Chromatic-Choosability of Hypergraphs with High Chromatic Number*

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Submitted: Aug 7, 2018; Accepted: Dec 5, 2018; Published: Jan 11, 2019

Abstract

It was conjectured by Ohba and confirmed by Noel, Reed and Wu that, for any graph G, if $|V(G)| \leqslant 2\chi(G) + 1$ then G is chromatic-choosable; i.e., it satisfies $\chi_l(G) = \chi(G)$. This indicates that the graphs with high chromatic number are chromatic-choosable. We observe that this is also the case for uniform hypergraphs and further propose a generalized version of Ohba's conjecture: for any r-uniform hypergraph H with $r \geqslant 2$, if $|V(H)| \leqslant r\chi(H) + r - 1$ then $\chi_l(H) = \chi(H)$. We show that the condition of the proposed conjecture is sharp by giving two classes of r-uniform hypergraphs H with $|V(H)| = r\chi(H) + r$ and $\chi_l(H) > \chi(H)$. To support the conjecture, we prove that $\chi_l(H) = \chi(H)$ for two classes of r-uniform hypergraphs H with $|V(H)| = r\chi(H) + r - 1$.

Mathematics Subject Classifications: 05C15

1 Introduction

For a graph or a hypergraph G, a vertex coloring of G is *proper* if every edge contains a pair of vertices with different colors. For a positive integer k, a *k-list assignment* of G is a mapping L which assigns to each vertex v a set $L(v)$ of k permissible colors. Given a k-list assignment L, an *L-coloring* of G is a proper vertex coloring in which the color of

*Supported by the National Natural Science Foundation of China under Grant No. 11471273 and 11561058.

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every vertex v is chosen from its list $L(v)$. We say that G is *L-colorable* if G has an L-coloring. A graph G is called *k-choosable* if for any k-list assignment L, G is L-colorable. The *list chromatic number* (or *choice number*) $\chi_l(G)$ is the minimum k for which G is k-choosable. It is obvious that $\chi_l(G) \geqslant \chi(G)$, where $\chi(G)$ is the chromatic number of G. A graph G is *chromatic-choosable* if $\chi_l(G) = \chi(G)$. The notion of list coloring was introduced independently by Vizing [25] and by Erdős, Rubin and Taylor [8] initially for ordinary graphs and then was extended to hypergraphs [2, 3, 13, 20, 22, 23].

List coloring for graphs has been extensively studied, much of the earlier fundamental work on which was surveyed in Alon [1], Tuza [24] and Kratochvíl-Tuza-Voigt [14]. One direction of interests on list coloring focused on the estimation or asymptotic behaviour of the list chromatic number $\chi_l(G)$ compared to the degrees of its vertices. In [8], Erdős, Rubin and Taylor proved that the list chromatic number of the complete bipartite graph $K_{d,d}$ grows roughly like the binary logarithm of d (the degree of $K_{d,d}$). More generally, Alon [1] showed that the list chromatic number of any graph grows with the average degree. However, this is not the case for hypergraphs. It was shown that, when $r \geqslant 3$, it is not true in general that the list chromatic number of r-uniform hypergraphs grows with its average degree [2]. Even so, it was also shown that similar property holds for many classes of hypergraphs [2, 12, 23], including all the simple uniform hypergraphs (here, a hypergraph is *simple* if different edges have at most one vertex in common) [22].

Another direction of interests on list coloring focused on the difference between the chromatic number $\chi(G)$ and list chromatic number $\chi_l(G)$. It was shown that $\chi_l(G)$ can be much larger than $\chi(G)$ for both the ordinary graphs [8] and hypergraphs [12]. This yields a natural question: which graphs are chromatic-choosable? A well known example concerning this question is the List Coloring Conjecture (attributed in particular to Vizing, see [11]), which says that every line-graph is chromatic-choosable. This conjecture was later extended to claw-free graphs [9].

In addition to particular classes of the graphs that might be chromatic-choosable, the graphs with 'high chromatic number' (compared to the number of the vertices in the graph) also received much attention. A trivial fact is that every complete graph is chromatic-choosable. In [18], Ohba showed that, for any graph G, if $|V(G)| \leqslant \chi(G) + \sqrt{2\chi(G)}$ then $\chi_l(G) = \chi(G)$. Further, in the same paper, Ohba conjectured that if $|V(G)| \leqslant 2\chi(G) + 1$ then $\chi_l(G) = \chi(G)$. This conjecture was confirmed by Noel, Reed and Wu [17].

In this paper we focus on the chromatic-choosability of the uniform hypergraphs with high chromatic number, where a hypergraph H is *r-uniform* if every edge of H has cardinality r. In particular, we propose the following generalized version of Ohba's conjecture, which is inspired by a recent Ohba-like conjecture for d-improper colorings given by Yan et al. [27] (See Conjecture 2 below).

Conjecture 1. Let $r \geqslant 2$ and H be an r-uniform hypergraph. If

$$|V(H)| \leqslant r\chi(H) + r - 1$$

then $\chi_l(H) = \chi(H)$.

The rest of the paper is organized as follows. In Section 2, we revisit both the conjecture of Yan et al. [27] and some recent supporting results [26] on this conjecture from the point view of hypergraph coloring. We show that Conjecture 1 implies the conjecture of Yan et al. Moreover, we observe that arguments from a proof of Wang et al. [26] on d-improper colorings can be modified only slightly to obtain a weak form of Conjecture 1: if $|V(H)| \leqslant (r - \frac{1}{2})\chi(H) + \frac{r}{2} - 1$ then $\chi_l(H) = \chi(H)$. Therefore, r-uniform hypergraphs with high chromatic number are chromatic-choosable. We also give a simplified but equivalent form of Conjecture 1 using the notion of r-complete multipartite hypergraph defined in [6]. In Section 3, we show that the condition of Conjecture 1 is sharp by giving two classes of r-uniform hypergraphs H with $|V(H)| = r\chi(H)+r$ and $\chi_l(H) > \chi(H)$. Finally, to support our conjecture, in Section 4 we prove that $\chi_l(H) = \chi(H)$ for two classes of r-uniform hypergraphs H with $|V(H)| = r\chi(H) + r - 1$.

2 Preliminaries

2.1 Improper colorings and hypergraphs

For a graph G and a set C of colors, a coloring $f\colon V(G) \to C$ is a *d-improper coloring* if each color class induces a subgraph with maximum degree at most d. Let $\chi^d(G)$ and $\chi_l^d(G)$ denote the d-improper chromatic number and d-improper list chromatic number of G, respectively. Yan et al. [27] proposed an Ohba-like conjecture for d-improper colorings.

Conjecture 2. [27] For any graph G, if

$$|V(G)| \leqslant (d+2)\chi^d(G) + (d+1)$$

then $\chi_l^d(G) = \chi^d(G)$.

