

Shell Tableaux: A set partition analog of vacillating tableaux

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Abstract

Schur–Weyl duality is a fundamental framework in combinatorial representation theory. It intimately relates the irreducible representations of a group to the irreducible representations of its centralizer algebra. We investigate the analog of Schur–Weyl duality for the group of unipotent upper triangular matrices over a finite field. In this case, the character theory of these upper triangular matrices is “wild” or unattainable. Thus we employ a generalization, known as supercharacter theory, that creates a striking variation on the character theory of the symmetric group with combinatorics built from set partitions. In this paper, we present a combinatorial formula for calculating a restriction and induction of supercharacters based on statistics of set partitions and seashell inspired diagrams. We use these formulas to create a graph that encodes the decomposition of a tensor space, and develop an analog of Young tableaux, known as shell tableaux, to index paths in this graph.

Mathematics Subject Classifications: 05E10

1 Introduction

Schur–Weyl duality forms an archetypal situation in combinatorial representation theory involving two actions that complement each other. In the basic setup, a G -module M of a finite group G is tensored together k times to form the tensor space

$$M^{\otimes k} = \underbrace{M \otimes \cdots \otimes M}_{k \text{ factors}}.$$

The commuting actions of G and its centralizer algebra $Z_k = \text{End}_G(M^{\otimes k})$ on $M^{\otimes k}$ produce a decomposition

$$M^{\otimes k} \cong \bigoplus_{\lambda} G^{\lambda} \otimes Z_k^{\lambda} \quad \text{as a } (G, Z_k)\text{-bimodule}$$

where the G^λ are irreducible G -modules and the Z_k^λ are irreducible Z_k -modules. This bimodule decomposition intimately relates the irreducible representations of G with the irreducible representations of Z_k .

In the classical situation, the general linear group $GL_n(\mathbb{C})$ of $n \times n$ matrices over the field \mathbb{C} of complex numbers acts on the tensor space $V^{\otimes k}$ of an n dimensional vector space V , and its centralizer algebra is the symmetric group S_k on the k tensor factors. More recently, the study of new versions of Schur–Weyl duality has led to many remarkable discoveries about algebras of operators on tensor space that are full centralizers of each other. For example,

1. the Brauer algebra is the centralizer of the symplectic and orthogonal groups acting on the tensor space $(\mathbb{C}^n)^{\otimes k}$ [9];
2. the Temperley-Lieb algebra is the centralizer of the special linear Lie group of degree two acting on the tensor space $(\mathbb{C}^2)^{\otimes k}$ [14];
3. the partition algebra is the centralizer of the symmetric group acting on the tensor space $V^{\otimes k}$ of its permutation representation V [13].

This paper focuses on a unipotent analog of Schur–Weyl duality.

For a positive integer n and a power of a prime $q = p^r$, consider the finite group of unipotent $n \times n$ upper-triangular matrices

$$U_n = \left\{ \begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & 1 \end{bmatrix} \right\}$$

with ones on the diagonal and entries $*$ in the finite field \mathbb{F}_q with q elements. Since U_n is a Sylow p -subgroup of $GL_n(\mathbb{F}_q)$, then every p -group of $GL_n(\mathbb{F}_q)$ is conjugate to a subgroup of U_n . Embedding every finite p -group in $S_n \subseteq GL_n(\mathbb{F}_q)$ as permutation matrices, it follows that every p -group is isomorphic to a subgroup of U_n . This is akin to how every finite group is isomorphic to a subgroup of S_n , so it is not unreasonable to hope that the representation theories of U_n and S_n have comparable structures.

Unlike the combinatorially rich representation theory of S_n [16], the representation theory of U_n is well-known to be intractable or “wild” [12]. Nevertheless, André [2, 3, 4, 5] and Yan [19] constructed a workable approximation that has been useful in studying Fourier analysis [11], random walks [7], and Hopf algebras [1]. In [11] Diaconis and Isaacs generalize this idea to arbitrary finite groups to develop the notion of supercharacter theory. Supercharacter theory approximates the character theory of a finite group by replacing conjugacy classes with certain unions of conjugacy classes called “superclasses” and irreducible characters with certain linear combinations of irreducible characters called “supercharacters”.

We study a coarsening of André and Yan’s traditional super-representation theory on U_n [8] where there is a one-to-one correspondence between

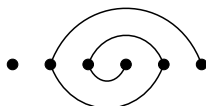
$$\left\{ \begin{array}{c} \text{supercharacters} \\ \text{of } U_n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Set partitions of} \\ \{1, 2, \dots, n\} \end{array} \right\}.$$

It is becoming ever more apparent that the set partition combinatorics of this super-representation theory is analogous to the classical partition combinatorics of the representation theory of the symmetric group, but with some important differences.

We first study the decomposition of $V^{\otimes k}$ where $V = \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1}$ as a U_n -supermodule. Much like the partition algebra, we have

$$V^{\otimes k} \cong \underbrace{(\text{Ind}_{U_{n-1}}^{U_n} \text{Res}_{U_{n-1}}^{U_n}) \cdots (\text{Ind}_{U_{n-1}}^{U_n} \text{Res}_{U_{n-1}}^{U_n})}_{k \text{ times}}(\mathbb{1})$$

where the trivial supercharacter $\mathbb{1}$ is restricted and induced k times. We provide a combinatorial formula calculating a restriction of supercharacters from U_n to U_{n-1} where the coefficients of the supercharacters of U_{n-1} are a product of powers of q and $q - 1$ based on statistics of set partitions and seashell inspired diagrams. For example, a shell formed by two set partitions is shown below.



Using Frobenius reciprocity, we obtain a corresponding formula for inducing supercharacters. Together these formulas are known as branching rules. As opposed to the representation theory of the symmetric group, they depend on the embedding of U_{n-1} in U_n .

We then use the branching rules to create a graph that encodes the decomposition of $V^{\otimes k}$ known as the Bratteli diagram. Since we are approximating by supercharacters, the Bratteli diagram produces a decomposition of a subalgebra of the centralizer algebra that treats supermodules as irreducibles. For the partition algebra, paths in the Bratteli diagram are indexed by a set of combinatorial objects called vacillating tableaux. We create an analog of vacillating tableaux, known as shell tableaux, built from a generalization of shells. Next, we construct a bijection between shell tableaux and paths in the Bratteli diagram. When $q = 2$, we remove a condition on shell tableaux to produce a bijection with weighted paths in the Bratteli diagram. In contrast with the symmetric group, these weights account for the multiplicities in our Bratteli diagram. On the whole, the shell combinatorics developed from this paper may help compute in other algebraic structures related to the supercharacter theory of U_n , such as the Hopf algebra of symmetric functions in noncommuting variables.

2 Preliminaries

This section reviews a supercharacter theory for the group of unipotent upper triangular matrices and the combinatorics of its representation theory based on set partitions.

2.1 A supercharacter theory for U_n

A *supercharacter theory* of a group G consists of a set of *superclasses* \mathcal{K} and a set of *supercharacters* \mathcal{X} such that

- (a) the set \mathcal{K} is a partition of G into unions of conjugacy classes,
- (b) the set \mathcal{X} is a set of characters such that each irreducible character of G is a constituent of exactly one supercharacter,
- (c) $|\mathcal{K}| = |\mathcal{X}|$,
- (d) the supercharacters are constant on superclasses.

Every group G has two “trivial” supercharacter theories: the usual character theory, and the supercharacter theory with $\mathcal{K} = \{\{1\}, G - \{1\}\}$ and $\mathcal{X} = \{\mathbb{1}, \chi_{\text{reg}} - \mathbb{1}\}$ where $\mathbb{1}$ is the trivial character of G and χ_{reg} is the regular character. While many finite groups have several supercharacter theories [11], preference is given to supercharacter theories that strike a balance between computability and producing better approximations of the usual character theory.

We focus on the supercharacter theory on U_n given in [18] that is a slight coarsening of the traditional supercharacter theory of André and Yan.

Let U_n be the subgroup of unipotent upper-triangular matrices of the general linear group $GL_n(\mathbb{F}_q)$ over the finite field \mathbb{F}_q with q elements, B_n be the normalizer of U_n in $GL_n(\mathbb{F}_q)$ consisting of upper triangular matrices, and

$$\mathfrak{u}_n = U_n - 1$$

be the nilpotent \mathbb{F}_q -algebra of strictly upper triangular matrices. The subgroup B_n acts by left and right multiplication on \mathfrak{u}_n , and the superclasses are given by the two-sided orbits

$$\begin{array}{ccc} B_n \mathfrak{u}_n B_n & \longleftrightarrow & \mathcal{K} \\ B_n x B_n & \mapsto & 1 + B_n x B_n. \end{array}$$

Following the construction in [8], fix a nontrivial homomorphism $\vartheta : \mathbb{F}_q^+ \rightarrow \mathbb{C}^\times$. The \mathbb{F}_q -vector space of $n \times n$ matrices $\mathfrak{gl}_n(\mathbb{F}_q)$ decomposes in terms of upper triangular matrices \mathfrak{b}_n and strictly lower triangular matrices \mathfrak{l}_n as

$$\mathfrak{gl}_n = \mathfrak{b}_n \oplus \mathfrak{l}_n.$$

Identifying \mathfrak{l}_n with $\mathfrak{gl}_n/\mathfrak{b}_n$ makes \mathfrak{l}_n a canonical set of coset representatives in $\mathfrak{gl}_n/\mathfrak{b}_n$. For $v \in \mathfrak{gl}_n$ define

$$\bar{v} = (v + \mathfrak{b}_n) \cap \mathfrak{l}_n.$$

Then for $v \in \mathfrak{l}_n$,

$$\mathbb{C}\text{-span}\{\overline{av} \mid a \in B_n\}$$

is U_n -supermodule with left action

$$uw = \vartheta(\text{tr}((u-1)w)(\overline{uw})) \quad \text{for } u \in U_n, w \in \mathfrak{l}_n$$

and right action

$$wu = \vartheta(\text{tr}(w(u^{-1}-1))(\overline{wu^{-1}})) \quad \text{for } u \in U_n, w \in \mathfrak{l}_n.$$

The two-sided orbits from extending these actions on \mathfrak{l}_n to the normalizer subgroup B_n yields corresponding supercharacters given by,

$$\begin{aligned} B_n \mathfrak{l}_n B_n &\longleftrightarrow \mathcal{X} \\ B_n v B_n &\mapsto g \mapsto \frac{|B_n v|}{|B_n v B_n|} \sum_{w \in B_n v B_n} \vartheta(\text{tr}((g-1)w)). \end{aligned}$$

In constructing the supercharacters of U_n it is more common to construct a module structure on the dual \mathfrak{u}_n^* , where $\mathfrak{u}_n = U_n - 1$ as in [11]. However, the actions of B_n on \mathfrak{l}_n are a translation of the actions on \mathfrak{u}_n^* that make studying modules more straightforward [8].

By elementary row and column operations we may choose orbit representatives for the two-sided action of B_n on \mathfrak{u}_n and \mathfrak{l}_n so that there is a one to one correspondence between

$$\begin{aligned} \left\{ \begin{array}{c} \text{superclasses} \\ \text{of } U_n \end{array} \right\} &\longleftrightarrow \left\{ u \in U_n \mid \begin{array}{l} u-1 \text{ has at most one } 1 \\ \text{in every row and column} \end{array} \right\} \\ \left\{ \begin{array}{c} \text{supercharacters} \\ \text{of } U_n \end{array} \right\} &\longleftrightarrow \left\{ v \in \mathfrak{l}_n \mid \begin{array}{l} v \text{ has at most one } 1 \\ \text{in every row and column} \end{array} \right\}. \end{aligned}$$

These representatives are indexed by set partitions.

If instead of considering the orbits of the full subgroup B_n , we consider the U_n orbits on the group \mathfrak{u}_n and its dual \mathfrak{u}_n^* , then we obtain the traditional supercharacter theory of André and Yan. In this case the combinatorics depends on the finite field \mathbb{F}_q and is based on \mathbb{F}_q^\times -colored set partitions.

