Vizing's 2-factor Conjecture Involving Toughness and Maximum Degree Conditions

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Abstract

Let G be a simple graph, and let $\Delta(G)$ and $\chi'(G)$ denote the maximum degree and chromatic index of G, respectively. Vizing proved that $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G)+1$. We say G is Δ-critical if $\chi'(G) = \Delta(G)+1$ and $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. In 1968, Vizing conjectured that if G is a Δ-critical graph, then G has a 2-factor. Let G be an n-vertex Δ-critical graph. It was proved that if $\Delta(G) \geqslant n/2$, then G has a 2-factor; and that if $\Delta(G) \geqslant 2n/3 + 13$, then G has a hamiltonian cycle, and thus a 2-factor. It is well known that every 2-tough graph with at least three vertices has a 2-factor. We investigate the existence of a 2-factor in a Δ-critical graph under "moderate" given toughness and maximum degree conditions. In particular, we show that if G is an n-vertex Δ-critical graph with toughness at least $3/2$ and with maximum degree at least $n/3$, then G has a 2-factor. We also construct a family of graphs that have order n, maximum degree $n-1$, toughness at least $3/2$, but have no 2-factor. This implies that the Δ-criticality in the result is needed. In addition, we develop new techniques in proving the existence of 2-factors in graphs.

Mathematics Subject Classifications: 05C38, 05C42

1 Introduction

In this paper, we consider only simple, undirected, and finite graphs. Let G be a graph. The notation Δ is fixed for the maximum degree of G throughout the paper. A *k-vertex* of G is a vertex of degree exactly k in G. Denote by V_Δ the set of Δ-vertices in G, and

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by $\chi'(G)$ the chromatic index of G. The graph G is called *critical* if $\chi'(G) > \Delta$ and $\chi'(H) < \chi'(G)$ for every proper subgraph H of G. It is clear that if G is critical then G must be connected. In 1965, Vizing [15] showed that a graph of maximum degree Δ has chromatic index either Δ or $\Delta+1$. If $\chi'(G) = \Delta$, then G is said to be of class 1; otherwise, it is said to be of class 2. Holyer [8] showed that it is NP-complete to determine whether an arbitrary graph is of class 1. A critical graph G is called Δ-*critical* if $\chi'(G) = \Delta + 1$. So Δ-critical graphs are class 2 graphs. Motivated by the classification problem, Vizing studied critical class 2 graphs, or Δ-critical graphs, and made two well-known conjectures.

The first conjecture [16] is on the independence number $\alpha(G)$ of G, that is, the size of a maximum independent set in G.

Conjecture 1 (Vizing's Independence Number Conjecture)**.** *Let G be a Δ-critical graph of order n. Then $\alpha(G) \leqslant n/2$.*

Furthermore, Vizing [14] conjectured that the following statement is true.

Conjecture 2 (Vizing's 2-Factor Conjecture)**.** *Let G be a Δ-critical graph. Then G contains a 2-factor.*

As each cycle C satisfying $\alpha(C) \leqslant |V(C)|/2$, Conjecture 2 implies Conjecture 1.

For the Independence Number Conjecture, Brinkmann et al. [2], in 2000, proved that if G is a critical graph, then $\alpha(G) < 2n/3$; and the upper bound is further improved when the maximum degree is between 3 and 10. Luo and Zhao [11], in 2008, by improving the result of Brinkmann et al., showed that if G is an n-vertex Δ-critical graph, then $\alpha(G) < (5\Delta - 6)n/(8\Delta - 6) < 5n/8$ if $\Delta \geqslant 6$. In 2009, Woodall [17] further improved the upper bound to $3n/5$. By restricting the problem to graphs with large maximum degrees, in 2006, Luo and Zhao [10] showed that Vizing's Independence Number Conjecture is true if $\Delta(G) \geqslant n/2$.

Compared to the progresses on the first Conjecture, the progresses on Vizing's 2-Factor Conjecture has been slow. In 2004, Grünewald and Steffen [7] established Vizing's 2-Factor Conjecture for graphs with the deficiency $\sum_{v \in V(G)}(\Delta(G) - d_G(v))$ small; in particular, for overfull graphs (graphs of an odd order and with the deficiency $\sum_{v \in V(G)}(\Delta(G) - d_G(v)) < \Delta(G)$). In 2012, Luo and Zhao [12] proved that if G is an n-vertex Δ-critical graph with $\Delta \geqslant \frac{6n}{7}$, then G contains a hamiltonian cycle, and thus a 2-factor with exactly one component. Continuing the investigation on the existence of a hamiltonian cycle in Δ-critical graphs with "very large" maximum degrees, Luo and Zhao [9] in 2016 showed that an n-vertex Δ-critical graph with $\Delta \geqslant \frac{4n}{5}$ is hamiltonian. The lower bound on $\Delta(G)$ assuring an n-vertex Δ-critical graph to be hamiltonian, has been improved to $\frac{2n}{3} + 12$, in [3]. Just finding 2-factors, Chen and Shan [5] proved the following result.