For a graph G and an integer $r \geqslant 2$, we construct an r-uniform hypergraph $G^{(r)}$ as follows:
1). $V(G^{(r)}) = V(G)$, and
2). $E(G^{(r)}) = \{S \subseteq V(G)\colon\ |S| = r \text{ and } \Delta(G[S]) = r-1\}$, where $\Delta(G[S])$ is the maximum degree of $G[S]$, i.e., the subgraph of G induced by S.

Proposition 3. *For any graph G and nonnegative integer d, we have $\chi(G^{(d+2)}) = \chi^d(G)$ and $\chi_l(G^{(d+2)}) = \chi_l^d(G)$.*

Proof. It is easy to see that a coloring f of G is d-improper if and only if f is proper when regarded as a coloring of $G^{(d+2)}$. Thus, the assertion holds. □

Theorem 4. *Conjecture 1 implies Conjecture 2.*

Proof. Let G be a graph with at most $(d+2)\chi^d(G) + (d+1)$ vertices. Let $r = d+2$ and $H = G^{(r)}$. By Proposition 3, $\chi(H) = \chi^d(G)$. Note that H and G have the same vertex set. Thus, $|V(H)| \leqslant (d+2)\chi^d(G) + (d+1) = r\chi(H) + r - 1$. If Conjecture 1 is true, then $\chi_l(H) = \chi(H)$. This implies $\chi_l^d(G) = \chi^d(G)$ by Proposition 3. The proof is completed. □

To support Conjecture 2, Yan et al. [27] also proved the following result.

Theorem 5. *(Theorem 1, [27]) For any graph G and integer $d \geqslant 0$, if*

$$|V(G)| \leqslant (d+1)\chi^d(G) + \sqrt{(d+1)\chi^d(G)} - d$$

then $\chi_l^d(G) = \chi^d(G)$.

Wang et al. [26] further improved Theorem 5 as follows.

Theorem 6. *(Theorem 2, [26]) Let $d \geqslant 1$. For any graph G, if*

$$|V(G)| \leqslant \left(d + \frac{3}{2}\right)\chi^d(G) + \frac{d}{2}$$

then $\chi_l^d(G) = \chi^d(G)$.

For $n \geqslant 1$, write $[n] = \{1, 2, \ldots, n\}$. Let $\mathcal{H}_n^r$ denote the set of all r-uniform hypergraphs on $[n]$ and let $\mathcal{H}_n^{(r)} = \bigcup \left\{G^{(r)}\right\}$ where the union is taken over all graphs G on $[n]$. We notice that $\mathcal{H}_n^{(r)}$ is usually a proper subset of $\mathcal{H}_n^r$. Using Proposition 3, we may restate Theorem 6 as follows:

Theorem 7. *Let $r \geqslant 3$. For any $H \in \mathcal{H}_n^{(r)}$, if*

$$n \leqslant \left(r - \frac{1}{2}\right)\chi(H) + \frac{r}{2} - 1$$

then $\chi_l(H) = \chi(H)$.

Furthermore, we may rewrite the original proof of Theorem 6 in the setting of hypergraph coloring. Then in terms of hypergraph coloring, the key point of the proof is to find a coloring f of $G^{(r)}$ such that each color class of f has size at most $r-1$ or is contained in some color class $V_i, i \in \{1, 2, \ldots, \chi\}$, where $\chi = \chi(G^{(r)})$ and $V_1, V_2, \ldots, V_\chi$ are the color classes of $G^{(r)}$ induced by an arbitrary proper χ-coloring of $G^{(r)}$. This means that, though $\mathcal{H}_n^{(r)}$ is in general a proper subset of $\mathcal{H}_n^r$, the existence of such f does not rely on any structural property specified by the hypergraphs in the class $\mathcal{H}_n^{(r)}$ and hence, the proof for the hypergraphs in $\mathcal{H}_n^{(r)}$ can be extended for that in $\mathcal{H}_n^r$. Therefore, Theorem 7 holds even if we replace $H \in \mathcal{H}_n^{(r)}$ by $H \in \mathcal{H}_n^r$. We write this observation as the following theorem but omit its proof. Interested readers may get a full proof through consulting [26].

Theorem 8. *For any r-uniform hypergraph H, if*

$$|V(H)| \leqslant \left(r - \frac{1}{2}\right)\chi(H) + \frac{r}{2} - 1$$

then $\chi_l(H) = \chi(H)$.

2.2 Reduction to multipartite hypergraphs

For k positive integers $p_1, p_2, \ldots, p_k$, let $V_1, V_2, \ldots, V_k$ be k disjoint sets of size p_1,p_2,...,p_k, respectively. Following [6], we define the r-complete k-partite hypergraph $K^r_{p_1,p_2,\ldots,p_k}$ with partite sets V_1,V_2,...,V_k as follows:
1). $V\left(K^r_{p_1,p_2,\ldots,p_k}\right) = V_1 \cup V_2 \cup \cdots \cup V_k$, and
2). $E\left(K^r_{p_1,p_2,\ldots,p_k}\right) = \left\{S \subseteq \bigcup_{i=1}^k V_i \colon\ |S| = r, S \nsubseteq V_i \text{ for any } i \in \{1, 2, \ldots, k\}\right\}$.

We note that the notion of k-partite hypergraph here means that each edge may contain two or more vertices from a partite set, which is different from others that used in some literatures. Nevertheless, when $r = 2$, $K^r_{p_1,p_2,\ldots,p_k}$ agrees with the usual complete k-partite graph $K_{p_1,p_2,\ldots,p_k}$. Further, if there are two p_i's, say p_1 and p_2, which are less than $r-1$, then $K^r_{p_1,p_2,\ldots,p_k}$ is isomorphic to $K^r_{p_1+1,p_2-1,\ldots,p_k}$ (or $K^r_{p_1+1,p_3,\ldots,p_k}$ if $p_2 = 1$). Therefore, in the following we always assume that $p_i \geqslant r-1$ for all $i \in \{1, 2, \ldots, k\}$ with at most one exception. For simplicity, if $p_1 = \cdots = p_s = p$ for some s with $1 \leqslant s \leqslant k$, we write $K^r_{p_1,p_2,\ldots,p_k}$ as $K^r_{p*s,p_{s+1},\ldots,p_k}$.