2.2 Set Partition Combinatorics

Define $[n] = \{1, 2, \dots, n\}$. A *set partition* λ of $[n]$ is a subset $\{(i, j) \in [n] \times [n] \mid i < j\}$ such that if $(i, k), (j, l) \in \lambda$, then $i = j$ if and only if $k = l$. We represent each set partition $\lambda \vdash [n]$ diagrammatically as a set of arcs on a row of n nodes so that if $(i, j) \in \lambda$, then there is an arc connecting the i th node to the j th node. For example,

$$\{(1, 3), (3, 5), (2, 6)\} \longleftrightarrow \begin{array}{cccccc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ 1 & 2 & 3 & 4 & 5 & 6 \end{array} \quad \text{or} \quad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}.$$

In these diagrams it is natural to draw the arcs above or below the nodes. We will use both orientations to compare set partitions. We typically refer to the pair (i, j) as an *arc* in λ and write $(i, j) = i \frown j$ or $(i, j) = i \smile j$ to specify the arc. For each arc $(i, j) \in \lambda$ we call i the *left endpoint* and j the *right endpoint*. The sets of left and right endpoints of λ are given by

$$\begin{aligned} le(\lambda) &= \{i \in [n] \mid (i, j) \in \lambda, \text{ for some } j \in [n]\} \\ re(\lambda) &= \{j \in [n] \mid (i, j) \in \lambda, \text{ for some } i \in [n]\}. \end{aligned}$$

We say two arcs *conflict* if they have the same left or right endpoints. Thus no arcs conflict in a set partition.

We obtain the more traditional definition of set partitions by taking $\text{part}(\lambda)$ for $\lambda \vdash [n]$ to be the set of equivalence classes on $[n]$ given by the reflexive transitive closure of $i \sim j$ if $(i, j) \in \lambda$. For instance,

$$\text{part} \left(\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \bullet_1 \quad \bullet_2 \quad \bullet_3 \quad \bullet_4 \quad \bullet_5 \quad \bullet_6 \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \right) = \{\{1, 3, 5\}, \{2, 6\}, \{4\}\}.$$

Note the connected components of the diagram are the parts of the set partition and the arcs are the adjacent pairs of elements in each part.

There are some natural statistics on set partitions [10]. For a set partition $\lambda \vdash [n]$ the *dimension* is

$$\dim(\lambda) = \sum_{i \frown j \in \lambda} j - i - 1.$$

For a pair of set partitions $\lambda, \mu \vdash [n]$ define

$$\begin{aligned} \text{Crs}(\lambda, \mu) &= \{((i, k), (j, l)) \in \lambda \times \mu \mid i < j < k < l\}, & \text{crs}(\lambda, \mu) &= |\text{Crs}(\lambda, \mu)|, \\ \text{Nst}_\mu^\lambda &= \{((i, l), (j, k)) \in \lambda \times \mu \mid i < j < k < l\}, & \text{nst}_\mu^\lambda &= |\text{Nst}_\mu^\lambda| \end{aligned}$$

as the *crossing set*, *crossing number*, *nesting set*, and *nesting number* respectively. To illustrate, if

$$\lambda = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \quad \text{and} \quad \mu = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array},$$

then we have

$$\dim(\lambda) = 3, \quad \text{crs}(\lambda, \lambda) = 1, \quad \text{nst}_\lambda^\lambda = 0, \quad \dim(\mu) = 4, \quad \text{crs}(\mu, \mu) = 0, \quad \text{nst}_\mu^\mu = 1.$$

Superimposing λ and μ , where the arcs of λ are dashed

$$\lambda \cup \mu = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}$$

yields

$$\text{Crs}(\lambda, \mu) = \{(1 \curvearrowright 4, 2 \curvearrowright 6), (1 \curvearrowright 4, 3 \curvearrowright 5)\}, \quad \text{Nst}_\mu^\lambda = \emptyset$$

but,

$$\text{Crs}(\mu, \lambda) = \emptyset, \quad \text{Nst}_\lambda^\mu = \{(2 \curvearrowright 6, 3 \curvearrowright 5)\}.$$

While it is not generally true that $\text{Crs}(\lambda, \mu) = \text{Crs}(\mu, \lambda)$, it follows from the definition of a crossing number that for all set partitions $\lambda, \mu, \nu \vdash [n]$,

$$\text{Crs}(\lambda, \mu \cup \nu) = \text{Crs}(\lambda, \mu) \cup \text{Crs}(\lambda, \nu) \quad (1)$$

$$\text{Crs}(\lambda \cup \mu, \nu) = \text{Crs}(\lambda, \nu) \cup \text{Crs}(\mu, \nu). \quad (2)$$

2.3 An uncolored supercharacter theory

We describe the correspondence between set partitions and the superclasses and supercharacters of U_n . Given a set partition $\lambda \vdash [n]$, we construct a representative u_λ of a superclass of U_n by

$$(u_\lambda)_{i,j} = \begin{cases} 1 & \text{if } i \curvearrowright j \in \lambda \text{ or } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, the correspondence between λ and u_λ is given as follows

$$\lambda = \begin{array}{c} \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\ \curvearrowright \quad \curvearrowright \quad \curvearrowright \end{array} \longleftrightarrow u_\lambda - 1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The corresponding superclass \mathcal{K}_λ is

$$\mathcal{K}_\lambda = 1 + B_n(u_\lambda - 1)B_n.$$

Similarly, a representative v_λ for the two-sided action of B_n on \mathfrak{l}_n is

$$(v_\lambda)_{k,j} = \begin{cases} 1 & \text{if } j \curvearrowright k \in \lambda, \\ 0 & \text{otherwise} \end{cases}$$

so that

$$V^\lambda \cong \mathbb{C}\text{-span}\{\overline{av_\lambda} \mid a \in B_n\}$$

and for $g \in U_n$, the corresponding supercharacter χ^λ is defined as

$$\chi^\lambda(g) = \frac{|B_nv_\lambda|}{|B_nv_\lambda B_n|} \sum_{v \in B_nv_\lambda B_n} \vartheta(\text{tr}((g-1)v)).$$

Amazingly, many properties of these supercharacters can be determined using statistics of set partitions.

Proposition 1 ([8, Proposition 2.1]). *For $\lambda, \mu \vdash [n]$, we have*

$$\chi^\lambda(u_\mu) = \begin{cases} \frac{(-1)^{|\lambda \cap \mu|} q^{\dim(\lambda)} (q-1)^{|\lambda - \mu|}}{q^{\text{nst}_\mu^\lambda}} & \text{if } i < j < k, i \frown k \in \lambda \\ & \text{then } i \frown j, j \frown k \notin \mu, \\ & \text{otherwise.} \\ 0 & \end{cases}$$

In particular the trivial supercharacter $\mathbb{1}$ is the supercharacter χ^\emptyset corresponding to the empty set partition of $[n]$, and the degree of each supercharacter is

$$\chi^\lambda(1) = q^{\dim(\lambda)} (q-1)^{|\lambda|}.$$

It also follows from the formula that supercharacters factor as tensor products of arcs

$$\chi^\lambda = \bigodot_{i \frown j \in \lambda} \chi^{i \frown j} \quad \text{where } (\chi \odot \psi)(g) = \chi(g)\psi(g). \quad (3)$$

With respect to the inner product the supercharacters form an orthogonal set.

Proposition 2. *For $\lambda, \mu \vdash [n]$, we have*

$$\langle \chi^\lambda, \chi^\mu \rangle = \delta_{\lambda\mu} (q-1)^{|\lambda|} q^{\text{crs}(\lambda, \lambda)}.$$

Proposition 2 can be proved from [18, (2.3)]. The crossing number $\text{crs}(\lambda, \lambda)$ helps measure how close a supercharacter is to being irreducible.

3 Branching Rules

An important property of the supercharacters of U_n is that their restriction to any subgroup is a linear combination of supercharacters with nonnegative integer coefficients [11]. However, the coefficients in the restriction decompositions are not well understood [18]. We provide a combinatorial formula for calculating the restriction of supercharacters of U_n to U_{n-1} . Using Frobenius reciprocity, we obtain a corresponding formula for inducing supercharacters. Since these formulas depend on the number of nonzero elements in the field \mathbb{F}_q , fix

$$t = q - 1.$$

3.1 Restriction

We consider the restriction of supercharacters from U_n to U_{n-1} by embedding $U_{n-1} \subseteq U_n$ as

$$U_{n-1} = \{u \in U_n \mid (u-1)_{ij} \neq 0 \text{ implies } i < j < n\}.$$

Since supercharacters decompose into tensor products of arcs (3), for $\lambda \vdash [n]$, we have

$$\chi^\lambda = \bigodot_{i \frown l \in \lambda} \chi^{i \frown l} \quad \text{and} \quad \text{Res}_{U_{n-1}}^{U_n}(\chi^\lambda) = \bigodot_{i \frown l \in \lambda} \text{Res}_{U_{n-1}}^{U_n}(\chi^{i \frown l}).$$

Consequently we compute restrictions for each $\chi^{i\curvearrowright l}$ and use the tensor product to glue together the resulting restrictions.

The restriction of the supercharacter $\chi^{i\curvearrowright l}$ is given using the formulas in [18] for computing restrictions in André and Yan's traditional supercharacter theory.

Proposition 3. *For $1 \leq i < l \leq n$, the restriction $\text{Res}_{U_{n-1}}^{U_n}(\chi^{i\curvearrowright l})$ is given by*

$$\text{Res}_{U_{n-1}}^{U_n}(\chi^{i\curvearrowright l}) = \begin{cases} \chi^{i\curvearrowright l} & \text{if } l \neq n, \\ t\left(\mathbb{1} + \sum_{i < k < l} \chi^{i\curvearrowright k}\right) & \text{if } l = n. \end{cases}$$

Proof. By the formulas for restriction of colored arcs [18, Theorem 4.5], for $l \neq n$, we have

$$\text{Res}_{U_{n-1}}^{U_n}(\chi^{i\curvearrowright l}) = \sum_{a \in \mathbb{F}_q^\times} \text{Res}_{U_{n-1}}^{U_n}(\chi^{i\curvearrowright a l}) = \sum_{a \in \mathbb{F}_q^\times} \chi^{i\curvearrowright a l} = \chi^{i\curvearrowright l},$$

and for $l = n$, we have

$$\text{Res}_{U_{n-1}}^{U_n}(\chi^{i\curvearrowright l}) = \sum_{a \in \mathbb{F}_q^\times} \text{Res}_{U_{n-1}}^{U_n}(\chi^{i\curvearrowright a l}) = \sum_{a \in \mathbb{F}_q^\times} \left(\mathbb{1} + \sum_{\substack{i < k < l \\ b \in \mathbb{F}_q^\times}} \chi^{i\curvearrowright b k} \right) = t\left(\mathbb{1} + \sum_{i < k < l} \chi^{i\curvearrowright k}\right). \quad \square$$

Intuitively, restricting an arc corresponds to removing the last node and reattaching the arc in all possible ways.

We now use the tensor product to glue together the resulting restrictions. For $1 \leq i < l$, define

$$\chi^{i\curvearrowright \times l} = t\left(\mathbb{1} + \sum_{i < k < l} \chi^{i\curvearrowright k}\right) \quad \text{and} \quad \chi^{i \times \curvearrowright l} = t\left(\mathbb{1} + \sum_{i < j < l} \chi^{j \curvearrowright l}\right).$$

Using the formulas in [18] for the colored supercharacter theory yields the following proposition.

