On almost-equidistant sets II

Alexandr Polyanskii*

Laboratory of Advanced Combinatorics and Network Applications
Moscow Institute of Physics and Technology (State University)
Dolgoprudny, Russia

Laboratory 4
Institute for Information Transmission Problems RAS
Moscow, Russia

Caucasus Mathematical Center
Adyghe State University
Maykop, Russia
alexander.polyanskii@yandex.ru

Submitted: Jul 31, 2018; Accepted: Apr 12, 2019; Published: May 3, 2019

Abstract

A set in $\mathbb{R}^d$ is called *almost-equidistant* if for any three distinct points in the set, some two are at unit distance apart. First, we give a short proof of the result of Bezdek and Lángi claiming that an almost-equidistant set lying on a $(d-1)$-dimensional sphere of radius r, where $r < 1/\sqrt{2}$, has at most $2d+2$ points. Second, we prove that an almost-equidistant set V in $\mathbb{R}^d$ has $O(d)$ points in two cases: if the diameter of V is at most 1 or if V is a subset of a d-dimensional ball of radius at most $1/\sqrt{2} + cd^{-2/3}$, where $c < 1/2$. Also, we present a new proof of the result of Kupavskii, Mustafa and Swanepoel that an almost-equidistant set in $\mathbb{R}^d$ has $O(d^{4/3})$ elements.

Mathematics Subject Classifications: 05C55, 52C99, 51K99

1 Introduction

A set in $\mathbb{R}^d$ is called *almost-equidistant* if among any three points in the set, some two are at unit distance apart. The natural conjecture [12, Conjecture 12] claims that an almost-equidistant set in $\mathbb{R}^d$ has $O(d)$ points.

*Supported by the Russian Foundation for Basic Research, grant 18-31-00149 mol_a.

Using an elegant linear algebraic argument, Rosenfeld [13] proved that an almost-equidistant set on a $(d-1)$-dimensional sphere of radius $1/\sqrt{2}$ has at most $2d$ points. Note that the set of the vertices of two unit $(d-1)$-simplices lying on the same $(d-1)$-dimensional sphere of radius $1/\sqrt{2}$ is almost-equidistant. Bezdek and Lángi [2, Theorem 1] generalized Rosenfeld's approach and showed that an almost-equidistant set on a $(d-1)$-dimensional sphere of radius at most $1/\sqrt{2}$ has at most $2d+2$ points; this bound is tight because the vertices of two unit d-simplices inscribed in the same sphere form an almost-equidistant set.

Balko, Pór, Scheucher, Swanepoel and Valtr [1, Theorem 6] showed that an almost equidistant set in $\mathbb{R}^d$ has $O(d^{3/2})$ points. This bound was improved by the author [12, Theorem 1] to $O(d^{13/9})$. Recently, Kupavskii, Mustafa, Swanepoel [8] further improved to $O(d^{4/3})$. For more references we refer interested readers to [1, Section 1].

The first goal of this paper is to give a short proof of the result of Bezdek–Lángi [2] using a lifting argument and the fact that an almost-equidistant set on a $(d-1)$-dimensional sphere has at most $2d$ points. The second aim is to confirm the conjecture in two cases: for almost-equidistant sets of diameter 1 (see Section 4) and for almost-equidistant sets lying in a d-dimensional ball of radius $1/\sqrt{2}+cd^{-2/3}$, where $c<1/2$ (see Section 5). The third goal is to give a new proof of the upper bound $O(d^{4/3})$ for the size of an almost-equidistant set in $\mathbb{R}^d$ (see Section 6). Also, we discuss several open problems related to almost-equidistant sets (see Section 7).

2 Preliminaries

Let $\{\mathbf{v}_1,\dots,\mathbf{v}_n\}\subset\mathbb{R}^d$ be an almost-equidistant set. Consider the matrix

$$\mathbf{U} := \|\mathbf{v}_i-\mathbf{v}_j\|^2+\mathbf{I}_n-\mathbf{J}_n, \tag{1}$$

where $\mathbf{J}_n$ is the n-by-n matrix of ones and $\mathbf{I}_n$ is the identity matrix of size n. We need two simple facts proved in [12, Corollary 4 and Lemma 5]. We join them in the following lemma.

Lemma 1. *The matrix $\mathbf{U}$ satisfies the following two properties.*

1. The equalities $\operatorname{tr}(\mathbf{U})=\operatorname{tr}(\mathbf{U}^3)=0$ hold.

2. The matrix $\mathbf{U}$ has at most one eigenvalue larger than 1 and at least $n-d-2$ eigenvalues equal to 1.

We use Lemma 1 combined with the following lemma several times.