Proposition 9. *If $p_i \geqslant r-1$ for all $i \in \{1, 2, \ldots, k\}$ with at most one exception, then $\chi\left(K^r_{p_1,p_2,\ldots,p_k}\right) = k$.*

Proof. Since $\chi\left(K^r_{p_1,p_2,\ldots,p_k}\right) \leqslant k$ always holds, it suffices to show the reversed inequality. By the assumption of the proposition, $K^r_{(r-1)*(k-1),1}$ is a subgraph of $K^r_{p_1,p_2,\ldots,p_k}$ and therefore, $\chi\left(K^r_{p_1,p_2,\ldots,p_k}\right) \geqslant \chi\left(K^r_{(r-1)*(k-1),1}\right)$. Further, notice that each r-subset of $V\left(K^r_{(r-1)*(k-1),1}\right)$ is an edge. We have

$$\chi\left(K^r_{(r-1)*(k-1),1}\right) \geqslant \left\lceil \frac{(r-1)(k-1)+1}{r-1} \right\rceil = k.$$

Therefore, $\chi\left(K^r_{p_1,p_2,\ldots,p_k}\right) \geqslant k$ and the proposition follows. □

It is easy to see that Conjecture 1 is true if and only if it is true for all r-complete multipartite hypergraphs. Thus, in view of Proposition 9 we can restate Conjecture 1 as follows.

Conjecture 10. Let $r \geqslant 2$ and let $p_1, p_2, \ldots, p_k$ be k positive integers such that $p_i \geqslant r-1$ for all $i \in \{1, 2, \ldots, k\}$ with at most one exception. If $\sum_{i=1}^{k} p_i \leqslant rk + r - 1$, then $\chi_l\left(K^r_{p_1,p_2,\ldots,p_k}\right) = k$.

3 Sharpness of Conjecture 1

It is well known that the condition of Ohba's Conjecture is sharp. Indeed, in [7] it was proved that the complete k-partite graph G on $2k+2$ vertices is not chromatic-choosable if k is even and either every part of G has size 2 or 4, or every part of G has size 1 or 3. In the following, we give an analogue of the former for r-uniform hypergraphs with $r \geqslant 3$ and a partial generalization of the latter when $G = K_{3,3}$ to r-uniform hypergraphs with $r \geqslant 2$, indicating that the upper bound $r\chi(H) + r - 1$ in Conjecture 1 is also sharp.

Theorem 11. *For any integer $r \geqslant 3$, if k is divisible by $r-1$ then*

$$\chi_l\left(K^r_{2r,r*(k-1)}\right) > \chi\left(K^r_{2r,r*(k-1)}\right) = k.$$

Proof. Let $V_1, V_2, \dots, V_k$ be the k partite sets of $K^r_{2r,r*(k-1)}$, where

$$V_1 = \{u_1, v_1, u_2, v_2, \dots, u_r, v_r\} \text{ and } V_i = \{w_{i,1}, w_{i,2}, \dots, w_{i,r}\}$$

for $i \in \{2, 3, \dots, k\}$. Let $C_1, C_2, \dots, C_r$ be r disjoint color sets of size $\frac{k}{r-1}$. Let L be the k-list assignment of $K^r_{2r,r*(k-1)}$ defined by

$$L(w_{i,j}) = L(u_j) = L(v_j) = \bigcup_{t=1, t\neq j}^{r} C_t,\ \ i = 2, 3, \dots, k;\ \ j = 1, 2, \dots, r.$$

We show that $K^r_{2r,r*(k-1)}$ is not L-colorable. Suppose to the contrary that $f\colon\ V \to \bigcup_{i=1}^r C_i$ is an L-coloring of $K^r_{2r,r*(k-1)}$, where $V = V_1 \cup V_2 \cup \cdots \cup V_k$. Define

$$S = \left\{ c \in \bigcup_{j=1}^r C_j\colon\ f^{-1}(c) \nsubseteq V_1 \right\} \text{ and } T = \left\{ c \in \bigcup_{j=1}^r C_j\colon\ f^{-1}(c) \subseteq V_1 \right\}.$$

Clearly, $S \cup T$ is a partitioning of $\bigcup_{j=1}^r C_j$.

Claim: $\left|f^{-1}(c)\right| \leqslant r-1$ *for each* $c \in S$.

Suppose to the contrary that $|f^{-1}(c)| > r-1$. Since $f^{-1}(c) \nsubseteq V_1$, there exists an r-subset W of $f^{-1}(c)$ such that $W \nsubseteq V_1$. For any $i \in \{2, 3, \dots, k\}$, by the definition of L we have $L(w_{i,1}) \cap L(w_{i,2}) \cap \cdots \cap L(w_{i,r}) = \emptyset$. This means that V_i has at least one vertex which is not assigned the color c by f. Therefore, $W \neq V_i$, or equivalently, $W \nsubseteq V_i$ as $|W| = |V_i|$. Combining with $W \nsubseteq V_1$, we have $W \nsubseteq V_i$ for all $i \in \{1, 2, \dots, k\}$. Thus, W is an edge of $K^r_{2r,r*(k-1)}$. Further, since f is a proper coloring, the edge W is not monochromatic under f, which contradicts the fact that $W \subseteq f^{-1}(c)$. This proves the claim.

Let $\ell = |\bigcup_{c\in T} f^{-1}(c)|$. Then $|\bigcup_{c\in S} f^{-1}(c)| = |V| - \ell = rk + r - \ell$. It follows from the above claim that $|S| \geqslant \lceil \frac{rk+r-\ell}{r-1} \rceil$. Since $\bigcap_{j=1}^r L(u_j) = \emptyset$ and $\bigcap_{j=1}^r L(v_j) = \emptyset$, any $2r-1$ vertices in V_1 share no common color in their lists. Thus, $|f^{-1}(c)| \leqslant 2r-2$ for each $c \in T$ since $f^{-1}(c) \subseteq V_1$. Therefore, $|T| \geqslant \lceil \frac{\ell}{2r-2} \rceil$. Since $|\bigcup_{j=1}^r C_j| = \frac{rk}{r-1}$ and $S \cup T$ is a partitioning of $\bigcup_{j=1}^r C_j$, we have

$$\left\lceil \frac{rk + r - \ell}{r-1} \right\rceil + \left\lceil \frac{\ell}{2r-2} \right\rceil \leqslant |S| + |T| \leqslant \frac{rk}{r-1}.$$

As k is a multiple of $r-1$, the above inequality can be reduced to

$$\left\lceil \frac{r-\ell}{r-1} \right\rceil + \left\lceil \frac{\ell}{2r-2} \right\rceil \leqslant 0.$$

On the other hand, notice that $\ell \leqslant |V_1| = 2r$. If $\ell \leqslant 2r-1$ then

$$\left\lceil \frac{r-\ell}{r-1} \right\rceil + \left\lceil \frac{\ell}{2r-2} \right\rceil \geqslant \frac{r-\ell}{r-1} + \frac{\ell}{2r-2} = \frac{2r-\ell}{2r-2} > 0,$$

a contradiction. If $\ell = 2r$ then

$$\left\lceil \frac{r-\ell}{r-1} \right\rceil + \left\lceil \frac{\ell}{2r-2} \right\rceil = \left\lceil \frac{-r}{r-1} \right\rceil + \left\lceil \frac{2r}{2r-2} \right\rceil \geqslant -1+2 > 0,$$

where '$\geqslant$' holds as $r \geqslant 3$. This is again a contradiction and hence completes the proof of the theorem. □

Remark 12. Although Theorem 11 is somewhat of an analogue of the $K_{4,2,2,\ldots,2}$ example, the "divisibility" condition is different. For graphs, one required k to be divisible by r, which is 2, but for hypergraphs it is $r-1$.

