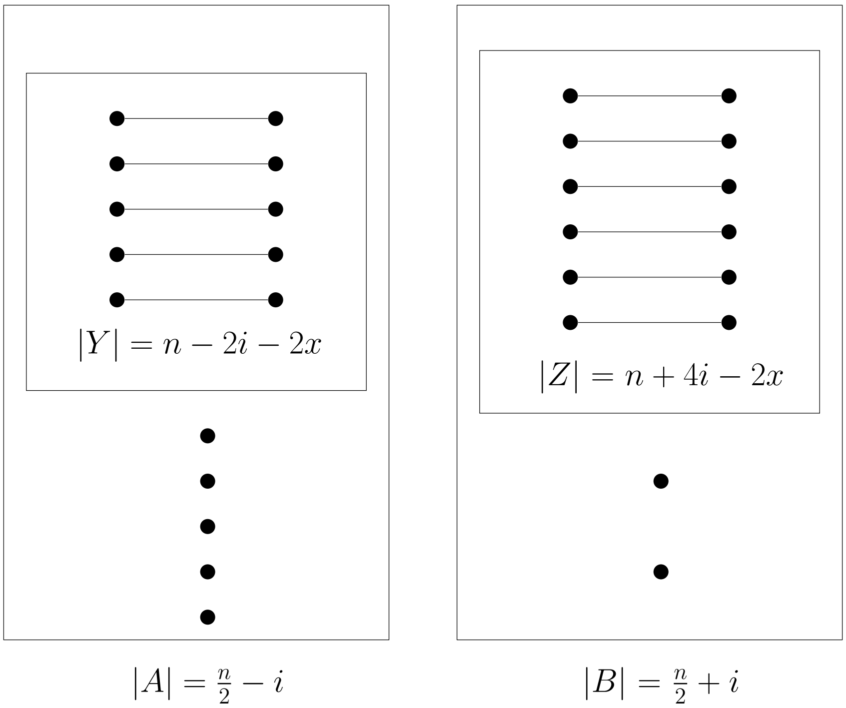

Figure 1: A MAI set A.

4 The set up of the proof

4.1 Restating the theorem

We will use Theorem 1 of McKay in the following stronger form.

Theorem 16 (McKay [12])**.** *For every $\epsilon > 0$, there exists an $N > 0$ such that for each $n > N$,*

$$|\{F|F \in \mathcal{F}_3(n) \,:\, \alpha(\pi(F)) > 0.45537n\}| \; < \epsilon \cdot (3n-1)!!.$$

We will show that "almost all" cubic labeled graphs of girth at least 16 have independence ratio at most 0.454. In view of Theorem 4, the following more technical statement implies Theorem 2.

Theorem 17. *For every $\epsilon > 0$, there is an $N > 0$ such that for each $n > N$,*

$$|\{F \in \mathcal{G}'_{16}(n) \,:\, \alpha(\pi(F)) > 0.454n\}| \; < \epsilon\,(3n-1)!!. \tag{7}$$

A referee asked whether one can derive from Theorem 17 that *a random cubic graph $G \in G_{n,3}$ asymptotically almost surely satisfies $\alpha(G) \leqslant 0.454n$.* We do not see how to derive this from the statement of the theorem but think that one can modify our proof to show this fact.

The rest of the paper is a proof of Theorem 17. By definition, every graph has a MAI set. So, for large n, nonnegative integers $x \geqslant 0.454n$ and $i \leqslant \frac{n}{2} - x$, and each set A of size $\frac{n}{2} - i$ with a fixed matching of size $\frac{n}{2} - i - x$ we will estimate the total *x-weight* of configurations $F \in \mathcal{G}'_{16}(n)$ in which A forms a MAI set. The idea of the weight (used by McKay in [12]) is to decrease overcount of the configurations containing a given MAI set, but guarantee that the total weight of each configuration containing at least one MAI set with independence number x would be at least 1.

4.2 Setup of the proof of Theorem 17

An *AI-pair* on $[n]$ is a pair (A, R) consisting of a set $A \subset [n]$ and a matching R on a subset of A such that $E(G[A]) = R$. The *independence number*, $\alpha(A, R)$, of an AI-pair (A, R) is $|A| - |R|$. Let $\mathcal{P}(n, x)$ denote the family of all AI-pairs (A, R) on $[n]$ with $\alpha(A, R) = x$.

A *preimage* of an AI-pair (A, R) on $[n]$ is a pair $(\hat{A}, \hat{R})$ where $\hat{A} = A \times [3]$ and $\hat{R}$ is a matching on a subset of $\hat{A}$ with $|\hat{R}| = |R|$ such that for each edge $(i, j)(i', j') \in \hat{R}$, $ii' \in R$. In other words, each edge $e \in R$ is obtained from an edge in $\hat{e} \in \hat{R}$ by ignoring the second coordinates of the ends of $\hat{e}$, and this mapping is one-to-one.

By the x-weight of a configuration F we mean

$$\omega_x(F) := \text{the reciprocal of the number of preimages } (\hat{A}, \hat{R}) \subseteq F \text{ of AI-pairs } (A, R) \text{ on } [n] \text{ such that } A \text{ is an AI set in } \pi(F) \text{ with } E(\pi(F)[A]) = R \text{ and } \alpha(A, R) = x. \tag{8}$$

By the definition of x-weight, each pairing $F \in \mathcal{G}'_{16}(n)$ with $\alpha(\pi(F)) = x$ contributes exactly 1 to

$$\sigma(n, x, 16) := \sum_{(A,R)\in\mathcal{P}(n,x)} \{\omega_x(F') : F' \in \mathcal{G}'_{16}(n) \text{ and } (\hat{A}, \hat{R}) \textit{ is an induced subpairing of } F'\}. \tag{9}$$

It follows that

$$\sigma(n, x, 16) \geqslant |\{F' \in \mathcal{G}'_{16}(n) \text{ with } \alpha(\pi(F')) = x\}| \,. \tag{10}$$

Lemma 18. *Let n be a positive even integer and x be an integer with $0.454n < x \leqslant 0.45537n$. The number of pairings $F \in \mathcal{G}'_{16}(n)$ such that $\pi(F)$ has a MAI set A with $|A'| = x$ is at most*

$$\begin{aligned} q(x, n) := \sum_{i=0}^{\frac{n}{2}-x} \binom{n}{\frac{n}{2} - i} \cdot \frac{(\frac{n}{2} - i)! \cdot 3^{(n-2x-2i)}}{(2x + i - \frac{n}{2})! \cdot 2^{\frac{n}{2}-x-i} \cdot (\frac{n}{2} - x - i)!} \\ \cdot \frac{(\frac{n}{2} + i)! \cdot 3^{n-2x+4i}}{(2x - 3i - \frac{n}{2})! \cdot 2^{\frac{n}{2}-x+2i} \cdot (\frac{n}{2} - x + 2i)!} \\ \cdot \sum_{j=0}^{n-2i-2x} \binom{n - 2i - 2x}{j} \cdot \binom{n - 2x + 4i}{j} \cdot 2^{2j} \cdot j! \cdot \left(\frac{1}{1.618}\right)^j \\ \cdot \frac{(3(2x - \frac{n}{2} - 3i))! \cdot (3(2x - \frac{n}{2} + i))!}{(3(2x - \frac{n}{2} - 3i) - 2(n - 2i - 2x) + j)!}. \end{aligned}$$

Proof. By (10), it is enough to show that $\sigma(n, x, 16) \leqslant q(x, n)$. Below we describe a procedure of constructing for every AI-pair (A, R) on $[n]$ with $\alpha(A, R) = x$ all pairings in $F \in \mathcal{G}'_{16}(n)$ for which A is a MAI set. Not every obtained pairing will be in $\mathcal{G}'_{16}(n)$ and some pairings will have independence number larger than x, but every $F \in \mathcal{G}'_{16}(n)$ such that A is a MAI set in $\pi(F)$ will be a result of this procedure.

0. Choose nonnegative integers n, x, i, j such that n is even, $0.454n < x \leqslant 0.45537n$, $i \leqslant \frac{n}{2} - x$, and $j \leqslant \frac{n}{2} - x - i$.

