On almost-equidistant sets II

Alexandr Polyanskii*

Laboratory of Advanced Combinatorics and Network Applications
Moscow Institute of Physics and Technology (State University)
Dolgoprudny, Russia

Laboratory 4
Institute for Information Transmission Problems RAS
Moscow, Russia

Caucasus Mathematical Center
Adyghe State University
Maykop, Russia
alexander.polyanskii@yandex.ru

Submitted: Jul 31, 2018; Accepted: Apr 12, 2019; Published: May 3, 2019
© The author. Released under the CC BY-ND license (International 4.0).

Abstract

A set in $\mathbb{R}^d$ is called almost-equidistant if for any three distinct points in the set, some two are at unit distance apart. First, we give a short proof of the result of Bezdek and Lángi claiming that an almost-equidistant set lying on a $(d-1)$-dimensional sphere of radius $r$, where $r < 1/\sqrt{2}$, has at most $2d+2$ points. Second, we prove that an almost-equidistant set $V$ in $\mathbb{R}^d$ has $O(d)$ points in two cases: if the diameter of $V$ is at most 1 or if $V$ is a subset of a $d$-dimensional ball of radius at most $1/\sqrt{2} + cd^{-2/3}$, where $c < 1/2$. Also, we present a new proof of the result of Kupavskii, Mustafa and Swanepoel that an almost-equidistant set in $\mathbb{R}^d$ has $O(d^{4/3})$ elements.

Mathematics Subject Classifications: 05C55, 52C99, 51K99

1 Introduction

A set in $\mathbb{R}^d$ is called almost-equidistant if among any three points in the set, some two are at unit distance apart. The natural conjecture [12, Conjecture 12] claims that an almost-equidistant set in $\mathbb{R}^d$ has $O(d)$ points.

*Supported by the Russian Foundation for Basic Research, grant 18-31-00149 mol.a.
Using an elegant linear algebraic argument, Rosenfeld [13] proved that an almost-equidistant set on a \((d-1)\)-dimensional sphere of radius \(1/\sqrt{2}\) has at most \(2d\) points. Note that the set of the vertices of two unit \((d-1)\)-simplices lying on the same \((d-1)\)-dimensional sphere of radius \(1/\sqrt{2}\) is almost-equidistant. Bezdek and Lángi [2, Theorem 1] generalized Rosenfeld’s approach and showed that an almost-equidistant set on a \((d-1)\)-dimensional sphere of radius at most \(1/\sqrt{2}\) has at most \(2d + 2\) points; this bound is tight because the vertices of two unit \(d\)-simplices inscribed in the same sphere form an almost-equidistant set.

Balko, Pór, Scheucher, Swanepoel and Valtr [1, Theorem 6] showed that an almost-equidistant set in \(\mathbb{R}^d\) has \(O\left(d^{3/2}\right)\) points. This bound was improved by the author [12, Theorem 1] to \(O(d^{13/9})\). Recently, Kupavskii, Mustafa, Swanepoel [8] further improved to \(O(d^{4/3})\). For more references we refer interested readers to [1, Section 1].

The first goal of this paper is to give a short proof of the result of Bezdek–Lángi [2] using a lifting argument and the fact that an almost-equidistant set on a \((d-1)\)-dimensional sphere has at most \(2d\) points. The second aim is to confirm the conjecture in two cases: for almost-equidistant sets of diameter 1 (see Section 4) and for almost-equidistant sets lying in a \(d\)-dimensional ball of radius \(1/\sqrt{2} + cd^{-2/3}\), where \(c < 1/2\) (see Section 5). The third goal is to give a new proof of the upper bound \(O(d^{4/3})\) for the size of an almost-equidistant set in \(\mathbb{R}^d\) (see Section 6). Also, we discuss several open problems related to almost-equidistant sets (see Section 7).