Proposition 4. *For $1 \leq i < l \leq n$ and $1 \leq j < k \leq n$ such that $(i, l) \neq (j, k)$,*

$$\chi^{i\curvearrowright l} \odot \chi^{j\curvearrowright k} = \begin{cases} \chi^{\{i\curvearrowright l, j\curvearrowright k\}} & \text{if } k \neq l, i \neq j, \\ \chi^{i\curvearrowright l} \odot \chi^{j\curvearrowright \times k} & \text{if } i < j < k = l, \\ \chi^{i\curvearrowright l} \odot \chi^{j \times \curvearrowright k} & \text{if } i = j < k < l. \end{cases}$$

Proof. Let $1 \leq i < l \leq n$ and $1 \leq j < k \leq n$ such that $(i, l) \neq (j, k)$. For $k \neq l$ and $i \neq j$, the tensor product $\chi^{i\curvearrowright l} \odot \chi^{j\curvearrowright k}$ is given by

$$\chi^{i\curvearrowright l} \odot \chi^{j\curvearrowright k} = \sum_{a \in \mathbb{F}_q^\times} \sum_{b \in \mathbb{F}_q^\times} \chi^{i\curvearrowright a l} \odot \chi^{j\curvearrowright b k} = \sum_{a \in \mathbb{F}_q^\times} \sum_{b \in \mathbb{F}_q^\times} \chi^{\{i\curvearrowright a l, j\curvearrowright b k\}} = \chi^{\{i\curvearrowright l, j\curvearrowright k\}},$$

for $i < j < k = l$, we have

$$\chi^{i\curvearrowright l} \odot \chi^{j\curvearrowright l} = \sum_{a \in \mathbb{F}_q^\times} \sum_{b \in \mathbb{F}_q^\times} \chi^{i\curvearrowright a l} \odot \chi^{j\curvearrowright b l} = \sum_{a \in \mathbb{F}_q^\times} \sum_{b \in \mathbb{F}_q^\times} \chi^{i\curvearrowright a l} \odot \left(\mathbb{1} + \sum_{\substack{j < k < l \\ c \in \mathbb{F}_q^\times}} \chi^{j\curvearrowright c k} \right) = \chi^{i\curvearrowright l} \odot \chi^{j\curvearrowright \times l},$$

and for $i = j < k < l$, we obtain

$$\chi^{i \curvearrowright l} \odot \chi^{i \curvearrowright k} = \sum_{a \in \mathbb{F}_q^\times} \sum_{b \in \mathbb{F}_q^\times} \chi^{i \curvearrowright a l} \odot \chi^{i \curvearrowright b k} = \sum_{a \in \mathbb{F}_q^\times} \sum_{b \in \mathbb{F}_q^\times} \chi^{i \curvearrowright a l} \odot \left(\mathbb{1} + \sum_{\substack{i < j < k \\ c \in \mathbb{F}_q^\times}} \chi^{j \curvearrowright c k} \right) = \chi^{i \curvearrowright l} \odot \chi^{i \times k}$$

by the tensor formulas for colored arcs [18, Lemma 4.6]. \square

Thus the tensor product provides a rule for resolving conflicting arcs that have the same right endpoint by removing the smaller arc and reattaching it in all possible ways.

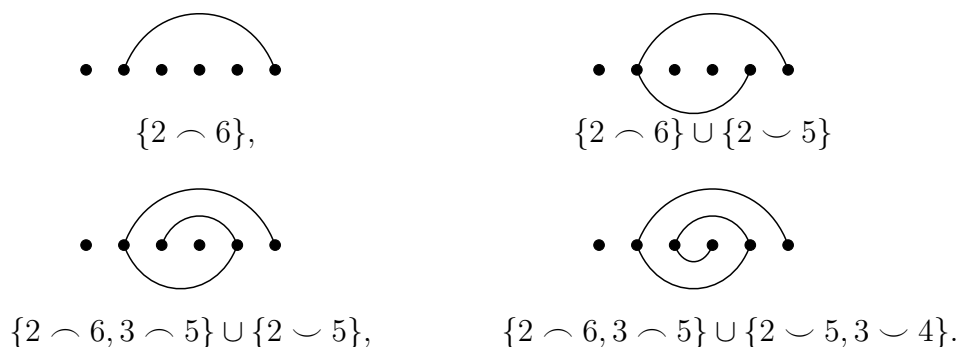
Next we work toward providing a combinatorial description of the coefficients in the tensor product based on statistics of set partitions and seashell inspired diagrams.

Definition 5. Let $s' \in \{s, s+1\}$ for $s \in \mathbb{Z}_{\geq 1}$ and $1 \leq i \leq l \leq n$. A *shell* of size n and width $l - i$ is a set of arcs on n nodes of the form

$$\bigcup_{r=1}^s \{i_r \curvearrowright l_r\} \cup \bigcup_{r=1}^{s'-1} \{i_r \curvearrowleft l_{r+1}\}$$

where $i = i_1 < \dots < i_s \leq l_{s'} < \dots < l_1 = l$.

For example, some shells of size 6 and width $6 - 2$ are

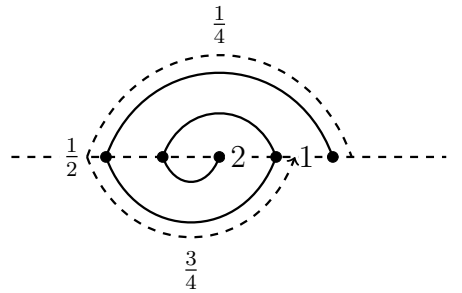


A *whorl* is pair of consecutive arcs $(i \curvearrowright l, i \curvearrowleft j)$ in a shell corresponding to a 360° rotation in the spiral configuration. Following the notation of Definition 5, the number of whorls of a shell is

$$\left\lceil \frac{s + s' - 1}{2} \right\rceil$$

as each arc is half a whorl. We use the convention that whorls are counted from the right endpoint l spiraling inward. For instance, in the shell below we count the two whorls

$(1 \curvearrowright 5, 1 \curvearrowright 4)$ and $(2 \curvearrowright 4, 2 \curvearrowright 3)$ as follows



If the whorls of a shell are given by $(i_1 \curvearrowright l_1, i_1 \curvearrowright l_2), \dots, (i_s \curvearrowright l_s, i_s \curvearrowright l_{s+1})$ we say the pair $(i_1 \curvearrowright l_1, i_1 \curvearrowright l_2)$ is the *outer whorl* and the other whorls are *inner whorls*.

We can use shells to determine the partitions that appear in the restriction of a supercharacter. Note that a set partition of $[n-1]$ is a set partition of $[n]$. We represent this diagrammatically by embedding the leftmost $n-1$ nodes in a row of n nodes. By drawing the arcs of a partition $\mu \vdash [n-1]$ below the nodes and identifying these nodes with the nodes of a partition $\lambda \vdash [n]$, we characterize the partitions with nonzero coefficients in the restriction of λ as the partitions $\mu \vdash [n-1]$ such that the symmetric difference between λ and μ form a shell.

Definition 6. For $\lambda \vdash [n]$ and $1 \leq i < l \leq n$ with $i \notin le(\lambda)$, the *shell set* $C^{\lambda, i \curvearrowright l}$ of $\lambda \cup \{i \curvearrowright l\}$ is

$$C^{\lambda, i \curvearrowright l} = \{\mu \vdash [n] \mid ((\lambda \cup \{i \curvearrowright l\}) - \mu) \cup (\mu - (\lambda \cup \{i \curvearrowright l\})) \text{ is a shell of width } l - i\}.$$

This corresponds to all the ways to reattach the arc $i \curvearrowright l$ and “straighten” the resulting diagram by resolving all the conflicting arcs that share the same right endpoint.

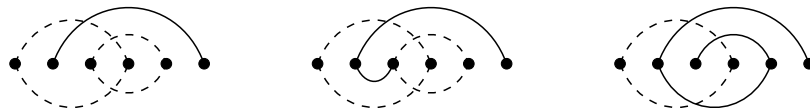
Example 7. Suppose $\lambda = \{1 \curvearrowright 4, 3 \curvearrowright 5\} \vdash [6]$. Consequently, we have

$$\lambda \cup \{2 \curvearrowright 6\} = \text{diagram with nodes 1 to 6 and arcs } (1,4), (3,5), (2,6)$$

and

$$C^{\lambda, 2 \curvearrowright 6} = \left\{ \text{diagram 1}, \text{diagram 2}, \text{diagram 3} \right\}.$$

The seashells created by the symmetric differences between $\lambda \cup \{2 \curvearrowright 6\}$ and $\mu \in C^{\lambda, 2 \curvearrowright 6}$ are shown as solid lines



while the arcs in $\lambda \cap \mu$ are dashed.

It will be of interest to examine the shell sets $C^{\lambda, i \curvearrowright l}$ by considering the right endpoints $re(\lambda)$.

Definition 8. For each $j \curvearrowright k \in \lambda$ with $i < j < k$ define $\lambda|_{j \mapsto i}$ as the set partition obtained by replacing $j \curvearrowright k$ with $i \curvearrowright k$ and leaving everything else in λ the same. That is,

$$\lambda|_{j \mapsto i} = \lambda \cup \{i \curvearrowright k\} - \{j \curvearrowright k\}.$$

With this notation we can describe the shell set $C^{\lambda, i \curvearrowright l}$ as a union of shells with half a whorl, shells with one whorl, and shells with greater than one whorl.

Lemma 9. For $\lambda \vdash [n]$, and $i \notin le(\lambda)$, the shell set is given by

$$\begin{aligned} C^{\lambda, i \curvearrowright l} = & \{\lambda\} \cup \{\lambda \cup \{i \curvearrowright k\} \mid i < k < l, k \notin re(\lambda)\} \\ & \cup \{\mu \in C^{\lambda|_{j \mapsto i}, j \curvearrowright k} \mid i < j < k < l, j \curvearrowright k \in \lambda\}. \end{aligned}$$

Proof. By definition $\{\lambda, \lambda \cup \{i \curvearrowright k\} \mid i < k < l, k \notin re(\lambda)\} \subseteq C^{\lambda, i \curvearrowright l}$, so it suffices to show that

$$C^{\lambda, i \curvearrowright l} \setminus \{\lambda, \lambda \cup \{i \curvearrowright k\} \mid i < k < l, k \notin re(\lambda)\} = \{\mu \in C^{\lambda|_{j \mapsto i}, j \curvearrowright k} \mid i < j < k < l, j \curvearrowright k \in \lambda\}.$$

There exist $j = j_1 < \dots < j_s < k_{s'} < \dots < k_1 = k$ with $s' \in \{s, s+1\}$ such that

$$\begin{aligned} (\lambda|_{j \mapsto i} \cup \{j \curvearrowright k\}) - \mu &= \{j_1 \curvearrowright k_1, j_2 \curvearrowright k_2, \dots, j_s \curvearrowright k_s\}, \text{ and} \\ \mu - (\lambda|_{j \mapsto i} \cup \{j \curvearrowright k\}) &= \{j_1 \curvearrowright k_2, j_2 \curvearrowright k_3, \dots, j_{s'-1} \curvearrowright k_{s'}\} \end{aligned}$$

if and only if there exist $i < j = j_1 < \dots < j_s < k_{s'} < \dots < k_1 = k < l$ such that

$$\begin{aligned} (\lambda \cup \{i \curvearrowright l\}) - \mu &= \{i \curvearrowright l, j_1 \curvearrowright k_1, \dots, j_s \curvearrowright k_s\}, \text{ and} \\ \mu - (\lambda \cup \{i \curvearrowright l\}) &= \{i \curvearrowright k, j_1 \curvearrowright k_2, \dots, j_{s'-1} \curvearrowright k_{s'}\}. \end{aligned}$$

Thus $\mu \in C^{\lambda|_{j \mapsto i}, j \curvearrowright k}$ for some $i < j < k < l, j \curvearrowright k \in \lambda$ if and only if $\mu \in C^{\lambda, i \curvearrowright l} \setminus \{\lambda, \lambda \cup \{i \curvearrowright k\} \mid i < k < l, k \notin re(\lambda)\}$ as desired. \square

Definition 10. For each $\mu \in C^{\lambda, i \curvearrowright l}$ define the *shell coefficient* of $\lambda \cup \{i \curvearrowright l\}$ and μ as

$$c_{\mu}^{\lambda, i \curvearrowright l} = \frac{t^{|(\lambda \cup \{i \curvearrowright l\}) - \mu|} q^{\text{crs}((\lambda \cup \{i \curvearrowright l\}) \cap \mu, (\lambda \cup \{i \curvearrowright l\}) - \mu)}}{q^{\text{crs}((\lambda \cup \{i \curvearrowright l\}) \cap \mu, \mu - (\lambda \cup \{i \curvearrowright l\}))}}$$

where $t = q - 1$ and $\text{crs}(\cdot, \cdot)$ is the crossing number of two set partitions given in Section 2.2.

We can associate each shell coefficient $c_{\mu}^{\lambda, i \curvearrowright l}$ to the shell created by the symmetric difference of $\lambda \cup \{i \curvearrowright l\}$ and μ . The next lemma shows the shell coefficient is the product of the shell coefficient of the outer whorl with the shell coefficient of the inner whorls.