1. Choose a set $A \subset [n]$ with $|A| = \frac{n}{2} - i$. There are $\binom{n}{\frac{n}{2}-i}$ ways to do it.

2. Choose a matching R on A with $|R| = \frac{n}{2} - x - i$. There are

$$\frac{(\frac{n}{2} - i)!}{(2x + i - \frac{n}{2})! \cdot 2^{\frac{n}{2}-x-i} \cdot (\frac{n}{2} - x - i)!}$$

ways to do it. Then there are $3^{n-2x-2i}$ ways to decide which point of each chosen end of an edge in R will be the end of the corresponding edge in F.

3. Similarly to Step 2, we have

$$\frac{(\frac{n}{2} + i)!}{(2x - 3i - \frac{n}{2})! \cdot 2^{\frac{n}{2}-x+2i} \cdot (\frac{n}{2} - x + 2i)!}$$

ways to construct a matching R' of $\frac{n}{2} - x + 2i$ edges on $B := [n] - A$, since $|B| = \frac{n}{2} + i$. After that there are $3^{n-2x+4i}$ ways to decide which point of each chosen end of an edge in R' will be the end of the corresponding edge in F.

4. Let Y (respectively, Z) be the set of vertices covered by the matching R (respectively, R'). By Lemma 13, if A is a MAI-set in $\pi(F)$, then the set of edges connecting Y with Z is a matching. If this matching, say M has j edges, then there are $\binom{n-2i-2x}{j}$ ways to choose the set of the ends of M in Y and $\binom{n-2x+4i}{j} j!$ ways to choose the ends of M in Z. Since there are 2 free points left for each vertex in Y and Z, we have 2^{2j} ways to choose which point of each vertex in Y and Z to be used to form an edge in M.

5. By Lemma 14 each pairing $F \in \mathcal{G}'_{16}(n)$ containing a MAI set A with j edges between Y and Z contains at least $I(2, 8)^j$ distinct MAI sets of the same cardinality. By Lemma 7, $I(2, 8)^j \geqslant 1.618^j$. Hence by (8), $\omega_x(F) \leqslant 1.618^{-j}$.

6. Now we choose for each remaining free point p from vertices in Y a free point q in a vertex in $B - Z$ and add edge pq. There are

$$\frac{(3(2x - \frac{n}{2} - 3i))!}{(3(2x - \frac{n}{2} - 3i) - 2(n - 2i - 2x) + j)!}$$

ways to do it.

7. Similarly to Step 6, we choose for each remaining free point q from vertices in Z a free point p in a vertex in $A - Y$ and add edge pq. There are

$$\frac{3(2x - \frac{n}{2} + i))!}{(3(2x - \frac{n}{2} + i) - 2(n - 2x + 4i) + j)!}$$

ways to do it.

8. Finally, there are $3(2x-\frac{n}{2}+i)-2(n-2x+4i)+j=10x-\frac{7n}{2}-5i+j$ free points left in A and $10x-\frac{7n}{2}-5i+j$ free points left in B. We have $(10x-\frac{7n}{2}-5i+j)!$ ways to complete a pairing on W_n. □

In the proofs below we will use Stirling's formula: For every $n \geqslant 1$,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leqslant n! \leqslant \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}. \tag{11}$$

We will also use the notation $\frac{\partial}{\partial j}$ to denote the partial derivative with respect to j. Moreover, we use the domain $x \geqslant 0$ and define $\ln(0) = -\infty$ when we consider $\ln x$.

Lemma 19. *Let n be a positive even integer and x be an integer satisfying $0.454n < x \leqslant 0.45537n$. Let*

$$\Omega = \{(\chi,\zeta,\xi) : 0.454 < \chi \leqslant 0.45537,\ 0 \leqslant \zeta \leqslant \frac{1}{2}-\chi,\ 0 \leqslant \xi \leqslant 1-2\chi-2\zeta\}. \tag{12}$$

Let

$$\begin{aligned} f(\chi,\zeta) :=& 3^{\frac{1}{2}-4\chi+2\zeta}\cdot(1-2\chi-2\zeta)^{1-2\chi-2\zeta}\cdot(1-2\chi+4\zeta)^{1-2\chi+4\zeta}\cdot(6\chi-\frac{3}{2}+3\zeta)^{6\chi-\frac{3}{2}+3\zeta}\\ &\cdot(6\chi-\frac{3}{2}-9\zeta)^{6\chi-\frac{3}{2}-9\zeta}\cdot(2\chi+\zeta-\frac{1}{2})^{-(2\chi+\zeta-\frac{1}{2})}\cdot 2^{-(1-2\chi+\zeta)}\\ &\cdot(\frac{1}{2}-\chi-\zeta)^{-(\frac{1}{2}-\chi-\zeta)}\cdot(\frac{1}{2}-\chi+2\zeta)^{-(\frac{1}{2}-\chi+2\zeta)}\cdot(2\chi-3\zeta-\frac{1}{2})^{-(2\chi-3\zeta-\frac{1}{2})}, \end{aligned}$$

$$\begin{aligned} g(\chi,\zeta,\xi) :=& 2^{2\xi}\cdot(\frac{1}{1.618})^{\xi}\cdot\xi^{-\xi}\cdot(1-2\chi-2\zeta-\xi)^{-(1-2\chi-2\zeta-\xi)}\\ &\cdot(1-2\chi+4\zeta-\xi)^{-(1-2\chi+4\zeta-\xi)}\cdot(-\frac{7}{2}+10\chi-5\zeta+\xi)^{\frac{7}{2}-10\chi+5\zeta-\xi}, \end{aligned}$$

and

$$h(\chi,\zeta,\xi) := f(\chi,\zeta)\cdot g(x,\zeta,\xi).$$

Then

$$\frac{q(x,n)}{(3n-1)!!} = O(n^6)\cdot\max\{(h(\chi,\zeta,\xi))^n\ :\ (\chi,\zeta,\xi)\in\Omega\}. \tag{13}$$

Proof. We write $q(x,n)$ as a double sum of i and j and let $r(x,n,i,j)$ be the function inside the double sum of $q(x,n)$, i.e.,

$$q(x,n) = \sum_{i=0}^{\frac{n}{2}-x}\sum_{j=0}^{n-2x-2i} r(x,n,i,j).$$

Then certainly,

$$q(x,n) \leqslant n^2\cdot\max\{r(x,n,i,j) : 0 \leqslant i \leqslant \frac{n}{2}-x,\ 0 \leqslant j \leqslant n-2x-2i\}.$$

So, it is enough to estimate $r(x,n,i,j)$. We know that

$$r(x,n,i,j) = \frac{n!}{(\frac{n}{2}-i)!\cdot(\frac{n}{2}+i)!}\cdot\frac{(\frac{n}{2}-i)!\cdot 3^{n-2x-2i}}{(2x+i-\frac{n}{2})!\cdot 2^{\frac{n}{2}-x-i}\cdot(\frac{n}{2}-x-i)!}$$

$$\cdot \frac{(\frac{n}{2}+i)! \cdot 3^{n-2x+4i}}{(2x-3i-\frac{n}{2})! \cdot 2^{\frac{n}{2}-x+2i} \cdot (\frac{n}{2}-x+2i)!} \cdot \frac{(n-2i-2x)!}{j! \cdot (n-2i-2x-j)!}$$
$$\cdot \frac{(n-2x+4i)!}{j! \cdot (n-2x+4i-j)!} \cdot 2^{2j} \cdot j! \cdot (\frac{1}{1.618})^j \cdot \frac{(6x-\frac{3n}{2}-9i)! \cdot (6x-\frac{3n}{2}+3i)!}{(10x-\frac{7n}{2}-5i+j)!}.$$

Recall that

$$(3n-1)!! \geqslant \frac{(3n)!!}{3n} \geqslant \frac{\sqrt{(3n)!}}{3n}.$$

Therefore,

$$\frac{r(x,n,i,j)}{(3n-1)!!} \leqslant \frac{n! \cdot (3n)}{((3n)!)^{\frac{1}{2}}} \cdot \frac{3^{n-2x-2i}}{(2x+i-\frac{n}{2})! \cdot 2^{\frac{n}{2}-x-i} \cdot (\frac{n}{2}-x-i)!}$$
$$\cdot \frac{3^{n-2x+4i}}{(2x-3i-\frac{n}{2})! \cdot 2^{\frac{n}{2}-x+2i} \cdot (\frac{n}{2}-x+2i)!} \cdot \frac{(n-2i-2x)!}{j! \cdot (n-2i-2x-j)!}$$
$$\cdot \frac{(n-2x+4i)!}{(n-2x+4i-j)!} \cdot 2^{2j} \cdot (\frac{1}{1.618})^j \cdot \frac{(6x-\frac{3n}{2}-9i)! \cdot (6x-\frac{3n}{2}+3i)!}{(10x-\frac{7n}{2}-5i+j)!}.$$

