On Cartesian Products which Determine Few Distinct Distances

Cosmin Pohoata

Department of Mathematics
California Institute of Technology
Pasadena, U.S.A.

apohoata@caltech.edu

Submitted: Mar 21, 2018; Accepted: Nov 18, 2018; Published: Jan 11, 2019

Abstract

Every set of points $\mathcal{P}$ determines $\Omega(|\mathcal{P}|/\log|\mathcal{P}|)$ distances. A close version of this was initially conjectured by Erdős in 1946 and rather recently proved by Guth and Katz. We show that when near this lower bound, a point set $\mathcal{P}$ of the form $A \times A$ must satisfy $|A - A| \ll |A|^{2-\frac{2}{7}} \log^{\frac{1}{7}} |A|$. This improves recent results of Hanson and Roche-Newton.

Mathematics Subject Classifications: 05D99

1 Introduction

Let $\mathcal{P}$ be a set of points in plane, and let $\Delta(\mathcal{P})$ denote the set of squares of distances spanned by $\mathcal{P}$. In other words,

$$\Delta(\mathcal{P}) = \left\{(p_1 - q_1)^2 + (p_2 - q_2)^2 : (p_1, p_2), (q_1, q_2) \in \mathcal{P}\right\}.$$

In [7], Guth and Katz showed that $\Delta(\mathcal{P}) \gg |\mathcal{P}|/\log|\mathcal{P}|$, where $\gg$ represents the usual Vinogradov symbol. When $P = A \times B$ for some finite sets of reals A and B, $\Delta(A \times B) = (A - B)^2 + (A - B)^2$, so this says that

$$|(A - B)^2 + (A - B)^2| \gg \frac{|A||B|}{\log|A||B|}.$$

In [5], Erdős originally conjectured that all sets $\mathcal{P}$ should determine $\Omega(|\mathcal{P}|/\sqrt{\log|\mathcal{P}|})$ distinct distances, so the Guth-Katz bound is almost optimal. Nonetheless, very little is known for sets that achieve this bound. It is widely believed that sets with $O(|\mathcal{P}|/\log|\mathcal{P}|)$ distinct distances should come from some type of lattice. This is very well-motivated by the following beautiful result of Bernays [2], which generalizes a classical theorem of Landau.

Theorem 1. *Let $f(x,y) = ax^2 + bxy + cy^2$ for integers $a, b, c \in \mathbb{Z}$, such that the determinant $b^2 - 4ac$ is not an integer square. Then, the number of integers between 1 and n that can be expressed as $f(u,v)$ with $u, v \in \mathbb{Z}$ is $O(n/\sqrt{\log n})$.*

Using Theorem 1, one can easily check that sets with $O(n/\sqrt{\log n})$ distinct distances are given by $\sqrt{n} \times \sqrt{n}$ subsets of the integer lattice, the (equilateral) triangular lattice, or, more exotically, by the rectangular lattice

$$\mathcal{L}_r = \left\{ (i, j\sqrt{r}) \mid i, j \in \mathbb{Z},\ 1 \leqslant i, j \leqslant n \right\},$$

for every integer $r > 1$. We refer the reader to [14] for a more detailed presentation of this discussion, where Sheffer also points out that unlike the first two examples, the latices $\mathcal{L}_r$ do not span squares or equilateral triangles.

In this paper, we will only take a look at sets that come from cartesian products, and show that whenever they determine few distinct distances they must exhibit some additive structure. Specifically, when $\mathcal{P} = A \times A$, we show that when the Guth-Katz bound is close to being tight, we have that

$$|A - A| \ll |A|^{2-\frac{2}{7}} \log^{\frac{1}{7}} |A|.$$

In light of the bipartite distance problems discussed by Brunner and Sharir in [3] and by Sheffer and the author in [2], we also consider the problem of showing that if there are few distinct distances between two cartesian products $A \times A$ and $B \times B$, then one of A or B has additive structure. We state both of these results more formally below.

