Extremal H-free planar graphs

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Abstract

Given a graph H, a graph is H*-free* if it does not contain H as a subgraph. We continue to study the topic of "extremal" planar graphs initiated by Dowden [J. Graph Theory 83 (2016) 213–230], that is, how many edges can an H-free planar graph on n vertices have? We define $ex_{\mathcal{P}}(n,H)$ to be the maximum number of edges in an H-free planar graph on n vertices. We first obtain several sufficient conditions on H which yield $ex_{\mathcal{P}}(n,H) = 3n-6$ for all $n \geqslant |V(H)|$. We discover that the chromatic number of H does not play a role, as in the celebrated Erdős-Stone Theorem. We then completely determine $ex_{\mathcal{P}}(n,H)$ when H is a wheel or a star. Finally, we examine the case when H is a (t,r)-fan, that is, H is isomorphic to $K_1 + tK_{r-1}$, where $t \geqslant 2$ and $r \geqslant 3$ are integers. However, determining $ex_{\mathcal{P}}(n,H)$, when H is a planar subcubic graph, remains wide open.

Mathematics Subject Classifications: 05C10, 05C35

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1 Introduction

All graphs considered in this paper are finite and simple. We use K_t, C_t and P_t to denote the complete graph, cycle, and path on t vertices, respectively. Given a graph G, we will use $V(G)$ to denote the vertex set, $E(G)$ the edge set, $|G|$ the number of vertices, $e(G)$ the number of edges, $\delta(G)$ the minimum degree, $\Delta(G)$ the maximum degree. For a vertex $v \in V(G)$, we will use $N_G(v)$ to denote the set of vertices in G which are adjacent to v. Let $d_G(v) = |N_G(v)|$ denote the degree of the vertex v in G and $N_G[v] = N_G(v) \cup \{v\}$. A vertex is a k-*vertex* in G if it has degree k. We use $n_k(G)$ to denote the number of k-vertices in G. For any set $S \subset V(G)$, the subgraph of G induced on S, denoted $G[S]$, is the graph with vertex set S and edge set $\{xy \in E(G) : x, y \in S\}$. We denote by $G \setminus S$ the subgraph of G induced on $V(G) \setminus S$. If $S = \{v\}$, then we simple write $G \setminus v$. For any two disjoint sets A and B of $V(G)$, we use $e_G(A, B)$ to denote the number of edges between A and B in G. The *join* $G + H$ (resp. *union* $G \cup H$) of two vertex-disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$). For a positive integer t, we use tG to denote disjoint union of t copies of a graph G. Given two isomorphic graphs G and H, we may (with a slight but common abuse of notation) write $G = H$. For any positive integer k, let $[k] := \{1, 2, \ldots, k\}$.

Given a graph H, a graph is H-free if it does not contain H as a subgraph. One of the best known results in extremal graph theory is Turán's Theorem [15], which gives the maximum number of edges that a K_t-free graph on n vertices can have. The celebrated Erdős-Stone Theorem [7] then extends this to the case when K_t is replaced by an arbitrary graph H with at least one edge, showing that the maximum number of edges possible is $(1+o(1))\binom{n}{2}\left(\frac{\chi(H)-2}{\chi(H)-1}\right)$, where $\chi(H)$ denotes the chromatic number of H. This latter result has been called the "fundamental theorem of extremal graph theory" [1]. Turán-type problems when host graphs are hypergraphs are notoriously difficult. A large quantity of work in this area has been carried out in determining the maximum number of edges in a k-uniform hypergraph on n vertices without containing k-uniform linear paths and cycles (see, for example, [9, 10, 12]). Surveys on Turán-type problems of graphs and hypergraphs can be found in [8, 11].

In this paper, we continue to study the topic of "extremal" planar graphs, that is, how many edges can an H-free planar graph on n vertices have? We define $ex_{\mathcal{P}}(n, H)$ to be the maximum number of edges in an H-free planar graph on n vertices. Dowden [3] initiated the study of $ex_{\mathcal{P}}(n, H)$ and proved the following results, where each bound is tight.

Theorem 1 ([3]). *Let n be a positive integer.*

(a) $ex_{\mathcal{P}}(n, C_4) \leqslant 15(n-2)/7$ *for all* $n \geqslant 4$.
(b) $ex_{\mathcal{P}}(n, C_5) \leqslant (12n-33)/5$ *for all* $n \geqslant 11$.

Let Θ_4 and Θ_5 be obtained from C_4 and C_5, respectively, by adding an additional edge joining two non-consecutive vertices. The present authors [13] studied $ex_{\mathcal{P}}(n, H)$ when

$H \in \{\Theta_4, \Theta_5, C_6\}$ and when H is a path on at most 9 vertices. Results from [13] are summarized below.

Theorem 2 ([13]). *Let n be a positive integer. Then*

(a) $ex_{\mathcal{P}}(n, \Theta_4) \leqslant 12(n-2)/5$ *for all* $n \geqslant 4$*, with equality when* $n \equiv 12 \pmod{20}$.
(b) $ex_{\mathcal{P}}(n, \Theta_5) \leqslant 5(n-2)/2$ *for all* $n \geqslant 5$*, with equality when* $n \equiv 50 \pmod{120}$.
(c) $ex_{\mathcal{P}}(n, C_6) \leqslant 18(n-2)/7$ *for all* $n \geqslant 6$.
(d) $ex_{\mathcal{P}}(n, P_9) \leqslant \max\{9n/4, (5n-8)/2\}$.

It seems quite non-trivial to determine $ex_{\mathcal{P}}(n, C_t)$ for all $t \geqslant 7$. In this paper, we first investigate planar graphs H satisfying $ex_{\mathcal{P}}(n, H) = 3n-6$ for all $n \geqslant |H|$. This partially answers a question of Dowden [3]. As observed in [3], for all $n \geqslant 6$, the planar triangulation $2K_1 + C_{n-2}$ is K_4-free. Hence, $ex_{\mathcal{P}}(n, H) = 3n-6$ for all graphs H which contains K_4 as a subgraph and $n \geqslant \max\{|H|, 6\}$. In particular, $ex_{\mathcal{P}}(n, K_5^-) = 3n-6$ for all $n \geqslant 6$, where K_p^- denotes the graph obtained from K_p by deleting one edge. Proposition 3 below describes several sufficient conditions on K_4-free planar graphs H such that $ex_{\mathcal{P}}(n, H) = 3n-6$ for all $n \geqslant |H|$.

Proposition 3. *Let H be a K_4-free planar graph and let $n \geqslant |H|$ be an integer. Then $ex_{\mathcal{P}}(n, H) = 3n-6$ if one of the following holds.*

(a) $\chi(H) = 4$ *and* $n \geqslant |H| + 2$.
(b) $\Delta(H) \geqslant 7$.
(c) $\Delta(H) = 6$ *and either* $n_6(H) + n_5(H) \geqslant 2$ *or* $n_6(H) + n_5(H) = 1$ *and* $n_4(H) \geqslant 5$.
(d) $\Delta(H) = 5$ *and either H has at least three* 5*-vertices or H has exactly two adjacent* 5*-vertices.*
(e) $\Delta(H) = 4$ *and* $n_4(H) \geqslant 7$.
(f) H is 3*-regular with* $|H| \geqslant 9$ *or H has at least three vertex-disjoint cycles or H has exactly one vertex u of degree* $\Delta(H) \in \{4, 5, 6\}$ *such that* $\Delta(H[N(u)]) \geqslant 3$.
(g) $\delta(H) \geqslant 4$ *or H has exactly one vertex of degree at most* 3.

Proposition 3 implies that $ex_{\mathcal{P}}(n, H) = 3n-6$ for all H with $n \geqslant |H| + 2$ and either $\chi(H) = 4$ or $\chi(H) = 3$ and $\Delta(H) \geqslant 7$. Note that $\Theta_4 = K_4^-$, and both K_4^- and $K_1 + 2K_2$ have chromatic number 3. Theorem 2(a) and Theorem 5 (see below) then demonstrate that the chromatic number of H does not play a role, as in the Erdős-Stone Theorem.