Introducing new variables $\chi := \frac{x}{n}$, $\zeta := \frac{i}{n}$, and $\xi := \frac{j}{n}$ and using Stirling's formula (11), we get

$$\frac{r(x,n,i,j)}{(3n-1)!!} = O(n^4) \cdot \left(\frac{n}{e}\right)^n \cdot \left(\frac{n}{e}\right)^{(1-2\zeta-2\chi)n} \cdot \left(\frac{n}{e}\right)^{(1-2\chi+4\zeta)n} \cdot \left(\frac{n}{e}\right)^{(6\chi-\frac{3}{2}-9\zeta)n} \cdot \left(\frac{n}{e}\right)^{(6\chi-\frac{3}{2}+3\zeta)n}$$
$$\cdot \left(\frac{n}{e}\right)^{-\frac{3}{2}n} \cdot \left(\frac{n}{e}\right)^{-(2\chi+\zeta-\frac{1}{2})n} \cdot \left(\frac{n}{e}\right)^{-(\frac{1}{2}-\chi-\zeta)n} \cdot \left(\frac{n}{e}\right)^{-(2\chi-3\zeta-\frac{1}{2})n} \cdot \left(\frac{n}{e}\right)^{-(\frac{1}{2}-\chi+2\zeta)n} \cdot \left(\frac{n}{e}\right)^{-\xi n}$$
$$\cdot \left(\frac{n}{e}\right)^{-(1-2\zeta-2\chi-\xi)n} \cdot \left(\frac{n}{e}\right)^{-(1-2\chi+4\zeta-\xi)n} \cdot \left(\frac{n}{e}\right)^{-(10\chi-\frac{7}{2}-5\zeta+\xi)n} \cdot (f(\chi,\zeta) \cdot g(\chi,\zeta,\xi))^n.$$

Therefore,

$$\frac{r(x,n,i,j)}{(3n-1)!!} = O(n^4) \cdot (h(\chi,\zeta,\xi))^n.$$

This proves the lemma. □

Recall that the domain of $h(\chi,\zeta,\xi)$ is Ω defined in (12). Our main goal now is to show that

$$\max_{(\chi,\zeta,\xi)\in\Omega} h(\chi,\zeta,\xi) \leqslant 0.999983 < 1. \tag{14}$$

We do this in the next section, and then Theorem 17 easily follows.

5 Proof of (14)

In order to find the maximum value of $h(\chi,\zeta,\xi)$ for a fixed χ, we will maximize

$$\ln(h(\chi,\zeta,\xi)).$$

We first find the value of ξ in terms of χ and ζ that maximizes $\ln(g(\chi,\zeta,\xi))$. By definition,

$$\ln(g(\chi,\zeta,\xi)) = \xi \ln(\frac{4}{1.618}) - (\xi \ln(\xi) + (1-2\zeta-2\chi-\xi)\ln(1-2\zeta-2\chi-\xi)$$

$$+(1-2\chi+4\zeta-\xi)\ln(1-2\chi+4\zeta-\xi)+(10\chi-\frac{7}{2}-5\zeta+\xi)\ln(10\chi-\frac{7}{2}-5\zeta+\xi)).$$

Hence

$$\begin{aligned}\frac{\partial\ln(g(\chi,\zeta,\xi))}{\partial\xi} =& \ln(1-2\chi-2\zeta-\xi)+\ln(1-2\chi+4\zeta-\xi)\\ &-\ln\left(10\chi-5\zeta+\xi-\frac{7}{2}\right)-\ln(\xi)+\ln\left(\frac{4}{1.618}\right)\\ =& \ln\left(\frac{(1-2\chi-2\zeta-\xi)\cdot(1-2\chi+4\zeta-\xi)\cdot\frac{4}{1.618}}{\xi\cdot(10\chi-5\zeta+\xi-\frac{7}{2})}\right).\end{aligned}$$

In order to solve

$$\frac{\partial\ln(g(\chi,\zeta,\xi))}{\partial\xi}=0,$$

we solve the equivalent equation

$$p(\xi):=4\cdot(1-2\chi-2\zeta-\xi)\cdot(1-2\chi+4\zeta-\xi)-1.618\cdot\xi\cdot(10\chi-5\zeta+\xi-\frac{7}{2})=0,$$

where $p(\xi)$ has domain $0\leqslant\xi\leqslant 1-2\chi-2\zeta$. By the quadratic formula, the roots are

$$\xi_1=\frac{-b-\sqrt{b^2-4ac}}{2a}\quad\text{and}\quad\xi_2=\frac{-b+\sqrt{b^2-4ac}}{2a},$$

where

$$\begin{aligned}a &= 2.382,\\ b &= -0.18\chi+0.09\zeta-2.337, &(15)\\ c &= 16\chi^2-32\zeta^2-16\chi+8\zeta-16\chi\zeta+4. &(16)\end{aligned}$$

Moreover, for fixed χ and ζ satisfying $0.454\leqslant\chi\leqslant 0.45537$ and $\chi+\zeta\leqslant\frac{1}{2}$, $p(\xi)$ is a parabola opening upward with $\xi_1\leqslant 1-2\chi-2\zeta\leqslant\xi_2$ because $p(1-2\chi-2\zeta)\leqslant 0$, and $g(\chi,\zeta,\xi)$ is a continuous function on ξ. Therefore, the maximum of $g(\chi,\zeta,\xi)$ can only be attained at $\xi=\xi_1$.

Let $g_1(\chi,\zeta)=g(\chi,\zeta,\xi_1(\chi,\zeta))$. For each fixed χ, consider the maximum of

$$h_1(\chi,\zeta):=f(\chi,\zeta)\cdot g_1(\chi,\zeta).$$

By definition,

$$\begin{aligned}\ln(h_1)=&(\frac{1}{2}-4\chi+2\zeta)\ln(3)+(1-2\chi-2\zeta)\ln(1-2\chi-2\zeta)+(1-2\chi+4\zeta)\ln(1-2\chi+4\zeta)\\ &+(6\chi-\frac{3}{2}+3\zeta)\ln(6\chi-\frac{3}{2}+3\zeta)+(6\chi-\frac{3}{2}-9\zeta)\ln(6\chi-\frac{3}{2}-9\zeta)+\xi_1(\chi,\zeta)\cdot\ln(\frac{4}{1.618})\\ &-(2\chi+\zeta-\frac{1}{2})\ln(2\chi+\zeta-\frac{1}{2})-(1-2\chi+\zeta)\ln(2)-(\frac{1}{2}-\chi-\zeta)\ln(\frac{1}{2}-\chi-\zeta)\\ &-(\frac{1}{2}-\chi+2\zeta)\ln(\frac{1}{2}-\chi+2\zeta)-(2x-3\zeta-\frac{1}{2})\ln(2\chi-3\zeta-\frac{1}{2})-\xi_1(\chi,\zeta)\cdot\ln(\xi_1(\chi,\zeta))\\ &-(1-2\zeta-2\chi-\xi_1(\chi,\zeta))\ln(1-2\zeta-2\chi-\xi_1(\chi,\zeta))\\ &-(1-2\chi+4\zeta-\xi_1(\chi,\zeta))\ln(1-2\chi+4\zeta-\xi_1(\chi,\zeta))\end{aligned}$$

$$- (10\chi - \frac{7}{2} - 5\zeta + \xi_1(\chi,\zeta)) \ln(10\chi - \frac{7}{2} - 5\zeta + \xi_1(\chi,\zeta)),$$