Theorem 13. *For any integer $r \geqslant 2$,*

$$\chi_l\left(K^r_{(r+1)*r}\right) > \chi\left(K^r_{(r+1)*r}\right) = r.$$

Proof. Let $H = K^r_{(r+1)*r}$ with r partite sets $V_1, V_2, \ldots, V_r$, where

$$V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,r+1}\} \text{ for } i \in \{1,2,\ldots,r\}.$$

Let L be the r-list assignment of H defined by $L(v_{i,j}) = \{1,2,\ldots,r+1\} \setminus \{j\}$ for $i \in \{1,2,\ldots,r\}$ and $j \in \{1,2,\ldots,r+1\}$. We show that H is not L-colorable.

Suppose to the contrary that $f\colon\ V(H) \to \{1,2,\ldots,r+1\}$ is an L-coloring of H. Then we have

$$|f^{-1}(1)| + |f^{-1}(2)| + \cdots + |f^{-1}(r+1)| = |V(H)| = (r+1)r. \tag{1}$$

On the other hand, for any $i \in \{1,2,\ldots,r\}$, the lists of all $r+1$ vertices in V_i have an empty intersection. Thus, $|f^{-1}(k)| \leqslant r$ for $k \in \{1,2,\ldots,r+1\}$. This, combining with (1), implies that $|f^{-1}(k)| = r$ for $k \in \{1,2,\ldots,r+1\}$. Therefore, for each $k \in \{1,2,\ldots,r+1\}$, $f^{-1}(k)$ must be contained in V_i for some $i \in \{1,2,\ldots,r\}$ since otherwise $f^{-1}(k)$ is an edge in H. By the pigeonhole principle, there exist two color classes, say $f^{-1}(1)$ and $f^{-1}(2)$, contained in the same partite set, say V_1. Consequently, $|f^{-1}(1)| + |f^{-1}(2)| = 2r > r+1 = |V_1|$. This is a contradiction and hence completes the proof. □

4 Support for Conjecture 1

We begin with some lemmas that are necessary for our forthcoming argument.

For an r-hypergraph H and a subset $X \subseteq V(H)$, we denote by $H[X]$ the subgraph of H induced by X, i.e., $H[X] = (X, \{e\colon\ e \in E(H), e \subseteq X\})$. For a list assignment L of H, let $L(X) = \bigcup_{v \in X} L(v)$ and let L_X denote L restricted to X. We may omit the subscript of L_X when there is no ambiguity. For example, when $H[X]$ is L_X-colorable we simply say that $H[X]$ is L-colorable. For a color set C, let $L \setminus C$ be the list assignment of H defined by $(L \setminus C)(v) = L(v) \setminus C$ for each vertex $v \in V(H)$.

Lemma 14. *Let $X \cup Y = V(H)$ be a partitioning of the vertex set of an r-uniform hypergraph H and f be an L-coloring of $H[X]$. If there is a color set C such that $C \supseteq f(X)$ and $H[Y]$ is $L \setminus C$-colorable, then H is L-colorable.*

Proof. Let g be an $L \setminus C$-coloring of $H[Y]$. Define a coloring h of H by $h(v) = f(v)$ if $v \in X$, and $h(v) = g(v)$ if $v \in Y$. Thus, for any edge e of H, if $e \subseteq X$ or $e \subseteq X$ then e has two vertices colored differently by h since f and g are proper. Further, if e contains vertices of both X and Y, say $x \in X$ and $y \in Y$, then x and y are colored differently by h since C and $L \setminus C$ are disjoint. Therefore, h is an L-coloring of H. □

Lemma 15. *Let L be a list assignment of an r-uniform hypergraph H. If $(r-1)|L(X)| \geqslant |X|$ for every nonempty subset $X \subseteq V(H)$, then H is L-colorable.*

Proof. Consider the bipartite graph B with vertex partition $V(B) = (V(H), C)$, where C consists of $(r-1)$ copies of $L(V(H))$ and each $v \in V(H)$ is adjacent to the $(r-1)$ copies of $L(v)$. Clearly, for each $X \subseteq V(H)$, we have $N_B(X) = (r-1)L(X)$ and hence $|N_B(X)| \geqslant |X|$ by the condition of the lemma. Thus, by Hall's Matching Theorem, there exists a matching M that saturates $V(H)$. We associate M with an L-coloring f_M of H where $f_M(v)$ is defined to be the color matched to v by M. We can see that each vertex v is colored by a color from its own list $L(v)$, and each color class of H induced by f_M contains at most $r-1$ vertices. This means that each edge of H contains at least two vertices with different colors since H is r-uniform. Thus, f_M is proper and therefore, H is L-colorable. □

Lemma 16. *For a list assignment L of an r-uniform hypergraph H, if $H[X]$ is L-colorable for each nonempty subset $X \subseteq V(H)$ with $(r-1)|L(X)| < |X|$, then H is L-colorable.*

Proof. If $(r-1)|L(X)| \geqslant |X|$ for each nonempty subset $X \subseteq V(H)$, then we are done by Lemma 15. We now assume that X is a maximal nonempty subset of $V(H)$ such that $(r-1)|L(X)| < |X|$. Let $C = L(X)$, let $Y = V(H) \setminus X$ and let S be an arbitrary nonempty subset of Y. Then by the maximality of X, $(r-1)|L(X \cup S)| \geqslant |X \cup S|$. On the other hand, notice that $|L(X \cup S)| = |L(X)| + |(L \setminus C)(S)|$ and $|X \cup S| = |X| + |S|$ as $X \cap S = \emptyset$. So we have $(r-1)|(L \setminus C)(S)| \geqslant |S|$. Consequently, $H[Y]$ is $L \setminus C$-colorable by Lemma 15. Let f be any L-coloring of $H[X]$. Clearly, $L(X) \supseteq f(X)$, that is, $C \supseteq f(X)$. Therefore, H is L-colorable by Lemma 14. □

The following lemma is an extension of a widely used lemma in list colorings of graphs [15, 21], namely the "Small Pot Lemma" [4, 5, 16, 19].

