On monochromatic pairs with nondecreasing diameters

Adam O'Neal   Michael W. Schroeder*

Department of Mathematics
Marshall University
Huntington, WV, U.S.A.

{onleala,schroederm}@marshall.edu

Submitted: Jul 14, 2018; Accepted: Apr 18, 2019; Published: May 3, 2019

Abstract

Let n, m, r, t be positive integers and $\Delta : [n] \to [r]$. We say Δ is (m,r,t)-permissible if there exist t disjoint m-sets $B_1, \ldots, B_t$ contained in $[n]$ for which

(a) $|\Delta(B_i)| = 1$ for each $i = 1, 2, \ldots, t$,

(b) $\max(B_i) < \min(B_{i+1})$ for each $i = 1, 2, \ldots, t-1$, and

(c) $\max(B_i) - \min(B_i) \leqslant \max(B_{i+1}) - \max(B_{i+1})$ for each $i = 1, 2, \ldots, t-1$.

Let $f(m,r,t)$ be the smallest such n so that all colorings Δ are (m,r,t)-permissible. In this paper, we show that $f(2,2,t) = 5t - 4$.

Mathematics Subject Classifications: 05D10, 11B75, 11B50

1 Introduction

In this paper we study a Ramsey-type integer coloring problem posed by Bialostocki et al. [2] in 1995, which we restate using the definition of *permissibility* given below.

Definition 1. Let n, m, r, and t be positive integers and $\Delta : [n] \to [r]$ be a coloring. We say Δ is (m,r,t)*-permissible* if there exists a collection of disjoint m-sets $\{B_1, B_2, \ldots, B_t\}$ each contained in $[n]$ such that

- $|\Delta(B_i)| = 1$ for each $i = 1, 2, \ldots, t$,
- $\max(B_i) < \min(B_{i+1})$ for each $i = 1, 2, \ldots, t-1$, and
- $\max(B_i) - \min(B_i) \leqslant \max(B_{i+1}) - \min(B_{i+1})$ for each $i = 1, 2, \ldots, t-1$.

*Corresponding author

We say the collection $\{B_1, B_2, \ldots, B_t\}$ is *permissible* in Δ if they satisfy the above conditions, thereby realizing Δ as (m, r, t)-permissible.

The first condition requires the m-sets be monochromatic; note two m-sets can be associated with different colors. The second condition is a non-overlapping property establishing precedence between the m-sets, and the third condition requires the ranges of each m-set, called their *diameters*, form a nondecreasing sequence.

In this language the question posed by Bialostocki, Erdős, and Lefmann [2] is as follows:

Question 2. Given positive integers m, r, and t, does there exist an integer n such that for all $\Delta : [n] \to [r]$, Δ is (m, r, t)-permissible? If so, what is the minimum possible value for n?

The answer to the first question is yes; it follows from a result of van der Waerden involving arithmetic progressions [11]. Bialostocki et al. define $f(m, r, t)$ as the smallest integer n for which all r-colorings of $[n]$ are (m, r, t)-permissible. Several infinite families of parameters have $f(m, r, t)$ determined or bounded.

Observation 3. Note that for all positive integers m, r, and t, $f(m, r, t) \geqslant mt$ since the union of t disjoint m-sets realizing a coloring as (m, r, t)-permissible must have cardinality no less than $f(m, r, t)$. This inequality is an equality if and only if $m = 1$ or $r = 1$. Additionally $f(m, r, 1) = (m - 1)r + 1$ by the pigeonhole principle.

Based on the above observation, interest in this question focuses on parameter families in which m, r, and t are at least 2.

Theorem 4 (Bialostocki et al. [2])**.** *Let m, r, and t be integers at least 2.*

(a) $f(m, 2, 2) = 5m - 3$.

(b) $f(m, 3, 2) = 9m - 7$.

(c) *If $r \geqslant 4$, then*

$$3r(m - 1) + 3 \leqslant f(m, r, 2) < ((2m - 2) - r + 1) \cdot (2 + \log_2 r) - 1.$$

(d) $f(m, 2, t) \leqslant cmt^2$ *for some constant c.*

(e) $8m - 4 \leqslant f(m, 2, 3) \leqslant 10m - 6$.

(f) $f(2, r, t) \leqslant (r(t - 1) + 1)(r + 1)$.

Also in 1995 Bollobás, Erdős, and Jin [5] discovered results for the analogous question involving strictly increasing diameters, as well as established bounds for $f(2, r, 2)$ and gave asymptotic results for $f(2, r, t)$:

Theorem 5 (Bollobás et al. [5])**.**

(a) *Let* $r \geqslant 2$*. Then* $4r - 5\log_2 r < f(2, r, 2) \leqslant 4r + 1$*.*

(b) *For a fixed* t*,* $f(2, r, t)$ *is linear in* r*.*

In 2005 Grynkiewicz showed the lower bound of the inequality in Theorem 4(c) is an equality when $r = 4$.

Theorem 6 (Grynkiewicz [9])**.** *Let* m *be a positive integer at least* 2*. Then*

$$f(m, 4, 2) = 12m - 9.$$

In 2015 Bernstein et al. found the value of $f(m, 2, 3)$, first bounded in Theorem 4(e).