2 Preliminaries

Let \(\{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d\) be an almost-equidistant set. Consider the matrix

\[
U := \|v_i - v_j\|^2 + I_n - J_n,
\]

where \(J_n\) is the \(n\times n\) matrix of ones and \(I_n\) is the identity matrix of size \(n\). We need two simple facts proved in [12, Corollary 4 and Lemma 5]. We join them in the following lemma.

Lemma 1. The matrix \(U\) satisfies the following two properties.
1. The equalities \(\text{tr}(U) = \text{tr}(U^3) = 0\) hold.
2. The matrix \(U\) has at most one eigenvalue larger than 1 and at least \(n - d - 2\) eigenvalues equal to 1.

We use Lemma 1 combined with the following lemma several times.

Lemma 2. Let \(\lambda_0, 1, \ldots, 1, \lambda_1, \ldots, \lambda_k\) be the eigenvalues of an Hermitian matrix \(A\) of size \(n\), indexed in nondecreasing order. Assume \(\text{tr}(A) = \text{tr}(A^3) = 0\).
1. If \(\lambda_0 = 1\), then \(n \leq 2k\).
2. If \(\lambda_0 + \lambda_k \leq 0\), then \(n \leq 2k\).
3. If \(n \geq 2k\) and \(\lambda_0 > 1\), then

\[
\lambda_0^3 > \frac{(n-k)^3}{k^2} - (n-k-1).
\]
Proof. 1. Suppose to the contrary that \( n > 2k \). Introducing the notation \( l := n - 2k \), we can rewrite the equations \( \text{tr}(A) = \text{tr}(A^3) = 0 \) as

\[
\sum_{i=1}^{k} (-\lambda_i) = \sum_{i=1}^{k} (-\lambda_i)^3 = k + l.
\]

To finish the proof, we need Lemma 1 in [2]. For the sake of completeness we provide its proof here.

Lemma 3. Let \( x_1, \ldots, x_m \) be real numbers such that \( x_i \geq -2 \) for \( i = 1, \ldots, m \). If \( \sum_{i=1}^{m} x_i = (m + l) \), where \( l \geq 0 \), then

\[
\sum_{i=1}^{m} x_i^3 \geq \frac{(m + l)^3}{m^2}.
\]

Here the equality is possible if and only if the \( x_i \) are equal to \( 1 + l/m \).

Remark 4. Also, note that \( (m + l)^3/m^2 \geq m + 3l \).

Proof. Consider functions \( f, g : [-2, +\infty) \to \mathbb{R} \) such that

\[
f(x) = x^3 \text{ for any } x \geq -2, \quad g(x) = \begin{cases} 3x - 2, & \text{for any } -2 \leq x \leq 1, \\ x^3, & \text{for any } 1 \leq x. \end{cases}
\]

For \(-2 \leq x \leq 1\) we have \( g(x) \leq f(x) \) because in this range \( g(x) \) has the value of a tangent line to \( f(x) \) at \( x = 1 \) and the second point of the intersection of that tangent line and \( f(x) \) is at \( x = -2 \). Further, \( g(x) \) is a convex function in the range \(-2 \leq x \). By Jensen’s Inequality, we obtain

\[
\sum_{i=1}^{m} x_i^3 = \sum_{i=1}^{m} f(x_i) \geq \sum_{i=1}^{m} g(x_i) \geq mg\left(\frac{\sum_{i=1}^{m} x_i}{m}\right) = m\left(\frac{\sum_{i=1}^{m} x_i}{m}\right)^3 = \frac{(m + l)^3}{m^2} \geq m + 3l.
\]

The equality case we leave as an exercise.

Clearly, Lemma 3 implies that

\[
\sum_{i=1}^{k} (-\lambda_i)^3 \geq k + 3l,
\]

and this is a contradiction with the inequality \( l > 0 \).