Lemma 17. *For an r-uniform hypergraph H, if H is L-colorable for every k-list assignment L such that $(r-1)|L(V(H))| < |V(H)|$ then H is k-choosable.*

Proof. Suppose to the contrary that H is not k-choosable and let L' be a k-list assignment such that H is not L'-colorable. We show that there is a k-list assignment L with $(r-1)|L(V(H))| < |V(H)|$ such that H is not L-colorable.

By Lemma 16, there is a nonempty subset $X \subseteq V(H)$ such that $(r-1)|L'(X)| < |X|$ and $H[X]$ is not L'-colorable. Choose $x \in X$ and define the list assignment L of H by

$L(v) = L'(v)$ if $v \in X$ and $L(v) = L'(x)$ otherwise. Clearly, H is not L-colorable since $H[X]$ is not L'-colorable and therefore, not L-colorable. Moreover, $L(V(H)) = L'(X)$ and hence $(r-1)|L(V(H))| < |X| \leqslant |V(H)|$. This contradicts the condition of the lemma. $\square$

In [10] (before Ohba's Conjecture was formulated), Gravier and Maffray showed that $K_{3,2*(k-1)}$ is chromatic-choosable. The following two theorems are the generalizations of this result to uniform hypergraphs and therefore, give support to Conjecture 1.

For a color c of L and a vertex subset X of H, the *multiplicity* of c in X is defined by $|\{v : v \in X, c \in L(v)\}|$, that is, the total times of c that appears in the lists of the vertices in X. For a list assignment L, the multiplicity of c in X is denoted by $\eta_{L,X}(c)$, or simply $\eta_X(c)$ when the list assignment is clear.

Theorem 18. *$\chi_l\big(K^r_{2r-1,r*(k-1)}\big) = k$ for $r \geqslant 2$ and $k \geqslant 1$.*

Proof. We prove it by contradiction. Suppose k is the minimal positive integer such that $K^r_{2r-1,r*(k-1)}$ is not k-choosable. Note that if $k = 1$ then $K^r_{2r-1,r*(k-1)}$ contains no edges and therefore is trivially 1-choosable. Thus $k \geqslant 2$. Write $H = K^r_{2r-1,r*(k-1)}$. Since H is not k-choosable, Lemma 17 implies that there exists a k-list assignment L such that $(r-1)|L(V(H))| < |V(H)|$ and H is not L-colorable. Let $V_1, V_2, \ldots, V_k$ be all partite sets of H, where $|V_1| = 2r-1$ and $|V_i| = r$ for $i \in \{2, 3, \ldots, k\}$. As $(r-1)|L(V(H))| < |V(H)| = rk + r - 1$ and $L(V_i) \subseteq L(V(H))$, we have $(r-1)|L(V_i)| \leqslant rk + r - 2$ and hence

$$|L(V_i)| \leqslant \left\lfloor \frac{rk + r - 2}{r - 1} \right\rfloor, \ i = 1, 2, \ldots, k. \tag{2}$$

Claim 1: *$\bigcap_{v \in V_i} L(v) = \emptyset$ for each $i \in \{2, 3, \ldots, k\}$.*

Suppose to the contrary that there exists a color $c^* \in \bigcap_{v \in V_i} L(v)$ for some $i \in \{2, 3, \ldots, k\}$. We use c^* to color all vertices in V_i and let $Y = V(H) \setminus V_i$. Note that $H[Y] = K^r_{2r-1,r*(k-2)}$. By the minimality of k, $H[Y]$ is $(k-1)$-choosable. Therefore, $H[Y]$ is $L \setminus \{c^*\}$-colorable since $(L \setminus \{c^*\})(v)$ contains at least $k-1$ colors for each $v \in Y$. So by Lemma 14, H is L-colorable. This is a contradiction and hence Claim 1 follows.

Let

$$\xi = \left\lceil \frac{(2r-1)k}{\lfloor \frac{rk+r-2}{r-1} \rfloor} \right\rceil. \tag{3}$$

Claim 2: *L has a color $\bar{c}$ such that $\eta_{V_1}(\bar{c}) \geqslant \xi$.*

Clearly, $\sum_{c \in L(V_1)} \eta_{V_1}(c) = \sum_{v \in V_1} |L(v)| = (2r-1)k$. Let $\bar{c}$ be the color such that $\eta_{V_1}(\bar{c})$ is maximum. By (2) we have

$$\eta_{V_1}(\bar{c}) \geqslant \frac{\sum_{v \in V_1} |L(v)|}{|L(V_1)|} \geqslant \frac{(2r-1)k}{\lfloor \frac{rk+r-2}{r-1} \rfloor},$$

which implies $\eta_{V_1}(\bar{c}) \geqslant \xi$. Thus, Claim 2 follows.

Claim 3: *$|L(V_i)| = \lfloor \frac{rk+r-2}{r-1} \rfloor$ for each $i \in \{2, 3, \ldots, k\}$.*

Suppose to the contrary that $|L(V_i)| \neq \lfloor \frac{rk+r-2}{r-1} \rfloor$ for some $i \in \{2,3,\ldots,k\}$. Then, by (2), $|L(V_i)| \leqslant \lfloor \frac{rk+r-2}{r-1} \rfloor - 1$ and hence $|L(V_i)| \leqslant \frac{rk-1}{r-1}$. Let c_i be the color in $L(V_i)$ such that $\eta_{V_i}(c_i)$ is maximum. By an argument similar to the proof of Claim 2, we have

$$\eta_{V_i}(c_i) \geqslant \frac{\sum_{c\in L(V_i)} \eta_{V_i}(c)}{|L(V_i)|} = \frac{rk}{|L(V_i)|} \geqslant \frac{rk}{\frac{rk-1}{r-1}} > r-1.$$

This means that all vertices in V_i have a common color in their lists. This contradicts Claim 1 and therefore, Claim 3 follows.

Claim 4: $\xi \geqslant r + (r-1)\left(\frac{rk+r-2}{r-1} - \lfloor \frac{rk+r-2}{r-1} \rfloor\right)$ and in particular, $\xi \geqslant r$.

Write $k-1 = (r-1)p + q$, where $p = \lfloor \frac{k-1}{r-1} \rfloor$ and $0 \leqslant q \leqslant r-2$. Then $\frac{rk+r-2}{r-1} = k + 1 + \frac{k-1}{r-1} = rp + q + 2 + \frac{q}{r-1}$. Thus, the first inequality in the claim is reduced to

$$\left\lceil \frac{(2r-1)((r-1)p+q+1)}{rp+q+2} \right\rceil \geqslant r+q, \tag{4}$$

that is,

$$(2r-1)((r-1)p+q+1) > (rp+q+2)(r+q-1). \tag{5}$$

Let $\Delta = (2r-1)((r-1)p+q+1) - (rp+q+2)(r+q-1) = -q^2 + (r-2-pr)q + (1+p-2pr+pr^2)$. In order to show $\Delta > 0$ we consider the quadratic function $f(x) = -x^2 + (r-2-pr)x + (1+p-2pr+pr^2)$. Note that $0 \leqslant q \leqslant r-2$ and $\Delta = f(q)$. As $f(x)$ is strictly concave on the interval $[0, r-2]$, the minimum value of $f(x)$ must be attained at $x=0$ or $r-2$. Direct calculation leads to $f(0) = 1+p-2pr+pr^2 = rp(r-2)+p+1 > 0$ and $f(r-2) = p+1 > 0$. Therefore, $f(x) > 0$ on $[0, r-2]$ and hence $\Delta > 0$. This proves Claim 4.