Lemma 11. Let $\lambda \vdash [n]$, $i \notin \text{le}(\lambda)$, and $h \frown l, j \frown k \in \lambda$ with $1 \leq h < i < j < k < l \leq n$. If $\mu \in C^{\lambda|_{j \rightarrow i}, j \frown k}$ then

$$c_{\mu}^{\lambda, i \frown l} = c_{\lambda \cup \{i \frown k\}}^{\lambda, i \frown l} c_{\mu}^{\lambda|_{j \rightarrow i}, j \frown k}.$$

Proof. Let $\mu \in C^{\lambda|_{j \rightarrow i}, j \frown k}$. By construction $i \notin \text{le}(\lambda|_{j \rightarrow i})$, so $i \frown l \notin \mu$. Thus we have

$$(\lambda \cup \{i \frown l\}) - \mu = \{i \frown l\} \cup (\lambda - \mu),$$

hence

$$(\lambda \cup \{i \frown l\}) \cap \mu = \lambda \cap \mu.$$

Substituting this and applying the crossing number equation (1), it follows that

$$\begin{aligned} c_{\mu}^{\lambda, i \frown l} &= \frac{t^{|\lambda \cup \{i \frown l\} - \mu|} q^{\text{crs}((\lambda \cup \{i \frown l\}) \cap \mu, (\lambda \cup \{i \frown l\}) - \mu)}}{q^{\text{crs}((\lambda \cup \{i \frown l\}) \cap \mu, \mu - (\lambda \cup \{i \frown l\}))}} \\ &= \frac{t^{|\{i \frown l\} \cup (\lambda - \mu)|} q^{\text{crs}(\lambda \cap \mu, \{i \frown l\} \cup (\lambda - \mu))}}{q^{\text{crs}(\lambda \cap \mu, \mu - (\lambda \cup \{i \frown l\}))}} \\ &= \frac{t^{i \frown l} q^{\text{crs}(\lambda \cap \mu, i \frown l)} t^{|\lambda - \mu|} q^{\text{crs}(\lambda \cap \mu, \lambda - \mu)}}{q^{\text{crs}(\lambda \cap \mu, \mu - (\lambda \cup \{i \frown l\}))}}. \end{aligned}$$

Similarly since $j \frown k \in \lambda$ and $i \frown k \in \mu - \lambda$, we have

$$\mu - (\lambda \cup \{i \frown l\}) = \{i \frown k\} \cup (\mu - (\lambda|_{j \rightarrow i} \cup \{j \frown k\}))$$

and thus

$$\lambda - \mu = \lambda|_{j \rightarrow i} \cup \{j \frown k\} - \mu.$$

By the crossing number equation (1),

$$\begin{aligned} c_{\mu}^{\lambda, i \frown l} &= \frac{t^{i \frown l} q^{\text{crs}(\lambda \cap \mu, i \frown l)} t^{|\lambda|_{j \rightarrow i} \cup j \frown k - \mu|} q^{\text{crs}(\lambda \cap \mu, (\lambda|_{j \rightarrow i} \cup j \frown k) - \mu)}}{q^{\text{crs}(\lambda \cap \mu, \{i \frown k\} \cup (\mu - (\lambda|_{j \rightarrow i} \cup j \frown k)))}} \\ &= \frac{t^{i \frown l} q^{\text{crs}(\lambda \cap \mu, i \frown l)}}{q^{\text{crs}(\lambda \cap \mu, i \frown k)}} \cdot \frac{t^{|\lambda|_{j \rightarrow i} \cup j \frown k - \mu|} q^{\text{crs}(\lambda \cap \mu, (\lambda|_{j \rightarrow i} \cup j \frown k) - \mu)}}{q^{\text{crs}(\lambda \cap \mu, \mu - (\lambda|_{j \rightarrow i} \cup j \frown k))}}. \end{aligned}$$

Moreover any arc in λ that crosses with $i \frown k$ or $i \frown l$ must be in μ , implying

$$\begin{aligned} c_{\mu}^{\lambda, i \frown l} &= \frac{t^{i \frown l} q^{\text{crs}(\lambda, i \frown l)}}{q^{\text{crs}(\lambda, i \frown k)}} \cdot \frac{t^{|\lambda|_{j \rightarrow i} \cup j \frown k - \mu|} q^{\text{crs}((\lambda|_{j \rightarrow i} \cup j \frown k) \cap \mu, (\lambda|_{j \rightarrow i} \cup j \frown k) - \mu)}}{q^{\text{crs}((\lambda|_{j \rightarrow i} \cup j \frown k) \cap \mu, \mu - (\lambda|_{j \rightarrow i} \cup j \frown k))}} \\ &= c_{\lambda \cup \{i \frown k\}}^{\lambda, i \frown l} c_{\mu}^{\lambda|_{j \rightarrow i}, j \frown k}. \end{aligned} \quad \square$$

Theorem 12. For $\lambda \vdash [n]$, $i \notin \text{le}(\lambda)$, and $1 \leq i < l \leq n$, we have

$$\chi^{\lambda} \odot \chi^{i \frown l} = \sum_{\mu \in C^{\lambda, i \frown l}} c_{\mu}^{\lambda, i \frown l} \chi^{\mu} \quad \text{where} \quad c_{\mu}^{\lambda, i \frown l} = \frac{t^{|\lambda \cup \{i \frown l\} - \mu|} q^{\text{crs}((\lambda \cup \{i \frown l\}) \cap \mu, (\lambda \cup \{i \frown l\}) - \mu)}}{q^{\text{crs}((\lambda \cup \{i \frown l\}) \cap \mu, \mu - (\lambda \cup \{i \frown l\}))}}.$$

where $C^{\lambda, i \frown l}$ is the shell set of $\lambda \cup \{i \frown l\}$ and $c_{\mu}^{\lambda, i \frown l}$ is the shell coefficient of $\lambda \cup \{i \frown l\}$ and μ .

Before proving the theorem we state a lemma about the q -analog of a crossing number. In general, the q -analog of a nonnegative integer n is

$$[n]_q = \frac{q^n - 1}{q - 1}.$$

Lemma 13. For $\lambda \vdash [n]$, and $1 \leq j < l \leq n$ where $j \notin le(\lambda)$, we have

$$\sum_{\substack{i \frown k \in \lambda \\ i < j < k < l}} q^{\text{crs}(\lambda, j \frown k)} = \frac{q^{\text{crs}(\lambda, j \frown l)} - 1}{q - 1} = [\text{crs}(\lambda, j \frown l)]_q.$$

Proof. Let $\lambda \vdash [n]$, $1 \leq j < l \leq n$, and $j \notin le(\lambda)$. If the set of arcs in λ that cross with $j \frown k$ is given by

$$\{i \frown k \in \lambda \mid i < j < k < l\} = \{i_1 \frown k_1, i_2 \frown k_2, \dots, i_r \frown k_r\},$$

then for $1 \leq s \leq r$

$$\{i \frown k \in \lambda \mid i < j < k < k_s\} = \{i_1 \frown k_1, i_2 \frown k_2, \dots, i_{s-1} \frown k_{s-1}\}.$$

By the definition of the crossing number

$$\sum_{\substack{i < j < k < l \\ i \frown k \in \lambda}} q^{\text{crs}(\lambda, j \frown k)} = \sum_{s=1}^r q^{\#\{i \frown k \in \lambda \mid i < j < k < k_s\}} = \sum_{s=1}^r q^{s-1} = \frac{q^r - 1}{q - 1} = \frac{q^{\text{crs}(\lambda, j \frown l)} - 1}{q - 1}. \quad \square$$

We are now ready to prove Theorem 12.

Proof. We induct on $l - i$. For the base case assume $l - i = 1$. Then $C^{\lambda, i \frown l} = \{\lambda\}$, and we obtain

$$\chi^\lambda \odot \chi^{i \frown l} = \chi^\lambda \odot t\mathbb{1} = t\chi^\lambda = c_\lambda^{\lambda, i \frown l} \chi^\lambda$$

as desired.

Assume the formula holds for all $1 \leq k < j \leq n$ such that the difference $k - j < l - i$. Then, this yields

$$\begin{aligned} \chi^\lambda \odot \chi^{i \frown l} &= \chi^\lambda \odot t \left(\mathbb{1} + \sum_{i \frown k < l} \chi^{i \frown k} \right) \\ &= t\chi^\lambda + t \sum_{i \frown k < l} \chi^\lambda \odot \chi^{i \frown k} \\ &= t\chi^\lambda + t \left(\sum_{\substack{i \frown k < l \\ k \notin re(\lambda)}} \chi^\lambda \odot \chi^{i \frown k} + \sum_{\substack{h \frown i \frown k < l \\ h \frown k \in \lambda}} \chi^\lambda \odot \chi^{i \frown k} + \sum_{\substack{i \frown j \frown k < l \\ j \frown k \in \lambda}} \chi^\lambda \odot \chi^{i \frown k} \right), \end{aligned}$$

which by Proposition 4 is equal to

$$= t\chi^\lambda + t \sum_{\substack{i \frown k < l \\ k \notin re(\lambda)}} \chi^{\lambda \cup \{i \frown k\}} + t \sum_{\substack{h \frown i \frown k < l \\ h \frown k \in \lambda}} \chi^\lambda \odot \chi^{i \frown k} + t \sum_{\substack{i \frown j \frown k < l \\ j \frown k \in \lambda}} \chi^{\lambda \cup \{i \frown k\} - \{j \frown k\}} \odot \chi^{j \frown k}.$$

Recall from Definition 8 that $\lambda|_{j \mapsto i} = \lambda \cup \{i \frown k\} - \{j \frown k\}$ for each $j \frown k \in \lambda$ such that $i < j < k < l$. By the induction hypothesis the tensor product $\chi^\lambda \odot \chi^{i \frown l}$ is

$$= t\chi^\lambda + t \sum_{\substack{i < k < l \\ k \notin \text{re}(\lambda)}} \chi^{\lambda \cup \{i \frown k\}} + t \sum_{\substack{h < i < k < l \\ h \frown k \in \lambda}} \left(\sum_{\mu \in C^{\lambda, i \frown k}} c_\mu^{\lambda, i \frown k} \chi^\mu \right) + t \sum_{\substack{i < j < k < l \\ j \frown k \in \lambda}} \left(\sum_{\mu \in C^{\lambda|_{j \mapsto i}, j \frown k}} c_\mu^{\lambda|_{j \mapsto i}, j \frown k} \chi^\mu \right).$$

By Lemma 13, the coefficient of χ^λ will be

$$\begin{aligned} t + t \sum_{\substack{h < i < k < l \\ h \frown k \in \lambda}} c_\lambda^{\lambda, i \frown k} &= t \left(1 + \sum_{\substack{h < i < k < l \\ h \frown k \in \lambda}} t q^{\text{crs}(\lambda, i \frown k)} \right) = t \left(1 + t \cdot \frac{q^{\text{crs}(\lambda, i \frown l)} - 1}{t} \right) \\ &= t q^{\text{crs}(\lambda, i \frown l)} = c_\lambda^{\lambda, i \frown l}. \end{aligned}$$

Similarly for $\lambda \cup \{i \frown k'\}$ where $i < k' < l$ and $k' \notin \text{re}(\lambda)$, the coefficient of $\chi^{\lambda \cup \{i \frown k'\}}$ is

$$t + t \sum_{\substack{h < i < k' < k < l \\ h \frown k \in \lambda}} c_\mu^{\lambda, i \frown k} = t \left(1 + \sum_{\substack{h < i < k' < k < l \\ h \frown k \in \lambda}} t \frac{q^{\text{crs}(\lambda, i \frown k)}}{q^{\text{crs}(\lambda, i \frown k')}} \right) = t \left(1 + \sum_{\substack{h < i < k' < k < l \\ h \frown k \in \lambda}} t q^{\text{crs}(\lambda - \nu, i \frown k)} \right),$$

where $\nu = \{h \frown k \in \lambda \mid (h \frown k, i \frown k') \in \text{Crs}(\lambda, i \frown k')\}$. This is equivalent to

$$t \left(1 + t \cdot \frac{q^{\text{crs}(\lambda - \nu, i \frown l)} - 1}{t} \right) = t \left(1 + t \cdot \frac{q^{\text{crs}(\lambda, i \frown l) - \text{crs}(\lambda, i \frown k')} - 1}{t} \right) = \frac{t q^{\text{crs}(\lambda, i \frown l)}}{q^{\text{crs}(\lambda, i \frown k')}} = c_{\lambda \cup \{i \frown k'\}}^{\lambda, i \frown l}$$

by Lemma 13. If $j \frown k' \in \lambda$ is such that $i < j < k' < l$ then we have $\lambda|_{j \mapsto i} = \lambda \cup \{i \frown k'\} - \{j \frown k'\}$. Let $\nu = \{h \frown k \in \lambda \mid (h \frown k, i \frown k') \in \text{Crs}(\lambda, i \frown k')\}$. Using Lemma 11, the coefficient of χ^μ for each $\mu \in C^{\lambda|_{j \mapsto i}, j \frown k'}$ is

$$\begin{aligned} t c_\mu^{\lambda|_{j \mapsto i}, j \frown k'} + t \sum_{\substack{h < i < k' < k < l \\ h \frown k \in \lambda}} c_\mu^{\lambda, i \frown k} &= t c_\mu^{\lambda|_{j \mapsto i}, j \frown k'} + t \sum_{\substack{h < i < k' < k < l \\ h \frown k \in \lambda}} c_{\lambda \cup i \frown k'}^{\lambda, i \frown k} c_\mu^{\lambda|_{j \mapsto i}, j \frown k'} \\ &= t c_\mu^{\lambda|_{j \mapsto i}, j \frown k'} \left(1 + \sum_{\substack{h < i < k' < k < l \\ h \frown k \in \lambda}} t \frac{q^{\text{crs}(\lambda, i \frown k)}}{q^{\text{crs}(\lambda, i \frown k')}} \right) \\ &= t c_\mu^{\lambda|_{j \mapsto i}, j \frown k'} \left(1 + t \cdot \frac{q^{\text{crs}(\lambda - \nu, i \frown l)} - 1}{t} \right). \end{aligned}$$