2. Suppose to the contrary that \( n \geq 2k + 1 \). Introducing the notation \( l := n - 2k - 1 \), we can rewrite the equalities \( \text{tr}(A) = \text{tr}(A^3) = 0 \) as

\[
\sum_{i=1}^{k-1} (-\lambda_i) + (-\lambda_0 - \lambda_k) = \sum_{i=1}^{k-1} (-\lambda_i)^3 - \lambda_0^3 - \lambda_k^3 = k + l. \tag{2}
\]
By Lemma 3, we obtain
\[ \sum_{i=1}^{k-1} (-\lambda_i)^3 + (-\lambda_0 - \lambda_k)^3 \geq k + 3l. \] (3)

The second equality in (2) implies
\[ k + l + (-\lambda_0 - \lambda_k)^3 + \lambda_0^3 + \lambda_k^3 \geq k + 3l, \]
and so
\[ -3\lambda_0\lambda_k(\lambda_0 + \lambda_k) \geq 2l. \] (4)

Since \( \lambda_0 > 0, \lambda_k < 0, \lambda_0 + \lambda_k \leq 0 \) and \( l \geq 0 \), we get \( \lambda_0 + \lambda_k = 0 \) and \( l = 0 \). By Lemma 3, if \( \lambda_1 + \lambda_k = 0 \) then we have a strict inequality in (3), and thus we get a strict inequality in (4). This implies a contradiction with the equality \( l = 0 \).

3. Clearly, we can rewrite the equalities \( \text{tr}(A) = \text{tr}(A^3) = 0 \) as
\[ \sum_{i=1}^{k} (-\lambda_i) = \lambda_0 + n - k - 1 > n - k \]
and
\[ \lambda_0^3 = \sum_{i=1}^{k} (-\lambda_i)^3 - (n - k - 1). \]

Since \( n \geq 2k \), Lemma 3 implies that
\[ \lambda_0^3 > \frac{(n - k)^3}{k^2} - (n - k - 1). \]

**Corollary 5.** If the matrix \( U \) does not have an eigenvalue larger than 1, then \( n \leq 2d + 4 \).

**Proof.** The proof easily follows from Lemmas 1 and 2 (case 1).

We need the following technical lemma only in the proof of the fact that an almost-equidistant set has \( O(d^{1/3}) \) points; see Theorem 14.

**Lemma 6.** If \( \{w_0, w_1, \ldots, w_k\} \) is an almost-equidistant set in \( \mathbb{R}^d \) such that
\[ |\|w_i\|^2 - 1/2| \leq x \] (5)
for \( 0 \leq i \leq k \) and a positive \( x \), then
\[ \left| \sum_{1 \leq i \leq k} (\|w_0 - w_i\|^2 - 1) \right| \leq c \left( d^{1/2} + dx^{3/2} + dx \right) \]
for some positive constant \( c \) independent of \( d \) and \( x \).
Proof. We may assume without loss of generality that \( \|w_0 - w_i\| \neq 1 \) for \( 1 \leq i \leq k \). Since the set is almost-equidistant, the points \( w_1, \ldots, w_k \) form a regular unit \((k - 1)\)-simplex, and so \( k \leq d + 1 \leq 2d \).

We use the following theorem (see [4, Theorem 1]) several times.

**Theorem 7.** Let \( X = \{x_1, \ldots, x_n\} \) and \( Y = \{y_1, \ldots, y_n\} \) be two point-sets in \( \mathbb{R}^d \). Then

\[
\sum_{1 \leq i,j \leq n} \|x_i - y_j\|^2 = \sum_{1 \leq i,j \leq n} \|x_i - x_j\|^2 + \sum_{1 \leq i,j \leq n} \|y_i - y_j\|^2 + n^2 \|x - y\|^2,
\]

where \( x \) and \( y \) are the barycenters of \( X \) and \( Y \), respectively, that is,

\[
x = (x_1 + \cdots + x_n)/n, \quad y = (y_1 + \cdots + y_n)/n.
\]