By Proposition 3, $ex_{\mathcal{P}}(n, H)$ remains unknown for K_4-free planar graphs H with $\Delta(H) = 6$, $n_6(H) + n_5(H) = 1$ and $n_4(H) \leqslant 4$; or $\Delta(H) = 5$ and $n_5(H) \leqslant 2$ (and the two 5-vertices are not adjacent when $n_5(H) = 2$); or $\Delta(H) = 4$ and $n_4(H) \leqslant 6$; or $\Delta(H) \leqslant 3$. In particular, by Proposition 3(f), $ex_{\mathcal{P}}(n, H)$ remains unknown for K_4-free planar graphs H with exactly one vertex, say u, of degree $\Delta(H) \leqslant 6$ and $\Delta(H[N(u)]) \leqslant 2$. It seems non-trivial to determine $ex_{\mathcal{P}}(n, H)$ for all such H. We next study $ex_{\mathcal{P}}(n, W_k)$, where $W_k := K_1 + C_k$ is a *wheel* on $k+1 \geqslant 5$ vertices. Unlike the classic Turán number of W_k (see [4, 5] for more information), the planar Turán number of W_k can be completely determined. We establish this in Theorem 4.

Theorem 4. *Let n, k be integers with $n \geqslant k+1 \geqslant 5$. Then*

$$ex_{\mathcal{P}}(n, W_k) = \begin{cases} 3n-6 & \textit{if } k \geqslant 6, \textit{ or } k=5 \textit{ and } n \neq 7, \textit{ or } k=4 \textit{ and } n \geqslant 12 \\ 3n-7 & \textit{if } k=4 \textit{ and } n \in \{5,6\}, \textit{ or } k=5 \textit{ and } n=7 \\ 3n-8 & \textit{if } k=4 \textit{ and } 7 \leqslant n \leqslant 11. \end{cases}$$

A graph is a (t, r)*-fan* if it is isomorphic to $K_1 + tK_{r-1}$, where $t \geqslant 2$ and $r \geqslant 2$ are integers. The classical Turán number of (t, r)-fan, namely, $ex(n, K_1 + tK_{r-1})$, has also been studied when n is sufficiently large (see [6, 2] for more information). We next study $ex_{\mathcal{P}}(n, H)$ when H is a K_4-free (t, r)-fan, in particular, when $H \in \{K_1 + 2K_2, K_{1,t}, K_1 + 3K_2\}$. Theorem 5 below establishes a sharp upper bound for $ex_{\mathcal{P}}(n, K_1 + 2K_2)$, and Theorem 6 completely determines the value of $ex_{\mathcal{P}}(n, K_{1,t})$ for all $t \geqslant 3$. However, the upper bound for $ex_{\mathcal{P}}(n, K_1 + 3K_2)$ when $n \geqslant 15$ in Theorem 7 is not tight.

Theorem 5. *Let $n \geqslant 5$ be an integer. Then*

$$2n-3 \leqslant ex_{\mathcal{P}}(n, K_1+2K_2) \leqslant \frac{19n}{8} - 4.$$

Furthermore, $ex_{\mathcal{P}}(n, K_1 + 2K_2) = \frac{19n}{8} - 4$ if and only if n is divisible by 8.

Theorem 6. *Let n, t be integers with $n \geqslant t+1 \geqslant 4$. Then*

$$ex_{\mathcal{P}}(n, K_{1,t}) = \begin{cases} 3n-6 & \textit{if } t \geqslant 7, \textit{ or } t=6 \textit{ and } n \in \{7,8,9,10,12\} \\ 3n-7 & \textit{if } t=6 \textit{ and } n=11 \\ 3n-8 & \textit{if } t=6 \textit{ and } n \in \{13,14\}, \textit{ or } t=5 \textit{ and } n=7 \\ \left\lfloor \frac{(t-1)n}{2} \right\rfloor & \textit{if } t \in \{3,4\}, \textit{ or } t=5 \textit{ and } n \neq 7, \textit{ or } t=6 \textit{ and } n \geqslant 15. \end{cases}$$

Theorem 7. *Let $n \geqslant 7$ be an integer. Then*

$$\left\lfloor \frac{5n}{2} \right\rfloor \leqslant ex_{\mathcal{P}}(n, K_1+3K_2) < \frac{17n}{6} - 4$$

for all $n \geqslant 15$ and

$$ex_{\mathcal{P}}(n, K_1+3K_2) = \begin{cases} 3n-6 & \textit{if } n \in \{7,8,9,10,12\} \\ 3n-7 & \textit{if } n=11 \\ 3n-8 & \textit{if } n \in \{13,14\}. \end{cases}$$

We need to introduce more notation. Given a plane graph G and an integer $i \geqslant 3$, an i-face in G is a face of order i. Let f_i denote the number of i-faces in G. Let T_n denote a plane triangulation on $n \geqslant 3$ vertices, and let T_n^- be obtained from T_n with one edge removed. For every integer $n \geqslant 5$, let O_n denote the unique outerplane graph with $2n-3$ edges, maximum degree 4, and the outer face of order n; let O'_n be a different drawing of O_n with one unique inner face of order n; and finally, let O_n^* be the planar triangulation obtained from O_n and O'_n by identifying the outer face of O_n with the unique n-face of O'_n in such a way that O_n^* is a simple graph. The proof of Proposition 3 relies heavily on the fact that O_n^*, $K_1 + O_{n-1}$ and $2K_1 + C_{n-2}$ are planar triangulations.

Finally, we shall make use of the following lemma in the proof of Theorem 6 and Theorem 7.

Lemma 8 ([14]). *There does not exist a 4-regular planar graph on 7 vertices, or a 5-regular planar graph on 14 vertices, or a planar graph on $n \in \{11, 13\}$ vertices with exactly one vertex of degree 4 and $n-1$ vertices of degree 5.*

2 Proof of Proposition 3

Let H and n be given as in the statement. To prove (a), assume $\chi(H) = 4$ and $n \geqslant |H|+2$. Since the planar triangulation $2K_1+C_{n-2}$ has no subgraph on $|H|$ vertices with chromatic number 4, we see that $2K_1 + C_{n-2}$ is H-free. Hence, $ex_{\mathcal{P}}(n, H) = 3n-6$ when $\chi(H) = 4$ and $n \geqslant |H| + 2$.

To prove (b), assume that $\Delta(H) \geqslant 7$. Then $n \geqslant |H| \geqslant 8$, and the planar triangulation O_n^* is H-free because $\Delta(O_n^*) = 6$. Hence, $ex_{\mathcal{P}}(n, H) = 3n-6$ for all $n \geqslant |H|$.

To prove (c), assume $\Delta(H) = 6$. Then $n \geqslant |H| \geqslant 7$. Assume first $n_6(H)+n_5(H) \geqslant 2$. Let $x, y \in V(H)$ be such that $d_H(x) = 6$ and $d_H(y) \geqslant 5$. Then the planar triangulation $K_1 + O_{n-1}$ is H-free when $xy \notin E(H)$, and the planar triangulation $2K_1 + C_{n-2}$ is H-free when $xy \in E(H)$. Hence, $ex_{\mathcal{P}}(n, H) = 3n - 6$ when $n_6(H) + n_5(H) \geqslant 2$. Next assume that $n_6(H) + n_5(H) = 1$ and $n_4(H) \geqslant 5$. Then $n_5(H) = 0$. Note that the planar triangulation $2K_1 + C_{n-2}$ is H-free when $n_4(H) \geqslant 6$. We may further assume that $n_4(H) = 5$. Let $u, v_1, \ldots, v_5 \in V(H)$ with $d_H(u) = 6$ and $d_H(v_i) = 4$ for all $i \in [5]$. Let $M := H[\{u, v_1, \ldots, v_5\}]$. We may assume that $uv_1 \notin E(M)$ when $d_M(u) = 4$. Then the planar triangulation $2K_1 + C_{n-2}$ is H-free when $d_M(u) = 5$, or $d_M(u) \leqslant 3$, or $d_M(u) = 4$ and $d_M(v_1) \leqslant 3$; and the planar triangulation O_n^* is H-free when $N_M(u) = N_M(v_1) = \{v_2, v_3, v_4, v_5\}$. It follows that $ex_{\mathcal{P}}(n, H) = 3n-6$ for all $n \geqslant |H|$. This proves (c).