Theorem 2. *Suppose A is a finite set of real numbers and let $\Delta(A \times A)$ be the set of distances spanned by $A \times A$. Then,*

$$|A - A| \ll |\Delta(A \times A)|^{\frac{6}{7}} \log^{\frac{1}{7}} |A|,$$

or equivalently $|D| \ll |D^2 + D^2|^{\frac{6}{7}} \log^{\frac{1}{7}} |D|$, where D denotes the difference set $A - A$.

Theorem 3. *Suppose A and B are finite sets of real numbers and let $\Delta(A \times A, B \times B)$ be the set of distances between points in $A \times A$ and points in $B \times B$. Then,*

$$\min\{|A - A|, |B - B|, |A - B|\} \ll |\Delta(A \times A, B \times B)|^{1-\frac{13}{205}} \cdot Ł(A, B),$$

where

$$Ł(A, B) = \min\left\{ \log^{\frac{3}{205}} |A|, \log^{\frac{3}{205}} |B| \right\}.$$

In particular, if $|\Delta(A \times A)| \ll |A|^2$ holds in Theorem 2, then $|A - A| \ll |A|^{2-\frac{2}{7}} \log^{\frac{1}{7}} |A|$. This improves a recent theorem by Hanson [8], who showed that under this hypothesis we have that $|A - A| \ll |A|^{2-\frac{1}{8}}$. In the meantime this was also sharpened by Roche-Newton in [13], who showed $|A - A| \ll |A|^{2-\frac{2}{11}}$, but the estimate from Theorem 2 is stronger. Our proof will rely on the sum-product estimate of Solymosi from [16] as a black-box:

Theorem 4. *Let $S \subset \mathbb{R}$ be a set. Then*

$$|S+S|^2|SS| \geqslant \frac{|S|^4}{4\lceil \log |S| \rceil}.$$

The proof of Theorem 3 will rely on two results. The first one is the following Lemma by Balog [1], which comes from Solymosi's original idea for Theorem 4.

Lemma 5. *Let R, S, T be finite sets of real numbers. Then*

$$|RT+RT||ST+ST| \gg |R/S||T|^2.$$

The second one is the following Lemma due to Shkredov, which is Theorem 3 in [15] (and the statement of which should be in some sense compared to that of Theorem 2 above).

Lemma 6. *Let $A \subset \mathbb{R}$ be a finite set and let $D = A - A$. Then*

$$|D/D| \gg |D|^{1+\frac{1}{12}} \log^{-1/4} |D|.$$

Last but not least, we will also need the classical Plünnecke-Ruzsa inequality, for which a simple proof can be found in [11].

Lemma 7. *Let $A \subset \mathbb{R}$ be a finite set. Then*

$$|kA - \ell A| \leqslant \frac{|A+A|^{k+\ell}}{|A|^{k+\ell-1}}.$$

2 Proof of Theorem 2

If $D = A - A$, then $|\Delta(A \times A)| = |D^2 + D^2|$, where $D^2 = \{(x-y)^2 : x, y \in A\}$. We claim that

$$|D| \ll |D^2 + D^2|^{\frac{6}{7}} \log^{\frac{1}{7}} |D|.$$

We apply Theorem 4 for the set $S := D^2$. Using the observation that $|D^2D^2|$ is equal to $|DD|$ (up to a small constant), this yields

$$|D^2+D^2|^2|DD| \geqslant |D^2+D^2|^2|D^2D^2| \geqslant \frac{|D^2|^4}{4\lceil \log |D^2| \rceil} \gg \frac{|D|^4}{\log |D|}.$$

On the other hand for every four real numbers a_1, a_2, b_1, b_2, we have that

$$(b_1-a_1)^2 + (b_2-a_2)^2 - (b_1-a_2)^2 - (b_2-a_1)^2 = 2(a_2-a_1)(b_1-b_2),$$

which yields the inclusion

$$2 \cdot DD \subset 2D^2 - 2D^2.$$

We emphasize here that for $X \subset \mathbb{R}$ and $c \in \mathbb{Z}_{>0}$, the set $c \cdot X$ denotes the set of scalar multiples $\{cx : x \in X\}$, whereas cX denotes the sumset $\sum_{i=1}^{c} X$. The inclusion together with Lemma 7 then yield

$$\begin{aligned} |D^2 + D^2|^2 |DD| &= |D^2 + D^2|^2 |2 \cdot DD| \\ &\leqslant |D^2 + D^2|^2 |2D^2 - 2D^2| \\ &\ll |D^2 + D^2|^2 \left(\frac{|D^2 + D^2|^4}{|D|^3} \right). \end{aligned}$$