Theorem 7 (Bernstein et al. [1])**.** *Let* m *be a positive integer at least* 2*. Then*

$$f(m, 2, 3) = 8m - 5 + \left\lfloor \frac{2m-2}{3} \right\rfloor + \delta,$$

where $\delta = 1$ *if* $m \in \{2, 5\}$ *and* $\delta = 0$ *otherwise.*

Recent work has been done on generalizations of Question 2, as well as investigations into its relationship to a theorem by Erdős, Ginzburg, and Ziv (see [3, 4, 6, 7, 8, 10] for examples). With possibly the only exceptions being the bounds given in Theorem 4(d) and (f), as well as Theorem 5(b), all recent work in evaluating $f(m, r, t)$ focuses on parameter families with $t = 2$ and more recently $t = 3$ [1]. In our work we compute values for $f(2, 2, t)$ for arbitrarily large t which, as far as we can tell, is the first known set of actual values of $f(m, r, t)$ with $m, r \geqslant 2$ and arbitrarily large t, rather than asymptotic results.

Note that $f(2, 2, t)$ has been computed when $t \leqslant 3$; $f(2, 2, 1) = 3$ by Observation 3, $f(2, 2, 2) = 7$ by Theorem 4(a), and $f(2, 2, 3) = 12$ by Theorem 7. Additionally, $f(2, 2, t) \leqslant 6t - 3$ by Theorem 4(f). Note that the upper bound given in Theorem 4(f) is not an attempt at a sharp bound. The inequality is found by using the pigeonhole principle and, in fact, guarantees the existence of a permissible collection of t 2-sets which all have the same diameter.

In this paper we prove the following:

Theorem 8. *Let* $t \geqslant 4$ *be a positive integer. Then* $f(2, 2, t) = 5t - 4$*.*

In Section 2 we introduce notation for the problem and present some constructions which are useful in later proofs for finding collections of permissible sets. In Section 3 we first prove that $f(2, 2, t) \geqslant 5t - 4$ by presenting a coloring of $[5t - 5]$ which is not $(2, 2, t)$-permissible, then prove a weak upper bound on $f(2, 2, t)$ which gives the existence of a permissible collection of sets with additional structure. We finish the section with a proof that all colorings of $[5t - 4]$ with $t \geqslant 4$ are $(2, 2, t)$-permissible by showing there does not exist a coloring of $[5t - 4]$ which is not $(2, 2, t)$-permissible.

2 Definitions and Constructions

We begin this section with a series of definitions, terminology, and notation related to colorings with 2 colors (using the color set $\{a, b\}$ rather than $[2]$) and the identification of collections of 2-sets (abbreviated *pairs*) which are permissible.

Let n be a positive integer and $\Delta : [n] \to \{a, b\}$. At times we present Δ as the string $x_1 x_2 \cdots x_n$, where $x_i = \Delta(i)$ for each $i \in [n]$. We say $x_i x_{i+1} \cdots x_{i+k-1}$, where $x_j = \Delta(j)$ for each $j = i, i+1, \ldots, i+k-1$, is a *string of length k in Δ*. For $x \in \{a, b\}$, we use the abbreviation x^n to denote the string $xx \cdots x$ (n times) and similarly define $w^n = ww \cdots w$ (n times), where w is any word from the alphabet $\{a, b\}$. We say $\{i, i+1, \ldots, i+k-1\}$ is a *k-tuple* in Δ if $\Delta(i) = \Delta(i+1) = \cdots = \Delta(i+k-1)$. If $\Delta(i-1) \neq \Delta(i)$ (or $i = 1$) and $\Delta(i+k-1) \neq \Delta(i+k)$ (or $i+k-1 = n$), we say the k-tuple is *isolated*. Specifically, a *double* is a 2-tuple and a *triple* is a 3-tuple. A string $x_i x_{i+1} \cdots x_{i+k-1}$ in Δ of length k is *alternating* if $\Delta(j) \neq \Delta(j+1)$ for each $j = i, i+1, \ldots, i+k-2$. A pair $\{i, i+k\}$ in $[n]$ is a *monochromatic pair of diameter k*, abbreviated $\mathtt{D}_k$, in Δ if $\Delta(i) = \Delta(i+k)$. Note that a $\mathtt{D}_1$ is a double and a triple contains two $\mathtt{D}_1$s and a $\mathtt{D}_2$.

We now define the *alternating substring/triples* partition of $[n]$ with respect to Δ (abbreviated the AST partition of Δ) and give the process for constructing it. As we see later, this partition yields a permissible collection of pairs which have diameters 1 or 2, which is instrumental going forward.

Construction 9. Let $n > 0$ and $\Delta : [n] \to \{a, b\}$. By the following steps, we partition $[n]$ into alternating substrings and triples.

1. Find the maximum number of pairwise disjoint triples in Δ. Suppose that there are w of them; call them $T_1, T_2, \ldots, T_w$, and suppose the minimal element in T_i is τ_i for each $i \in [w]$. We additionally require that for all $i \in [w]$, if $\tau_i > 1$, then either $\Delta(\tau_i - 1) \neq \Delta(\tau_i)$ or $\tau_i - 1 \in T_{i-1}$. This ensures that we "frontload" the triples. In other words, we read through the string from left to right and define our triples in a greedy method. We define the collection of triples as $\mathcal{T} := \{T_1, T_2, \ldots, T_w\}$.

2. Let $\mathcal{U} = \{U_0, U_1, \ldots, U_w\}$ be a collection of subsets of $[n]$ defined as $U_0 = [\min(T_1) - 1]$, $U_i = [\max(T_i) + 1, \min(T_{i+1}) - 1]$ for each $i = 1, 2, \ldots, w-1$, and $U_w = [\max(T_w) + 1, n]$. Note that some of these sets may be empty if Δ contains consecutive triples, or begins or ends with a triple. It follows that $\mathcal{U} \cup \mathcal{T}$ is a partition of $[n]$.