and

$$\begin{aligned}\frac{\partial \ln(h_1)}{\partial \zeta} =& -4\ln(3) + \ln(2) - 3 + 2\ln(2\chi + \zeta - \frac{1}{2}) - \ln(\frac{1}{2} - \chi - \zeta) + 2\ln(\frac{1}{2} - \chi + 2\zeta) \\ &- 6\ln(2\chi - 3\zeta - \frac{1}{2}) + \ln(\frac{4}{1.618}) \cdot \frac{\partial \xi_1(\chi,\zeta)}{\partial \zeta} - \frac{\partial \xi_1(\chi,\zeta)}{\partial \zeta} \cdot (\ln(\xi_1(\chi,\zeta)) + 1) \\ &+ (2 + \frac{\partial \xi_1(\chi,\zeta)}{\partial \zeta}) \cdot (\ln(1 - 2\zeta - 2\chi - \xi_1(\chi,\zeta)) + 1) \\ &+ (\frac{\partial \xi_1(\chi,\zeta)}{\partial \zeta} - 4) \cdot (\ln(1 - 2\chi + 4\zeta - \xi_1(\chi,\zeta)) + 1) + (5 - \frac{\partial \xi_1(\chi,\zeta)}{\partial \zeta}) \\ &\cdot (\ln(10\chi - \frac{7}{2} - 5\zeta + \xi_1(\chi,\zeta)) + 1),\end{aligned}$$

where

$$\begin{aligned}\frac{\partial \xi_1(\chi,\zeta)}{\partial \zeta} &= \frac{1}{2a} \cdot (-\frac{\partial b}{\partial \zeta} - \frac{1}{2} \cdot (b^2 - 4ac)^{-\frac{1}{2}} \cdot (2b\frac{\partial b}{\partial \zeta} - 4a\frac{\partial c}{\partial \zeta})), \\ \frac{\partial b}{\partial \zeta} &= 0.09, \\ \frac{\partial c}{\partial \zeta} &= -16\chi - 64\zeta + 8.\end{aligned}$$

Lemma 20. *When* $\chi = 0.454$, *the maximum of* $h(\chi, \zeta, \xi)$ *over* $0 \leqslant \zeta \leqslant 0.046$ *and* $0 \leqslant \xi \leqslant 0.092 - 2\zeta$ *is at most* 0.999983.

Proof. Fix $\chi = 0.454$. For $0 \leqslant \zeta \leqslant 0.046$, denote

$$\xi_1'(\zeta) := \frac{\partial \xi_1(0.454,\zeta)}{\partial \zeta},\ \xi_1''(\zeta) := \frac{\partial^2 \xi_1(0.454,\zeta)}{\partial \zeta^2}, \text{ and } \xi_1(\zeta) := \xi_1(0.454,\zeta).$$

We have

$$\begin{aligned}\frac{\partial \ln(h_1(0.454,\zeta))}{\partial \zeta} =& -4\ln(3) + \ln(2) - 3 + 2\ln(0.408 + \zeta) - \ln(0.046 - \zeta) + 2\ln(0.046 + 2\zeta) \\ &- 6\ln(0.408 - 3\zeta) + \ln(\frac{4}{1.618}) \cdot \xi_1'(\zeta) - \xi_1'(\zeta) \cdot (\ln(\xi_1(\zeta)) + 1) + (2 + \xi_1'(\zeta)) \\ &\cdot (\ln(0.092 - 2\zeta - \xi_1(\zeta)) + 1) + (\xi_1'(\zeta) - 4) \\ &\cdot (\ln(0.092 + 4\zeta - \xi_1(\zeta)) + 1) + (5 - \xi_1'(\zeta)) \cdot (\ln(1.04 - 5\zeta + \xi_1(\zeta)) + 1),\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \ln(h_1(0.454,\zeta))}{\partial \zeta^2} =& \frac{1}{0.046 - \zeta} + \frac{4}{0.046 + 2\zeta} + \frac{2}{0.408 + \zeta} + \frac{18}{0.408 - 3\zeta} \\ &+ \ln(\frac{4}{1.618}) \cdot \xi_1''(\zeta) - \xi_1''(\zeta) \cdot (\ln(\xi_1(\zeta)) + 1) - (\xi_1'(\zeta))^2 \cdot \frac{1}{\xi_1(\zeta)} \\ &+ \xi_1''(\zeta) \cdot (\ln(0.092 - 2\zeta - \xi_1(\zeta)) + 1) - (2 + \xi_1'(\zeta))^2 \cdot \frac{1}{0.092 - 2\zeta - \xi_1(\zeta)} \\ &+ \xi_1''(\zeta) \cdot (\ln(0.092 + 4\zeta - \xi_1(\zeta)) + 1) - (\xi_1'(\zeta) - 4)^2 \cdot \frac{1}{0.092 + 4\zeta - \xi_1(\zeta)} \\ &- \xi_1''(\zeta) \cdot (\ln(1.04 - 5\zeta + \xi_1(\zeta)) + 1) - (\xi_1'(\zeta) - 5)^2 \cdot \frac{1}{1.04 - 5\zeta + \xi_1(\zeta)},\end{aligned}$$

where

$$\xi_1''(\zeta) = \frac{1}{2a} \cdot (b^2 - 4ac)^{-\frac{1}{2}} \cdot \left(\frac{1}{4} \cdot (b^2 - 4ac)^{-1} \cdot (2b\frac{\partial b}{\partial \zeta} - 4a\frac{\partial c}{\partial \zeta})^2 - \frac{1}{2} \cdot (2(\frac{\partial b}{\partial \zeta})^2 + 1024a) \right). \tag{17}$$

We will show that

$$\frac{\partial^2 \ln(h_1(0.454, \zeta))}{\partial \zeta^2} < 0, \text{ for all } 0 \leqslant \zeta < 0.046. \tag{18}$$

This will guarantee that if we find a solution $\zeta_0 \in [0, 0.046)$ of the equation $\frac{\partial \ln(h_1(0.454,\zeta))}{\partial \zeta} = 0$, then the maximum of $h_1(0.454, \zeta)$ over $\zeta \in [0, 0.046)$ is attained at ζ_0.

Claim 21. *For each* $\zeta \in [0, 0.046)$*,* $-27.336 \leqslant \xi_1''(\zeta) < -24.822$.

Proof: By (15) and (16), for $\chi = 0.454$, the function $\Delta(\zeta) := b^2 - 4ac$ is quadratic in ζ with derivative

$$\Delta'(\zeta) = 2b\frac{\partial b}{\partial \zeta} - 4a\frac{\partial c}{\partial \zeta},$$

which is linear in ζ and has minimum at $\zeta = 0$ and maximum at $\zeta = 0.046$. Therefore,

$$-7.45 \leqslant \Delta'(0) \leqslant \Delta'(\zeta) \leqslant \Delta'(0.046) \leqslant 20.61$$

for each $\zeta \in [0, 0.046)$. Also for such ζ,

$$\Delta''(\zeta) = 2(\frac{\partial b}{\partial \zeta})^2 - 4a\frac{\partial^2 c}{\partial \zeta^2} = 2 \cdot 0.09^2 - 4 \cdot 2.382 \cdot (-64) \in (609.8, 609.81),$$

so $\Delta(\zeta)$ is a parabola opening upward with minimum attained at the unique root ζ_ξ of the equation $\Delta'(\zeta) = 0$. By $\Delta'(0.012213) < -0.00039$, $\Delta'(0.012214) > 0.000219$, and the above statements,

$$0.012213 \leqslant \zeta_\xi \leqslant 0.012214.$$

Hence $\Delta(\zeta_\xi)$ satisfies

$$\begin{aligned} 5.4821 \leqslant 5.48214 - 0.0004 \cdot 0.000001 &\leqslant \Delta(0.012213) + \Delta'(0.012213) \cdot 0.000001 \\ &\leqslant \Delta(\zeta_\xi) \leqslant \Delta(0.012213) \leqslant 5.4822, \end{aligned}$$

and the maximum of $\Delta(\zeta)$ over $\zeta \in [0, 0.046)$ is attained at $\zeta = 0.046$ and satisfies

$$5.83019 \leqslant \Delta(0.046) \leqslant 5.8302.$$

Therefore, for each $\zeta \in [0, 0.046)$,

$$0.41415 \leqslant \frac{1}{\sqrt{5.8302}} \leqslant (\Delta(\zeta))^{-\frac{1}{2}} \leqslant \frac{1}{\sqrt{5.4821}} \leqslant 0.427098,$$

$$0.17152 \leqslant (\Delta(\zeta))^{-1} \leqslant 0.182412.$$

Thus by (17),

$$-27.336 \leqslant \frac{1}{2 \cdot 2.382} \cdot 0.427098 \cdot (0 - 0.5 \cdot 609.81) \leqslant \xi_1''(\zeta)$$

$$= \frac{1}{2a} \cdot (\Delta(\zeta))^{-\frac{1}{2}} \cdot \left(\frac{1}{4} \cdot (\Delta(\zeta))^{-1} \cdot (\Delta'(\zeta))^2 - \frac{1}{2} \cdot \Delta''(\zeta)\right)$$
$$\leqslant \frac{1}{2 \cdot 2.382} \cdot 0.41415 \cdot (0.25 \cdot 0.182412 \cdot (20.61)^2 - 0.5 \cdot 609.8) \leqslant -24.822.$$