Lemma 2. *Let $\lambda_0,1,\dots,1,\lambda_1,\dots,\lambda_k$ be the eigenvalues of an Hermitian matrix $\mathbf{A}$ of size n, indexed in nondecreasing order. Assume $\operatorname{tr}(\mathbf{A})=\operatorname{tr}(\mathbf{A}^3)=0$.*

1. If $\lambda_0=1$, then $n\leqslant 2k$.

2. If $\lambda_0+\lambda_k\leqslant 0$, then $n\leqslant 2k$.

3. If $n\geqslant 2k$ and $\lambda_0>1$, then

$$\lambda_0^3>\frac{(n-k)^3}{k^2}-(n-k-1).$$

Proof. 1. Suppose to the contrary that $n > 2k$. Introducing the notation $l := n - 2k$, we can rewrite the equations $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^3) = 0$ as

$$\sum_{i=1}^{k}(-\lambda_i) = \sum_{i=1}^{k}(-\lambda_i)^3 = k + l.$$

To finish the proof, we need Lemma 1 in [2]. For the sake of completeness we provide its proof here.

Lemma 3. *Let $x_1, \ldots, x_m$ be real numbers such that $x_i \geqslant -2$ for $i = 1, \ldots, m$. If $\sum_{i=1}^{m} x_i = (m + l)$, where $l \geqslant 0$, then*

$$\sum_{i=1}^{m} x_i^3 \geqslant \frac{(m+l)^3}{m^2}.$$

Here the equality is possible if and only if the x_i are equal to $1 + l/m$.

Remark 4. Also, note that $(m + l)^3/m^2 \geqslant m + 3l$.

Proof. Consider functions $f, g : [-2, +\infty) \to \mathbb{R}$ such that

$$f(x) = x^3 \text{ for any } x \geqslant -2, \;\; g(x) = \begin{cases} 3x - 2, & \text{for any } -2 \leqslant x \leqslant 1, \\ x^3, & \text{for any } 1 \leqslant x. \end{cases}$$

For $-2 \leqslant x \leqslant 1$ we have $g(x) \leqslant f(x)$ because in this range $g(x)$ has the value of a tangent line to $f(x)$ at $x = 1$ and the second point of the intersection of that tangent line and $f(x)$ is at $x = -2$. Further, $g(x)$ is a convex function in the range $-2 \leqslant x$. By Jensen's Inequality, we obtain

$$\begin{aligned} \sum_{i=1}^{m} x_i^3 = \sum_{i=1}^{m} f(x_i) \geqslant \sum_{i=1}^{m} g(x_i) \geqslant mg\left(\frac{\sum_{i=1}^{m} x_i}{m}\right) \\ = m\left(\frac{\sum_{i=1}^{m} x_i}{m}\right)^3 = \frac{(m+l)^3}{m^2} \geqslant m + 3l. \end{aligned}$$

The equality case we leave as an exercise. □

Clearly, Lemma 3 implies that

$$\sum_{i=1}^{k}(-\lambda_i)^3 \geqslant k + 3l,$$

and this is a contradiction with the inequality $l > 0$.

2. Suppose to the contrary that $n \geqslant 2k + 1$. Introducing the notation $l := n - 2k - 1$, we can rewrite the equalities $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^3) = 0$ as

$$\sum_{i=1}^{k-1}(-\lambda_i) + (-\lambda_0 - \lambda_k) = \sum_{i=1}^{k-1}(-\lambda_i)^3 - \lambda_0^3 - \lambda_k^3 = k + l. \tag{2}$$

By Lemma 3, we obtain

$$\sum_{i=1}^{k-1}(-\lambda_i)^3 + (-\lambda_0 - \lambda_k)^3 \geqslant k + 3l. \tag{3}$$

The second equality in (2) implies

$$k + l + (-\lambda_0 - \lambda_k)^3 + \lambda_0^3 + \lambda_k^3 \geqslant k + 3l,$$

and so

$$-3\lambda_0\lambda_k(\lambda_0 + \lambda_k) \geqslant 2l. \tag{4}$$

Since $\lambda_0 > 0$, $\lambda_k < 0$, $\lambda_0 + \lambda_k \leqslant 0$ and $l \geqslant 0$, we get $\lambda_0 + \lambda_k = 0$ and $l = 0$. By Lemma 3, if $\lambda_1 + \lambda_k = 0$ then we have a strict inequality in (3), and thus we get a strict inequality in (4). This implies a contradiction with the equality $l = 0$.