3. Partition each U_i into consecutive alternating substrings of maximal length. Note that if there are s such maximal substrings in U_i, then there will be $s-1$ doubles contained in U_i. Thus in total we have $v + w + 1$ such maximal alternating substrings S_i, $i \in [v + w + 1]$, where we use the convention that an empty set U_i corresponds to an empty substring. Thus the substrings S_i partition the terms of $\bigcup_{i=0}^{w} U_i$ as $\bigcup_{i=1}^{v+w+1} S_i$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_{v+w+1}\}$ be the collection of these alternating substrings, and for each $i \in [v + w + 1]$, let $k_i = |S_i|$ be the *length* of S_i.

Observe that $\mathcal{S} \cup \mathcal{T}$ is a partition of $[n]$, and so we define $(\mathcal{S}, \mathcal{T}; v, w)$ as the *AST partition of* Δ. Since $\mathcal{S} \cup \mathcal{T}$ is a partition of $[n]$, note $n = 3w + k_1 + \cdots + k_{v+w+1}$.

Example 10. Let $\Delta_1 : [16] \to \{a, b\}$ be given by *abaaababaabaaaba*. Then the AST partition of Δ_1 has $v = 1$ and $w = 2$ with $\mathcal{T} = \{\{3, 4, 5\}, \{12, 13, 14\}\}$ and $\mathcal{S} = \{\{1, 2\}, \{6, 7, 8, 9\}, \{10, 11\}, \{15, 16\}\}$. One can see this visually as

$$\Delta_1 : \ \underset{S_1}{\underline{ab}}\ \underset{T_1}{\underline{aaa}}\ \underset{S_2}{\underline{baba}}\ \underset{S_3}{\underline{ab}}\ \underset{T_2}{\underline{aaa}}\ \underset{S_4}{\underline{ba}}$$

Let $\Delta_2 : [36] \to \{a, b\}$ be given as *abababababaababaaabaabbababbbaaaaab*. Then the AST partition of Δ_2 has $v = 4$ and $w = 3$ with the parts indicated below.

$$\Delta_2 : \ \underset{S_1}{\underline{abababababa}}\ \underset{S_2}{\underline{abab}}\ \underset{T_1}{\underline{aaa}}\ \underset{S_3}{\underline{ba}}\ \underset{S_4}{\underline{ab}}\ \underset{S_5}{\underline{baba}}\ \underset{T_2}{\underline{bbb}}\ \underset{S_6}{\underline{\quad}}\ \underset{T_3}{\underline{aaa}}\ \underset{S_7}{\underline{a}}\ \underset{S_8}{\underline{ab}}$$

Definition 11. Let $(\mathcal{S}, \mathcal{T}; v, w)$ be the AST partition of $\Delta : [n] \to \{a, b\}$ for some positive integer n. Observe that $\mathcal{S} \cup \mathcal{T}$ is a partition of $[n]$ into sets of consecutive integers, and these $v+2w+1$ parts have an implied order. With this in mind, we say Δ is of *type* γ, where $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{v+2w+1})$ is a $(v + 2w + 1)$-tuple, which is an ordering of $k_1, k_2, \ldots, k_{v+w+1}$ and w copies of the symbol τ where the order of the ordinates reflects the relative order of the alternating substrings and triples in the AST partition of Δ. We say Δ *contains* $\mathbf{y} = (y_1, y_2, \ldots, y_\ell)$ for some ℓ and symbols y_i, $i \in [\ell]$ if, for some j, $y_i = \gamma_{i+j}$ for each $i \in [\ell]$. Specifically, we say Δ *ends* with $\mathbf{y}$ if Δ contains $\mathbf{y}$ and $j = v + 2w + 1 - \ell$. For convenience, we use $\overline{a}$ to mean "is congruent to a mod 3 and at least a".

Example 12. Let Δ_1 and Δ_2 be the colorings from Example 10. Then Δ_1 is type $(2, \tau, 4, 2, \tau, 2)$ and Δ_2 is type $(11, 4, \tau, 2, 2, 4, \tau, 0, \tau, 1, 2)$. We may also say Δ_1 contains $(\tau, \overline{1}, 2, \tau)$ and ends with $(\tau, 2)$, while Δ_2 contains $(\overline{1}, \tau, 2)$ and ends with $(\tau, \overline{1}, \overline{2})$.

Observation 13. Let $(\mathcal{S}, \mathcal{T}; v, w)$ be the AST partition of $\Delta : [n] \to \{a, b\}$. If $v + w \geqslant t$, then Δ contains w disjoint triples – each of which contain a double – and v disjoint doubles which are disjoint from the triples. Therefore Δ contains $v + w \geqslant t$ disjoint $\mathtt{D}_1$s, which implies Δ is $(2, 2, t)$-permissible.

Observation 14. Let $k > 0$ and $w : [k] \to \{a, b\}$ be a coloring represented by an alternating string, i.e. $w = abab \ldots a$ if k is odd and $w = abab \cdots ab$ if k is even. Then $\{\{3i - 2, 3i\} \mid i = 1, \ldots, \lfloor k/3 \rfloor\}$ is a permissible collection of $\lfloor k/3 \rfloor$ $\mathtt{D}_2$s. In this manner, all alternating strings of length k contain a permissible collection of $\lfloor k/3 \rfloor$ $\mathtt{D}_2$s.