Let $X = \{v \in V_1 \colon \bar{c} \in L(v)\}$. Let $Y = V(H) \setminus X$, $V_1' = V_1 \setminus X$ and $L' = L_Y \setminus \{\bar{c}\}$. Then by Claims 2 and 4, we have $|X| \geqslant \xi \geqslant r$ and therefore,

$$|V_1'| = |V_1 \setminus X| \leqslant 2r - 1 - \xi \leqslant r-1. \tag{6}$$

Clearly, $|L'(v)| = |L(v)| = k$ for each $v \in V_1'$, and $|L'(v)| \geqslant |L(v)| - 1 = k-1$ for each $v \in V_i$, $i \in \{2,3,\ldots,k\}$.

Claim 5: *$H[Y]$ is L'-colorable.*

Let S be an arbitrary nonempty subset of Y. By Lemma 15, it suffices to show that $(r-1)|L'(S)| \geqslant |S|$. To this end, we consider two cases.

Case 1: $V_i \nsubseteq S$ for any $i \in \{2,3,\ldots,k\}$.

In this case, we have

$$|S \cap (V(H) \setminus V_1)| \leqslant (|V_2| - 1) + \cdots + (|V_k| - 1) = (r-1)(k-1). \tag{7}$$

Notice that $|L'(S)| \geqslant |L'(v)| \geqslant k-1$ for any vertex v in S. So by (7), if $S \cap V_1' = \emptyset$ then $(r-1)|L'(S)| \geqslant (r-1)(k-1)| \geqslant |S \cap (V(H) \setminus V_1)| = |S|$, as desired. Now we assume that $S \cap V_1' \neq \emptyset$. Then by (6) and (7) we have $|S| = |(S \cap V_1') \cup (S \cap (V(H) \setminus V_1))| \leqslant (r-1) + (r-1)(k-1) = (r-1)k$. Let $v \in S \cap V_1'$. Then $|L'(v)| = k$ and hence $|L'(S)| \geqslant k$. Again we have $(r-1)|L'(S)| \geqslant |S|$.

Case 2: $V_i \subseteq S$ for some $i \in \{2, 3, \ldots, k\}$.

By Claim 3, $L'(V_i) \geqslant \lfloor \frac{rk+r-2}{r-1} \rfloor - 1$. On the other hand, by the first inequality in (6), $|S| \leqslant |V_1'| + |V_2| + \cdots + |V_k| \leqslant 2r - 1 - \xi + r(k-1)$. Therefore, by Claim 4,

$$\begin{aligned}(r-1)|L'(S)| - |S| &\geqslant (r-1)\left(\left\lfloor \frac{rk+r-2}{r-1} \right\rfloor - 1\right) - (2r - 1 - \xi + r(k-1)) \\ &= \xi + (r-1)\left\lfloor \frac{rk+r-2}{r-1} \right\rfloor - rk - 2r + 2 \\ &\geqslant r + (r-1)\left(\frac{rk+r-2}{r-1} - \left\lfloor \frac{rk+r-2}{r-1} \right\rfloor\right) \\ &\quad +(r-1)\left\lfloor \frac{rk+r-2}{r-1} \right\rfloor - rk - 2r + 2 \\ &= 0.\end{aligned}$$

Thus, $(r-1)|L'(S)| \geqslant |S|$, as desired.

From the above two cases, Claim 5 follows.

Finally, by Claim 5 and Lemma 14, H is L-colorable. This is a contradiction and hence completes the proof of this theorem. □

In the previous section we have known that $\chi_l\big(K^r_{2r,r*(k-1)}\big) \geqslant k+1$ if k is divisible by $r-1$. By Theorem 18, we have $k \leqslant \chi_l\big(K^r_{2r,r*(k-1)}\big) \leqslant \chi_l\big(K^r_{2r-1,r*(k-1)}\big) + 1 = k+1$. This gives that

$$\chi_l\left(K^r_{2r,r*(k-1)}\right) = k+1$$

if k is divisible by $r-1$. We propose the following problem.

Problem 19. If k is not divisible by $r-1$, determine when does $\chi_l\big(K^r_{2r,r*(k-1)}\big)$ equal k and when does it equal $k+1$?

The following result gives the second generalization of $K_{3,2*(k-1)}$ for supporting our conjecture.

Theorem 20. *$\chi_l\big(K^r_{(r+1)*(r-1),r*(k-r+1)}\big) = k$ for $r \geqslant 2$ and $k \geqslant r-1$.*

Before proving it, we need first to show that $\chi_l\big(K^r_{(r+1)*(r-1)}\big) = r-1$. In fact, we prove the following more general result.

Proposition 21. *$\chi_l\big(K^r_{(r+1)*k}\big) = k$ for $r \geqslant 2$ and $k \leqslant r-1$.*

Proof. If $r=2$ then $k=1$ and the assertion trivially holds. We may assume that $r \geqslant 3$. We prove the proposition by induction on k. Since $\chi_l\big(K^r_{(r+1)*k}\big) \geqslant \chi\big(K^r_{(r+1)*k}\big) = k$, it suffices to show that $K^r_{(r+1)*k}$ is k-choosable. If $k=1$ then $K^r_{(r+1)*k}$ contains no edges and hence is 1-choosable. Let $1 < k \leqslant r-1$ and assume that $K^r_{(r+1)*t}$ is t-choosable for any $t < k$. For simplicity, let $H = K^r_{(r+1)*k}$ and let $V_1, V_2, \ldots, V_k$ be the k partite sets of H. We need to show that H is k-choosable.

Let L be any k-list assignment of H such that

$$(r-1)|L(V(H))| < |V(H)| = (r+1)k. \tag{8}$$

By Lemma 17, to show that H is k-choosable, it suffices to show that H is L-colorable. If there is some V_i such that all vertices in V_i have a common color c^* in their lists, then we can color each vertex in V_i by c^* and remove c^* from the lists of all other vertices in H. Using induction on k and Lemma 14, one can easily verify that H is L-colorable.