3. Clearly, we can rewrite the equalities $\mathrm{tr}(\mathbf{A}) = \mathrm{tr}(\mathbf{A}^3) = 0$ as

$$\sum_{i=1}^{k}(-\lambda_i) = \lambda_0 + n - k - 1 > n - k$$

and

$$\lambda_0^3 = \sum_{i=1}^{k}(-\lambda_i)^3 - (n - k - 1).$$

Since $n \geqslant 2k$, Lemma 3 implies that

$$\lambda_0^3 > \frac{(n-k)^3}{k^2} - (n - k - 1).$$

□

Corollary 5. *If the matrix* $\mathbf{U}$ *does not have an eigenvalue larger than* 1*, then* $n \leqslant 2d+4$.

Proof. The proof easily follows from Lemmas 1 and 2 (*case 1*). □

We need the following technical lemma only in the proof of the fact that an almost-equidistant set has $O(d^{4/3})$ points; see Theorem 14.

Lemma 6. *If* $\{\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_k\}$ *is an almost-equidistant set in* $\mathbb{R}^d$ *such that*

$$\big|\|\mathbf{w}_i\|^2 - 1/2\big| \leqslant x \tag{5}$$

for $0 \leqslant i \leqslant k$ *and a positive* x*, then*

$$\left|\sum_{1\leqslant i\leqslant k}\left(\|\mathbf{w}_0 - \mathbf{w}_i\|^2 - 1\right)\right| \leqslant c\left(d^{1/2} + dx^{1/2} + dx\right)$$

for some positive constant c *independent of* d *and* x*.*

Proof. We may assume without loss of generality that $\|\mathbf{w}_0 - \mathbf{w}_i\| \neq 1$ for $1 \leqslant i \leqslant k$. Since the set is almost-equidistant, the points $\mathbf{w}_1, \dots, \mathbf{w}_k$ form a regular unit $(k-1)$-simplex, and so $k \leqslant d+1 \leqslant 2d$.

We use the following theorem (see [4, Theorem 1]) several times.

Theorem 7. *Let $X = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ and $Y = \{\mathbf{y}_1, \dots, \mathbf{y}_n\}$ be two point-sets in $\mathbb{R}^d$. Then*

$$\sum_{1 \leqslant i,j \leqslant n} \|\mathbf{x}_i - \mathbf{y}_j\|^2 = \sum_{1 \leqslant i < j \leqslant n} \|\mathbf{x}_i - \mathbf{x}_j\|^2 + \sum_{1 \leqslant i < j \leqslant n} \|\mathbf{y}_i - \mathbf{y}_j\|^2 + n^2 \|\mathbf{x} - \mathbf{y}\|^2,$$

where $\mathbf{x}$ and $\mathbf{y}$ are the barycenters of X and Y, respectively, that is,

$$\mathbf{x} = (\mathbf{x}_1 + \dots + \mathbf{x}_n)/n, \ \mathbf{y} = (\mathbf{y}_1 + \dots + \mathbf{y}_n)/n.$$

Theorem 7 applied to $\{\mathbf{w}_0, \dots, \mathbf{w}_0\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ implies

$$\sum_{1 \leqslant i \leqslant k} \|\mathbf{w}_0 - \mathbf{w}_i\|^2 = \frac{k-1}{2} + k\|\mathbf{w}_0 - \mathbf{o}'\|^2 = \frac{k-1}{2} + k\|\mathbf{w}_0\|^2 + k\|\mathbf{o}'\|^2 + O(1) \cdot k\|\mathbf{w}_0\|\|\mathbf{o}'\|, \quad (6)$$

where $\mathbf{o}'$ is the center of the simplex $\mathbf{w}_1 \dots \mathbf{w}_k$. Theorem 7 applied to $\{\mathbf{o}, \dots, \mathbf{o}\}$ and $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$, where $\mathbf{o}$ is the origin, and combined with (5) yields

$$k\|\mathbf{o}'\|^2 = \sum_{1 \leqslant i \leqslant k} \|\mathbf{w}_i\|^2 - \frac{k-1}{2} = \frac{k}{2} - \frac{k-1}{2} + O(1) \cdot kx = \frac{1}{2} + O(1) \cdot kx. \quad (7)$$