Theorem 7 applied to \( \{w_0, \ldots, w_0\} \) and \( \{w_1, \ldots, w_k\} \) implies

\[
\sum_{1 \leq i \leq k} \|w_0 - w_i\|^2 = \frac{k - 1}{2} + k \|w_0 - o'\|^2 = \frac{k - 1}{2} + k \|w_0\|^2 + k \|o'\|^2 + O(1) \cdot k \|w_0\| \|o'\|, \tag{6}
\]

where \( o' \) is the center of the simplex \( w_1 \ldots w_k \). Theorem 7 applied to \( \{o, \ldots, o\} \) and \( \{w_1, \ldots, w_k\} \), where \( o \) is the origin, and combined with (5) yields

\[
k \|o'\|^2 = \sum_{1 \leq i \leq k} \|w_i\|^2 - \frac{k - 1}{2} = \frac{k}{2} - \frac{k - 1}{2} + O(1) \cdot kx = 1 \cdot 2 + O(1) \cdot kx. \tag{7}
\]

By (5) and (7), we obtain

\[
(6) = \frac{k - 1}{2} + \frac{k}{2} + \frac{1}{2} + O(1) \cdot kx + O(1) \cdot k(1 + x^{1/2})(k^{-1/2} + x^{1/2})
\]

\[
= k + O(1) \cdot (k^{1/2} + kx^{1/2} + kx).
\]

The fact \( k \leq 2d \) finishes the proof of the lemma.

---

### 3 A simple proof of the result of Bezdek and Lángi

**Theorem 8** (Bezdek, Lángi, 1999). For \( r \leq 1/\sqrt{2} \), an almost-equidistant set lying on a \((d - 1)\)-dimensional sphere \( S \) of radius \( r \) contains at most \( 2d + 2 \) points.

**Proof.** Assume that \( S \) is embedded in \( \mathbb{R}^{d+1} \) and its center \( o \) and a point \( o' \in \mathbb{R}^{d+1} \) are such that the line \( o'o \) is orthogonal to the affine hull of \( S \) and \( |o' - o|^2 = 1/2 - r \). Now, we get that the almost-equidistant set lies on the \( d \)-dimensional sphere of radius \( 1/\sqrt{2} \) with center \( o' \). By Rosenfeld’s theorem [13], the size of the almost-equidistant sets is at most \( 2(d + 1) \). \( \square \)
4 Almost-equidistant diameter sets

A subset of $\mathbb{R}^d$ is called an almost-equidistant diameter set if it is almost-equidistant and has diameter 1; see [12, Proposition 13 and Problem 14]. The next theorem is about the maximal size of such sets.

**Theorem 9.** An almost-equidistant diameter set in $\mathbb{R}^d$ has at most $2d + 4$ points.

**Proof.** Clearly, the matrix $U$ (see (1)) for an almost-equidistant diameter set in $\mathbb{R}^d$ has non-positive entries. Lemma 1 and Corollary 5 imply that we can assume without loss of generality that $U$ has exactly one eigenvalue $\lambda$ larger than 1.

We need the following week form of the Perron–Frobenius Theorem; see [11, 5].

**Theorem 10** (Perron–Frobenius Theorem). If an $n$-by-$n$ matrix has non-negative entries, then it has a non-negative real eigenvalue, which has maximum absolute value among all eigenvalues.

By the Perron–Frobenius Theorem, the matrix $U$ has a negative eigenvalue $\lambda'$ such that $|\lambda'| \geq \lambda$. Therefore, Lemmas 1 and 2 (case 2) imply that the almost-equidistant diameter set has at most $2d + 2$ points. $\square$

5 Almost-equidistant sets in small balls

**Theorem 11.** Let $0 \leq c_0 < 1/2$ be a fixed constant. If an almost-equidistant set lies in a $d$-dimensional ball of radius $\sqrt{1/2 + c_0/(d + 1)^{2/3}}$, then its cardinality is $O(d)/(1 - 2c_0)$.