To prove (d), assume $\Delta(H) = 5$ and $n_5(H) \geqslant 2$. Then $n \geqslant |H| \geqslant 6$. Let u, v be two distinct 5-vertices in H. Then either $n_5(H) \geqslant 3$ or $n_5(H) = 2$ with $uv \in E(H)$. Note that the planar triangulation $2K_1 + C_{n-2}$ has exactly two non-adjacent vertices of degree at least 5 when $n \geqslant 7$ and has maximum degree 4 when $n = 6$. Hence, $2K_1 + C_{n-2}$ is H-free, and so $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geqslant |H|$. This proves (d).

To prove (e) and (f). Assume $\Delta(H) = 4$ and $n_4(H) \geqslant 7$, or H is 3-regular with $|H| \geqslant 9$, or H has at least three vertex-disjoint cycles, or H has exactly one vertex u of degree $\Delta(H) \in \{4, 5, 6\}$ such that $\Delta(H[N(u)]) \geqslant 3$. Then the planar triangulation $2K_1 + C_{n-2}$ is H-free. Hence, $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geqslant |H|$.

It remains to prove (g). Assume $\delta(H) \geqslant 4$ or H has exactly one vertex of degree at most 3. Then $n \geqslant |H| \geqslant 5$. Note that the planar triangulation $K_1 + O_{n-1}$ is 3-degenerate and every subgraph of $K_1 + O_{n-1}$ has at least two vertices of degree at most 3, because every subgraph of O_{n-1} has at least two vertices of degree at most 2. Hence, $K_1 + O_{n-1}$ is H-free, and so $ex_{\mathcal{P}}(n, H) = 3n - 6$ for all $n \geqslant |H|$.

This completes the proof of Proposition 3. □

3 Proof of Theorem 4

Let n, k be given as in the statement. Assume $k \geqslant 7$. By Proposition 3(b), $ex_{\mathcal{P}}(n, W_k) = 3n-6$ for all $n \geqslant k+1$. Assume next $k \in \{5, 6\}$. Since the planar triangulation $2K_1+C_{n-2}$ is W_k-free when $n \geqslant k+3$ or $n = k+1$, we see that $ex_{\mathcal{P}}(n, W_k) = 3n-6$ when $n \geqslant k+3$ or $n = k+1$. We next determine $ex_{\mathcal{P}}(n, W_k)$ when $n = k+2$. For $k = 6$ and $n = 8$, the plane triangulation on 8 vertices depicted in Figure 1(a) is W_6-free and so $ex_{\mathcal{P}}(n, W_6) = 3n - 6$ when $n = 8$. For $k = 5$ and $n = 7$, note that the plane graph with 7 vertices and 14 edges

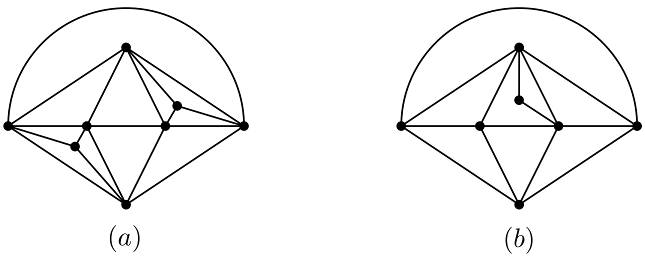


Figure 1: W_k-free plane graphs with $n = k + 2$ vertices and $3n - 12 + k$ edges, where $k \in \{5, 6\}$.

given in Figure 1(b) is W_5-free. Thus, $ex_{\mathcal{P}}(7, W_5) \geqslant 3 \cdot 7 - 6 - 1 = 14$. On the other hand, all plane triangulations on seven vertices[1] are depicted in Figure 2, each containing a copy of W_5. Hence, $ex_{\mathcal{P}}(n, W_5) = 3n - 7$ when $n = 7$.

It remains to consider the case when $k = 4$. To show $ex_{\mathcal{P}}(n, W_4) = 3n-6$ for all $n \geqslant 12$, assume first that $n = 5t + 2$ for some integer $t \geqslant 2$. Let L_t be a plane triangulation on $n = 5t + 2$ vertices constructed as follows: for each $i \in [t]$, let C^i be a cycle with vertices $u_{i,1}, u_{i,2}, \ldots, u_{i,5}$ in order. Let L_t be the plane triangulation obtained from disjoint union of $C^1, \ldots, C^t$ by adding edges $u_{i,j}u_{i+1,j}$ and $u_{i,j}u_{i+1,j+1}$ for all $i \in [t-1]$ and $j \in [5]$, where

[1]To find all plane triangulations on 7 vertices, let H be a plane triangulation on 7 vertices. Then $e(H) = 15$ and H must be 3-connected with maximum degree 5 or 6. Let $u \in V(H)$ be a vertex of maximum degree. If $d(u) = 6$, then H has at most two vertices of degree 6 and $H - u$ has neither K_4 nor $K_{2,3}$ minor. Thus $H - u$ must be outer planar and so H is isomorphic to one of the plane triangulations in Figure 2(a,b,e). If $d(u) = 5$, then the degree sequence of H is either $(5, 5, 5, 4, 4, 4, 3)$ or $(5, 5, 4, 4, 4, 4, 4)$. Since H has no $K_{3,3}$ minor, one can then check that H is isomorphic to one of the plane triangulations in Figure 2(c,d).

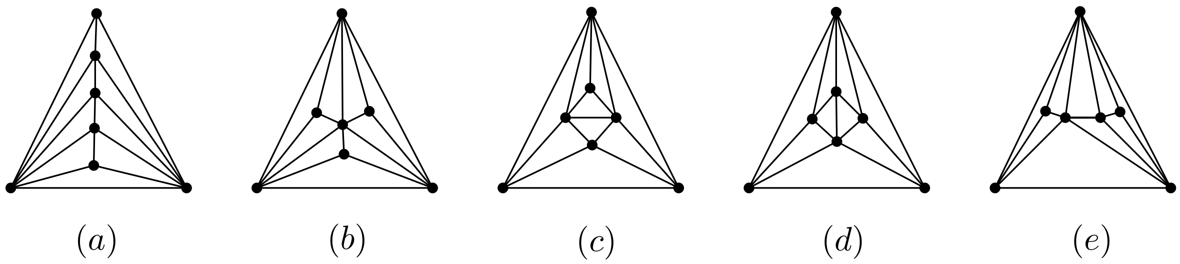


Figure 2: All plane triangulations on 7 vertices. Furthermore, each T_7 has a copy of W_4 and W_5, and each T_7^- has a copy of W_4.

all arithmetic on the index $j+1$ here is done modulo 5, and finally adding two new non-adjacent vertices u and v such that u is adjacent to all vertices of C^1 and v is adjacent to all vertices of C^t. The graph L_t when $t = 3$ is depicted in Figure 3. It is worth noting that L_t is K_4-free, $d_{L_t}(u) = d_{L_t}(v) = 5$, $d_{L_t}(u_{1,j}) = d_{L_t}(u_{t,j}) = 5$, $d_{L_t}(u_{i,j}) = 6$ for $2 \leqslant i \leqslant t-1$ and $j \in [5]$. Furthermore, the subgraph induced by the neighborhood of each vertex in L_t is isomorphic to either C_5 or C_6. Hence, L_t is W_4-free and so $ex_{\mathcal{P}}(n, W_4) = 3n - 6$ when $n = 5t + 2$ for some integer $t \geqslant 2$.

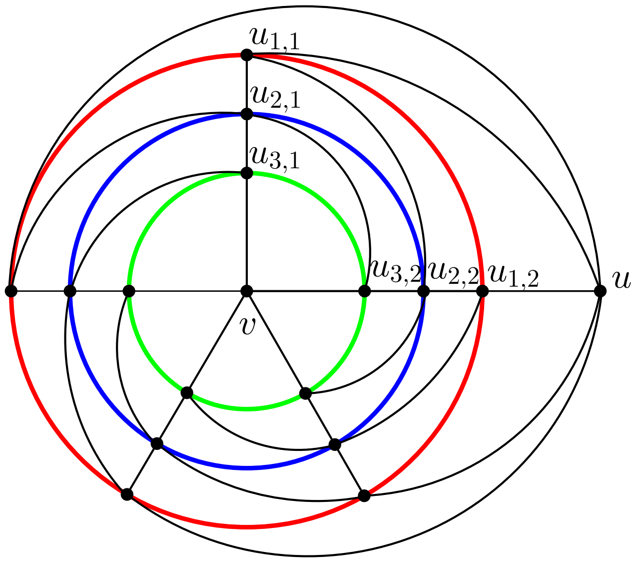


Figure 3: The plane triangulation L_t when $t = 3$, where C^1, C^2 and C^3 are in red, blue and green, respectively.