This proves Claim 21. ■

Claim 22. *For each* $\zeta \in [0, 0.046)$, $-0.91445 \leqslant \xi_1'(0.046) \leqslant \xi_1'(\zeta) \leqslant \xi_1'(0) \leqslant 0.31359.$

Proof: By Claim 21, $\xi_1'(\zeta)$ is a decreasing function on $0 \leqslant \zeta \leqslant 0.046$. ■

To prove (18), we write $\frac{\partial^2 \ln(h_1(0.454,\zeta))}{\partial \zeta^2}$ in a form

$$\frac{\partial^2 \ln(h_1(0.454, \zeta))}{\partial \zeta^2} = A_1(\zeta) + A_2(\zeta) + A_3(\zeta) + A_4(\zeta) + A_5(\zeta), \tag{19}$$

and then bound these expressions separately so that the sum of the upper bounds will be negative for each $\zeta \in [0, 0.046)$. By definition

$$\begin{aligned}\frac{\partial^2 \ln(h_1)}{\partial \zeta^2} =& \frac{1}{0.046 - \zeta} - \xi_1''(\zeta) \cdot \ln(\xi_1(\zeta)) - \frac{(\xi_1'(\zeta))^2}{\xi_1(\zeta)} + \frac{4}{0.046 + 2\zeta} + \frac{2}{0.408 + \zeta} \\ &+ \frac{18}{0.408 - 3\zeta} + \ln(\frac{4}{1.618}) \cdot \xi_1''(\zeta) + \xi_1''(\zeta) \cdot \ln(0.092 - 2\zeta - \xi_1(\zeta)) - \frac{(2 + \xi_1'(\zeta))^2}{0.092 - 2\zeta - \xi_1(\zeta)} \\ &+ \xi_1''(\zeta) \cdot \ln(0.092 + 4\zeta - \xi_1(\zeta)) - \frac{(\xi_1'(\zeta) - 4)^2}{0.092 + 4\zeta - \xi_1(\zeta)} \\ &- \xi_1''(\zeta) \cdot \ln(1.04 - 5\zeta + \xi_1(\zeta)) - \frac{(\xi_1'(\zeta) - 5)^2}{1.04 - 5\zeta + \xi_1(\zeta)}.\end{aligned}$$

Let

$$A_1(\zeta) := \frac{1}{0.046 - \zeta} - (\xi_1'(\zeta))^2 \cdot \frac{1}{\xi_1(\zeta)} - \frac{(2 + \xi_1'(\zeta))^2}{0.092 - 2\zeta - \xi_1(\zeta)}, \tag{20}$$
$$A_2(\zeta) := \xi_1''(\zeta) \cdot \ln(0.092 - 2\zeta - \xi_1(\zeta)) - \xi_1''(\zeta) \cdot \ln(\xi_1(\zeta)), \tag{21}$$
$$A_3(\zeta) := \ln(\frac{4}{1.618}) \cdot \xi_1''(\zeta), \tag{22}$$
$$A_4(\zeta) := \frac{4}{0.046 + 2\zeta} + \frac{2}{0.408 + \zeta} + \frac{18}{0.408 - 3\zeta}, \text{ and} \tag{23}$$
$$\begin{aligned}A_5(\zeta) :=& \xi_1''(\zeta) \cdot \ln(0.092 + 4\zeta - \xi_1(\zeta)) - \frac{(\xi_1'(\zeta) - 4)^2}{0.092 + 4\zeta - \xi_1(\zeta)} \\ &- \xi_1''(\zeta) \cdot \ln(1.04 - 5\zeta + \xi_1(\zeta)) - \frac{(\xi_1'(\zeta) - 5)^2}{1.04 - 5\zeta + \xi_1(\zeta)},\end{aligned} \tag{24}$$

so that (19) holds.

Claim 23. *For each* $\zeta \in [0, 0.046)$, $A_1(\zeta) < 0$.

Proof: Since $0.092 - 2\zeta - \xi_1(\zeta) \geqslant 0$ and $\xi_1(\zeta) \geqslant 0$, by Claim 22,

$$\begin{aligned}A_1(\zeta) &= \frac{1}{0.046 - \zeta} - (\xi_1'(\zeta))^2 \cdot \frac{1}{\xi_1(\zeta)} - (2 + \xi_1'(\zeta))^2 \cdot \frac{1}{0.092 - 2\zeta - \xi_1(\zeta)} \\ &\leqslant \frac{1}{0.046 - \zeta} - (\xi_1'(\zeta))^2 \cdot \frac{1}{0.092 - 2\zeta} - (2 + \xi_1'(\zeta))^2 \cdot \frac{1}{0.092 - 2\zeta}\end{aligned}$$

$$= \frac{1}{0.046 - \zeta} - \frac{(\xi_1'(\zeta) + 1)^2 + 1}{0.046 - \zeta} = -\frac{(\xi_1'(\zeta) + 1)^2}{0.046 - \zeta} < 0.$$

■

Claim 24. *For each* $\zeta \in [0, 0.046)$, $A_2(\zeta) < 0$.

Proof: Let $\zeta \in [0, 0.046)$. By Claim 21, inequality $A_2(\zeta) < 0$ is equivalent to

$$0.092 - 2\zeta - \xi_1(\zeta) > \xi_1(\zeta).$$

Let $y(\zeta) = 0.092 - 2\zeta - 2\xi_1(\zeta)$. By Claim 22,

$$y'(\zeta) = -2 - 2\xi_1'(\zeta) < 0.$$

Therefore, $y(\zeta) > y(0.046) = 0$ for each $\zeta \in [0, 0.046)$. This proves the claim. ■

Claim 25. *For each* $\zeta \in [0, 0.046)$, $A_3(\zeta) \leqslant -22.46$.

Proof: This follows from the definition (22), since $\xi_1''(\zeta) \leqslant -24.822$ by Claim 21. ■

Claim 26. *The function* $A_4'(\zeta)$ *has exactly one root* d_ζ *in the interval* $[0, 0.046]$. *Furthermore,* $d_\zeta \in (0.0355167, 0.0355168)$, *and* $A_4(\zeta)$ *is decreasing on* $[0, d_\zeta]$ *and increasing on* $[d_\zeta, 0.046]$.

Proof: By Definition (23),

$$A_4'(\zeta) = -\frac{2}{(\zeta + 0.023)^2} - \frac{2}{(\zeta + 0.408)^2} + \frac{6}{(\zeta - 0.136)^2}$$

and

$$A_4''(\zeta) = \frac{4}{(\zeta + 0.023)^3} + \frac{4}{(\zeta + 0.408)^3} - \frac{12}{(\zeta - 0.136)^3}.$$

The last expression is positive for all $\zeta \in [0, 0.046]$, so function $A_4'(\zeta)$ may have at most one root on $[0, 0.046]$. On the other hand, $A_4'(0.0355167) < -0.002$ and $A_4'(0.0355168) > 0.0006$. This proves the claim. ■

Claim 27. *For each* $\zeta \in [0, 0.046)$, $A_4(\zeta) + A_5(\zeta) \leqslant 20$.