Definition 15. Let $\Delta : [n] \to \{a, b\}$ with AST partition $(\mathcal{S}, \mathcal{T}; v, w)$. Each of the w triples in $\mathcal{T}$ contain a $\mathtt{D}_2$, while each alternating substring in $\mathcal{S}$ contains a permissible collection of $\lfloor k/3 \rfloor$ $\mathtt{D}_2$s. Hence Δ has a permissible collection of $w + \lfloor k_1/3 \rfloor + \cdots + \lfloor k_{v+w+1}/3 \rfloor$ $\mathtt{D}_2$s. As previously indicated in Observation 14, we "frontload" when selecting $\mathtt{D}_2$s from an alternating substring by reading from left to right. We call this collection of $\mathtt{D}_2$s of Δ the *canonical* $\mathtt{D}_2$s of Δ.

Example 16. Let Δ_1 and Δ_2 be the colorings defined in Examples 10 and 12. There are 3 canonical $\mathtt{D}_2$s in Δ_1, which are $\{3, 5\}$, $\{6, 8\}$, and $\{12, 14\}$, seen below:

$$\Delta_1 : \ \underset{S_1}{\underline{ab}}\ \underset{T_1}{\underline{aaa}}\ \underset{S_2}{\underline{baba}}\ \underset{S_3}{\underline{ab}}\ \underset{T_2}{\underline{aaa}}\ \underset{S_4}{\underline{ba}}$$

In a similar manner there are 8 canonical $\mathtt{D}_2$s in Δ_2, which are illustrated below.

$$\Delta_2 : \underbrace{ababababababa}_{S_1}\ \underbrace{abab}_{S_2}\ \underbrace{aaa}_{T_1}\ \underbrace{ba}_{S_3}\ \underbrace{ab}_{S_4}\ \underbrace{baba}_{S_5}\ \underbrace{bbb}_{T_2}\ \underbrace{}_{S_6}\ \underbrace{aaa}_{T_3}\ \underbrace{a}_{S_7}\ \underbrace{ab}_{S_8}$$

3 Main Results

We prove Theorem 8 in two stages. First, we give what will be the sharp lower bound on $f(2,2,t)$, then produce a weaker upper bound by showing when n is large enough, there is a permissible collection of pairs with bounded diameters contained in any 2-coloring of $[n]$. We conclude the section by establishing properties that a non-permissible coloring of $[5t-4]$ must have, then showing that no coloring satisfies all of the properties.

3.1 Bounds on $f(2,2,t)$

We first show that $f(2,2,t) \geqslant 5t-4$ by demonstrating the existence of a coloring of $[5t-5]$ which is not $(2,2,t)$-permissible. This is derived by setting $m = r = 2$ in the following theorem.

Theorem 17. *Let $m,r,t \geqslant 2$ and Δ be the r-coloring of $[(mr+1)(t-1)]$ given by $((12\cdots r)^m 1)^{t-1}$. Then Δ is not (m,r,t)-permissible.*

Proof. The result is trivial for $t = 2$. Assume that $((12\cdots r)^m 1)^{t-2}$ is not $(m,r,t-1)$-permissible for some $t \geqslant 3$ and $((12\cdots r)^m 1)^{t-1}$ is (m,r,t)-permissible. Let $\{B_1,\ldots,B_t\}$ be a permissible collection of pairs in $((12\cdots r)^m 1)^{t-1}$.

Since $((12\cdots r)^m 1)^{t-2}$ is not $(m,r,t-1)$-permissible, we have $\min(B_2) \leqslant mr+1$; otherwise $\{B_2,\ldots,B_t\}$ is a permissible collection of $t-1$ pairs in Δ restricted to $[mr+2,(mr+1)(t-1)]$, implying that $((12\cdots r)^m 1)^{t-2}$ is $(m,r,t-1)$-permissible. Hence $B_1 = \{i, r+i, \ldots, (m-1)r+i\}$ for some $i \in [r]$, and thus $\operatorname{diam}(B_1) = m(r-1)$. Similarly, we have that $\max(B_{t-1}) > (rm+1)(t-2)$ and hence $\operatorname{diam}(B_t) = r(m-1)$ for reasons identical to those above,. Therefore $\operatorname{diam}(B_2) = r(m-1)$, and it follows that if $\min(B_2) = j$ for some $j \in [r]$ (and necessarily $r(m-1)+2 \leqslant \min(B_2) \leqslant mr+1$):

$$\Delta(\max(B_2)) = \Delta(\min(B_2) + r(m-1)) = \begin{cases} j-1 & \text{if } j \neq 1 \text{ and} \\ r & \text{if } j = 1. \end{cases}$$

Thus B_2 is not monochromatic, giving a contradiction. So $((12\cdots r)^m 1)^{t-1}$ is not (m,r,t)-permissible. It follows that $f(2,2,t) > 5t-5$. □

We now show that $f(2,2,t) \leqslant 5t-2$, a weaker result than what Theorem 8 would guarantee, but we show this bound can be realized by a permissible collection of pairs with bounded diameter.