**Proof.** Let $\{v_1, \ldots, v_n\}$ be an almost-equidistant set in $\mathbb{R}^d$ such that $n \geq 2d + 2$ and $\|v_i\|^2 = 1/2 + c_i$ for $1 \leq i \leq n$, where $c_i \leq c_0/(d + 1)^{2/3} = r$.

By Corollary 5 and Lemma 1, without loss of generality we can assume that the matrix $U$ (see (1)) for this set has exactly one eigenvalue $\lambda$ larger than 1. Hence Lemmas 1 and 2 (case 3) for $U$ yield

$$
\lambda^3 > \frac{(n - d - 1)^3}{(d + 1)^2} - (n - d - 2).
$$

To finish the proof, we need the following special case of Weyl’s Inequality [14, Theorem 1].

**Theorem 12** (Weyl’s Inequality). If $\alpha, \beta$ and $\gamma$ are the largest eigenvalues of Hermitian matrices $A, B$ and $A + B$ respectively, then $\gamma \leq \alpha + \beta$.

Clearly, by (1), we have

$$
U = (c^t j + j^t c) - 2\langle v_i, v_j \rangle + I,
$$

where $c = (c_1, \ldots, c_n)$ and $j = (1, \ldots, 1)$. Obviously, the largest eigenvalues of the matrices $c^t j$ and $j^t c$ do not exceed $nr$. Therefore, from Weyl’s Inequality and positivity of the Gram matrix $\langle v_i, v_j \rangle$, we conclude that $\lambda \leq 2nr + 1$. This forces

$$
(2nr + 1)^3 \geq \lambda^3 > \frac{(n - d - 1)^3}{(d + 1)^2} - (n - d - 2),
$$

6
and so

\[(2c_0x + o(1))^3 \geq (x - 1)^3 - (x - 1 - o(1)),\]

where \(x = n/(d+1)\). Hence we get \(x = O(1)/(1 - 2c_0)\), and thus \(n = O(d)/(1 - 2c_0)\). \(\square\)

**Remark 13.** Interestingly, an almost-equidistant set lying in a \(d\)-dimensional ball of radius \(1/\sqrt{2}\) has at most \(2^{d+4}\) points. Indeed, in that case \(r\) is equal to 0, and hence (8) yields \(n < 2d + 2\) if \(U\) has an eigenvalue larger then 1. Therefore, by Corollary 5, we get \(n \leq 2d + 4\).

## 6 Almost-equidistant sets: general case

**Theorem 14** (Kupavskii, Mustafa, Swanepoel, 2018). The size of an almost-equidistant set in \(\mathbb{R}^d\) is \(O(d^{4/3})\).

**Proof.** Assume that \(\{v_1, \ldots, v_n\}\) is an almost-equidistant set in \(\mathbb{R}^d\). Let

\[f := \max_{i=1,\ldots,n} \left\{ \left| \sum_{j=1}^{n} (\|v_i - v_j\|^2 - 1) \right| \right\}. \tag{9}\]

Assuming that \(\sum_{i=1}^{n} v_i\) is the origin \(o\), we easily get

\[-f \leq n\|v_i\|^2 + \sum_{j=1}^{n} \|v_j\|^2 - n \leq f\]

for \(1 \leq i \leq n\). Summing up these inequalities for \(1 \leq i \leq n\), we obtain

\[-f/2 \leq \sum_{i=1}^{n} \|v_i\|^2 - n/2 \leq f/2.\]

The last two inequalities implies

\[-3f/(2n) \leq \|v_i\|^2 - 1/2 \leq 3f/(2n) \tag{10}\]

for \(1 \leq i \leq n\). By (9), (10) and Lemma 6, we have

\[f \leq O(1) \cdot \left( d^{1/2} + d(f/n)^{1/2} + df/n \right).\]

Therefore, we obtain either \(n = O(d)\) or

\[f = O(1) \cdot \left( d^{1/2} + d^2/n \right). \tag{11}\]