By (5) and (7), we obtain

$$\begin{aligned}(6) = \frac{k-1}{2} + \frac{k}{2} + \frac{1}{2} + O(1) \cdot kx + O(1) \cdot k(1 + x^{1/2})(k^{-1/2} + x^{1/2}) \\ = k + O(1) \cdot (k^{1/2} + kx^{1/2} + kx).\end{aligned}$$

The fact $k \leqslant 2d$ finishes the proof of the lemma. □

3 A simple proof of the result of Bezdek and Lángi

Theorem 8 (Bezdek, Lángi, 1999)**.** *For $r \leqslant 1/\sqrt{2}$, an almost-equidistant set lying on a $(d-1)$-dimensional sphere S of radius r contains at most $2d+2$ points.*

Proof. Assume that S is embedded in $\mathbb{R}^{d+1}$ and its center $\mathbf{o}$ and a point $\mathbf{o}' \in \mathbb{R}^{d+1}$ are such that the line $\mathbf{o}'\mathbf{o}$ is orthogonal to the affine hull of S and $|\mathbf{o}' - \mathbf{o}|^2 = 1/2 - r$. Now, we get that the almost-equidistant set lies on the d-dimensional sphere of radius $1/\sqrt{2}$ with center $\mathbf{o}'$. By Rosenfeld's theorem [13], the size of the almost-equidistant sets is at most $2(d+1)$. □

4 Almost-equidistant diameter sets

A subset of $\mathbb{R}^d$ is called an *almost-equidistant diameter set* if it is almost-equidistant and has diameter 1; see [12, Proposition 13 and Problem 14]. The next theorem is about the maximal size of such sets.

Theorem 9. *An almost-equidistant diameter set in $\mathbb{R}^d$ has at most $2d + 4$ points.*

Proof. Clearly, the matrix $\mathbf{U}$ (see (1)) for an almost-equidistant diameter set in $\mathbb{R}^d$ has non-positive entries. Lemma 1 and Corollary 5 imply that we can assume without loss of generality that $\mathbf{U}$ has exactly one eigenvalue λ larger than 1.

We need the following week form of the Perron–Frobenius Theorem; see [11, 5].

Theorem 10 (Perron–Frobenius Theorem)**.** *If an n-by-n matrix has non-negative entries, then it has a non-negative real eigenvalue, which has maximum absolute value among all eigenvalues.*

By the Perron–Frobenius Theorem, the matrix $\mathbf{U}$ has a negative eigenvalue λ' such that $|\lambda'| \geqslant \lambda$. Therefore, Lemmas 1 and 2 (*case 2*) imply that the almost-equidistant diameter set has at most $2d + 2$ points. □

5 Almost-equidistant sets in small balls

Theorem 11. *Let $0 \leqslant c_0 < 1/2$ be a fixed constant. If an almost-equidistant set lies in a d-dimensional ball of radius $\sqrt{1/2 + c_0/(d+1)^{2/3}}$, then its cardinality is $O(d)/(1 - 2c_0)$.*

Proof. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be an almost-equidistant set in $\mathbb{R}^d$ such that $n \geqslant 2d + 2$ and $\|\mathbf{v}_i\|^2 = 1/2 + c_i$ for $1 \leqslant i \leqslant n$, where $c_i \leqslant c_0/(d+1)^{2/3} = r$.

By Corollary 5 and Lemma 1, without loss of generality we can assume that the matrix $\mathbf{U}$ (see (1)) for this set has exactly one eigenvalue λ larger than 1. Hence Lemmas 1 and 2 (*case 3*) for $\mathbf{U}$ yield

$$\lambda^3 > \frac{(n-d-1)^3}{(d+1)^2} - (n-d-2).$$

To finish the proof, we need the following special case of Weyl's Inequality [14, Theorem 1].

Theorem 12 (Weyl's Inequality)**.** *If α, β and γ are the largest eigenvalues of Hermitian matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{A} + \mathbf{B}$ respectively, then $\gamma \leqslant \alpha + \beta$.*

Clearly, by (1), we have

$$\mathbf{U} = (c^t j + j^t c) - 2\langle \mathbf{v}_i, \mathbf{v}_j \rangle + \mathbf{I},$$

where $c = (c_1, \dots, c_n)$ and $j = (1, \dots, 1)$. Obviously, the largest eigenvalues of the matrices $c^t j$ and $j^t c$ do not exceed nr. Therefore, from Weyl's Inequality and positivity of the Gram matrix $\langle \mathbf{v}_i, \mathbf{v}_j \rangle$, we conclude that $\lambda \leqslant 2nr + 1$. This forces

$$(2nr+1)^3 \geqslant \lambda^3 > \frac{(n-d-1)^3}{(d+1)^2} - (n-d-2), \tag{8}$$

and so

$$(2c_0x + o(1))^3 \geqslant (x-1)^3 - (x - 1 - o(1)),$$

where $x = n/(d+1)$. Hence we get $x = O(1)/(1-2c_0)$, and thus $n = O(d)/(1-2c_0)$. $\square$