Theorem 1.1 ([5])**.** *Let G be an n-vertex Δ-critical graph. Then G has a 2-factor if $\Delta \geqslant n/2$.*

As a measure of graph connectivity and "resilience" under removal of vertices, graph toughness is a useful condition in finding factors in graphs. To be precise, we recall the definition of toughness below. The number of components of G is denoted by $c(G)$. Let $t \geqslant 0$ be a real number. The graph is said to be t-*tough* if $|S| \geqslant t \cdot c(G-S)$ for each $S \subseteq V(G)$ with $c(G-S) \geqslant 2$. The *toughness* $\tau(G)$ is the largest real number t for which G is t-tough, or is ∞ if G is complete. A *toughset* in G is a cutset $S \subseteq V(G)$ of G such that $\frac{|S|}{c(G-S)} = \tau(G)$. Enomoto et al. [6] proved the classic result below.

Theorem 1.2 (Enomoto et al. [6])**.** *Every k-tough graph has a k-factor if $k|V(G)|$ is even and $|V(G)| \geqslant k+1$.*

Combining the result in Theorem 1.1 and the result in Theorem 1.2 when restricted to 2-factors, one might wonder — can we get something in between, i.e., is it possible to find 2-factors in an n-vertex Δ-critical graph G, under the condition that $\Delta(G) < n/2$ but $\Delta(G) \geqslant cn$ for some positive constant c, and $\tau(G) < 2$ but $\tau(G) \geqslant d$ for some positive constant d? Particularly, we prove the following result.

Theorem 1.3. *Let G be an n-vertex Δ-critical graph. Then G has a 2-factor if $\tau(G) \geqslant 3/2$ and $\Delta \geqslant n/3$.*

The remaining of the paper is organized as follows: in Section 2, we recall some graph terminologies and present several lemmas; in Section 3, we recall Tutte's 2-factor Theorem and develop techniques for showing the existence of 2-factors upon applying Tutte's 2-factor Theorem; in Section 4, we prove Theorem 1.3. In the last Section, we construct a family of graphs that have order n, maximum degree $n-1$, toughness at least 3/2, but have no 2-factor. This implies that the Δ-criticality in Theorem 1.3 is needed.

2 Notation and Lemmas

Let G be a graph. For $x \in V(G)$ we denote by $d_G(x)$ the degree of x in G. For disjoint subsets of vertices S and T in G, we denote by $E_G(S,T)$, the set of edges that has one end vertex in S and the other in T, and let $e_G(S,T) = |E_G(S,T)|$. If $S = \{s\}$ is a singleton, we write $e_G(s,T)$ instead of $e_G(\{s\},T)$. If $H \subseteq G$ is a subgraph of G, and $T \subseteq V(G)$ with $T \cap V(H) = \emptyset$, we write $E_G(H,T)$ and $e_G(H,T)$ for notational simplicity. A *matching* in G is a set of independent edges. If M is a matching of G, then let $V(M)$ denote the set of end vertices of the edges in M. For $X \subseteq V(G)$, M is said to *saturate* X if $X \subseteq V(M)$. If G is a bipartite graph with partite sets A and B, we denote G by $G[A,B]$ to emphasize the two partite sets.

To prove Theorem 1.3, we present two lemmas below.

Lemma 2.1 (Vizing's Adjacency Lemma)**.** *Let G be a Δ-critical graph. Then for any edge $xy \in E(G)$, x is adjacent to at least $\Delta - d_G(y) + 1$ Δ-vertices z with $z \neq y$.*

The following lemma is a generalization of a result in [10].

Lemma 2.2. *Let G be a Δ-critical graph and T be an independent set in G. Let $S = V(G) - T$, and let $H = G - E(G[S])$ be the bipartite graph with partite sets S and T.*

For each $x \in S$, let σ_x be the number of non-Δ-degree neighbors of x from S. Assume that there are δ_0 Δ-vertices in T. Then for each edge $xy \in E(H)$ with $x \in S$ and $y \in T$, $d_H(y) \geqslant d_H(x) + 1 - \delta_0 + \sigma_x$.