In the following, we assume that $\bigcap_{v\in V_i} L(v) = \emptyset$ for any $i \in \{1, 2, \ldots, k\}$. As $|V_i| = r+1$ we have $\eta_{V_i}(c) \leqslant r$ for each $c \in L(V_i)$. For each $i \in \{1, 2, \ldots, k\}$, let $C_i = \{c \in L(V_i)\colon\ \eta_{V_i}(c) = r\}$. Thus, for each color $c \in L(V_i) \setminus C_i$, we have $\eta_{V_i}(c) \leqslant r-1$ and hence,

$$r|C_i| + (r-1)(|L(V_i)| - |C_i|) \geqslant \sum_{v\in V_i} |L(v)| = (r+1)k \tag{9}$$

for each $i \in \{1, 2, \ldots, k\}$. Equivalently, $|C_i| \geqslant (r+1)k - (r-1)|L(V_i)|$. Since $|L(V_i)| \leqslant |L(V(H))|$, we have $|C_i| > 0$ by (8).

Let I be a maximal subset of $\{1, 2, \ldots, k\}$ such that $\{C_i\colon\ i \in I\}$ has a system of distinct representatives and let $s = |I|$. Since C_i is nonempty, $s \geqslant 1$. With no loss of generality, we may assume that $I = \{1, 2, \ldots, s\}$. Let $(c_1, c_2, \ldots, c_s)$ be a system of distinct representatives of $(C_1, C_2, \ldots, C_s)$. Notice that $\eta_{V_i}(c_i) = r$ and $|V_i| = r+1$. For each $i \in \{1, 2, \ldots, s\}$, let v_i be the only vertex in V_i such that $c_i \notin L(v_i)$. Let $H' = H[\{v_1, \ldots, v_s\} \cup V_{s+1} \cup \cdots \cup V_k]$ and define a list assignment L' on the hypergraph H' by $L'(v) = L(v) \setminus \{c_1, \ldots, c_s\}$ for any $v \in V(H')$. For each $i \in \{1, 2, \ldots, s\}$, we use c_i to color all vertices in V_i except v_i. By Lemma 14, to show that H is L-colorable, it suffices to show that H' is L'-colorable.

For each $i \in \{1, 2, \ldots, s\}$, as $s \leqslant k$ and $c_i \notin L(v_i)$, we have $|L'(v_i)| \geqslant L(v_i) - (s-1) = k - (s-1) \geqslant 1$. If $s = k$ then $|V(H')| = k < r$ and hence H' contains no edges. In this case, H' is trivially L'-colorable. Thus, we assume that $s \leqslant k-1$. For each $p \in \{s+1, s+2, \ldots, k\}$, by the maximality of I, we have $C_p \subseteq \{c_1, c_2, \ldots, c_s\}$ and hence $|C_p| \leqslant s$.

Let S be an arbitrary subset of $V(H')$. We consider three cases:

Case 1: $v_i \notin S$ for any $i \in \{1, 2, \ldots, s\}$.

In this case, $H'[S]$ is an induced subgraph of $K^r_{(r+1)*(k-s)}$. Further, by the induction hypothesis, $K^r_{(r+1)*(k-s)}$ is $(k-s)$-choosable. Therefore, $H'[S]$ is $(k-s)$-choosable. As $|L'(v)| \geqslant |L(v)| - s = k - s$ for each $v \in S$, $H'[S]$ is L'-colorable.

Case 2: $v_i \in S$ for some $i \in \{1, 2, \ldots, s\}$ and $V_p \not\subseteq S$ for any $p \in \{s+1,\ s+2, \ldots, k\}$.

In this case, $|S| \leqslant r(k-s) + s$. As $|L'(v_i)| \geqslant k - s + 1$ and $k \leqslant r-1$, we have

$$(r-1)|L'(S)| - |S| \geqslant (r-1)(k-s+1) - (r(k-s)+s) = r-1-k \geqslant 0,$$

that is, $(r-1)|L'(S)| \geqslant |S|$.

Case 3: $v_i \in S$ for some $i \in \{1, 2, \ldots, s\}$ and $V_p \subseteq S$ for some $p \in \{s+1,\ s+2, \ldots, k\}$.

By (9), we have $(r-1)|L(V_p)| \geqslant (r+1)k - |C_p|$ and hence $(r-1)|L(V_p)| \geqslant (r+1)k - s$ as $|C_p| \leqslant s$. Therefore,

$$(r-1)|L'(S)| \geqslant (r-1)|L'(V_p)| \geqslant (r-1)(|L(V_p)| - s) \geqslant (r+1)k - rs.$$

On the other hand, $|S| \leqslant |V(H')| = (r+1)k - rs$. Thus, $(r-1)|L'(S)| \geqslant |S|$.

By the above three cases, for any $S \subseteq V(H')$, either $(r-1)|L'(S)| \geqslant |S|$ or $H'[S]$ is L'-colorable. It follows from Lemma 16 that H' is L'-colorable. Thus, H is L-colorable and hence k-choosable. This proves the proposition by induction. $\square$

Proof of Theorem 20. We prove the theorem by induction on k. If $k = r-1$ then the assertion holds by Proposition 21. Now let $k \geqslant r$ and assume that $K^r_{(r+1)*(r-1),r*(k-r)}$ is $(k-1)$-choosable. We are going to show that $K^r_{(r+1)*(r-1),r*(k-r+1)}$ is k-choosable. Write $H = K^r_{(r+1)*(r-1),r*(k-r+1)}$ and let $V_1, V_2, \ldots, V_k$ be the partite sets of H with $|V_i| = r+1$ for $i \in \{1, 2, \ldots, r-1\}$ and $|V_i| = r$ for $i \in \{r, r+1, \ldots, k\}$.

Let L be any k-list assignment of H such that

$$(r-1)|L(V(H))| < |V(H)| = rk + r - 1. \tag{10}$$

By Lemma 17, it suffices to show that H is L-colorable.

For some $i \in \{1, 2, \ldots, k\}$, if all vertices in V_i have a common color, say c^*, in their lists, then we can color each vertex in V_i by c^*. Let H' be the subgraph of H induced by $V(H) \setminus V_i$. That is, $H' = K^r_{(r+1)*(r-2),r*(k-r+1)}$ if $i \leqslant r-1$ or $K^r_{(r+1)*(r-1),r*(k-r)}$ if $i > r-1$, both of which are subgraphs of $K^r_{(r+1)*(r-1),r*(k-r)}$. Further, by the induction hypothesis, $K^r_{(r+1)*(r-1),r*(k-r)}$ is $(k-1)$-choosable and so is H'. Let L' be the list assignment of H' defined by $L'(v) = L(v) \setminus \{c^*\}$ for each $v \in V(H')$. Then $|L'(v)| \geqslant k-1$ and hence H' is L'-colorable. Thus, H is L-colorable by Lemma 14.