Next assume that $n = 5t + 2 + i$ for some $i \in [4]$, where $t \geqslant 2$ is an integer. Note that the plane triangulation L_t on $5t + 2$ vertices constructed above contains at least four pairwise vertex-disjoint faces. Let $F_1, \ldots, F_i$ be any i pairwise vertex-disjoint faces of L_t, and let L_t^i be the plane triangulation obtained from L_t by adding one vertex, say x_j, of degree 3 to each F_j for all $j \in [i]$. Clearly, L_t^i is a plane triangulation on $n = 5t + 2 + i$

vertices. By the choice of $F_1, \dots, F_i$, we see that $x_1, \dots, x_i$ are pairwise non-adjacent in L_t^i, and no two of $x_1, \dots, x_i$ have common neighbors in L_t^i. We next show that L_t^i is W_4-free for all $i \in [4]$. Suppose that L_t^i contains a copy of W_4 for some $i \in [4]$. Let H be a W_4 in L_t^i. Then H must contain exactly one, say x_1, of $x_1, \dots, x_i$, because L_t is W_4-free, and no two of $x_1, \dots, x_i$ are adjacent or have common neighbors in L_t^i. Let $y, z \in V(H)$ be the two neighbors of x_1 such that $yz \notin E(H)$. By the choice of x_1, we see that $yz \in E(L_t)$. But then $L_t[V(H \setminus x_1)] = K_4$ and so L_t contains K_4 as a subgraph, a contradiction. Therefore, $ex_{\mathcal{P}}(n, W_4) = 3n - 6$ for all $n \geqslant 12$.

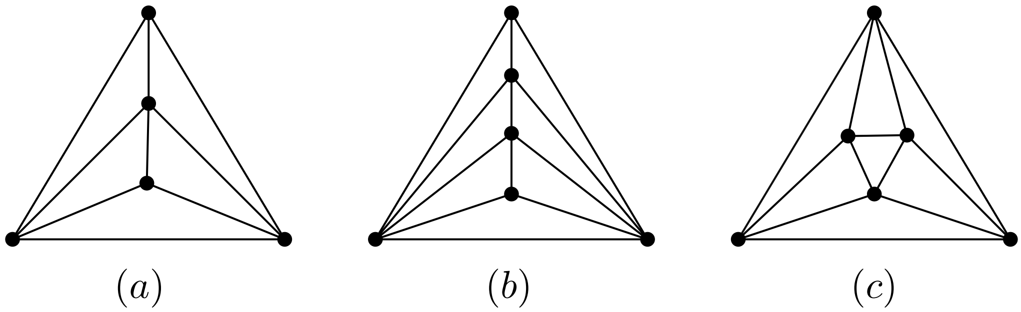


Figure 4: All plane triangulations on 5 and 6 vertices.

We next show that $ex_{\mathcal{P}}(n, W_4) = 3n - 7$ when $n \in \{5, 6\}$. Note that all plane triangulations on $n \in \{5, 6\}$ vertices are depicted in Figure 4, each containing a copy of W_4. Thus, $ex_{\mathcal{P}}(n, W_4) \leqslant 3n - 7$. On the other hand, for all $n \in \{5, 6\}$, the planar graph $K_2 + (K_2 \cup K_{n-4})$ has $3n - 7$ edges and is W_4-free. Hence, $ex_{\mathcal{P}}(n, W_4) = 3n - 7$ when $n \in \{5, 6\}$.

Finally, we show that $ex_{\mathcal{P}}(n, W_4) = 3n - 8$ for all $n \in \{7, 8, 9, 10, 11\}$. The plane graph J, given in Figure 5, is W_4-free with $n = 11$ vertices and $3n - 8$ edges. Let B be the set of

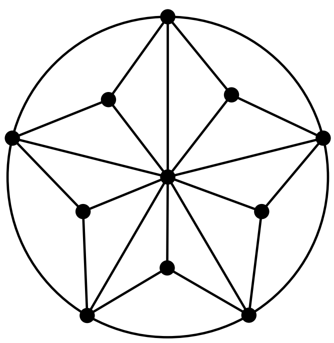


Figure 5: Graph J.

all vertices of degree 3 in J. Then $|B| = 5$. For each $n \in \{7, 8, 9, 10\}$, let J_n be the plane graph obtained from J by deleting $11 - n$ vertices in B. Then J_n is an induced subgraph of

J. Clearly, J_n is W_4-free with n vertices and $3n-8$ edges. Hence, $ex_{\mathcal{P}}(n, W_4) \geqslant 3n-8$ for all $n \in \{7, 8, 9, 10, 11\}$. We next show that $ex_{\mathcal{P}}(n, W_4) \leqslant 3n-8$ for all $n \in \{7, 8, 9, 10, 11\}$. Suppose this is not true. Let G be a W_4-free planar graph on $n \in \{7, 8, 9, 10, 11\}$ vertices with $e(G) \geqslant 3n-7$. We choose such a G with n minimum. Then $G = T_n$ or $G = T_n^-$. Since each T_7, depicted in Figure 2, contains a copy of W_4, and each T_7^- also contains a copy of W_4, it follows that $n \in \{8, 9, 10, 11\}$. Let $u \in V(G)$ with $d_G(u) = \delta(G)$. Then $\delta(G) \leqslant 4$, else $e(G) \geqslant \frac{5n}{2} > 3n-6$ because $n \leqslant 11$, a contradiction. Next, if $\delta(G) \leqslant 3$, then $e(G \setminus u) \leqslant 3(n-1)-8$ by minimality of n and the fact that $ex_{\mathcal{P}}(n, W_4) \leqslant 3n-8$ when $n = 7$. Thus, $e(G) = e(G \setminus u) + d_G(u) \leqslant 3(n-1)-8+3 = 3n-8$, a contradiction. This proves that $\delta(G) = 4$. Since $N_G[u]$ does not contain a copy of W_4 in G, we see that $G \neq T_n$. Thus $G = T_n^-$. We may assume that G is a plane drawing of T_n^- such that the outer face is a 3-face. Let $x_1, y_1 \in V(G)$ be such that $G + x_1y_1 = T_n$. Then x_1 and y_1 must lie on the boundary of the unique 4-face, say F, in G. Let x_1, x_2, y_1, y_2 be the vertices on the boundary of F in order. Then $d_G(v) \geqslant 5$ for all $v \in V(G) \setminus \{x_1, x_2, y_1, y_2\}$, because $G = T_n^-$ and $N_G[u]$ does not contain a copy of W_4 in G for any $u \in V(G)$ with $d_G(u) = 4$. Thus $2(3n-7) = 2e(G) \geqslant 4 \cdot 4 + 5 \cdot (n-4)$, which implies that $n \in \{10, 11\}$. Suppose each vertex in $\{x_1, x_2, y_1, y_2\}$ has degree 4 in G. Since $G = T_n^-$, there must exist four distinct vertices $z_1, z_2, z_3, z_4 \in V(G) \setminus \{x_1, x_2, y_1, y_2\}$ such that $G[A]$ is isomorphic to the graph given in Figure 6(a), where $A = \{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4\}$. But then G contains $K_{3,3}$ as

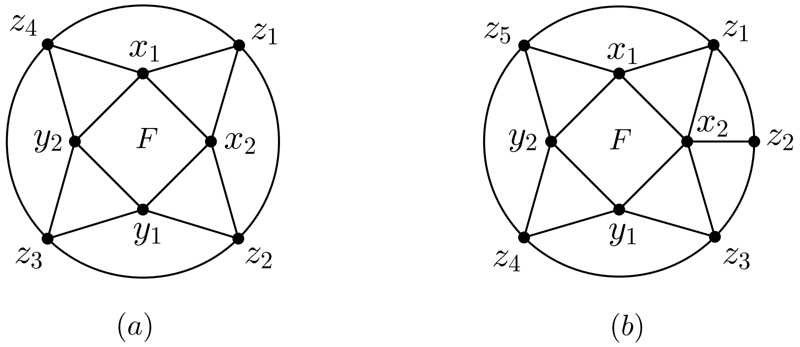


Figure 6: The graph $G[A]$.