Theorem 18. *Let $t \geqslant 1$ be an integer and $\Delta : [5t-2] \to \{a,b\}$. Then Δ is $(2,2,t)$-permissible, and there is a permissible collection of t pairs having diameter at most 2.*

Proof. Let $(\mathcal{S}, \mathcal{T}; v, w)$ be the AST partition of Δ. By Observation 13 if $v + w \geqslant t$, then Δ is t-permissible as it contains at least t disjoint $\mathtt{D}_1$s. Now suppose $v+w \leqslant t-1$. Recall that for each $i \in [v+w+1]$, $k_i = |S_i|$, and define $k_i' \in \{0, 1, 2\}$ so that $k_i' = k_i \bmod 3$. Note $(k_i - k_i')/3 = \lfloor k_i/3 \rfloor$ for each $i \in [v+w+1]$. Let σ denote the number of canonical $\mathtt{D}_2$s of Δ. Recall $5t-2 = 3w+k_1+k_2+\cdots+k_{v+w+1}$ and $\sigma = w + \lfloor k_1/3 \rfloor + \lfloor k_2/3 \rfloor + \cdots + \lfloor k_{v+w+1}/3 \rfloor$. So

$$\begin{aligned} 3\sigma &= 3w + (k_1 - k_1') + (k_2 - k_2') + \cdots + (k_{v+w+1} - k_{v+w+1}') \\ &= (5t-2) - (k_1' + k_2' + \cdots + k_{v+w+1}') \\ &\geqslant 5t - 2 - 2(v + w + 1) \\ &\geqslant 5t - 2 - 2t = 3t - 2. \end{aligned} \tag{1}$$

So $\sigma \geqslant t - \frac{2}{3}$, and since σ and t are integers, $\sigma \geqslant t$. By choosing any t of the σ canonical $\mathtt{D}_2$s of Δ, we have a permissible collection of t pairs in Δ. So Δ is $(2, 2, t)$-permissible and is realized by a permissible collection of pairs with diameter at most 2. ☐

3.2 Colorings of $[5t - 4]$ which are not $(2, 2, t)$-permissible

For the remainder of this section, let Δ be a coloring of $[5t - 4]$, $(\mathcal{S}, \mathcal{T}; v, w)$ be the AST partition of Δ, $\mathcal{D}$ be the set of canonical $\mathtt{D}_2$s of Δ, and $\sigma = |\mathcal{D}|$. We now outline a series of properties that Δ must satisfy to *not* be $(2, 2, t)$-permissible. We conclude this section by showing no coloring can satisfy all such conditions, which establishes $f(2, 2, t) \leqslant 5t - 4$, and thus proves Theorem 8.

As it will be relevant in the proof of Theorem 8, take note that all of the following lemmas and observations are valid for $t \geqslant 2$, and we indicate where appropriate what the largest diameter for a realization of permissibility is for a coloring. This will be leveraged in the final proof. We begin an observation about the number of canonical $\mathtt{D}_2$s in Δ provided it is not $(2, 2, t)$-permissible.

Observation 19. Let $t \geqslant 2$ and suppose Δ is not $(2, 2, t)$-permissible. By a similar argument found in (1), we find $3\sigma \geqslant 5t - 4 - 2(v + w + 1)$. So $\sigma \geqslant t - \frac{4}{3}$ and hence $\sigma \geqslant t - 1$. By Observation 13, we have $\sigma = t - 1$, and hence Δ contains exactly $t - 1$ canonical $\mathtt{D}_2$s.

The previous observation allows us to classify the value of $v + w$ in the AST partition of Δ given it is not $(2, 2, t)$-permissible.

Lemma 20. *Let $t \geqslant 2$. If Δ is not $(2, 2, t)$-permissible, then $v + w = t - 1$. Therefore Δ has exactly t alternating substrings in its AST partition.*

Proof. By Observation 13, $v + w \leqslant t - 1$. Suppose $v + w \leqslant t - 2$. Then in a manner similar to what is shown in (1) and Observation 19, we find

$$3\sigma \geqslant 5t - 4 - 2(v + w + 1) \geqslant 5t - 4 - 2(t - 2) = 3t.$$

So $\sigma \geqslant t$ and therefore Δ contains at least t canonical $\mathtt{D}_2$s. Hence Δ is $(2, 2, t)$-permissible with a realization whose largest diameter is 2. ☐

Adapting the proof for Theorem 18 allows us to determine the congruence for $|S_i|$ for each $i = 1, 2, \ldots, v + w + 1$.

Lemma 21. *Let $t \geqslant 2$. If Δ is not $(2,2,t)$-permissible, there exists $j \in \{1, 2, \ldots, v+w+1\}$ such that $|S_j| \equiv 1 \bmod 3$ and $|S_k| \equiv 2 \bmod 3$ whenever $k \neq j$, $1 \leqslant k \leqslant v + w + 1$.*

Proof. By Lemma 20, we have $t = v + w + 1$ and therefore Δ has t alternating substrings in its AST partition. Again for each $i \in [t]$, define $k_i' \in \{0, 1, 2\}$ so that $k_i' \equiv k_i \bmod 3$. By Observation 19 we have $\sigma = t - 1$. Combining this with a similar computation to (1) we get

$$3(t-1) = 3\sigma = 3w + \sum_{i=1}^{t}(k_i - k_i') = 3w + \sum_{i=1}^{t} k_i - \sum_{i=1}^{t} k_i' = 5t - 4 - \sum_{i=1}^{t} k_i'.$$

So $k_1' + \cdots + k_t' = 2t - 1$. Therefore exactly one term in this sum is 1 while the remaining terms are each 2. Hence Δ is $(2,2,t)$-permissible with a realization whose largest diameter is 2. $\square$

We now classify the k-tuples which may exist in Δ.