Remark 13. Interestingly, an almost-equidistant set lying in a d-dimensional ball of radius $1/\sqrt{2}$ has at most $2d+4$ points. Indeed, in that case r is equal to 0, and hence (8) yields $n < 2d+2$ if $\mathbf{U}$ has an eigenvalue larger then 1. Therefore, by Corollary 5, we get $n \leqslant 2d+4$.

6 Almost-equidistant sets: general case

Theorem 14 (Kupavskii, Mustafa, Swanepoel, 2018). *The size of an almost-equidistant set in $\mathbb{R}^d$ is $O(d^{4/3})$.*

Proof. Assume that $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is an almost-equidistant set in $\mathbb{R}^d$. Let

$$f := \max_{i=1,\dots,n} \left\{ \Big| \sum_{j=1}^{n} \left(\|\mathbf{v}_i - \mathbf{v}_j\|^2 - 1 \right) \Big| \right\}. \tag{9}$$

Assuming that $\sum_{i=1}^n \mathbf{v}_i$ is the origin $\mathbf{o}$, we easily get

$$-f \leqslant n\|\mathbf{v}_i\|^2 + \sum_{j=1}^{n} \|\mathbf{v}_j\|^2 - n \leqslant f$$

for $1 \leqslant i \leqslant n$. Summing up these inequalities for $1 \leqslant i \leqslant n$, we obtain

$$-f/2 \leqslant \sum_{i=1}^{n} \|\mathbf{v}_i\|^2 - n/2 \leqslant f/2.$$

The last two inequalities implies

$$-3f/(2n) \leqslant \|\mathbf{v}_i\|^2 - 1/2 \leqslant 3f/(2n) \tag{10}$$

for $1 \leqslant i \leqslant n$. By (9), (10) and Lemma 6, we have

$$f \leqslant O(1) \cdot \left(d^{1/2} + d(f/n)^{1/2} + d(f/n) \right).$$

Therefore, we obtain either $n = O(d)$ or

$$f = O(1) \cdot \left(d^{1/2} + d^2/n \right). \tag{11}$$

Suppose to the contrary that $n \geqslant Cd^{4/3}$, where C is a positive constant. Hence, by (10) and (11), the almost-equidistant set lies in a ball of radius $1/\sqrt{2} + O(d^{-2/3})/C^2$. Hence if C is big enough, then we can apply Theorem 11 to the almost-equidistant set, and thus $n = O(d)$, a contradiction. $\square$

Remark 15. It is possible to prove Theorem 14 using Lemma 6 combined with the Gershgorin Circle Theorem [6]; see [12, Section 3.2] for a similar argument.

7 Discussion

7.1 Almost-equidistant diameter sets

A graph (V, E) is called a *diameter graph* if its vertex set $V \subseteq \mathbb{R}^d$ is a set of points of diameter 1 and a pair of vertices forms an edge if they are at unit distance apart. Of course, the set of vertices of two cliques in a diameter graph is an almost-equidistant diameter set. For instance, in [9, the last paragraph of Section 3] there is given an example of diameter graph in $\mathbb{R}^d$ consisting of two cliques *without common vertices* such that they have $d+1$ and $\lfloor \frac{d+1}{2} \rfloor$ vertices respectively. We believe that the vertex set of this diameter graph has the maximal size among almost-equidistant diameter sets in $\mathbb{R}^d$.

Conjecture 16. An almost-equidistant diameter set in $\mathbb{R}^d$ has at most $\left\lfloor \frac{3(d+1)}{2} \right\rfloor$ points.

There is the following conjecture [7, Conjecture 5.5] that arose in the context of study of cliques in diameter graphs.

Conjecture 17 (Schur)**.** Let S_1 and S_2 be two unit simplices in $\mathbb{R}^d$ forming a set of diameter 1 such that they have k and m vertices respectively. Then S_1 and S_2 share at least $\min\{0, k + 2m - 2d - 2\}$ vertices for $k \geqslant m$.