Applying Lemmas 13 and 11 yields

$$\begin{aligned} t c_\mu^{\lambda|_{j \mapsto i}, j \frown k'} \left(1 + t \cdot \frac{q^{\text{crs}(\lambda - \nu, i \frown l)} - 1}{t} \right) &= t c_\mu^{\lambda|_{j \mapsto i}, j \frown k'} \left(1 + t \cdot \frac{q^{\text{crs}(\lambda, i \frown l) - \text{crs}(\lambda, i \frown k')} - 1}{t} \right) \\ &= \frac{t q^{\text{crs}(\lambda, i \frown l)}}{q^{\text{crs}(\lambda, i \frown k')}} c_\mu^{\lambda|_{j \mapsto i}, j \frown k'} \\ &= c_{\lambda \cup \{i \frown k'\}}^{\lambda, i \frown l} c_\mu^{\lambda|_{j \mapsto i}, j \frown k'} \end{aligned}$$

$$= c_{\mu}^{\lambda, i \curvearrowright l}.$$

Substituting this into the equation for $\chi^{\lambda} \odot \chi^{i \curvearrowright l}$ and applying Lemma 9 we obtain

$$\begin{aligned} \chi^{\lambda} \odot \chi^{i \curvearrowright l} &= c_{\lambda}^{\lambda, i \curvearrowright l} \chi^{\lambda} + \sum_{\substack{i < k < l \\ k \notin \text{re}(\lambda)}} c_{\lambda \cup \{i \curvearrowright k\}}^{\lambda, i \curvearrowright l} \chi^{\lambda \cup \{i \curvearrowright k\}} + \sum_{\mu \vdash [n]} \sum_{\substack{i < j < k < l \\ j \curvearrowright k \in \lambda \\ \mu \in C^{\lambda | j \mapsto i, j \curvearrowright k}}} c_{\mu}^{\lambda, i \curvearrowright l} \chi^{\mu} \\ &= \sum_{\mu \in C^{\lambda, i \curvearrowright l}} c_{\mu}^{\lambda, i \curvearrowright l} \chi^{\mu}. \end{aligned} \quad \square$$

This combinatorial description of the coefficients in the tensor product leads to a combinatorial description of the coefficients in the restriction to U_{n-1} .

Corollary 14. For $\lambda \vdash [n]$, the restriction $\text{Res}_{U_{n-1}}^{U_n}(\chi^{\lambda})$ is given by

$$\text{Res}_{U_{n-1}}^{U_n}(\chi^{\lambda}) = \sum_{\mu \vdash [n-1]} c_{\mu}^{\lambda} \chi^{\mu}$$

where

$$c_{\mu}^{\lambda} = \begin{cases} \delta_{\lambda\mu} & \text{if } n \notin \text{re}(\lambda), \\ \frac{t^{|\lambda-\mu|} q^{\text{crs}(\lambda \cap \mu, \lambda - \mu)}}{q^{\text{crs}(\lambda \cap \mu, \mu - \lambda)}} & \text{if } \mu \in C^{\lambda - \{i \curvearrowright n\}, i \curvearrowright n}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Applying Propositions 3, 4, and Theorem 12 respectively,

$$\begin{aligned} \text{Res}_{U_{n-1}}^{U_n}(\chi^{\lambda}) &= \bigodot_{i \curvearrowright l \in \lambda} \text{Res}_{U_{n-1}}^{U_n}(\chi^{i \curvearrowright l}) = \bigodot_{\substack{j \curvearrowright l \in \lambda \\ l \neq n}} \chi^{j \curvearrowright l} \odot \chi^{i \curvearrowright n} \\ &= \chi^{\lambda - \{i \curvearrowright n\}} \odot \chi^{i \curvearrowright n} = \sum_{\mu \in C^{\lambda - \{i \curvearrowright n\}, i \curvearrowright n}} c_{\mu}^{\lambda} \chi^{\mu} \end{aligned}$$

where

$$c_{\mu}^{\lambda} = c_{\mu}^{\lambda - \{i \curvearrowright n\}, i \curvearrowright n} = \frac{t^{|\lambda-\mu|} q^{\text{crs}(\lambda \cap \mu, \lambda - \mu)}}{q^{\text{crs}(\lambda \cap \mu, \mu - \lambda)}}. \quad \square$$

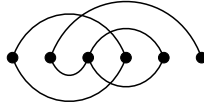
Example 15. Similar to Example 7, let



so that

$$C^{\lambda - \{2 \curvearrowright 6\}, 2 \curvearrowright 6} = \left\{ \begin{array}{c} \text{Diagram 1: } \bullet \bullet \bullet \bullet \bullet \bullet \text{ with arcs } (1,4), (2,3), (3,5) \\ \text{Diagram 2: } \bullet \bullet \bullet \bullet \bullet \bullet \text{ with arcs } (1,4), (2,3), (3,5), (4,6) \\ \text{Diagram 3: } \bullet \bullet \bullet \bullet \bullet \bullet \text{ with arcs } (1,4), (2,3), (3,5), (4,6), (5,6) \end{array} \right\}.$$

Drawing the arcs of $\mu = \{1 \curvearrowright 4, 2 \curvearrowright 3 \curvearrowright 5\}$ below the nodes of λ as shown below



illustrates that

$$c_\mu^\lambda = \frac{t^1 \cdot q^1}{q^0} = tq$$

since

$$\begin{aligned} \lambda - \mu &= \{2 \curvearrowright 6\}, & \text{Crs}(\lambda \cap \mu, \lambda - \mu) &= \{(1 \curvearrowright 4, 2 \curvearrowright 6)\}, \\ \text{crs}(\lambda \cap \mu, \lambda - \mu) &= 1, & \text{crs}(\lambda \cap \mu, \mu - \lambda) &= 0. \end{aligned}$$

We can calculate the other coefficients in the same manner to obtain

$$\text{Res}_{U_5}^{U_6}(\chi^\lambda) = tq\chi^{\{1 \curvearrowright 4, 3 \curvearrowright 5\}} + tq\chi^{\{2 \curvearrowright 3 \curvearrowright 5, 1 \curvearrowright 4\}} + t^2q\chi^{\{1 \curvearrowright 4, 2 \curvearrowright 5\}}.$$

3.2 Induction and Superinduction

While the restriction of a supercharacter of U_n is a nonnegative integer linear combination of supercharacters, an induced supercharacter may not be a sum of supercharacters. In fact, the induced character may not even be a superclass function; for an example see [11, Section 6]. If instead we generalize to superinduction by averaging over superclasses in the same way that induction averages over conjugacy classes, then the constructed function will be a linear combination of supercharacters with rational coefficients [11, Lemma 6.7].

Suppose $H \subseteq G$ and χ is a superclass function of H . If \mathcal{K}_g is the superclass containing $g \in G$, then the *superinduction* $\text{SInd}_H^G(\chi)$ is

$$\text{SInd}_H^G(\chi)(g) = |G : H| \frac{1}{|\mathcal{K}_g|} \sum_{x \in \mathcal{K}_g} \dot{\chi}(x) \text{ where } \dot{\chi}(x) = \begin{cases} \chi(x) & \text{if } x \in H \\ 0 & \text{if } x \notin H. \end{cases}$$

A nice property of superinduction is that the analog of Frobenius reciprocity holds.

Proposition 16 (Frobenius Reciprocity [15, Lemma 5.2]). *Let H be a subgroup of G . Suppose φ is a superclass function of G and θ is a class function of H . Then*

$$\langle \text{SInd}_H^G(\theta), \varphi \rangle_G = \langle \theta, \text{Res}_H^G(\varphi) \rangle_H.$$

However, superinduced characters are not necessarily characters so it is useful to know when superinduction is equivalent to induction.

[17, Section 3.2] examines some cases when this occurs for a larger class of p -groups known as algebra groups. If J is a finite dimensional nilpotent associative algebra over \mathbb{F}_q , then the algebra group based on J is $G = \{1 + x \mid x \in J\}$ under the multiplication $(1 + x)(1 + y) = 1 + x + y + xy$. In particular, Marberg and Thiem show if we embed U_{n-1} into U_n by

$$U_{n-1} = \{u \in U_n \mid (u - 1)_{ij} \neq 0 \text{ implies } i < j < n\}$$

then for any superclass function χ of U_{n-1} ,

$$\text{SInd}_{U_{n-1}}^{U_n}(\chi) = \text{Ind}_{U_{n-1}}^{U_n}(\chi).$$

They also provide some conditions when superinduction is the same as induction.

Proposition 17 ([17, Theorem 3.1]). *Let H be a subalgebra group of an algebra group G , and suppose*

1. *no two superclasses of H are in the same superclass of G , and*
2. *$x(h-1)+1 \in H$ for all $x \in G, h \in H$.*

Then the superinduction of any superclass function χ of H is

$$\text{SInd}_H^G(\chi) = \text{Ind}_H^G(\chi).$$

If we embed U_{n-1} into U_n by

$$U_{n-1} = \{u \in U_n \mid u_{n-1,n} = 0 \text{ and } u_{i,n-1} = 0 \text{ for } i < n-1\}$$

then we have the following corollary.

Corollary 18. *Let $U_{n-1} = \{u \in U_n \mid u_{n-1,n} = 0 \text{ and } u_{i,n-1} = 0 \text{ for } i < n-1\}$. Then the superinduction any superclass function χ of U_{n-1} is*

$$\text{SInd}_{U_{n-1}}^{U_n}(\chi) = \text{Ind}_{U_{n-1}}^{U_n}(\chi).$$

Proof. It suffices to show the hypotheses of the previous theorem hold. Because there is an injective function from superclasses of U_{n-1} to U_n then no two superclasses of U_{n-1} are in the same superclass of U_n .

Let $x \in U_n$, $h \in U_{n-1}$ and $u = x(h-1)+1$. Since $h_{i,n-1}-1=0$ we have $u_{i,n-1}=0$ for $i < n-1$. Similarly, $x_{n-1,j}=0$ for $j < n-1$ and $h_{n-1,j}-1=0$ for $j \geq n-1$ implies $u_{n-1,n}=0$. This shows $u \in U_{n-1}$. Therefore, $\text{SInd}_{U_{n-1}}^{U_n}(\chi) = \text{Ind}_{U_{n-1}}^{U_n}(\chi)$ for any superclass function χ of U_{n-1} by Proposition 17. \square

Unlike in the representation theory of the symmetric group, the decomposition of induced characters depends on the embedding of U_{n-1} into U_n . If we instead consider right modules, then superinduction is equivalent to induction for the following embeddings

$$U_{n-1} = \{u \in U_n \mid (u-1)_{ij} \neq 0 \text{ implies } 1 < i < j\}$$

and

$$U_{n-1} = \{u \in U_n \mid u_{1,2} = 0 \text{ and } u_{2,j} = 0 \text{ for } 2 < j\}$$

[17, Section 3.1]. However, it is not known if superinduction is the same as induction for other embeddings. In our case we use the embedding of $U_{n-1} \subseteq U_n$ obtained by removing the last column so that superinduction is in fact induction.