Lemma 22. *If Δ is not $(2,2,t)$-permissible, then Δ does not contain an isolated 4-tuple or isolated k-tuple for each $k \geqslant 6$.*

Proof. First, observe that if Δ contains a k-tuple for some $k \geqslant 6$, then Δ has at least two consecutive triples in its AST partition. Hence there is an empty substring $S_i \in \mathcal{S}$ for some $i \in [v + w + 1]$, giving that $k_i \equiv 0 \bmod 3$. So Δ is $(2,2,t)$-permissible by Lemma 21 with a realization whose largest diameter is 2 – a contradiction.

Now suppose Δ contains the isolated 4-tuple $\{\ell, \ell+1, \ell+2, \ell+3\}$ for some $\ell \in [n-3]$. For each $i \in [w]$, let T_i' be the set containing the smallest two elements in T_i. It follows from Construction 9 that $T_j = \{\ell, \ell+1, \ell+2\}$ for some $j \in \{1, 2, \ldots, w\}$ and $\ell + 3$ does not belong to any double or triple identified in the AST partition. Then $\{P_1, P_2, \ldots, P_v, T_1', T_2', \ldots, T_w', \{\ell+2, \ell+3\}\}$ is a permissible collection of $v+w+1$ doubles, and since $v + w + 1 = t$ by Lemma 20, Δ is $(2,2,t)$-permissible with a realization whose largest diameter is 1 – a contradiction. $\square$

With the previous two lemmas, we now establish that Δ can have at most one triple and, if Δ contains a triple, give conditions on the lengths of the alternating substrings surrounding the triple.

Lemma 23. *If Δ is not $(2,2,t)$-permissible and Δ contains any triples, then Δ contains $(\overline{1}, \tau, \overline{2})$ or $(\overline{2}, \tau, \overline{1})$. Furthermore Δ contains at most one triple.*

Proof. Observe that Δ cannot begin with a triple, end with a triple, or have two consecutive triples in its AST partition; otherwise one of its alternating substrings has length 0, which contradicts Lemma 21. Therefore every triple in the AST partition of Δ is preceded and followed by a nonempty alternating substring. Furthermore by Lemma 21, it follows that if Δ contains a triple in its AST partition, then Δ must contain $(\overline{1}, \tau, \overline{2})$, $(\overline{2}, \tau, \overline{1})$, or $(\overline{2}, \tau, \overline{2})$.

Assume Δ contains (x, τ, y), where $x \equiv y \equiv 2 \bmod 3$, and let S_L, T, and S_R denote the corresponding alternating substrings and triple, and let i be the integer such that $T = \{i, i+1, i+2\}$. Since $\Delta(i-2) \neq \Delta(i-1)$, it following by the AST construction that $\Delta(i-1) \neq \Delta(i)$. Since $\Delta(i+3) \neq \Delta(i+4)$, it follows by Lemma 22 that $\Delta(i+2) \neq \Delta(i+3)$. Thus $S_L \cup \{i\}$ and $S_R \cup \{i+2\}$ are alternating substrings of lengths $x+1$ and $y+1$, respectively. Hence $S_L \cup \{i\}$ contains a permissible collection of $\lfloor x/3 \rfloor + 1$ $\mathtt{D}_2$s, and similarly $S_R \cup \{i+2\}$ contains a permissible collection of $\lfloor y/3 \rfloor + 1$ $\mathtt{D}_2$s. Removing those $\mathtt{D}_2$s from $\mathcal{D}$ contained in S_L, S_R, and T, then combining them with the permissible collections of $\mathtt{D}_2$s contained in $S_L \cup \{i\}$ and $S_R \cup \{i+2\}$ produces a permissible collection of t $\mathtt{D}_2$s, which is a contradiction. So Δ does not contain $(\overline{2}, \tau, \overline{2})$. So if Δ contains any triples, then Δ contains $(\overline{2}, \tau, \overline{1})$ or $(\overline{1}, \tau, \overline{2})$.

By Lemma 21, there is exactly one substring in the AST partition of Δ which has length congruent to $1 \bmod 3$. Therefore $w \leqslant 2$, and if $w = 2$ then Δ must contain $(\overline{2}, \tau, \overline{1}, \tau, \overline{2})$. Assume that $w = 2$ and Δ contains (x, τ, y, τ, z), where $y \equiv 1 \bmod 3$ and $x \equiv z \equiv 2 \bmod 3$. Let $S_L, \{i, i+1, i+2\}, S_M, \{j, j+1, j+2\}, S_R$ be the associated alternating substrings and triples. In a manner similar to the previous argument, $S_L \cup \{i\}, \{i+2\} \cup S_M \cup \{j\}$, and $\{j+2\} \cup S_R$ are alternating substrings. By removing the $\mathtt{D}_2$s in $\mathcal{D}$ contained in $S_L \cup \{i, i+1, i+2\} \cup S_M \cup \{j, j+1, j+2\} \cup S_R$ and adding those contained in $S_L \cup \{i\}, \{i+2\} \cup S_M \cup \{j\}$, and $\{j+2\} \cup S_R$, we produce a permissible collection of t $\mathtt{D}_2$s in Δ – a contradiction. Thus $w \leqslant 1$.

Note that in each of the above alterations, each coloring was found to be $(2, 2, t)$-permissible with a realization whose largest diameter is 2. $\square$

We now give an observation which classifies the possible final substring in Δ and is useful in the arguments which follow.

Observation 24. Suppose Δ is not $(2, 2, t)$-permissible.