We now assume that $\bigcap_{v \in V_i} L(v) = \emptyset$ for each $i \in \{1, 2, \ldots, k\}$. The following discussion is much similar to the proof of Proposition 21. For each $i \in \{1, 2, \ldots, r-1\}$ let $C_i = \{c \in L(V_i) \colon \eta_{V_i}(c) = r\}$. Then (9) holds for each $i \in \{1, 2, \ldots, r-1\}$ and, therefore, $|C_i| \geqslant (r+1)k - (r-1)|L(V_i)|$. Since $|L(V_i)| \leqslant |L(V(H))|$ and $k \geqslant r$, it follows by (10) that $|C_i| > 1$.

Let I be a maximal subset of $\{1, 2, \ldots, r-1\}$ such that $\{C_i \colon i \in I\}$ has a system of distinct representatives, and let $s = |I|$. It is clear that $1 \leqslant s \leqslant r-1$ as $C_i \neq \emptyset$ for each $i \in \{1, 2, \ldots, r-1\}$. With no loss of generality, we assume that $I = \{1, 2, \ldots, s\}$ and $(c_1, c_2, \ldots, c_s)$ is a system of distinct representatives of $(C_1, C_2, \ldots, C_s)$. For each $i \in \{1, 2, \ldots, s\}$, let v_i be the only vertex of V_i such that $c_i \notin L(v_i)$. Let $H' = H[\{v_1, \ldots, v_s\} \cup V_{s+1} \cup \cdots \cup V_k]$ and define $L'(v) = L(v) \setminus \{c_1, c_2 \ldots, c_s\}$ for every $v \in V(H')$. It suffices to show that H' is L'-colorable by Lemma 14.

For each $i \in \{1, 2, \ldots, s\}$, since $c_i \notin L(v_i)$, we have

$$|L'(v_i)| \geqslant |L(v_i)| - (s-1) = k - s + 1. \tag{11}$$

For each $p \in \{r, r+1, \ldots, k\}$, since $\bigcap_{v \in V_p} L(v) = \emptyset$, each color of $L(V_p)$ appears at most $r-1$ times in V_p. Therefore,

$$|L(V_p)| \geqslant \frac{\sum_{v \in V_p} |L(v)|}{r-1} = \frac{rk}{r-1}. \tag{12}$$

As $|L'(V_p)| \geqslant (|L(V_p)| - s)$, (12) implies

$$(r-1)|L'(V_p)| \geqslant rk - (r-1)s. \tag{13}$$

If $s < r-1$, then for each $q \in \{s+1, s+2, \cdots, r-1\}$, we have $C_q \subseteq \{c_1, c_2, \ldots, c_s\}$ by the maximality of I. Thus $|C_q| \leqslant s$. It follows from (9) (regard i as q) that $(r-1)|L(V_q)| \geqslant (r+1)k - |C_q| \geqslant (r+1)k - s$. Thus,

$$(r-1)|L'(V_q)| \geqslant (r-1)(|L(V_q)| - s) \geqslant (r+1)k - rs. \tag{14}$$

Let S be an arbitrary subset of $V(H')$. We will show that either $H'[S]$ is L'-colorable or $(r-1)|L(S)| \geqslant |S|$.

First assume that $s < r-1$ and $V_q \subseteq S$ for some $q \in \{s+1, s+2, \ldots, r-1\}$. Note that $|S| \leqslant |V(H')| = rk + (r-1) - rs$, $|L'(S)| \geqslant |L'(V_q)|$ and $k \geqslant r$. It follows from (14) that

$$(r-1)|L'(S)| \geqslant (r+1)k - rs \geqslant rk + r - rs > |S|, \tag{15}$$

as desired. In the following, we always assume that $V_q \nsubseteq S$ for any $q \in \{s+1, s+2, \ldots, r-1\}$, unless $s = r-1$. Under this assumption, we have $|S \cap V_i| \leqslant r$ for all $i \in \{s+1, s+2, \ldots, k\}$. We consider three cases:

Case 1: $v_i \notin S$ for any $i \in \{1, 2, \ldots, s\}$.

In this case, $H'[S]$ is an induced subgraph of $K^r_{r*(k-s)}$ and hence of $K^r_{2r-1,r*(k-s-1)}$. Thus, $H'[S]$ is $(k-s)$-choosable by Theorem 18. Since $|L'(v)| \geqslant |L(v)| - s = k - s$ for each $v \in S$, $H'[S]$ is L'-colorable, as desired.

Case 2: $v_i \in S$ for some $i \in \{1, 2, \ldots, s\}$ and $V_p \nsubseteq S$ for any $p \in \{r, r+1, \ldots, k\}$.

Combining with our assumption that $V_q \nsubseteq S$ for $q \in \{s+1, s+2, \ldots, r-1\}$, we have $V_j \nsubseteq S$ for any $j \in \{s+1, s+2, \ldots, k\}$. Thus,

$$S \leqslant |V(H')| - (k-s) = (rk + (r-1) - rs) - (k-s) = (r-1)(k+1-s).$$

As $v_i \in S$, we have $|L'(S)| \geqslant |L'(v_i)|$, implying that $|L'(S)| \geqslant k+1-s$ by (11). Thus, $(r-1)|L'(S)| \geqslant |S|$.

Case 3: $v_i \in S$ for some $i \in \{1, 2, \ldots, s\}$ and $V_p \subseteq S$ for some $p \in \{r, r+1, \ldots, k\}$.

In this case, again by our assumption that $V_p \nsubseteq S$ for any $p \in \{s+1, s+2, \ldots, r-1\}$, we have $|S| \leqslant |V(H')| - (r-1-s) = (rk + (r-1) - rs) - (r-1-s) = rk - (r-1)s$. Since $V_p \subseteq S$, so by (13) we have $(r-1)|L'(S)| \geqslant (r-1)|L'(V_p)| \geqslant rk - (r-1)s \geqslant |S|$.

By the above three cases, for any $S \subseteq V(H')$, either $(r-1)|L'(S)| \geqslant |S|$ or $H'[S]$ is L'-colorable. Therefore, H' is L'-colorable by Lemma 16. This completes the proof of Theorem 20.

Corollary 22. $\chi_l(K^r_{(r+1)*r}) = r+1$ *for* $r \geqslant 2$.

Proof. By Theorem 13, $\chi_l(K^r_{(r+1)*r}) \geqslant r+1$. On the other hand, using Theorem 20 for $k = r$, we have $\chi_l(K^r_{(r+1)*(r-1),r}) = r$. Thus $\chi_l(K^r_{(r+1)*r}) \leqslant r+1$. This proves the corollary. □

Acknowledgments

The authors thank the referee for her/his valuable and detailed suggestions. In particular, Problem 19 is suggested by the referee.

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