We now derive a corresponding formula for induction from restriction.

Corollary 19. For $\mu \vdash [n-1]$, the induction $\text{Ind}_{U_{n-1}}^{U_n}(\chi^\mu)$ is given by

$$\text{Ind}_{U_{n-1}}^{U_n}(\chi^\mu) = \sum_{\lambda \vdash [n]} d_\mu^\lambda \chi^\lambda,$$

where

$$d_\mu^\lambda = \begin{cases} \delta_{\lambda\mu} & \text{if } n \notin \text{re}(\lambda) \\ \frac{t^{|\mu-\lambda|} q^{\text{crs}(\mu-\lambda, \lambda \cap \mu)}}{q^{\text{crs}(\lambda-\mu, \lambda \cap \mu)}} & \text{if } \mu \in C^{\lambda-\{i \curvearrowright n\}, i \curvearrowright n} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\lambda \vdash [n]$ and $\mu \vdash [n-1]$. Frobenius reciprocity, Proposition 16, shows

$$\langle \chi^\lambda, \text{SInd}_{U_{n-1}}^{U_n}(\chi^\mu) \rangle_{U_n} = \langle \text{Res}_{U_{n-1}}^{U_n}(\chi^\lambda), \chi^\mu \rangle_{U_{n-1}}.$$

Thus if

$$\text{Ind}_{U_{n-1}}^{U_n}(\chi^\mu) = \sum_{\gamma} d_\mu^\gamma \chi^\gamma \quad \text{and} \quad \text{Res}_{U_{n-1}}^{U_n}(\chi^\lambda) = \sum_{\nu} c_\nu^\lambda \chi^\nu$$

then the inner product, Proposition 2, yields

$$q^{\text{crs}(\lambda, \lambda)} t^{|\lambda|} d_\mu^\lambda = q^{\text{crs}(\mu, \mu)} t^{|\mu|} c_\mu^\lambda.$$

Therefore, the coefficient d_μ^λ is

$$\begin{aligned} d_\mu^\lambda &= \frac{t^{|\mu|-|\lambda|} q^{\text{crs}(\mu, \mu)}}{q^{\text{crs}(\lambda, \lambda)}} c_\mu^\lambda \\ &= \frac{t^{|\mu|-|\lambda|} q^{\text{crs}(\mu, \mu)}}{q^{\text{crs}(\lambda, \lambda)}} \cdot \frac{t^{|\lambda-\mu|} q^{\text{crs}(\lambda \cap \mu, \lambda-\mu)}}{q^{\text{crs}(\lambda \cap \mu, \mu-\lambda)}} \\ &= \frac{t^{|\mu-\lambda|} q^{\text{crs}(\mu, \mu) - \text{crs}(\lambda \cap \mu, \mu-\lambda)}}{q^{\text{crs}(\lambda, \lambda) - \text{crs}(\lambda \cap \mu, \lambda-\mu)}} \end{aligned}$$

since $|\mu-\lambda| = |\mu| - |\lambda| + |\lambda-\mu|$. From the crossing number equation (1) we obtain

$$\begin{aligned} d_\mu^\lambda &= \frac{t^{|\mu-\lambda|} q^{\text{crs}(\mu-(\lambda \cap \mu), \mu-(\mu-\lambda))}}{q^{\text{crs}(\lambda-(\lambda \cap \mu), \lambda-(\lambda-\mu))}} \\ &= \frac{t^{|\mu-\lambda|} q^{\text{crs}(\mu-\lambda, \lambda \cap \mu)}}{q^{\text{crs}(\lambda-\mu, \lambda \cap \mu)}}. \end{aligned}$$

□

Together Corollaries 14 and 19 for decomposing restricted and induced supercharacters are known as *branching rules*, which we restate due to their importance.

Theorem 20 (Branching Rules). For $\lambda \vdash [n]$, the restriction $\text{Res}_{U_{n-1}}^{U_n}(\chi^\lambda)$ is given by

$$\text{Res}_{U_{n-1}}^{U_n}(\chi^\lambda) = \sum_{\mu \vdash [n-1]} c_\mu^\lambda \chi^\mu$$

where

$$c_\mu^\lambda = \begin{cases} \delta_{\lambda\mu} & \text{if } n \notin \text{re}(\lambda), \\ \frac{t^{|\lambda-\mu|} q^{\text{crs}(\lambda \cap \mu, \lambda - \mu)}}{q^{\text{crs}(\lambda \cap \mu, \mu - \lambda)}} & \text{if } \mu \in C^{\lambda - \{i \prec n\}, i \prec n}, \\ 0 & \text{otherwise.} \end{cases}$$

For $\mu \vdash [n-1]$, the induction $\text{Ind}_{U_{n-1}}^{U_n}(\chi^\mu)$ is given by

$$\text{Ind}_{U_{n-1}}^{U_n}(\chi^\mu) = \sum_{\lambda \vdash [n]} d_\mu^\lambda \chi^\lambda,$$

where

$$d_\mu^\lambda = \begin{cases} \delta_{\lambda\mu} & \text{if } n \notin \text{re}(\lambda) \\ \frac{t^{|\mu-\lambda|} q^{\text{crs}(\mu-\lambda, \lambda \cap \mu)}}{q^{\text{crs}(\lambda-\mu, \lambda \cap \mu)}} & \text{if } \mu \in C^{\lambda - \{i \prec n\}, i \prec n} \\ 0 & \text{otherwise.} \end{cases}$$

This enables us to quickly compute meaningful examples of restricting and inducing a supercharacter multiple times. While these formulas allow us to better understand restriction and induction, they are also useful for Schur–Weyl duality.

4 Shell Tableaux

We use the branching rules to create a graph known as the Bratteli diagram. For the symmetric group, paths in the Bratteli diagram are indexed by a set of combinatorial objects called Young tableaux [16]. Building from the combinatorics of the previous section, we create an analog of Young tableaux known as shell tableaux and construct a bijection between shell tableaux and paths in the Bratteli diagram.

For $k \in \mathbb{Z}_{\geq 1}$, consider

$$V^k = \underbrace{(\text{Ind}_{U_{n-1}}^{U_n} \text{Res}_{U_{n-1}}^{U_n}) \cdots (\text{Ind}_{U_{n-1}}^{U_n} \text{Res}_{U_{n-1}}^{U_n})}_{k \text{ times}}(\mathbb{1})$$

where $\mathbb{1}$ is the trivial supercharacter of U_n that is restricted and induced k times. This is reminiscent of the situation in the partition algebra where the permutation representation of the symmetric group is isomorphic to restricting and then inducing the trivial character. Note that if $V = \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \mathbb{1}$ then

$$V = \mathbb{C}U_n \otimes_{\mathbb{C}U_{n-1}} \text{Res}_{U_{n-1}}^{U_n}(\mathbb{1}) = \text{Ind}_{U_{n-1}}^{U_n} \text{Res}_{U_{n-1}}^{U_n}(\mathbb{1})$$

by the definition of induction. More broadly, we have the following generalization of the tensor identity from [13, (3.18)].

Lemma 21. *Let H be a subgroup of a group G . For a G -module M , the map*

$$\tau : \mathbb{C}G \otimes_{\mathbb{C}H} \text{Res}_H^G(M) \longrightarrow (\mathbb{C}G \otimes_{\mathbb{C}H} \mathbb{1}) \otimes M$$

$$\begin{aligned} g \otimes m &\mapsto (g \otimes \mathbb{1}) \otimes gm \\ g \otimes g^{-1}m &\mapsto (g \otimes \mathbb{1}) \otimes m \end{aligned}$$

is a G -module isomorphism.

Iterating this identity, we obtain

$$V^k \cong V^{\otimes k}.$$

For $\lambda \vdash [n]$, let U_n^λ denote the U_n -supermodule of the supercharacter χ^λ and let

$$\begin{aligned} \hat{Z}_k &= \{ \lambda \vdash [n] \mid \text{Hom}_{U_n}(U_n^\lambda, V^k) \neq \emptyset \} \\ \hat{Z}_{k+\frac{1}{2}} &= \{ \mu \vdash [n-1] \mid \text{Hom}_{U_{n-1}}(U_{n-1}^\mu, \text{Res}_{U_{n-1}}^{U_n}(V^k)) \neq \emptyset \}. \end{aligned}$$

The *Bratteli diagram* $\Lambda(n)$ is the graph with

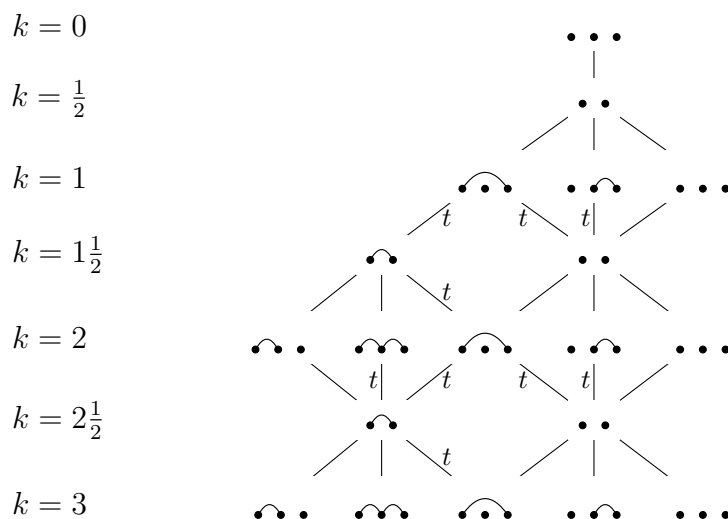
- (a) vertices $\{(\lambda, k) \mid k \in \mathbb{Z}_{\geq 0}, \lambda \in \hat{Z}_k\} \cup \{(\mu, k + \frac{1}{2}) \mid k \in \mathbb{Z}_{\geq 0}, \mu \in \hat{Z}_{k+\frac{1}{2}}\}$,
- (b) an edge $(\lambda, k) \rightarrow (\mu, k + \frac{1}{2})$ if $\langle \text{Res}_{U_{n-1}}^{U_n}(\chi^\lambda), \chi^\mu \rangle \neq 0$,
- (c) an edge $(\mu, k + \frac{1}{2}) \rightarrow (\lambda, k + 1)$ if $\langle \chi^\lambda, \text{Ind}_{U_{n-1}}^{U_n}(\chi^\mu) \rangle \neq 0$,
- (d) an edge labeling $m : E \rightarrow \mathbb{Z}_{\geq 1}$ on the set of edges E defined by

$$\begin{aligned} m((\lambda, k) \rightarrow (\mu, k + \tfrac{1}{2})) &= \frac{(q-1)^{|\lambda-\mu|} q^{\text{crs}(\lambda \cap \mu, \lambda - \mu)}}{q^{\text{crs}(\lambda \cap \mu, \mu - \lambda)}} \\ m((\mu, k + \tfrac{1}{2}) \rightarrow (\lambda, k + 1)) &= \frac{(q-1)^{|\mu-\lambda|} q^{\text{crs}(\mu - \lambda, \lambda \cap \mu)}}{q^{\text{crs}(\lambda - \mu, \lambda \cap \mu)}}. \end{aligned}$$

Recall from the branching rules, Theorem 20, that the edge labeling $m((\lambda, k) \rightarrow (\mu, k + \frac{1}{2}))$ is the restriction coefficient which specifies the multiplicity that χ^μ appears in $\text{Res}_{U_{n-1}}^{U_n}(\chi^\lambda)$. Similarly, the edge labeling $m((\lambda, k + \frac{1}{2}) \rightarrow (\mu, k + 1))$ is the induction coefficient which specifies the multiplicity that χ^λ appears in $\text{Ind}_{U_{n-1}}^{U_n}(\chi^\mu)$.

When drawing the Bratteli diagram, we place all the vertices (λ, l) in the l th row and

simply write λ . For example, the Bratteli diagram for $\Lambda(3)$ up to row 3 is



where $t = q - 1$.