(a) $\Delta(5t-4) \neq \Delta(5t-6)$. Otherwise $\Delta|_{[5t-7]}$ is a coloring which is $(2, 2, t-1)$-permissible with a permissible collection of $(t-1)$ pairs of diameter at most 2 by Theorem 18. The collection of these $(t-1)$ sets with $\{5t-6, 5t-4\}$ yields a permissible collection of t pairs in Δ with a largest diameter of 2.

With this in mind, since Δ cannot end with a triple, the alternating substring which ends Δ cannot have length exceeding 2. So $k_{v+w+1} = 1$ or $k_{v+w+1} = 2$ and thus Δ must end with type (1) or (2).

(b) $\Delta(5t-11) \neq \Delta(5t-9)$. Otherwise $\Delta|_{[5t-9]}$ is $(2, 2, t-1)$-permissible with a permissible collection of $t-1$ pairs whose largest diameter is at most 2. Since either $\{5t-8, 5t-6\}$, $\{5t-8, 5t-4\}$, or $\{5t-6, 5t-4\}$ is a monochromatic pair, we can produce a permissible collection of t pairs, which means that Δ is $(2, 2, t)$-permissible, and this realization has a largest diameter of either 2 or 4.

Now we give a proof for the main result.

Proof of Theorem 8. Suppose Δ is not $(2,2,t)$-permissible. By Observation 24(a), Δ must end with type (1) or (2). By Lemma 23, Δ may contain at most 1 triple and if Δ contains a triple, then $(\overline{2},\tau,\overline{1})$ or $(\overline{1},\tau,\overline{2})$ is contained in Δ. That said there are 12 types with which Δ may end; see Figure 1.

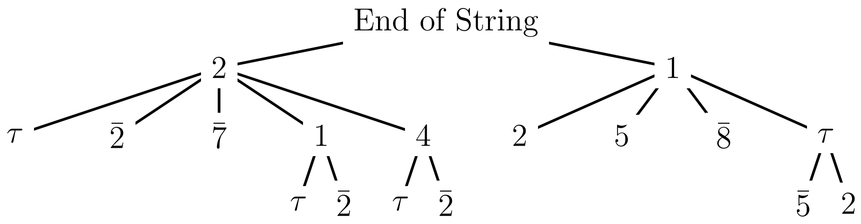

Figure 1: The possible endings for a coloring which is not (2,2,t)-permissible. For example, the left-most leaf corresponds to type $(\tau,2)$, while the right-most leaf corresponds to type $(2,\tau,1)$.

We now show in each case above that Δ is, in fact, $(2,2,t)$-permissible and therefore prove Theorem 8. Without loss of generality, we assume $\Delta(5t-4)=b$. Suppose Δ ends with type:

- $(\tau,2)$. Since Δ cannot end with *bbbab* by Observation 24(a), we have Δ ends with *aaaab*. Since $t \geqslant 2$, there is a nonempty alternating substring S_{t-1} which precedes the triple. If the last letter in S_{t-1} is an a, then by the AST construction Δ has two consecutive triples, which violates Lemma 23. So Δ ends with *baaaab*, in which case we have an isolated quadruple in Δ, which contradicts Lemma 22. Thus a coloring with this end type is $(2,2,t)$-permissible with a realization having a largest diameter at most 2.

- $(\overline{2},2)$. Then Δ ends with *baab*, and all of the corresponding indices in $[5t-7,5t-4]$ are preceded by the $t-1$ canonical $\mathtt{D}_2$s of Δ. Then $\mathcal{D}\cup\{\{5t-7,5t-4\}\}$ is a permissible collection of t pairs and hence Δ is $(2,2,t)$-permissible with a realization whose largest diameter is 3.

- $(\overline{7},2)$, $(\overline{8},1)$, or $(\overline{5},\tau,1)$. By Observation 24(b), Δ is $(2,2,t)$-permissible with a realization having a largest diameter at most 4. Note this argument implies that, if Δ is a coloring of $[5t-4]$ with $t\geqslant 3$ (rather than 4) and ending with any of these three types, then Δ is also $(2,2,t)$-permissible. This will be relevant in the last subcase of this proof.

- $(\tau,1,2)$. Since $t\geqslant 3$, by Lemmas 21 and 23, Δ ends with type $(\overline{2},\tau,1,2)$. So Δ ends with either *abaaaaab*, *baaaaaab*, *babbbaab*, or *abbbbaab*. Then Δ is $(2,2,t)$-permissible by Observation 24(b) for the first or third case and by Lemma 22 for the second or fourth case. Thus Δ is $(2,2,t)$-permissible with a realization having a largest diameter at most 4.

- $(\overline{2}, 1, 2)$. In this case Δ ends with *baaab*, meaning the triples were not correctly identified in its AST partition. So Δ cannot end with this type.

- $(\tau, 4, 2)$. Since $t \geqslant 3$, by Lemmas 21 and 23, Δ must end with type $(\overline{2}, \tau, 4, 2)$. Then Δ ends with *abaaababaab*, *abbbbbabaab*, *baaaababaab*, or *babbbbabaab*. Note the second and third cases are not possible as the triples were not selected properly for its partition. In the first case, $\{5t-12, 5t-10\}$ and $\{5t-9, 5t-7\}$ belong to $\mathcal{D}$. The collection the pairs in $\mathcal{D}$, excluding $\{5t-12, 5t-10\}$ and $\{5t-9, 5t-7\}$, combined with $\{5t-14, 5t-12\}$, $\{5t-11, 5t-8\}$, and $\{5t-7, 5t-4\}$ is permissible, demonstrating that Δ is $(2, 2, t)$-permissible. Finally Δ is $(2, 2, t)$-permissible by Observation 24(b) for the fourth case. Thus Δ is $(2, 2, t)$-permissible with a realization having a largest diameter at most 4.