Putting the two bounds together, we conclude that

$$\frac{|D^2 + D^2|^6}{|D|^3} \gg \frac{|D|^4}{\log |D|},$$

which yields

$$|D| \ll |D^2 + D^2|^{\frac{6}{7}} \log^{\frac{1}{7}} |D|.$$

□

3 Proof of Theorem 3

For convenience, write again that $|\Delta(A \times A, B \times B)| = |(A - B)^2 + (A - B)^2|$. Since

$$(b_1 - a_1)^2 + (b_2 - a_2)^2 - (b_1 - a_2)^2 - (b_2 - a_1)^2 = 2(a_2 - a_1)(b_1 - b_2)$$

holds for every $a_1, a_2 \in A$, $b_1, b_2 \in B$, we have the inclusion

$$2 \cdot (A - A)(B - B) + 2 \cdot (A - A)(B - B) \subset 4(A - B)^2 - 4(A - B)^2.$$

On one hand Lemma 7 gives

$$\begin{aligned} |4(A - B)^2 - 4(A - B)^2| &\leqslant \frac{|(A - B)^2 + (A - B)^2|^8}{|(A - B)^2|^7} \\ &= \frac{|(A - B)^2 + (A - B)^2|^8}{|A - B|^7}. \end{aligned}$$

On the other hand, the above inclusion gives

$$\begin{aligned} |4(A - B)^2 - 4(A - B)^2|^2 &\geqslant |2 \cdot (A - A)(B - B) + 2 \cdot (A - A)(B - B)|^2 \\ &= |(A - A)(B - B) + (A - A)(B - B)|^2. \end{aligned}$$

Furthermore, Lemma 5 applied for $R = S = A - A$, $T = B - B$ tells us that

$$|(A - A)(B - B) + (A - A)(B - B)|^2 \gg \left| \frac{A - A}{A - A} \right| |B - B|^2.$$

By Lemma 6,

$$\left|\frac{A-A}{A-A}\right| \gg |A-A|^{1+\frac{1}{12}} \log^{-1/4}|A-A|,$$

so

$$|4(A-B)^2-4(A-B)^2|^2 \gg |A-A|^{1+\frac{1}{12}}|B-B|^2 \log^{-1/4}|A-A|.$$

We conclude that

$$\frac{|(A-B)^2+(A-B)^2|^{16}}{|A-B|^{14}} \gg |A-A|^{1+\frac{1}{12}}|B-B|^2 \log^{-1/4}|A-A|.$$

By using Lemma 6 for $R=S=B-B$ and $T=A-A$ instead, we can similarly get

$$\frac{|(A-B)^2+(A-B)^2|^{16}}{|A-B|^{14}} \gg |A-A|^2|B-B|^{1+\frac{1}{12}} \log^{-1/4}|B-B|.$$

Putting everything together, we get

$$\min\{|A-A|,|B-B|,|A-B|\} \ll |(A-B)^2+(A-B)^2|^{1-\frac{13}{205}} \cdot \text{Ł}(A,B),$$

where

$$\text{Ł}(A,B)=\min\left\{\log^{\frac{3}{205}}|A|, \log^{\frac{3}{205}}|B|\right\}.$$

This completes the proof. □

4 Concluding Remarks

Theorem 2 is still far from being optimal. We conjecture that when $|\Delta(A\times A)| = o(|A|^2)$,

$$|A-A| \ll |A|^{1+\epsilon},$$

for any $\epsilon > 0$. The ϵ in the conjecture is justfied by the following remark. Translate the set A so that it contains 0 so that now $|(A-A)^2+(A-A)^2| = o(|A|^2)$ implies $|A^2+A^2| = o(|A|^2)$. On the other hand, using an argument similar to the one of Elekes and Ruzsa from [4], one can show that for every $A, B \subset \mathbb{R}$ we have that

$$|A^2+B^2||A-A+B|^2|A-A-B|^2 \gg |A|^4|B|^2.$$

In particular, whenever $A=B$ and $|A-A| \ll A$, Lemma 7 yields $|A^2+A^2| \gg |A|^2$, so we get a contradiction.