Suppose to the contrary that \(n \geq Cd^{4/3}\), where \(C\) is a positive constant. Hence, by (10) and (11), the almost-equidistant set lies in a ball of radius \(1/\sqrt{2} + O(d^{-2/3})/C^2\). Hence if \(C\) is big enough, then we can apply Theorem 11 to the almost-equidistant set, and thus \(n = O(d)\), a contradiction. \(\square\)

**Remark 15.** It is possible to prove Theorem 14 using Lemma 6 combined with the Gershgorin Circle Theorem [6]; see [12, Section 3.2] for a similar argument.
7 Discussion

7.1 Almost-equidistant diameter sets

A graph \((V, E)\) is called a diameter graph if its vertex set \(V \subseteq \mathbb{R}^d\) is a set of points of diameter 1 and a pair of vertices forms an edge if they are at unit distance apart. Of course, the set of vertices of two cliques in a diameter graph is an almost-equidistant diameter set. For instance, in [9, the last paragraph of Section 3] there is given an example of diameter graph in \(\mathbb{R}^d\) consisting of two cliques without common vertices such that they have \(d + 1\) and \(\lfloor \frac{d+1}{2}\rfloor\) vertices respectively. We believe that the vertex set of this diameter graph has the maximal size among almost-equidistant diameter sets in \(\mathbb{R}^d\).

Conjecture 16. An almost-equidistant diameter set in \(\mathbb{R}^d\) has at most \(\lfloor \frac{3(d+1)}{2}\rfloor\) points.

There is the following conjecture [7, Conjecture 5.5] that arose in the context of study of cliques in diameter graphs.

Conjecture 17 (Schur). Let \(S_1\) and \(S_2\) be two unit simplices in \(\mathbb{R}^d\) forming a set of diameter 1 such that they have \(k\) and \(m\) vertices respectively. Then \(S_1\) and \(S_2\) share at least \(\min\{0, k + 2m - 2d - 2\}\) vertices for \(k \geq m\).

Clearly, this conjecture is closely related to Conjecture 16. Note that Conjecture 17 was confirmed in two special (but not trivial!) cases: \((k,m,d) = (d, d, d)\), where \(d \geq 2\), in [9] and \((k, m, d) = (5, 3, 4)\) in [10].

7.2 Two-distant almost-equidistant sets

A subset of \(\mathbb{R}^d\) is called a two-distant set if there are only two distances formed by any two distinct points of the set. The following question seems to be interesting.

Problem 18. What is the largest cardinality of a set in \(\mathbb{R}^d\) that is two-distant and almost-equidistant at the same time?

Consider a two-distant almost-equidistant set \(V \subseteq \mathbb{R}^d\) with distances 1 and \(a\) between points of \(V\), where \(a > 1\); the case \(a < 1\) is not interesting because of Theorem 9. Lemma 1 and Corollary 5 imply that we may assume without loss of generality that \(U\) for \(V\) has exactly one eigenvalue \(\lambda\) larger than 1. Note that the matrix \(U/(a^2 - 1)\) is the adjacency matrix of a triangle-free graph. By Lemma 1, Problem 18 is reduced to following question.

Problem 19. What is the minimal rank of the matrix \(A - \lambda_2 I_n\), where \(A\) is the adjacency matrix of a triangle-free graph on \(n\) vertices and \(\lambda_2\) is its second largest eigenvalue provided that \(\lambda_2 > 0\)?

It is worth pointing out that for every positive integer \(n\) there are a triangle-free graph on \(n\) vertices and an eigenvalue \(\lambda\) of the adjacency matrix \(A\) of the graph such that \(\text{rank}(A - \lambda I) = O(n^{3/4})\) and \(\lambda > 0\); see [3, the proof of Theorem 5, pages 94–95].
Acknowledgements

We are grateful to Zilin Jiang for bringing the Perron–Frobenius Theorem to our attention.

References