- $(\overline{2}, 4, 2)$ or $(5, 1)$. In the latter case, since $t \geqslant 3$, by Lemmas 21 and 23, Δ must end with type $(\overline{2}, 5, 1)$. So Δ ends with *abbabaab* or *abbababb*. In a similar manner to the above case, the collection of pairs in $\mathcal{D}$, excluding $\{5t-9, 5t-7\}$, combined with $\{5t-11, 5t-8\}$ and $\{5t-7, 5t-4\}$ is permissible, again demonstrating that Δ is $(2, 2, t)$-permissible with a realization having a largest diameter at most 3. Again note this argument holds for $t \geqslant 3$, similar to the colorings which end with $(\overline{8}, 1)$.

- $(2, 1)$. By Lemmas 21 and 23 and since $t \geqslant 3$, Δ must end with type $(\overline{2}, 2, 1)$. Then Δ ends with *baabb*. Note that each of the $(t-1)$ sets in $\mathcal{D}$ precedes $\{5t-8, 5t-5\}$. Hence $\mathcal{D} \cup \{\{5t-8, 5t-5\}\}$ is a set of t permissible sets, which demonstrates that Δ is $(2, 2, t)$-permissible with a realization having a largest diameter at most 3. Again note this argument holds for $t \geqslant 3$, similar to the colorings which end with $(\overline{8}, 1)$ or $(5, 1)$.

- $(2, \tau, 1)$. By Lemma 21 and since $t \geqslant 4$, Δ must end with type $(\overline{2}, \overline{2}, 2, \tau, 1)$. So Δ ends with *baabaaab*, *baabbbbb*, *abbaaaab*, or *abbabbbb*. Again note the second and third cases are not possible as the triples were not selected properly for its partition. In the first case, $\Delta|_{[5t-9]}$ ends with type $(\overline{2}, 1)$, or rather $(2, 1)$, $(5, 1)$, or $(\overline{8}, 1)$.

  Since $t-1 \geqslant 3$, we have $\Delta|_{[5t-9]}$ is therefore $(2, 2, t-1)$-permissible with a permissible collection of t pairs with largest diameter 4; let $\mathcal{D}'$ denote this collection. So $\mathcal{D}' \cup \{\{5t-8, 5t-4\}\}$ is a permissible collection of t pairs, demonstrating the $(2, 2, t)$-permissibility of Δ with a realization having a largest diameter at most 4. In the fourth case, Δ is $(2, 2, t)$-permissible with a realization having a largest diameter 1 by Lemma 22. □

We conclude with the observation that, in this work, we heavily use the binary nature of 2-colorings for constructing the AST partition used in our arguments. Hence we do not believe this method will translate nicely to $[r]$-colorings with $r \geqslant 3$ or the discovery of permissible m-sets with $m \geqslant 3$.

Acknowledgements

We also thank our anonymous referee for their careful reading, detailed comments, and thoughtful suggestions for improving our paper.

References

[1] D. I. Bernstein, D. J. Grynkiewicz, and C. Yerger. On three sets with nondecreasing diameter. *Discrete Math.*, 338(8):1328–1344, 2015.

[2] A. Bialostocki, P. Erdős, and H. Lefmann. Monochromatic and zero-sum sets of nondecreasing diameter. *Discrete Math.*, 137(1–3):19–34, 1995.

[3] A. Bialostocki and R. Sabar. On constrained 2-partitions of monochromatic sets and generalizations in the sense of Erdös-Ginzburg-Ziv. *Ars Combin.*, 76:277–286, 2005.

[4] A. Bialostocki and B. Wilson. On monochromatic sets of integers whose diameters form a monotone sequence. *Integers*, 9:A54, 717–723, 2009.

[5] B. Bollobás, P. Erdős, and G. Jin. Strictly ascending pairs and waves. In *Graph theory, combinatorics, and algorithms, Vol. 1, 2 (Kalamazoo, MI, 1992)*, Wiley-Intersci. Publ., pages 83–95. Wiley, New York, 1995.

[6] P. Erdős, A. Ginzburg, and A. Ziv. Theorem in the additive number theory. *Bull. Res. Counc. Israel Sect. F Math. Phys.*, 10F(1):41–43, 1961.

[7] D. Grynkiewicz and R. Sabar. Monochromatic and zero-sum sets of nondecreasing modified diameter. *Electron. J. Combin.*, 13(1):#R28, 2006.

[8] D. J. Grynkiewicz. On four colored sets with nondecreasing diameter and the Erdös-Ginzburg-Ziv theorem. *J. Combin. Theory Ser. A*, 100(1):44–60, 2002.

[9] D. J. Grynkiewicz. On four color monochromatic sets with nondecreasing diameter. *Discrete Math.*, 290(2-3):165–171, 2005.

[10] A. Schultz. On a modification of a problem of Bialostocki, Erdös, and Lefmann. *Discrete Math.*, 306(2):244–253, 2006.

[11] B. van der Waerden. Beweis einer Baudetschen Vermutung. *Nieuw. Arch. Wisk.*, 15:212–216, 1927.