A *path* P in the Bratteli diagram $\Lambda(n)$ to $\lambda \in \hat{Z}_k$ is a sequence

$$P = (\lambda^0, \lambda^{\frac{1}{2}}, \dots, \lambda^{k-\frac{1}{2}}, \lambda^k = \lambda)$$

such that for $0 \leq r \leq k-1$,

- (a) (λ^r, r) and $(\lambda^{r+\frac{1}{2}}, r + \frac{1}{2})$ are vertices in $\Lambda(n)$
- (b) $(\lambda^r, r) \rightarrow (\lambda^{r+\frac{1}{2}}, r + \frac{1}{2})$ and $(\lambda^{r+\frac{1}{2}}, r + \frac{1}{2}) \rightarrow (\lambda^{r+1}, r + 1)$ are edges in $\Lambda(n)$.

For instance,

$$P = \left(\bullet \bullet \bullet , \bullet \bullet , \overset{\frown}{\bullet \bullet \bullet} , \overset{\frown}{\bullet \bullet} , \overset{\frown}{\bullet \bullet \bullet} , \overset{\frown}{\bullet \bullet} , \overset{\frown}{\bullet \bullet \bullet} \right)$$

is a path in $\Lambda(3)$.

Taking the edge labeling into account, we say the *weight* $\text{wt}(P)$ of a path P is the product

$$\prod_{r=1}^{k-1} m((\lambda^r, r) \rightarrow (\lambda^{r+\frac{1}{2}}, r + \frac{1}{2})) m((\lambda^{r+\frac{1}{2}}, r + \frac{1}{2}) \rightarrow (\lambda^{r+1}, r + 1))$$

of its edge labels. The sum of the weights of the paths to $\lambda \in \hat{Z}_k$ is the multiplicity that χ^λ appears in V^k . The path given above has weight t^2 since $m((\lambda^1, 1) \rightarrow (\lambda^{1\frac{1}{2}}, \frac{1}{2})) = t$ and $m((\lambda^2, 2) \rightarrow (\lambda^{2\frac{1}{2}}, \frac{1}{2})) = t$.

Let $\mathcal{P}_k(\lambda)$ be the set of paths in $\Lambda(n)$ to $\lambda \in \hat{Z}_k$. There is a combinatorial way to encode paths in $\mathcal{P}_k(\lambda)$ using a generalization of shells.

Definition 22. Let $s' \in \{s, s+1\}$ for $s \in \mathbb{Z}_{\geq 1}$ and $1 \leq i \leq l \leq n$. A *generalized shell* of width $l-i$ is a set of arcs on n nodes of the form

$$\bigcup_{r=1}^s \{j \frown \min L_r \mid j \in I_r\} \cup \bigcup_{r=1}^{s'-1} \{\max I_r \smile m \mid m \in L_{r+1}\}$$

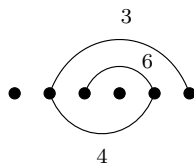
where $I_r, L_r \subseteq [n]$ with $\{i\} = I_1 < \cdots < I_s \leq L_{s'} < \cdots < L_1 = \{l\}$.

For subsets $I, L \subseteq [n]$ we say $I < L$ if $i < l$ for each $i \in I$ and $l \in L$. If $\max I = \min L$, we say $I \leq L$. It follows that a generalized shell with $|I_r| = 1$ and $|L_r| = 1$ for all r is simply a shell in the sense of Definition 5. Some generalized shells of size 6 and width $6-2$ are



$$\{2 \frown 6\} \cup \{2 \smile m \mid m \in \{3, 4\}\}, \quad \{2 \frown 6\} \cup \{j \frown 5 \mid j \in \{3, 4\}\} \cup \{2 \smile 5\}.$$

A *labeled shell* is a pair (ς, τ) for a generalized shell ς and a map $\tau : \varsigma \rightarrow \mathbb{Z}_{\geq 0}$. We say the labeling τ is *strict* if every pair of arcs $(i, l), (j, m) \in \varsigma$ with $\dim(i, j) > \dim(j, m)$ satisfies $\tau(i, j) < \tau(j, m)$, and $\tau(j, m) \neq \tau(i, l) + 1$ if $i = j$ or $l = m$. If $\tau(i, j) = a$, we write the labeled arc as $(i, j; a)$. When the orientation of the arc is specified we write $(i \frown j; a)$ or $(i \smile j; a)$. For example, in the case of the shell



$$\{(2 \frown 6; 3), (3 \frown 5; 6), (2 \smile 5; 4)\}.$$

From strictly labeled shells, we define the key notion shell tableaux.

Definition 23. A *shell tableau* $T = (\varsigma^1, \dots, \varsigma^k)$ of *length* k is a sequence of strictly labeled shells ς^r of size n and width $n - i_r$ such that

1. for $1 \leq r < k$, $\varsigma^r = \{(n \frown n; a)\}$ or $|\varsigma^r| \geq 2$, and $\varsigma^k = \{(i_k \frown n; a)\}$;
2. each arc has a distinct label in $\{1, 2, \dots, \sum_{r=1}^k |\varsigma^r|\}$;
3. the two smallest labels of each labeled shell ς^r are less than the smallest label in ς^{r+1} ;
4. for $l \neq m$ and $i < j \leq \min\{l, m\}$, if $(i, l; a) \in \varsigma^{r_l}$ then there exists a minimal $b > a$ such that $(i, m; b) \in \varsigma^{r_m}$ if and only if $(j, \min\{l, m\}; b+1) \in \varsigma^{r_{\min\{l, m\}}}$;
5. for $i \neq j$ and $\max\{i, j\} \leq l < m$, if $(j, m; a) \in \varsigma^{r_j}$ then there exists a minimal $b > a$ such that $(i, m; b) \in \varsigma^{r_i}$ if and only if $(\max\{i, j\}, l; b+1) \in \varsigma^{r_{\max\{i, j\}}}$.

Conditions 1–3 provide the basic set up of the shells and labeling that are analogous to the condition of increasing entires along the rows and columns in standard Young tableaux. Intuitively conditions 4 and 5 say a strictly labeled shell in a shell tableau has inner whorls if and only if its outer whorl conflicts with the outer whorl of another shell. As an example consider the tableau

$$T = \left(\begin{array}{c} \overset{1}{\curvearrowright} \\ \bullet \bullet \bullet \bullet \\ \underset{2}{\curvearrowright} \end{array}, \bullet \begin{array}{c} \overset{3}{\curvearrowright} \\ \bullet \bullet \bullet \bullet \\ \underset{4}{\curvearrowright} \end{array}, \bullet \begin{array}{c} \overset{5}{\curvearrowright} \\ \bullet \bullet \bullet \bullet \\ \underset{7}{\curvearrowright} \end{array}, \dots \bullet \begin{array}{c} \overset{8}{\curvearrowright} \\ \bullet \\ \underset{}{\curvearrowright} \end{array} \right)$$

of length 4. By condition 4, the inner half whorl $(3 \curvearrowright 5; 6)$ lies in ς^2 since $(2 \curvearrowright 5; 4) \in \varsigma^2$ conflicts with $(2 \curvearrowright 6; 5) \in \varsigma^3$.

Let \mathcal{ST}_k denote the set of shell tableaux of length k .

Definition 24. Define the map

$$\text{sh} : (\mathbb{Z}_{\geq 0}, \mathcal{ST}_k) \longrightarrow \text{Set of Arcs}$$

$$(a, T) \longmapsto \bigcup_{r=1}^k \left\{ (i, l) \in \varsigma^r \mid \begin{array}{l} i \neq l \text{ and } \tau((i, l)) \text{ is maximal} \\ \text{among all labels } b \in \varsigma^r \text{ with } b \leq a \end{array} \right\},$$

and $\text{sh}(T) = \text{sh}(|T|, T)$ be the *shape* of a shell tableau T .

For T in the example above, we have

$$\text{sh}(T) = \bullet \overset{\curvearrowright}{\bullet} \bullet \overset{\curvearrowright}{\bullet} \bullet \bullet$$

because $\tau(1, 4) = 2$ is the maximal label ς^1 , $\tau(3, 5) = 6$ is the maximal label in ς^2 and $\tau(2, 3)$ is the maximal label in ς^3 . For $\lambda \vdash [n]$, let $\mathcal{ST}_k(\lambda)$ denote the set of shell tableaux of shape λ .

Theorem 25. Let $\lambda \in \hat{Z}_k$. There is a bijection between $\mathcal{P}_k(\lambda)$ and $\mathcal{ST}_k(\lambda)$.

Proof. Let $\lambda \in \hat{Z}_k$. Given a path $P = (\lambda^0, \lambda^{\frac{1}{2}}, \lambda^1, \dots, \lambda^{k-\frac{1}{2}}, \lambda^k) \in \mathcal{P}_k(\lambda)$ we will recursively define a sequence

$$T_0, T_{\frac{1}{2}}, T_1, \dots, T_{k-\frac{1}{2}}, T_k$$

where T_j is a shell tableau of length j and shape λ^j , and $T_{j+\frac{1}{2}}$ is a shell tableau of length $j+1$ and shape $\lambda^{j+\frac{1}{2}}$. Let T_0 be the empty shell tableau of length 0.

1. If $\lambda^{j+\frac{1}{2}} = \lambda^j$, we define

$$T_{j+\frac{1}{2}} = (\varsigma_{j+\frac{1}{2}}^1, \varsigma_{j+\frac{1}{2}}^2, \dots, \varsigma_{j+\frac{1}{2}}^{j+1}),$$

where

$$\varsigma_{j+\frac{1}{2}}^r = \begin{cases} \varsigma_j^r & \text{if } r < j+1, \\ (n, n; |T_j| + 1) & \text{if } r = j+1. \end{cases}$$

2. If $\lambda^{j+\frac{1}{2}} \neq \lambda^j$, suppose

$$\bigcup_{s=1}^t \{i_s \frown l_s\} \cup \bigcup_{s=1}^t \{i_s \smile l_{s+1}\}$$

is the shell created by the symmetric difference of λ^r and $\lambda^{r+\frac{1}{2}}$. Since $\text{sh}(T_j) = \lambda^j$, for $1 \leq s \leq t$, $(i_s, l_s; a_s)$ is an arc with maximal label $a_s \leq |T_j|$ in a diagram $\varsigma_j^{r_s}$ of T_j . Let

$$T_{j+\frac{1}{2}} = (\varsigma_{j+\frac{1}{2}}^1, \varsigma_{j+\frac{1}{2}}^2, \dots, \varsigma_{j+\frac{1}{2}}^{j+1}),$$

where

$$\varsigma_{j+\frac{1}{2}}^r = \begin{cases} \varsigma_j^r & \text{if } r \neq r_s \text{ for any } s, \\ \varsigma_j^r \cup (i_s, l_{s+1}; |T_j| + s) & \text{if } r = r_s \text{ for some } s, \\ (n, n; |T_j| + t + 1) & \text{if } r = j + 1. \end{cases}$$

3. If $\lambda^{j+1} = \lambda^{j+\frac{1}{2}}$, define $T_{j+1} = T_{j+\frac{1}{2}}$.

4. If $\lambda^{j+1} \neq \lambda^{j+\frac{1}{2}}$, suppose

$$\bigcup_{s=1}^t \{i_s \frown l_s\} \cup \bigcup_{s=1}^{t-1} \{i_s \smile l_{s+1}\}$$

is the shell created by the symmetric difference of λ^r and $\lambda^{r+\frac{1}{2}}$. Since $\text{sh}(T_{j+\frac{1}{2}}) = \lambda^{j+\frac{1}{2}}$ then for $1 \leq s \leq t-1$, $(i_s, l_{s+1}; a_s)$ is an arc with maximal label $a_s \leq |T_{j+\frac{1}{2}}|$ in a distinct diagram $\varsigma_{j+\frac{1}{2}}^{r_s}$ in $T_{j+\frac{1}{2}}$. We define

$$T_{j+1} = (\varsigma_{j+1}^1, \varsigma_{j+1}^2, \dots, \varsigma_{j+1}^{j+1}),$$

where

$$\varsigma_{j+1}^r = \begin{cases} \varsigma_{j+\frac{1}{2}}^r & \text{if } r \neq r_s \text{ for any } s, \\ \varsigma_{j+\frac{1}{2}}^r \cup (i_{s+1}, l_{s+1}; |T_{j+\frac{1}{2}}| + s) & \text{if } r = r_s \text{ for some } s, \\ (i_1, l_1, |T_{j+\frac{1}{2}}|) & \text{if } r = j + 1. \end{cases}$$

In the above construction, we have $T_{\frac{1}{2}} = (\varsigma_{\frac{1}{2}}^1)$ where $\varsigma_{\frac{1}{2}}^1 = \{(n \frown n; 1)\}$ is a shell tableau of length 1 and shape \emptyset . If T_j is a shell tableau of length j and shape λ^j , then $T_{j+\frac{1}{2}}$ has length $j+1$ and

$$\text{sh}(T_{j+\frac{1}{2}}) = (\lambda^j \cap \lambda^{j+\frac{1}{2}}) \cup (\lambda^{j+\frac{1}{2}} - \lambda^j) = \lambda^{j+\frac{1}{2}}.$$

It is straightforward to check that $T_{j+\frac{1}{2}}$ satisfies conditions 1–4. Since T_j is a shell tableau, it suffices to prove condition 5 for the arcs $(i_s, l_{s+1}; |T_j| + s)$. For $s > 1$, consider $(i_s, l_s; a_s) \in \varsigma_{j+\frac{1}{2}}^{r_s}$. Then $(i_{s-1}, l_s; |T_j| + s - 1)$ lies in $\varsigma_{j+\frac{1}{2}}^{r_{s-1}}$ where $|T_j| + s - 1 > a_s$ is minimal, and $(i_s, l_{s+1}; |T_j| + s) \in \varsigma_{j+\frac{1}{2}}^{r_s}$. Thus condition 5 holds, so $T_{j+\frac{1}{2}}$ is in fact a shell tableau. A similar argument can be used to verify each T_j is a shell tableau of length j and shape λ^j .