Proof: Let

$$z_1(\zeta) = 0.092 + 4\zeta - \xi_1(\zeta) \quad \text{and} \quad z_2(\zeta) = 1.04 - 5\zeta + \xi_1(\zeta).$$

By Claim 22, $z_1'(\zeta) = 4 - \xi_1'(\zeta) > 0$ and $z_2'(\zeta) = -5 + \xi_1'(\zeta) < 0$ for each $\zeta \in [0, 0.046)$. So,

$$z_1(\zeta) \textit{ is increasing and } z_2(\zeta) \textit{ is decreasing on } [0, 0.046). \tag{25}$$

Since

$$z_1(\zeta) < z_1(0.046) < z_2(0.046) < z_2(\zeta)$$

for each $\zeta \in [0, 0.046)$, Definitions (23) and (24) together with Claim 21 yield

$$A_4(\zeta) + A_5(\zeta) = A_4(\zeta) + \xi_1''(\zeta) \cdot \ln(z_1(\zeta)) - \frac{(\xi_1'(\zeta) - 4)^2}{z_1(\zeta)} - \xi_1''(\zeta) \cdot \ln(z_2(\zeta)) - \frac{(\xi_1'(\zeta) - 5)^2}{z_2(\zeta)}$$

$$
\begin{aligned}
&\leqslant A_4(\zeta) - 27.336 \cdot (\ln(z_1(\zeta)) - \ln(z_2(\zeta))) - \frac{(\xi_1'(\zeta) - 4)^2}{z_1(\zeta)} - \frac{(\xi_1'(\zeta) - 5)^2}{z_2(\zeta)} \\
&=: Q(\zeta).
\end{aligned}
$$

Since $\zeta \in [0, 0.046)$, it belongs to the interval $[0.001k, 0.001(k+1))$ for some integer $0 \leqslant k \leqslant 45$. We consider 3 cases.

Case 1: $0 \leqslant k \leqslant 34$. Then by Claim 26 and (25), for each $\zeta \in [0.001k, 0.001(k+1))$,

$$
A_4(0.001k) \geqslant A_4(\zeta),
$$

$$
z_1(\zeta) \geqslant z_1(0.001k), \text{ and } z_2(0.001k) \geqslant z_2(\zeta).
$$

Therefore,

$$
\begin{aligned}
Q(\zeta) \leqslant & M_1(k) := A_4(0.001k) - 27.336 \cdot (\ln(z_1(0.001k)) - \ln(z_2(0.001k))) \\
& - \frac{(\xi_1'(0.001k) - 4)^2}{z_1(0.001(k+1))} - \frac{(\xi_1'(0.001k) - 5)^2}{z_2(0.001k)}.
\end{aligned}
$$

The bounds for $M_1(k)$ certifying that $M_1(k) < 20$ for each $0 \leqslant k \leqslant 34$ are given in Table 2 in Appendix 1.

Case 2: $k = 35$. Similarly to Case 1,

$$
\begin{aligned}
Q(\zeta) \leqslant & \max(A_4(0.035), A_4(0.036)) - 27.336 \cdot (\ln(z_1(0.035)) - \ln(z_2(0.035))) \\
& - \frac{(\xi_1'(0.035) - 4)^2}{z_1(0.036)} - \frac{(\xi_1'(0.035) - 5)^2}{z_2(0.035)} \\
< & 98.404 - 27.336 \cdot (-1.5 - (-0.135)) - 94 - 36.3 < 5.5 < 20.
\end{aligned}
$$

Case 3: $36 \leqslant k \leqslant 45$. Again, similarly to Case 1,

$$
\begin{aligned}
Q(\zeta) \leqslant M_3(k) := & A_4(0.001(k+1)) - 27.336 \cdot (\ln(z_1(0.001k)) - \ln(z_2(0.001k))) \\
& - \frac{(\xi_1'(0.001k) - 4)^2}{z_1(0.001(k+1))} - \frac{(\xi_1'(0.001k) - 5)^2}{z_2(0.001k)}.
\end{aligned}
$$

The bounds for $M_1(k)$ certifying that $M_1(k) < 20$ for each $36 \leqslant k \leqslant 45$ are given in Table 1 in Appendix 1. ■

Thus by (19) and Claims 23–27, for each $\zeta \in [0, 0.046)$,

$$
\frac{\partial^2 \ln(h_1(0.454, \zeta))}{\partial \zeta^2} = \sum_{i=1}^{5} A_i(\zeta) < -22.46 + 20 = -2.46 < 0.
$$

We also can check by plugging in the values that

$$
\frac{\partial \ln(h_1(0.454, 0.0228718))}{\partial \zeta} > 7.54 \cdot 10^{-8}, \text{ and } \frac{\partial \ln(h_1(0.454, 0.0228719))}{\partial \zeta} < -9 \cdot 10^{-6}.
$$

Thus, the derivative of $h_1(0.454, \zeta)$ equals 0 at a unique $\zeta_1 \in (0.0228718, 0.0228719)$.

Recall that $h_1(0.454, \zeta) > 0$ for $\zeta \in [0, 0.046)$. Therefore, after comparing the value $h_1(0.454, 0.0228719)$ with the boundary values $h_1(0.454, 0)$ and $h_1(0.454, 0.46)$, we conclude that the maximum of $h_1(0.454, \zeta)$ is attained at ζ_1. We can plug in numbers into a computer and obtain that

$$h_1(0.454, 0.0228718) \leqslant 0.999982,$$
$$\frac{\partial \ln(h_1(0.454, 0.0228718))}{\partial \zeta} \leqslant 1 \cdot 10^{-7},$$

and

$$\frac{\partial h_1(0.454, 0.0228718)}{\partial \zeta} = h_1(0.454, 0.0228718) \cdot \frac{\partial \ln(h_1(0.454, 0.0228718))}{\partial \zeta} \leqslant 1 \cdot 10^{-7},$$

which implies that

$$h_1(0.454, \zeta_1) \leqslant h_1(0.454, 0.0228718) + 1 \cdot 10^{-7} \cdot 0.0000001 \leqslant 0.999983. \qquad \square$$

The proof of the next lemma is similar but significantly simpler. It is mostly a routine bounding some expressions. So, we present the proof of Lemma 28 in Appendix 2.

Lemma 28. *For every*

$$(\chi, \zeta, \xi) \in \Omega = \{(\chi, \zeta, \xi) : 0.454 < \chi \leqslant 0.45537,\ 0 \leqslant \zeta \leqslant \frac{1}{2} - \chi,\ 0 \leqslant \xi \leqslant 1 - 2\chi - 2\zeta\},$$

we have

$$\frac{\partial \ln(h(\chi, \zeta, \xi))}{\partial \chi} < 0. \tag{26}$$

Since $h(\chi, \zeta, \xi) > 0$ for each $(\chi, \zeta, \xi) \in \Omega$, Lemma 28 yields that for each fixed ζ and ξ, the maximum of $h(\chi, \zeta, \xi)$ over $(\chi, \zeta, \xi) \in \Omega$ is attained at $\chi = 0.454$. By Lemma 20, this maximum is at most 0.999983. This yields (14).

6 Completion of the proof of Theorem 17

By (14) and Lemma 19, for all positive integers n and x such that n is even and $0.454n < x \leqslant 0.45537n$,

$$\frac{q(x, n)}{(3n-1)!!} \leqslant O(n^6) \cdot 0.999983^n.$$

It follows that

$$\frac{1}{(3n-1)!!} \sum_{x=\lceil 0.454n \rceil}^{\lfloor 0.45537n \rfloor} q(x, n) \leqslant O(n^7) \cdot 0.999983^n \to 0 \quad \text{as } n \to \infty. \tag{27}$$

Thus by Lemma 18, the number of pairings $F \in \mathcal{G}'_{16}(n)$ with $0.454n < \alpha(F) \leqslant 0.45537n$ is $o\left((3n-1)!!\right)$. Together with Theorem 16, this means that almost no pairings have independence ratio larger than 0.454. Thus by Corollary 5 we conclude that almost no n-vertex 3-regular graphs of girth at least 16 have independence ratio larger than 0.454. This proves Theorem 17 and thus also Theorem 2.

Acknowledgment. We thank Jan Volec for helpful discussion and bringing [4] to our attention. We thank a referee for the valuable comments.

References

[1] E.A. Bender and E.R. Canfield, The asymptotic number of non-negative integer matrices with given row and column sums, *Journal of Combinatorial Theory A*, 24 (1978), 296–307.

[2] B. Bollobás, The independence ratio of regular graphs, *Proc. Amer. Math. Soc.* 83 (1981), 433–436.

[3] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of regular graphs, *European J. Combinatorics* 1 (1980), 311–316.

[4] E. Csóka, Independent sets and cuts in large-girth regular graphs, arXiv:1602.02747.

[5] E. Csóka, B. Gerencsér, V. Harangi and B. Virág, Invariant Gaussian processes and independent sets on regular graphs of large girth, *Random Structures and Algorithms*, 47 (2015), 284–303.

[6] G. Hopkins and W. Staton, Girth and independence ratio, *Canad. Math. Bull.*, 25 (1982), no. 2, 179–186.

[7] C. Hoppen, Properties of graphs with large girth, Doctoral thesis, University of Waterloo, 2008.