Clearly, this conjecture is closely related to Conjecture 16. Note that Conjecture 17 was confirmed in two special (but not trivial!) cases: $(k, m, d) = (d, d, d)$, where $d \geqslant 2$, in [9] and $(k, m, d) = (5, 3, 4)$ in [10].

7.2 Two-distant almost-equidistant sets

A subset of $\mathbb{R}^d$ is called a *two-distant set* if there are only two distances formed by any two distinct points of the set. The following question seems to be interesting.

Problem 18. What is the largest cardinality of a set in $\mathbb{R}^d$ that is two-distant and almost-equidistant at the same time?

Consider a two-distant almost-equidistant set $V \subseteq \mathbb{R}^d$ with distances 1 and a between points of V, where $a > 1$; the case $a < 1$ is not interesting because of Theorem 9. Lemma 1 and Corollary 5 imply that we may assume without loss of generality that $\mathbf{U}$ for V has exactly one eigenvalue λ larger than 1. Note that the matrix $\mathbf{U}/(a^2 - 1)$ is the adjacency matrix of a triangle-free graph. By Lemma 1, Problem 18 is reduced to following question.

Problem 19. What is the minimal rank of the matrix $\mathbf{A} - \lambda_2 \mathbf{I}_n$, where $\mathbf{A}$ is the adjacency matrix of a triangle-free graph on n vertices and λ_2 is its second largest eigenvalue provided that $\lambda_2 > 0$?

It is worth pointing out that for every positive integer n there are a triangle-free graph on n vertices and an eigenvalue λ of the adjecancy matrix $\mathbf{A}$ of the graph such that $\operatorname{rank}(\mathbf{A} - \lambda \mathbf{I}) = O(n^{3/4})$ and $\lambda > 0$; see [3, the proof of Theorem 5, pages 94–95].

Acknowledgements

We are grateful to Zilin Jiang for bringing the Perron–Frobenius Theorem to our attention.

References

[1] M. Balko, A. Pór, M. Scheucher, K. Swanepoel, and P. Valtr. Almost-equidistant sets. arXiv:1706.06375v2, 2017.

[2] K. Bezdek, and Z. Lángi. Almost equidistant points on S^{d-1}. *Periodica Mathematica Hungarica*, 39(1-3):139–144, 1999.

[3] B. Codenotti, P. Pudlák, and G. Resta. Some structural properties of low-rank matrices related to computational complexity. *Theoretical Computer Science*, 235(1):89–107, 2000.

[4] M. Deza, and H. Maehara. A few applications of negative-type inequalities. *Graphs and Combinatorics*, 10(3):255-262, 1994.

[5] F. G. Frobenius. Über Matrizen aus nicht negativen Elementen. *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften*, May:456–477, 1912.

[6] S. Geršgorin. Über die Abgrenzung der Eigenwerte einer Matrix. *Bulletin de l'Académie des Sciences de l'URSS*, 6:749–754, 1931.

[7] G. Kalai Some old and new problems in combinatorial geometry I: around Borsuk's problem. In *Surveys in Combinatorics 2015*, volume 424 of London Mathematical Society Lecture Note Series, pages 147–174, Cambridge University Press. arXiv:1505.04952v1

[8] A. Kupavskii, N. H. Mustafa, and K. J. Swanepoel. Bounding the size of an almost-equidistant set in Euclidean space. *Combinatorics, Probability and Computing*, 28(2):1–7, 2018. arXiv:1708.01590v1.

[9] A. B. Kupavskii, and A. Polyanskii. Proof of Schur's Conjecture in $\mathbb{R}^D$. *Combinatorica*, 37(6):1181–1205, 2017. arXiv:1402.3694v4

[10] A. B. Kupavskii, and A. A. Polyanskii. On simplices in diameter graphs in $\mathbb{R}^4$. *Mathematical Notes*, 101(1-2):265-276, 2017.

[11] O. Perron. Zur Theorie der Matrices. *Mathematische Annalen*, 64(2):248–263, 1907.

[12] A. Polyanskii. On almost-equidistant sets. *Linear Algebra and its Applications*, 563:220–230, 2019. arXiv:1707.00295v3.

[13] M. Rosenfeld. Almost orthogonal lines in E^d. In *Applied geometry and discrete mathematics*, volume 4 of DIMACS: Series in Discrete Mathematics and Theoretical Computer Science, pages 489–492. American Mathematical Society, Providence, RI, 1991.

[14] H. Weyl. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen (mit einer Anwendung auf die Theorie der Hohlraumstrahlung). *Mathematische Annalen*, 71(4):441–479, 1912.