a minor, because $n \in \{10, 11\}$ and $d_G(v) \geqslant 5$ for any $v \in V(G) \setminus A$. Thus, we may assume that $d_G(x_2) \geqslant 5$. Then $2(3n-7) = 2e(G) \geqslant 4 \cdot 3 + 5 \cdot (n-3)$, which implies that $n = 11$, $d_G(v) = 4$ for all $v \in \{x_1, y_1, y_2\}$ and $\Delta(G) = 5$. Thus there exist five distinct vertices $z_1, z_2, z_3, z_4, z_5 \in V(G) \setminus \{x_1, x_2, y_1, y_2\}$ such that $G[A]$ is isomorphic to the graph given in Figure 6(b), where $A = \{x_1, x_2, y_1, y_2, z_1, z_2, z_3, z_4, z_5\}$. But then $e_G(A, V(G) \setminus A) = 6$, contrary to $e_G(V(G) \setminus A, A) \geqslant 8$ because $n = 11$ and $d_G(v) = 5$ for any $v \in V(G) \setminus A$.

This completes the proof of Theorem 4. □

4 Proof of Theorem 5

To establish the desired lower bound, note that the planar graph $K_2 + (n-2)K_1$ is $(K_1 + 2K_2)$-free for all $n \geqslant 5$. Hence, $ex_{\mathcal{P}}(n, K_1 + 2K_2) \geqslant 2n - 3$ for all $n \geqslant 5$. In particular, $ex_{\mathcal{P}}(5, K_1 + 2K_2) \geqslant 7$. We next show that every $(K_1 + 2K_2)$-free planar graph on $n \geqslant 5$ vertices has at most $19n/8 - 4$ edges. We proceed the proof by induction on n. Assume first $n = 5$. Then $ex_{\mathcal{P}}(5, K_1 + 2K_2) = 7$, because the only plane triangulation on five vertices, given in Figure 4(a), is not $(K_1 + 2K_2)$-free, and any T_5^- is not $(K_1 + 2K_2)$-free. Hence, $ex_{\mathcal{P}}(n, K_1 + 2K_2) = 7 < 19n/8 - 4$ when $n = 5$. So we may assume that $n \geqslant 6$. Let G be a $(K_1 + 2K_2)$-free plane graph on $n \geqslant 6$ vertices. Assume there exists a vertex $u \in V(G)$ with $d_G(u) \leqslant 2$. By the induction hypothesis, $e(G \setminus u) \leqslant 19(n-1)/8 - 4$ and so $e(G) = e(G \setminus u) + d_G(u) \leqslant 19(n-1)/8 - 4 + 2 < 19n/8 - 4$, as desired. So we may assume that $\delta(G) \geqslant 3$. Next, assume G is disconnected. Then each component of G either is isomorphic to K_4 or has at least six vertices because $\delta(G) \geqslant 3$. Let $G_1, \ldots, G_p, G_{p+1}, \ldots, G_{p+q}$ be all components of G such that $|G_1| = \cdots = |G_p| = 4$ and $6 \leqslant |G_{p+1}| \leqslant \cdots \leqslant |G_{p+q}|$, where $p \geqslant 0$ and $q \geqslant 0$ are integers with $p + q \geqslant 2$ and $|G_{p+1}| + \cdots + |G_{p+q}| = n - 4p$. Then $e(G_i) = 6$ for all $i \in [p]$, and $e(G_j) \leqslant \frac{19|G_j|}{8} - 4$ for all $j \in \{p+1, \ldots, p+q\}$ by the induction hypothesis. Therefore,

$$\begin{aligned} e(G) &\leqslant 6p + \frac{19(|G_{p+1}| + \cdots + |G_{p+q}|)}{8} - 4q \\ &= \frac{19n}{8} - \frac{7p}{2} - 4q \\ &\leqslant \frac{19n}{8} - \frac{7(p+q)}{2} < \frac{19n}{8} - 4, \end{aligned}$$

as desired. So we may further assume that G is connected. Then G has no faces of size at most two. Hence,

$$2e(G) = 3f_3 + \sum_{i \geqslant 4} if_i \geqslant 3f_3 + 4(f - f_3) = 4f - f_3,$$

which implies that

$$f \leqslant e(G)/2 + f_3/4. \tag{1}$$

Note that each 3-vertex is incident with at most three distinct 3-faces in G. Furthermore, since G is $(K_1 + 2K_2)$-free, we see that for all $j \geqslant 4$, each j-vertex is incident with at most two distinct 3-faces in G. Let $U \subseteq V(G)$ denote the set of 3-vertices each incident with exactly three distinct 3-faces in G. Then U must be an independent set in G because G is connected. Furthermore, no two vertices in U have a common neighbor in G, because G is $(K_1 + 2K_2)$-free. Thus, $4|U| \leqslant n$ and so $|U| \leqslant n/4$. It follows that

$$3f_3 \leqslant 3|U| + 2(n - |U|) = 2n + |U| \leqslant 9n/4, \tag{2}$$

which implies that $f_3 \leqslant 3n/4$. This, together with (1), further implies that $f \leqslant e(G)/2 + 3n/16$. By Euler's formula, $n - 2 = e(G) - f \geqslant e(G)/2 - 3n/16$. Hence, $e(G) \leqslant 19n/8 - 4$, as desired.

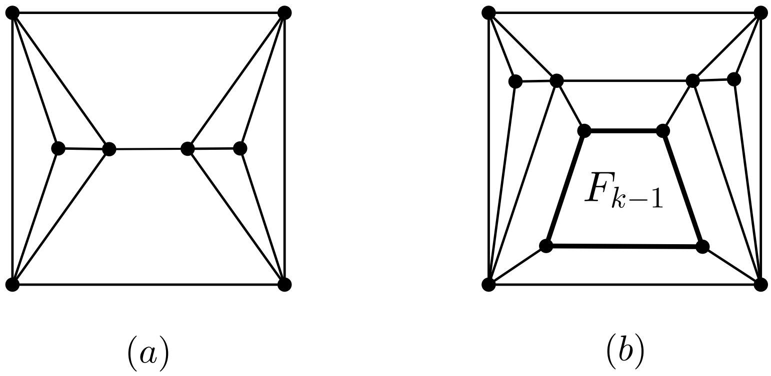


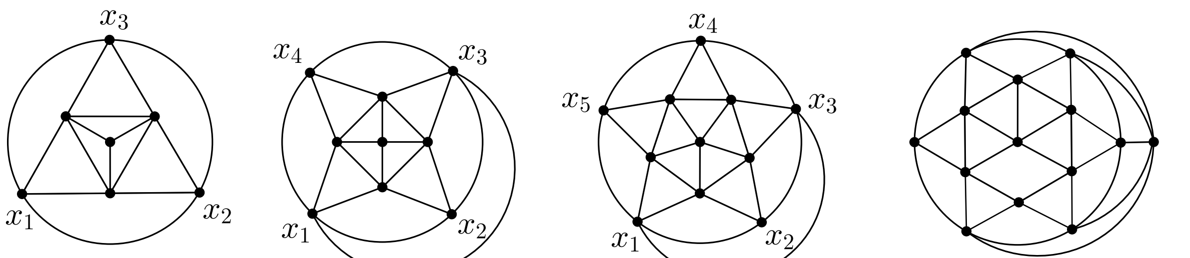
Figure 7: The construction of F_k.

From the proof above, we see that equality in $e(G) \leqslant 19n/8 - 4$ is achieved for n if and only if equalities hold in both (1) and (2), and in $4|U| \leqslant n$. This implies that $e(G) = 19n/8 - 4$ for n if and only if G is a connected, $(K_1 + 2K_2)$-free plane graph on n vertices satisfying: $\delta(G) \geqslant 3$; each 3-vertex in G is incident with exactly three distinct 3-faces; each vertex of degree at least 4 in G is incident with exactly two distinct 3-faces; each face is either a 3-face or a 4-face. We next construct such an extremal plane graph for n and $K_1 + 2K_2$. Let $n = 8(k + 1)$ for some integer $k \geqslant 0$. Let F_0 be the graph depicted in Figure 7(a), we then construct F_k of order n recursively for all $k \geqslant 1$ via the illustration given in Figure 7(b): the entire graph F_{k-1} is placed into the center quadrangle of Figure 7(b) (in such a way that the center bold quadrangle of Figure 7(b) is identified with the outer quadrangle of F_{k-1}). One can check that F_k is $(K_1 + 2K_2)$-free with $n = 8(k + 1)$ vertices and $19n/8 - 4$ edges for all $k \geqslant 0$.