[8] C. Hoppen and N. Wormald, Properties of regular graphs with large girth via local algorithms, *J. Combin. Theory Ser. B*, 121 (2016), 367–397.

[9] S. Janson, T. Łuczak and A. Rucinski, Random Graphs, Interscience Series in Discrete Mathematics and Optimization, *John Wiley*, 2000.

[10] F. Kardoš, D. Král and J. Volec, Fractional colorings of cubic graphs with large girth, *SIAM J. Discrete Math.* 25 (2011), 1454–1476.

[11] J. Lauer and N. Wormald, Large independent sets in regular graphs of large girth, *J. Combin. Theory Ser. B*, 97 (2007), 999–1009.

[12] B. D. McKay, Independent sets in regular graphs of high girth. Proceedings of the Singapore conference on combinatorial mathematics and computing (Singapore, 1986). *Ars Combin.* 23 (1987), A, 179–185.

[13] J. Shearer, A note on the independence number of triangle-free graphs, II, *J. Combin. Theory Ser. B*, 53 (1991), 300–307.

[14] N. C. Wormald, Some problems in the enumeration of labelled graphs, Doctoral Thesis, Univ. of Newcastle (1978).

Appendix 1: Tables for Claim 27

We define the following functions for the table.

$$T_1(k) := A_4(0.001k),\ T_2(k) := -\ln(z_1(0.001k)),\ T_3(k) := \ln(z_2(0.001k)),$$

$$T_4(k) := -\frac{(\xi_1'(0.001k) - 4)^2}{z_1(0.001(k+1))} \text{ and } T_5(k) := -\frac{(\xi_1'(0.001k) - 5)^2}{z_2(0.001k)}.$$

k	$T_1(k)$	$T_2(k)$	$T_3(k)$	$T_4(k)$	$T_5(k)$	$M_1(k)$
0	135.9762	2.553562	0.05277836	-166.7356	-20.83335	19.7
1	132.6679	2.507105	0.04831009	-161.790	-21.1686	19.6
2	129.6543	2.462392	0.04379588	-157.2333	-21.50947	19.5
3	126.903	2.419288	0.03923514	-153.0194	-21.85574	19.3
4	124.384	2.377674	0.03462729	-149.1122	-22.20758	19.1
5	122.0728	2.33745	0.02997174	-145.4794	-22.56505	18.8
6	119.9504	2.298492	0.02526786	-142.0935	-22.92824	18.5
7	117.9977	2.260743	0.02051507	-138.9302	-23.29722	18.2
8	116.1989	2.224115	0.01571273	-135.9683	-23.67205	17.8
9	114.5404	2.188538	0.01086022	-133.1892	-24.05282	17.5
10	113.0099	2.153948	0.005956888	-130.5765	-24.43961	17.1
11	111.5969	2.120286	0.001002109	-128.1157	-24.83248	16.7
12	110.292	2.0876	-0.004004782	-125.793	-25.23152	16.3
13	109.0867	2.055542	-0.009064451	-123.5994	-25.63682	15.8
14	107.9738	2.024367	-0.01417756	-121.5220	-26.04845	15.4
15	106.9466	1.993934	-0.01934483	-119.5525	-26.4664	15.0
16	106.000	1.964204	-0.02456692	-117.6823	-26.89104	14.5
17	105.1262	1.935144	-0.02984456	-115.9041	-27.32218	14.0
18	104.3229	1.90673	-0.03517846	-114.2111	-27.76000	13.6
19	103.585	1.878905	-0.04056935	-112.5970	-28.20460	13.1
20	102.9088	1.851668	-0.04601797	-111.0562	-28.65606	12.6
21	102.2906	1.824985	-0.05152507	-109.5836	-29.1144	12.1
22	101.7273	1.798832	-0.05709143	-108.1746	-29.57998	11.6
23	101.217	1.773185	-0.06271781	-106.8248	-30.05263	11.1
24	100.7543	1.748024	-0.06840502	-105.5304	-30.53255	10.7
25	100.3398	1.723329	-0.07415386	-104.2877	-31.01984	10.2
26	99.97009	1.699082	-0.07996514	-103.0934	-31.51462	9.7
27	99.64358	1.675265	-0.08583972	-101.9446	-32.016	9.2
28	99.3585	1.651862	-0.09177843	-100.8383	-32.52708	8.7
29	99.11297	1.628858	-0.09778215	-99.77206	-33.04501	8.2
30	98.90584	1.606239	-0.1038517	-98.74333	-33.5708	7.7
31	98.7358	1.583989	-0.1099881	-97.74996	-34.10486	7.2
32	98.6015	1.562098	-0.1161922	-96.78986	-34.64704	6.7
33	98.50187	1.54056	-0.122464	-95.86113	-35.19758	6.3
34	98.43615	1.519339	-0.1288073	-94.96198	-35.75661	5.8

Table 1: Upper bounds for expressions in $M_1(k)$.

k	$T_1(k+1)$	$T_2(k)$	$T_3(k)$	$T_4(k)$	$T_5(k)$	$M_3(k)$
36	98.43379	1.477873	-0.1417047	-93.24588	-36.90074	4.9
37	98.49569	1.457599	-0.1482617	-92.42593	-37.48615	4.4
38	98.58802	1.437619	-0.1548924	-91.62957	-38.08066	4.0
39	98.71033	1.417923	-0.1615978	-90.85551	-38.68445	3.6
40	98.86225	1.398503	-0.1683790	-90.1025	-39.29768	3.1
41	99.04347	1.379352	-0.1752370	-89.36971	-39.92054	2.7
42	99.25376	1.36046	-0.1821731	-88.65582	-40.55321	2.3
43	99.49293	1.341822	-0.1891883	-87.95995	-41.1958	1.9
44	99.76085	1.323429	-0.1962838	-87.28121	-41.84877	1.5
45	100.0576	1.305276	-0.2034610	-86.61873	-42.51206	1.1

Table 2: Upper bounds for expressions in $M_3(k)$.

Appendix 2: Proof of Lemma 28

By definition, the boundary, $\partial\Omega$, of Ω is

$$\begin{aligned}\partial\Omega =&\{(\chi,\zeta,\xi)\ :\ \xi=0, 2\chi+2\zeta\leqslant 1, 0.454\leqslant\chi\leqslant 0.45537, \zeta\geqslant 0\}\cup\\ &\{(\chi,\zeta,\xi)\ :\ \zeta=0, 2\chi+\xi\leqslant 1, 0.454\leqslant\chi\leqslant 0.45537, \xi\geqslant 0\}\cup\\ &\{(\chi,\zeta,\xi)\ :\ \chi=0.454, 2\zeta+\xi\leqslant 0.092, \zeta\geqslant 0, \xi\geqslant 0\}\cup\\ &\{(\chi,\zeta,\xi)\ :\ \chi=0.45537, 2\zeta+\xi\leqslant 0.08926, \zeta\geqslant 0, \xi\geqslant 0\}.\end{aligned}$$

We also will consider the 2-dimensional set

$$\Omega_1=\{(\chi,\zeta)\ :\ 0.454\leqslant\chi\leqslant 0.45537, 0\leqslant\zeta\leqslant 0.5-\chi\}.$$

Then the boundary of Ω_1 is

$$\begin{aligned}\partial\Omega_1 =&\{(\chi,\zeta)\ :\ 0.454\leqslant\chi\leqslant 0.45537, \zeta=0\}\cup\{(\chi,\zeta)\ :\ 0\leqslant\zeta\leqslant 0.046, \chi=0.454\}\\ &\cup\{(\chi,\zeta)\ :\ 0\leqslant\zeta\leqslant 0.04463, \chi=0.45537\}\cup\{(\chi,\zeta)\ :\ 0.454\leqslant\chi\leqslant 0.45537, \chi+\zeta=\frac{1}{2}\}.\end{aligned}$$

By the definition of h,

$$\begin{aligned}\frac{\partial\ln(h(\chi,\zeta,\xi))}{\partial\chi} =&4\ln(2\chi-3\zeta-\frac{1}{2})+4\ln(2\chi+\zeta-\frac{1}{2})-\ln(1-2\chi-2\zeta)\\ &-\ln(1-2\chi+4\zeta)+2\ln(1-2\chi-2\zeta-\xi)+2\ln(1-2\chi+4\zeta-\xi)\\ &-10\ln(10\chi-5\zeta+\xi-\frac{7}{2}).\end{aligned}$$

Similarly to the proof of Lemma 20, we present $\frac{\partial\ln(h(\chi,\zeta,\xi))}{\partial\chi}$ in the form $\sum_{j=1}^{6}B_j$, where

$$\begin{aligned}&B_1(\chi,\zeta):=4\ln\left(2\chi-3\zeta-\frac{1}{2}\right),\quad B_2(\chi,\zeta):=4\ln\left(2\chi+\zeta-\frac{1}{2}\right),\\ &B_3(\chi,\zeta,\xi):=2\ln(1-2\chi-2\zeta-\xi)-\ln(1-2\chi-2\zeta),\quad B_4(\chi,\zeta):=\ln(1-2\chi+4\zeta),\\ &B_5(\chi,\zeta,\xi):=2\ln(1-2\chi+4\zeta-\xi),\ \text{ and }\ B_6(\chi,\zeta,\xi):=\ln\left(10\chi-5\zeta+\xi-\frac{7}{2}\right),\end{aligned}$$

and then bound each of the terms separately.