For $\lambda \in \hat{Z}_k$, define

$$\begin{array}{ccc} \varphi : \mathcal{P}_k(\lambda) & \longrightarrow & \mathcal{ST}_k(\lambda) \\ P & \longmapsto & T_k. \end{array}$$

The map φ is bijective since the construction of the sequence of shell tableaux can be reversed as follows. Given a shell tableau $T = (\varsigma^1, \varsigma^2, \dots, \varsigma^k)$ of shape λ , let $T_k = T$.

1. If $\varsigma^j = \{(n \frown n; a)\}$, define $T_{j-\frac{1}{2}} = T_j$.
2. If $(i \frown n; a) \in \varsigma^j$ for $i < n$, let $(i_1, l_1; a_1), (i_2, l_2; a_2), \dots, (i_t, l_t; a_t)$ be the arcs in T_j with $i_s \frown l_s \in \varsigma_{i_s}^{r_s}$ and $a_s \geq a$. We define

$$T_{j-\frac{1}{2}} = (\varsigma_{j-\frac{1}{2}}^1, \varsigma_{j-\frac{1}{2}}^2, \dots, \varsigma_{j-\frac{1}{2}}^j),$$

where

$$\zeta_{j-\frac{1}{2}}^r = \begin{cases} \zeta_j^r & \text{if } r \neq r_s \text{ for any } s, \\ \zeta_j^r - (i_s, l_s; a_s) & \text{if } r = r_s \text{ for some } s, \\ (n, n, |T_j| - t + 1) & \text{if } r = j. \end{cases}$$

3. If $\varsigma^j = \{(n \smallfrown n; a)\}$, define $T_j = (\varsigma_j^1, \varsigma_j^2, \dots, \varsigma_j^{j-1})$ where $\varsigma_j^r = \varsigma_{j+\frac{1}{2}}^r$.
4. If $(i \smallfrown n; a) \in \varsigma^j$ for $i < n$, let $(i_1, l_1; a_1), (i_2, l_2; a_2), \dots, (i_t, l_t; a_t)$ be the arcs in T_j with $i_s \smallfrown l_s \in \varsigma_i^{r_s}$ and $a_s > a$. We define

$$T_j = (\varsigma_j^1, \varsigma_j^2, \dots, \varsigma_j^{j-1}),$$

where

$$\zeta_j^r = \begin{cases} \zeta_{j+\frac{1}{2}}^r & \text{if } r \neq r_s \text{ for any } s, \\ \zeta_{j+\frac{1}{2}}^r - (i_s, l_s; a_s) & \text{if } r = r_s \text{ for some } s. \end{cases}$$

Therefore the inverse of φ is

$$\begin{array}{ccc} \varphi^{-1} : \mathcal{ST}_k(\lambda) & \longrightarrow & \mathcal{P}_k(\lambda) \\ T & \mapsto & P = (\mathrm{sh}(T_1), \mathrm{sh}(T_2), \dots, \mathrm{sh}(T_k)). \end{array}$$

Example 26. For the path

[illegible]

the sequence of shell tableaux is

$$\begin{aligned} T_0 &= () \\ T_{\frac{1}{2}} &= \left(\begin{array}{c} \\ \bullet \end{array} \right) \end{aligned}$$

$$\begin{aligned}
T_1 &= \left(\overset{1}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \right) \\
T_{1\frac{1}{2}} &= \left(\overset{1}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{2}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{3}{\circ} \right) \\
T_2 &= \left(\overset{1}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{2}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{3}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \right) \\
T_{2\frac{1}{2}} &= \left(\overset{1}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{2}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{3}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{4}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{5}{\circ} \right) \\
T_3 &= \left(\overset{1}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{2}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{3}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{4}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{5}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \right) \\
T_{3\frac{1}{2}} &= \left(\overset{1}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{2}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{3}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{4}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{5}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{7}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{8}{\circ} \right) \\
T_4 &= \left(\overset{1}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{2}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{3}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{4}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{5}{\curvearrowright} \bullet \bullet \bullet \bullet \bullet \bullet \underset{7}{\curvearrowleft} \bullet \bullet \bullet \bullet \bullet \bullet \overset{8}{\circ} \right).
\end{aligned}$$

Note that each shell ς^r keeps track of the arc introduced at the r th row of the Bratteli diagram from inducing $\text{Res}_{U_{n-1}}^{U_n}(V^{\otimes r-1})$.

When $q = 2$, then $q - 1 = 1$ so that many of the edges in the Bratteli diagram have weight 1. In this case, we can account for the weights of paths in the Bratteli diagram by removing the second condition in the definition of a strict labeling. A *semi-strict shell tableau* is a shell tableau where we allow $\tau(j, m) = \tau(i, l) + 1$ for every pair of arcs $(i, l; \tau(i, l))$ and $(j, m; \tau(j, m))$ in a labeled shell with $\dim(i, l) > \dim(j, m)$ and $i = j$ or $k = l$. This is reminiscent of semi-standard Young tableaux where we allow the entries along the rows to be weakly increasing.

Suppose $\mathcal{SST}_k(\lambda)$ is the set of semi-strict shell tableaux of length k and shape λ . Recall the sum of the weights of paths to λ is the multiplicity of χ^λ in V^k . When $q = 2$, this is the number of semi-strict shell tableaux.

Proposition 27. *Let $q = 2$ and $\lambda \in \hat{Z}_k$. Then*

$$\sum_{P \in \mathcal{P}_k(\lambda)} w(P) = |\mathcal{SST}_k(\lambda)|.$$

Proof. Let $q = 2$ and $\lambda \in \hat{Z}_k$. Let $T = (\varsigma^1, \dots, \varsigma^k) \in \mathcal{ST}_k(\lambda)$ be the shell tableau corresponding to the path $P = (\lambda^0, \lambda^{\frac{1}{2}}, \dots, \lambda^{k-\frac{1}{2}}, \lambda^k)$ via the bijection in the proof of

Theorem 25. Since the weight of P is the product of its edge labels, it suffices to consider the label of a single edge. Recall the label of an edge $(\lambda^r, r) \rightarrow (\lambda^{r+\frac{1}{2}}, r + \frac{1}{2})$ in P is

$$m((\lambda^r, r) \rightarrow (\lambda^{r+\frac{1}{2}}, r + \frac{1}{2})) = \frac{2^{\text{crs}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^r - \lambda^{r+\frac{1}{2}})}}{2^{\text{crs}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^{r+\frac{1}{2}} - \lambda^r)}}.$$

Suppose

$$\bigcup_{s=1}^t \{i_s \frown l_s\} \cup \bigcup_{s=1}^t \{i_s \smile l_{s+1}\}$$

is the shell created by the symmetric difference of λ^r and $\lambda^{r+\frac{1}{2}}$ where $(i_s, l_{s+1}; a_s) \in \varsigma^{r_s}$. For $1 \leq s \leq t$, define the set

$$Y_s = \left\{ m \mid \begin{array}{l} (j \frown m, i_s \frown l_s) \in \text{Crs}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^r - \lambda^{r+\frac{1}{2}}), \\ (j \smile m, i_s \smile l_{s+1}) \notin \text{Crs}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^{r+\frac{1}{2}} - \lambda^r) \end{array} \right\}.$$

Note that

$$\sum_{s=1}^t |Y_s| = \text{crs}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^r - \lambda^{r+\frac{1}{2}}) - \text{crs}(\lambda^r \cap \lambda^{r+\frac{1}{2}}, \lambda^{r+\frac{1}{2}} - \lambda^r).$$

For each subset $X_s \subseteq Y_s$, add the arcs $(i_s, l; b_l) \in \varsigma^{r_s}$ for $l \in X_s$. There is a unique relabeling of the arcs with a distinct label in $\{1, 2, \dots, \sum_{r=1}^k |\varsigma^r| + \sum_{s=1}^t |X_s|\}$ so that the order of the labels of the original arcs in T is preserved, and every pair of arcs $(i, l; \tau(i, l))$ and $(j, m; \tau(j, m))$ in a labeled shell with $\dim(i, l) > \dim(j, m)$ satisfies $\tau(i, l) < \tau(j, m)$. Then each (X_1, \dots, X_t) determines one of the $2^{\sum_{s=1}^t |Y_s|}$ semi-strict shell tableaux. \square

Example 28. Consider the path P from Example 26,

$$P = \left(\dots, \dots, \overset{\curvearrowright}{\dots}, \overset{\curvearrowright}{\dots}, \overset{\curvearrowright}{\dots}, \overset{\curvearrowright}{\dots}, \overset{\curvearrowright}{\dots}, \overset{\curvearrowright}{\dots}, \overset{\curvearrowright}{\dots}, \overset{\curvearrowright}{\dots} \right)$$

and corresponding tableaux

$$T = \left(\overset{1}{\curvearrowright} \dots \overset{2}{\curvearrowright}, \dots \overset{3}{\curvearrowright} \overset{6}{\curvearrowright} \overset{4}{\curvearrowright}, \dots \overset{5}{\curvearrowright} \overset{7}{\curvearrowright}, \dots \overset{8}{\curvearrowright} \right).$$

The path P has weight 2 since $m((\lambda^3, 3) \rightarrow (\lambda^{3\frac{1}{2}}, 3\frac{1}{2})) = 2$. The shell created by the symmetric difference between λ^3 and $\lambda^{3\frac{1}{2}}$ is



and the set $Y_1 = \{4\}$ as $(1 \curvearrowright 4, 2 \curvearrowright 6) \in \text{Crs}(\lambda^3 \cap \lambda^{3\frac{1}{2}}, \lambda^3 - \lambda^{3\frac{1}{2}})$, but $(1 \curvearrowright 4, 2 \curvearrowright 3) \notin \text{Crs}(\lambda^3 \cap \lambda^{3\frac{1}{2}}, \lambda^{3\frac{1}{2}} - \lambda^3)$. The two semi-strict tableaux corresponding to \emptyset and Y_1 are

$$T = \left(\begin{array}{ccccccc} \overset{1}{\curvearrowright} & & \overset{3}{\curvearrowright} & & \overset{5}{\curvearrowright} & & \overset{8}{\curvearrowright} \\ \bullet & \cdots & \bullet & \cdots & \bullet & \cdots & \bullet \\ \underset{2}{\curvearrowright} & & \underset{4}{\curvearrowright} & & \underset{7}{\curvearrowright} & & \end{array} \right)$$

and

$$\tilde{T} = \left(\begin{array}{ccccccc} \overset{1}{\curvearrowright} & & \overset{3}{\curvearrowright} & & \overset{5}{\curvearrowright} & & \overset{9}{\curvearrowright} \\ \bullet & \cdots & \bullet & \cdots & \bullet & \cdots & \bullet \\ \underset{2}{\curvearrowright} & & \underset{4}{\curvearrowright} & & \underset{7}{\curvearrowright} & & \end{array} \right)$$

respectively.

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