This completes the proof of Theorem 5. $\square$

5 Proof of Theorem 6

By Proposition 3(b), $ex_{\mathcal{P}}(n, K_{1,t}) = 3n - 6$ for all $n \geqslant t + 1 \geqslant 8$. So we may assume that $t \leqslant 6$. We next show that $ex_{\mathcal{P}}(n, K_{1,6}) = 3n - 6$ for all $n \in \{7, 8, 9, 10, 12\}$. Let J_a, J_b, J_c be the plane graphs given in Figure 8. Let J'_a be the plane triangulation obtained from J_a by adding a new vertex adjacent to x_1, x_2, x_3, J'_b be the plane triangulation obtained from J_b by first deleting the edge x_1x_3 and then adding a new vertex adjacent to x_1, x_2, x_3, x_4, and J'_c be the plane triangulation obtained from J_c by first deleting the edge x_1x_3 and then adding one new vertex adjacent to x_1, x_2, x_3, x_4, x_5. Then the plane triangulations J_a, J'_a, J_b, J'_b and J'_c are $K_{1,6}$-free because each of them has maximum degree 5. Hence, $ex_{\mathcal{P}}(n, K_{1,6}) = 3n - 6$ for all $n \in \{7, 8, 9, 10, 12\}$. By Lemma 8, no plane triangulation on 11 vertices has maximum degree at most 5. Hence, every plane triangulation on $n \in \{11, 13, 14\}$ vertices has maximum degree at least 6. This implies that $ex_{\mathcal{P}}(n, K_{1,6}) \leqslant 3n - 7$ for all $n \in \{11, 13, 14\}$. Since J_c given in Figure 8 is a $K_{1,6}$-free plane graph with $n = 11$ vertices and $3n - 7$ edges, we have $ex_{\mathcal{P}}(n, K_{1,6}) = 3n - 7$ when



J_a J_b J_c J_d

Figure 8: The graphs J_a, J_b, J_c and J_d.

$n = 11$. By Lemma 8, there does not exist any planar graphs on $n \in \{13, 14\}$ vertices with $3n-7$ edges and maximum degree at most 5. It follows that $ex_{\mathcal{P}}(n, K_{1,6}) \leqslant 3n-8$ when $n \in \{13, 14\}$. Let J''_c be the plane graph obtained from J_c by first deleting the edge x_1x_3 and then adding two new adjacent vertices y_1, y_2 such that y_1 is adjacent to x_1, x_2, x_3 and y_2 is adjacent to x_4, x_5. Then J''_c and the graph J_d given in Figure 8 are $K_{1,6}$-free plane graph with $n \in \{13, 14\}$ vertices and $3n-8$ edges. Hence, $ex_{\mathcal{P}}(n, K_{1,6}) = 3n-8$ when $n \in \{13, 14\}$.

It is easy to see that $ex_{\mathcal{P}}(n, K_{1,3}) = n$ for all $n \geqslant 4$, because every $K_{1,3}$-free planar graph has maximum degree at most 2 and the planar graph C_n is $K_{1,3}$-free with n edges. We next show that $ex_{\mathcal{P}}(n, K_{1,4}) = \lfloor 3n/2 \rfloor$ for all $n \geqslant 5$. Clearly, $ex_{\mathcal{P}}(n, K_{1,4}) \leqslant \lfloor 3n/2 \rfloor$ for all $n \geqslant 5$, because every $K_{1,4}$-free planar graph has maximum degree at most 3. Next, for all $n \geqslant 5$, the planar graph obtained from C_n by adding a matching of size $\lfloor n/2 \rfloor$ is $K_{1,4}$-free with $\lfloor 3n/2 \rfloor$ edges. Hence, $ex_{\mathcal{P}}(n, K_{1,4}) = \lfloor 3n/2 \rfloor$ for all $n \geqslant 5$. To determine $ex_{\mathcal{P}}(n, K_{1,5})$ for all $n \geqslant 6$, since every $K_{1,5}$-free planar graph on $n \geqslant 6$ vertices has maximum degree at most 4, we have $ex_{\mathcal{P}}(n, K_{1,5}) \leqslant 2n$ for all $n \geqslant 6$. Let J''_a be the plane triangulation obtained from J_a by deleting the unique 3-vertex. Since J''_a is $K_{1,5}$-free plane graph on $n = 6$ vertices with $2n$ edges, we have $ex_{\mathcal{P}}(n, K_{1,5}) = 2n$ when $n = 6$. By Lemma 8, no planar graph with $n = 7$ vertices and $2n$ edges has maximum degree at most 4. Hence, $ex_{\mathcal{P}}(n, K_{1,5}) \leqslant 2n-1$ when $n = 7$. Let J'''_a be the plane graph obtained from J''_a by first deleting the edge x_1x_2 and then adding a new vertex adjacent to x_1, x_2 only. Note that J'''_a is a $K_{1,5}$-free plane graph on $n = 7$ vertices with $2n-1$ edges, we see that $ex_{\mathcal{P}}(n, K_{1,5}) = 2n-1$ when $n = 7$. Next, for all $n \geqslant 8$, let C be a cycle on $2\lfloor n/2 \rfloor$ vertices with vertices $u_1, \ldots, u_{\lfloor \frac{n}{2} \rfloor}, w_{\lfloor \frac{n}{2} \rfloor}, \ldots, w_1$ in order. Let H be the plane graph obtained from C by adding the path with vertices $w_1, u_2, w_2, u_3, \ldots, w_{\lfloor \frac{n}{2} \rfloor - 1}, u_{\lfloor \frac{n}{2} \rfloor}$ in order. When n is even, the planar graph $H + u_1u_{\lfloor \frac{n}{2} \rfloor} + u_1w_{\lfloor \frac{n}{2} \rfloor} + w_1w_{\lfloor \frac{n}{2} \rfloor}$ is $K_{1,5}$-free with $2n$ edges. When n is odd, let H' be obtained from H by first deleting the edge u_2u_3 and then adding a new vertex u adjacent to u_2 and u_3. Then the planar graph $H' + uu_1 + uu_{\lfloor \frac{n}{2} \rfloor} + w_1w_{\lfloor \frac{n}{2} \rfloor} + u_1w_{\lfloor \frac{n}{2} \rfloor}$ is $K_{1,5}$-free with $2n$ edges. Hence, $ex_{\mathcal{P}}(n, K_{1,5}) = 2n$ for all $n \geqslant 8$.

It remains to show that $ex_{\mathcal{P}}(n, K_{1,6}) = \lfloor 5n/2 \rfloor$ for all $n \geqslant 15$. Clearly, $ex_{\mathcal{P}}(n, K_{1,6}) \leqslant \lfloor 5n/2 \rfloor$ for all $n \geqslant 15$, because every $K_{1,6}$-free planar graph on $n \geqslant 15$ vertices has maximum degree at most 5. We next show that $ex_{\mathcal{P}}(n, K_{1,6}) \geqslant \lfloor 5n/2 \rfloor$ for all $n \geqslant 15$.