It is worth mentioning that even assuming the full-strength of the Erdős-Szemerédi conjecture [6], which says that for any $\epsilon > 0$ one has

$$\max|D^2+D^2|, |D^2D^2| \gg |D|^{2-\epsilon'},$$

our proof for Theorem 2 only gives

$$|A-A| \ll |A|^{2-\frac{4}{7}+\epsilon}.$$

Using the updates building on the Konyagin and Shkredov improvements of Solymosi's bound [9, 10], one can perhaps bring

$$|A - A| \ll |A|^{2-\frac{2}{7}} \log^{\frac{1}{7}} |A|$$

down to

$$|A - A| \ll |A|^{2-\frac{2}{7}-c} \log^{\frac{1}{7}} |A|$$

for some small constant $c > 0$, but significant improvements to Theorem 2 should perhaps first come from replacing the inequality $|DD| \leqslant |2D^2 - 2D^2|$ coming from $2 \cdot DD \subset 2D^2 - 2D^2$ with a more efficient argument. We believe that the Erdős-Szemerédi conjecture should imply the claim that cartesian products with $o(|A|^2)$ distinct distances satisfy $|A - A| \ll |A|^{1+\epsilon}$.

Acknowledgments

I would like to thank Oliver Roche-Newton, Misha Rudnev and Adam Sheffer for helpful conversations.

References

[1] A. Balog, A note on sum-product estimates, *Publ. Math. Debrecen.* **79** (2011), 3-4, 283-289.

[2] P. Bernays, Über die Darstellung von positiven, ganzen Zahlen durch die primitiven, binären quadratischen Formen einer nicht-quadratischen Diskriminante *Ph.D Thesis* (1912).

[3] A. Bruner and M. Sharir, Distinct distances between a collinear set and an arbitrary set of points, *Discrete Math.*, to appear.

[4] G. Elekes and I. Z. Ruzsa, Few sums, many products, *Studia Scientiarum Mathematicarum Hungarica* 40 (2003), 301–308.

[5] P. Erdős, On some metric and combinatorial geometric problems, *Discrete Math.* **60** (1986), 147–153.

[6] P. Erdős, E. Szemerédi, On sums and products of integers, *Studies in pure mathematics*, 213-218, Birkhäuser, Basel, 1983.

[7] L. Guth and N. H. Katz, On the Erdős distinct distances problem in the plane, *Ann. of Math.* **2** 181 (2015), 155–190.

[8] B. Hanson, The additive structure of cartesian products spanning few distinct distances, *Combinatorica* (2017), 1-6.

[9] S. Konyagin and I. D. Shkredov, On sum sets of sets, having small product set, *Proc. of the Steklov Institute of Mathematics*, **290** (2015), 288-299.

[10] S. Konyagin and I. D. Shkredov, New results on sum-products in $\mathbb{R}$, *Transactions of Steklov Mathematical Institute*, to appear; preprint: arXiv:1602.03473.

[11] G. Petridis, New proofs of Plünnecke-type estimates for product sets in groups, *Combinatorica* 32 (2012), no. 6, 721-733.

[12] C. Pohoata, A. Sheffer, Higher Distance Energies and Expanders with Structure, arXiv:1709.06696.

[13] O. Roche-Newton, On sets with few distinct distances, arXiv:1608.02775.

[14] A. Sheffer, Point sets with few distinct distances, blog post: https://adamsheffer.wordpress.com/2014/07/16/point-sets-with-few-distinct-distances/.

[15] I. D. Shkredov, Difference sets are not multiplicatively closed, Discrete Analysis 2016: 17, 20pp.

[16] J. Solymosi, Bounding multiplicative energy by the sumset, *Advances in Mathematics*, Volume 222, **2** (2009), 402-408.

[17] P. Ungar, $2N$ noncollinear points determine at least $2N$ directions, *J. Combinatorial Theory*, Ser. **A 33** (1982), 343-347.