Claim 29. *For all* $(\chi, \zeta) \in \Omega_1$, $B_1(\chi, \zeta) < -3.55$.

Proof: For each $(\chi, \zeta) \in \Omega_1$, we have $\chi - \frac{3}{2}\zeta - 0.25 > 0$, since $\chi \geqslant 0.454$ and $\zeta \leqslant 0.046$. As for each $(\chi, \zeta) \in \Omega_1$,

$$\frac{\partial B_1(\chi, \zeta)}{\partial \chi} = \frac{4}{\chi - \frac{3}{2}\zeta - 0.25} > 0 \qquad \textit{and} \qquad \frac{\partial B_1(\chi, \zeta)}{\partial \zeta} = \frac{-6}{\chi - \frac{3}{2}\zeta - 0.25} < 0,$$

the maximum is attained at a corner on the boundary $\partial\Omega_1$. Comparing the values of B_1 at the four corners of $\partial\Omega_1$, we see that the maximum is attained at $(\chi, \zeta) = (0.45537, 0)$ and $B_1(0.45537, 0) < -3.55$. ■

Claim 30. *For all* $(\chi, \zeta) \in \Omega_1$, $B_2(\chi, \zeta) < -3.14$.

Proof: For each $(\chi, \zeta) \in \Omega_1$, we have $2\chi + \zeta - \frac{1}{2} > 0$. As for each $(\chi, \zeta) \in \Omega_1$,

$$\frac{\partial B_2(\chi, \zeta)}{\partial \chi} = \frac{8}{2\chi + \zeta - \frac{1}{2}} > 0, \quad \textit{and} \quad \frac{\partial B_2(\chi, \zeta)}{\partial \zeta} = 4 \cdot (2\chi + \zeta - 0.5)^{-1} > 0,$$

the maximum is attained at a corner of the boundary $\partial\Omega_1$. Comparing the values of B_2 at the four corners of $\partial\Omega_1$, we see that the maximum is attained at $(\chi, \zeta) = (0.45537, 0.04463)$, and $B_2(0.45537, 0.04463) < -3.14$. ■

Claim 31. *For all* $(\chi, \zeta, \xi) \in \Omega$, $B_3(\chi, \zeta, \xi) < 0$.

Proof: We can write $B_3(\chi, \zeta, \xi)$ in the form

$$B_3(\chi, \zeta, \xi) = \ln(1 - 2\chi - 2\zeta - \xi) + \ln\left(\frac{1 - 2\chi - 2\zeta - \xi}{1 - 2\chi - 2\zeta}\right),$$

and observe that $\ln(1 - 2\chi - 2\zeta - \xi) < 0$ (since $2\chi + 2\zeta + \xi > 0$) and $\ln(\frac{1-2\chi-2\zeta-\xi}{1-2\chi-2\zeta}) \leqslant 0$ (since $1 - 2\chi - 2\zeta - \xi \leqslant 1 - 2\chi - 2\zeta$). ■

Claim 32. *For all* $(\chi, \zeta) \in \Omega_1$, $B_4(\chi, \zeta) < -1.28$.

Proof: For each $(\chi, \zeta) \in \Omega_1$, $-2\chi + 4\zeta + 1 > 0$. As for each $(\chi, \zeta) \in \Omega_1$,

$$\frac{\partial B_4(\chi, \zeta)}{\partial \chi} = \frac{-2}{-2\chi + 4\zeta + 1} < 0, \qquad \textit{and} \qquad \frac{\partial B_4(\chi, \zeta)}{\partial \zeta} = \frac{4}{-2\chi + 4\zeta + 1} > 0,$$

the maximum of B_4 is attained at a corner of the boundary $\partial\Omega_1$. Comparing the values of B_4 at the four corners of $\partial\Omega_1$, we see that the maximum is attained at $(\chi, \zeta) = (0.454, 0.046)$, and $B_4(0.454, 0.046) < -1.28$. ■

Claim 33. *For all* $(\chi, \zeta, \xi) \in \Omega$, $B_5(\chi, \zeta, \xi) < -2.57$.

Proof: For each $(\chi, \zeta, \xi) \in \Omega - \partial\Omega$, we have $2\chi - 4\zeta + \xi - 1 < 0$ since $2\chi + 2\zeta + \xi < 1 \leqslant 1 + 4\zeta + 2\zeta$. Since

$$\lim_{2\chi+\xi\to 1} B_5(\chi, 0, \xi) = -\infty,$$

the maximum of B_5 is not attained at $\zeta = 0, 2\chi + \xi = 1$. As for each $(\chi, \zeta, \xi) \in \Omega$,

$$\frac{\partial B_5(\chi, \zeta, \xi)}{\partial \chi} = \frac{4}{2\chi - 4\zeta + \xi - 1} < 0, \qquad \frac{\partial B_5(\chi, \zeta, \xi)}{\partial \zeta} = \frac{-8}{2\chi - 4\zeta + \xi - 1} > 0,$$

and

$$\frac{\partial B_5(\chi,\zeta,\xi)}{\partial \xi} = \frac{2}{2\chi - 4\zeta + \xi - 1} < 0,$$

the maximum of B_5 is attained at a corner of the boundary $\partial\Omega$. Comparing the values of B_5 at the corners of $\partial\Omega$, we see that the maximum is attained at $(\chi,\zeta,\xi) = (0.454, 0.046, 0)$ and $B_5(0.454, 0.046, 0) < -2.57$. ■

Claim 34. *For all* $(\chi,\zeta,\xi) \in \Omega,\ \ B_6(\chi,\zeta,\xi) < 0.14$.

Proof: For each $(\chi,\zeta,\xi) \in \Omega$, we have $10\chi - 5\zeta + \xi - \frac{7}{2} > 0$ since $10\chi - \frac{7}{2} \geqslant 1.04$ and $5\zeta \leqslant 0.23$. As for each $(\chi,\zeta,\xi) \in \Omega$,

$$\frac{\partial B_6(\chi,\zeta,\xi)}{\partial \chi} = \frac{10}{10\chi - 5\zeta + \xi - \frac{7}{2}} > 0, \qquad \frac{\partial B_6(\chi,\zeta,\xi)}{\partial \zeta} = \frac{-5}{10\chi - 5\zeta + \xi - \frac{7}{2}} < 0,$$

and

$$\frac{\partial B_6(\chi,\zeta,\xi)}{\partial \xi} = \frac{1}{10\chi - 5\zeta + \xi - \frac{7}{2}} > 0,$$

the maximum of B_5 is attained at a corner of the boundary $\partial\Omega$. Comparing the values of B_6 at the corners of $\partial\Omega$, we see that the maximum is attained at $(\chi,\zeta,\xi) = (0.45537, 0, 0.08926)$ and $B_6(0.454, 0, 0.08926) < 0.14$. ■

By Claims 29–34, for each $(\chi,\zeta,\xi) \in \Omega$,

$$\begin{aligned}\frac{\partial \ln(h(\chi,\zeta,\xi))}{\partial \chi} =& B_1(\chi,\zeta) + B_2(\chi,\zeta) + B_3(\chi,\zeta,\xi) + B_4(\chi,\zeta) + B_5(\chi,\zeta,\xi) + B_6(\chi,\zeta,\xi) \\ <& -3.55 - 3.14 + 0 - 1.28 - 2.57 + 0.14 < 0.\end{aligned}$$ □