Let $n := 4q + r \geqslant 15$, where $q \geqslant 3$ and $r \in \{0, 1, 2, 3\}$. Let $p \in \{q, q + 1\}$. Let C^1 and C^2 be two vertex-disjoint cycles with vertices $x_1, x_2, \ldots x_q$ in order and $y_1, y_2, \ldots y_p$ in order, respectively. Let C^3 be a cycle of length $q+p$ with vertices $b_1, a_1, b_2, a_2, \ldots b_q, a_q$ in order when $p = q$, and $b_1, a_1, b_2, a_2, \ldots b_q, a_q, b_{q+1}$ in order when $p = q + 1$. Let R_p be the plane graph on $2q + 2p$ vertices obtained from disjoint copies of C^1, C^2 and C^3 by making x_i adjacent to $\{a_i, b_i, b_{i+1}\}$, and y_j adjacent to $\{b_j, a_{j-1}, a_j\}$ for all $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant p$, where all arithmetic on the indices $i + 1$ and $j - 1$ here are done modulo

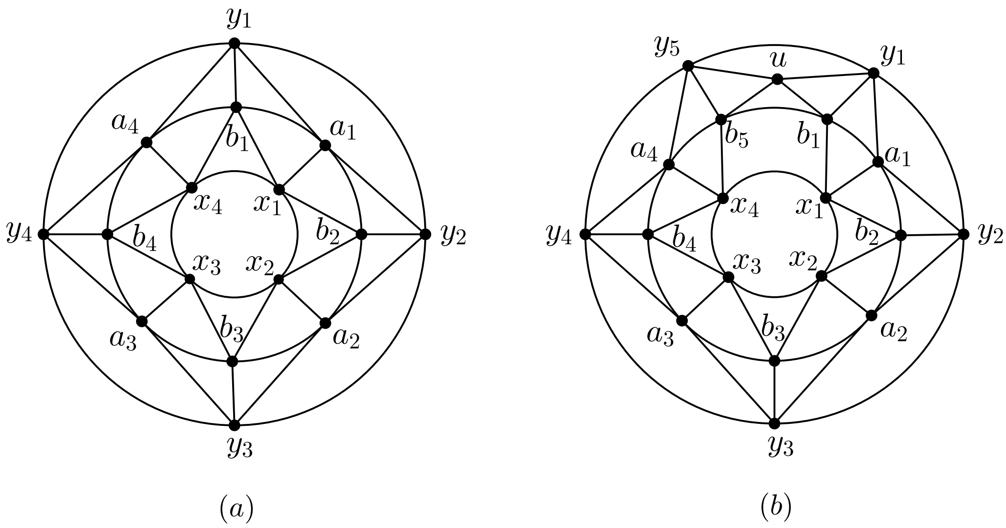


Figure 9: Almost 5-regular plane graphs on $4q + r$ vertices when $q = 4$ and $r \in \{0, 3\}$.

p. Then R_p is $K_{1,6}$-free planar graph with $10q$ edges when $p = q$ and $10q + 3$ edges when $p = q + 1$. The construction of R_p on $4q$ vertices when $q = 4$ is depicted in Figure 9(a). When $n = 4q$ for $q \geqslant 4$, the planar graph R_p with $p = q$ is $K_{1,6}$-free with $10q$ edges and so $ex_{\mathcal{P}}(n, K_{1,6}) = 10q = 5n/2$. When $n = 4q + 1$ for $q \geqslant 4$, let R^1 be obtained from $R_p - y_2y_3 - y_1y_p$ with $p = q$ by adding a new vertex u adjacent to y_2 and y_3. Then the planar graph $R^1 + uy_1 + uy_p$ is $K_{1,6}$-free with $10q + 2$ edges. Hence, $ex_{\mathcal{P}}(n, K_{1,6}) = 10q + 2 = \lfloor 5n/2 \rfloor$ when $n = 4q + 1$ for $q \geqslant 4$. When $n = 4q + 2$ for $q \geqslant 4$, let R^2 be obtained from $R_p - y_2y_3 - y_1y_p - x_2x_3 - x_1x_p - b_1a_p$ with $p = q$ by adding two new vertices u adjacent to y_2 and y_3 and v adjacent to x_2 and x_3. Then the planar graph $R^2 + uy_1 + uy_p + ua_q + vx_1 + vx_q + vb_1$ is $K_{1,6}$-free with $10q + 5$ edges. Hence, $ex_{\mathcal{P}}(n, K_{1,6}) = 10q + 5 = 5n/2$ when $n = 4q + 2$ for $q \geqslant 4$. When $n = 4q + 3$ for $n \geqslant 3$, the planar graph obtained from R_p with $p = q + 1$ by adding a new vertex u adjacent to y_1, b_1, y_p, b_p, given in Figure 9(b) when $q = 4$, is $K_{1,6}$-free with $10q + 7$ edges. Hence, $ex_{\mathcal{P}}(n, K_{1,6}) = 10q + 7 = \lfloor 5n/2 \rfloor$ when $n = 4q + 3$ for $q \geqslant 3$.

This completes the proof of Theorem 6. □

6 Proof of Theorem 7

Since the plane triangulations J_a, J'_a, J_b and J'_b constructed in the proof of Theorem 6 is (K_1+3K_2)-free, we see that $ex_{\mathcal{P}}(n, K_1+3K_2) = 3n-6$ for all $n \in \{7, 8, 9, 10\}$. To determine $ex_{\mathcal{P}}(11, K_1+3K_2)$, note that the plane graph J_c given in Figure 8 with $n = 11$ vertices and $3n-7$ edges is (K_1+3K_2)-free. Thus $ex_{\mathcal{P}}(n, K_1+3K_2) \geqslant 3n-7$ when $n = 11$. By Lemma 8, no plane triangulation on 11 vertices has maximum degree at most 5. Hence, every plane triangulation on 11 vertices must contain a vertex of degree at least 6 (and so contains a copy of K_1+3K_2), which implies that $ex_{\mathcal{P}}(n, K_1+3K_2) = 3n-7$ when $n = 11$. Since every $K_{1,6}$-free graph is certainly (K_1+3K_2)-free, by Theorem 6, $ex_{\mathcal{P}}(n, K_1+3K_2) = ex_{\mathcal{P}}(n, K_{1,6}) = 3n-6$ when $n = 12$, $ex_{\mathcal{P}}(n, K_1+3K_2) \geqslant ex_{\mathcal{P}}(n, K_{1,6}) = 3n-8$ when $n \in \{13, 14\}$, and $ex_{\mathcal{P}}(n, K_1+3K_2) \geqslant ex_{\mathcal{P}}(n, K_{1,6}) = \lfloor 5n/2 \rfloor$ when $n \geqslant 15$. Since every plane triangulation on $n \in \{13, 14\}$ vertices has maximum degree at least 6, we see that $ex_{\mathcal{P}}(n, K_1+3K_2) \leqslant 3n-7$ when $n \in \{13, 14\}$. By Lemma 8, every T_n^- with $n \in \{13, 14\}$ has maximum degree at least 6 and so contains a copy of K_1+3K_2. It follows that $ex_{\mathcal{P}}(n, K_1+3K_2) = 3n-8 < 17n/6-4$ when $n \in \{13, 14\}$.

We next show that every (K_1+3K_2)-free planar graph G on $n \geqslant 13$ vertices has at most $17n/6-4$ edges. We proceed the proof by induction on n. This is trivially true when $n \in \{13, 14\}$. So we may assume that $n \geqslant 15$. Assume there exists a vertex $u \in V(G)$ with $d_G(u) \leqslant 2$. By the induction hypothesis, $e(G \setminus u) \leqslant 17(n-1)/6-4$ and so $e(G) = e(G \setminus u) + d_G(u) \leqslant 17(n-1)/6-4+2 < 17n/6-4$, as desired. So we may assume that $\delta(G) \geqslant 3$. Assume next that G is disconnected. Let $G_1, \dots, G_p, G_{p+1}, \dots, G_{p+q}$ be all components of G such that $|G_1| \leqslant \cdots \leqslant |G_p| \leqslant 12$ and $13 \leqslant |G_{p+1}| \leqslant \cdots \leqslant |G_{p+q}|$, where $p \geqslant 0$ and $q \geqslant 0$ are integers with $p+q \geqslant 2$ and $|G_1| + \cdots + |G_{p+q}| = n$. Then $e(G_i) \leqslant 3|G_i|-6$ for all $i \in [p]$, and $e(G_j) \leqslant 17|G_j|/6-4$ for all $j \in \{p+1, \dots, p+q\}$ by the induction hypothesis. Therefore,

$$\begin{aligned} e(G) &\leqslant 3(|G_1| + \cdots + |G_p|) - 6p + \frac{17(|G_{p+1}| + \cdots + |G_{p+q}|)}{6} - 4q \\ &= \frac{17n}{6} - (6p+4q) + \frac{|G_1| + \cdots + |G_p|}{6} \\ &\leqslant \frac{17n}{6} - (6p+4q) + 2p = \frac{17n}{6} - 4(p+q) < \frac{17n}{6} - 4, \end{aligned}$$

as desired. So we may further assume that G is connected. Then G has no faces of size at most two. Hence,

$$2e(G) = 3f_3 + \sum_{i \geqslant 4} i f_i \geqslant 3f_3 + 4(f - f_3) = 4f - f_3,$$

which implies that $f \leqslant e(G)/2 + f_3/4$. Note that $n_3(G) \geqslant 0$ and $n_5(G) < n$; and for all $i \in \{3, 4, 5\}$, each i-vertex is incident with at most i 3-faces. Furthermore, for all $j \geqslant 6$, each j-vertex is incident with at most four 3-faces because G is (K_1+3K_2)-free and $n \geqslant 15$. It follows that

$$3f_3 \leqslant 3n_3(G) + 4n_4(G) + 5n_5(G) + 4(n - n_3(G) - n_4(G) - n_5(G)) = 4n - n_3(G) + n_5(G) < 5n,$$

which implies that $f_3 < 5n/3$. This, together with the fact that $f \leqslant e(G)/2+f_3/4$, further implies that $f < e(G)/2+5n/12$. By Euler's formula, $n-2 = e(G)-f > e(G)/2-5n/12$. Hence, $e(G) < 17n/6-4$.

This completes the proof of Theorem 7. □

7 Concluding remarks

The lower bound in Theorem 7 can be further improved when n is divisible by 24. To see this, let $n = 24(k+1)$ for some integer $k \geqslant 0$. Let R_5 be the 5-regular plane graph on twelve vertices given in Figure 10(a), and let G_0 be the plane graph obtained from two disjoint copies of R_5 by adding three independent edges between their outer faces, as depicted in Figure 10(b). We construct G_k of order n recursively for all $k \geqslant 1$ via the illustration given in Figure 10(c): the entire graph G_{k-1} is placed into the center quadrangle of Figure 10(c) (in such a way that the center bold quadrangle of Figure 10(c) is identified with the outer quadrangle of G_{k-1}). One can check that G_k is (K_1+3K_2)-free with $n = 24(k+1)$ vertices and $67n/24-4$ edges for all $k \geqslant 0$.

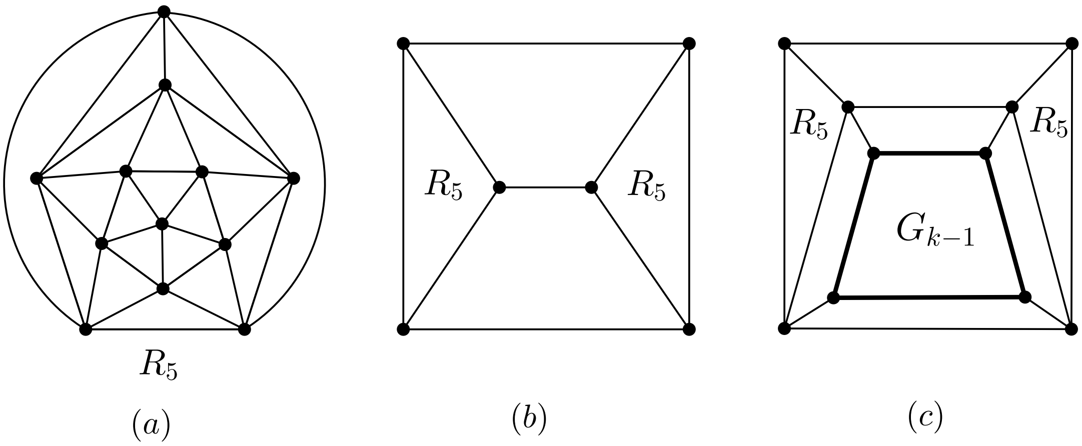


Figure 10: The construction of G_k.

As mentioned earlier, it seems non-trivial to determine $ex_{\mathcal{P}}(n, H)$ for all K_4-free planar graphs H with exactly one vertex, say u, satisfying $d_H(u) = \Delta(H) \leqslant 6$ and $\Delta(H[N(u)]) \leqslant 2$. We conclude this section by giving an upper bound (but not tight) for $ex_{\mathcal{P}}(n, K_1+H)$, where H is a disjoint union of paths.

Theorem 9. *Let $4 \leqslant t \leqslant 6$ be an integer and let H be a graph on t vertices such that H is a disjoint union of paths. Then $ex_{\mathcal{P}}(n, K_1+H) \leqslant \frac{13(t-1)n}{4t-2} - \frac{12(t-1)}{2t-1}$ for all $n \geqslant t+1$.*

Proof. Let t and H be given as in the statement. Since H is a subgraph of K_1+P_t, it suffices to show that $ex_{\mathcal{P}}(n, K_1+P_t) \leqslant \frac{13(t-1)n}{4t-2} - \frac{12(t-1)}{2t-1}$ for all $n \geqslant t+1$. Let G

be a $(K_1 + P_t)$-free planar graph on $n \geqslant t+1$ vertices. We next show that $e(G) \leqslant \frac{13(t-1)n}{4t-2} - \frac{12(t-1)}{2t-1}$ by induction on n. This is trivially true when $n = t+1$ because $e(G) \leqslant 3(t+1)-6 \leqslant \frac{13(t-1)(t+1)}{4t-2} - \frac{12(t-1)}{2t-1}$ for $5 \leqslant t \leqslant 6$ and $e(G) \leqslant 3(t+1)-7 \leqslant \frac{13(t-1)(t+1)}{4t-2} - \frac{12(t-1)}{2t-1}$ for $t = 4$. So we may assume that $n \geqslant t+2$. We may further assume that $\delta(G) \geqslant 3$ and G is connected. Hence,

$$2e(G) = 3f_3 + \sum_{i \geqslant 4} i f_i \geqslant 3f_3 + 4(f - f_3) = 4f - f_3,$$

which implies that $f \leqslant e(G)/2 + f_3/4$. Note that for all $3 \leqslant i \leqslant t-1$, each i-vertex is incident with at most i many 3-faces, and for all $j \geqslant t$, each j-vertex is incident with at most $(t-2)\lfloor \frac{j}{t-1} \rfloor$ many 3-faces, because G is $(K_1 + P_t)$-free. It follows that

$$\begin{aligned}
3f_3 &\leqslant \sum_{i=3}^{t-1} i \cdot n_i(G) + \sum_{j \geqslant t} (t-2) \left\lfloor \frac{j}{t-1} \right\rfloor \cdot n_j(G) \\
&\leqslant \sum_{i=3}^{t-1} \left(\frac{i \cdot n_i(G)}{t-1} + \frac{(t-2)i \cdot n_i(G)}{t-1} \right) + \frac{t-2}{t-1} \sum_{j \geqslant t} j \cdot n_j(G) \\
&= \sum_{i=3}^{t-1} \frac{i \cdot n_i(G)}{t-1} + \frac{t-2}{t-1} \sum_{\ell \geqslant 3} \ell \cdot n_\ell(G) \\
&= \sum_{i=3}^{t-1} \frac{i \cdot n_i(G)}{t-1} + \left(\frac{t-2}{t-1} \cdot 2e(G) \right) \\
&= \sum_{i=3}^{t-1} n_i(G) - \sum_{i=3}^{t-1} \frac{(t-1-i) \cdot n_i(G)}{t-1} + \left(\frac{t-2}{t-1} \cdot 2e(G) \right) \\
&< n + \frac{t-2}{t-1} \cdot 2e(G),
\end{aligned}$$

which implies that $f_3 < \frac{n}{3} + \frac{2(t-2)}{3(t-1)} \cdot e(G)$. This, together with the fact that $f \leqslant e(G)/2 + f_3/4$, further implies that $f \leqslant \frac{e(G)}{2} + \frac{n}{12} + \frac{(t-2)}{6(t-1)} \cdot e(G) = \frac{(4t-5)}{6(t-1)} \cdot e(G) + \frac{n}{12}$. By Euler's formula, $n - 2 = e(G) - f \geqslant \frac{(2t-1)}{6(t-1)} \cdot e(G) - \frac{n}{12}$. Hence, $e(G) \leqslant \frac{13(t-1)n}{4t-2} - \frac{12(t-1)}{2t-1}$.

This completes the proof of Theorem 9. □

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