Proof. Let $xy \in E(H)$ with $x \in S$ and $y \in T$. By Vizing's Adjacency Lemma, x is adjacent to at least $\Delta - d_G(y) + 1$ Δ-vertices in G. As T has δ_0 Δ-vertices, we know x is adjacent to at least $\Delta - d_G(y) + 1 - \delta_0$ Δ-vertices from S. Let σ_x be the number of all non Δ-degree neighbors of x in S. Then, $d_H(x) + \Delta - d_G(y) + 1 - \delta_0 + \sigma_x \leqslant d_G(x) \leqslant \Delta$. By noting that $d_G(y) = d_H(y)$, the inequality implies that $d_H(y) \geqslant d_H(x) + 1 - \delta_0 + \sigma_x$. □

3 Tutte's 2-factor Theorem and Biased Barriers

One of the main proof ingredients of Theorem 1.3 is to apply Tutte's 2-factor Theorem under a new setting that we develop in this section.

Let S and T be disjoint subsets of vertices of a graph G. Let D be a component of $G - (S \cup T)$. Then D is said to be an *odd component* (resp. *even component*) if $e_G(D, T) \equiv 1 \pmod 2$ (resp. $e_G(D, T) \equiv 0 \pmod 2$). Let $\mathcal{H}(S, T)$ be the set of odd components of $G - (S \cup T)$ and let $h(S, T) = |\mathcal{H}(S, T)|$. For $y \in T$, let $\mathcal{H}(y : S, T) = \{D \in \mathcal{H}(S, T), e_G(y, D) > 0\}$ and $h(y : S, T) = |\mathcal{H}(y : S, T)|$. Note that $e_G(y, V(G) - (S \cup T)) \geqslant h(y : S, T)$.

Let $\delta(S, T) = 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) - h(S, T)$. It is easy to see that $\delta(S, T) \equiv 0 \pmod 2$ for every S, $T \subseteq V(G)$ with $S \cap T = \emptyset$. We use the following criterion for the existence of a 2-factor, which is a restricted form of Tutte's f-factor Theorem.

Lemma 3.1 (Tutte [13]). *A graph G has a 2-factor if and only if $\delta(S, T) \geqslant 0$ for every S, $T \subseteq V(G)$ with $S \cap T = \emptyset$.*

An ordered pair (S, T) consists of disjoint subsets of vertices S and T in a graph G is called a *barrier* if $\delta(S, T) \leqslant -2$. By Lemma 3.1, if G does not have a 2-factor, then G has a barrier. We define a special barrier as below.

Definition 1. *Let G be a graph without a 2-factor. A barrier (S, T) of G is called a biased barrier if among all the barriers of G,*

(1) $|S|$ is maximum; and

(2) subject to (1), $|T|$ is minimum.

Properties of a minimum barrier (a barrier such that $|S \cup T|$ is minimum among all the barriers of G) has been established, for example, in [1, 4]. A biased barrier has similar nice properties as given in the lemma below.

Lemma 3.2. *Let G be a graph without a 2-factor, and let (S, T) be a biased barrier of G. Then each of the following holds.*

(1) The set T is independent in G.

(2) If D is an even component with respect to (S,T), then $e_G(T,D)=0$.

(3) If D is an odd component with respect to (S,T), then for any $y \in T$, $e_G(y,D) \leqslant 1$.

(4) If D is an odd component with respect to (S,T), then for any $x \in V(D)$, $e_G(x,T) \leqslant 1$.

Proof. Let $U = V(G) - (S \cup T)$ and $z \in T$ be a vertex. By the assumption that (S,T) is a biased barrier, we know that $\delta(S, T - \{z\}) \geqslant 0$. So,

$$\begin{aligned}
0 &\leqslant \delta(S,T-\{z\}) = 2|S| - 2|T| + 2 + \sum_{y\in T-\{z\}} d_{G-S}(y) - h(S,T-\{z\}) \\
&= 2|S| - 2|T| + 2 + \sum_{y\in T} d_{G-S}(y) - e_G(z,T-\{z\}) - e_G(z,U) - h(S,T-\{z\}) \\
&\leqslant 2|S| - 2|T| + 2 + \sum_{y\in T} d_{G-S}(y) - e_G(z,T-\{z\}) - e_G(z,U) - h(S,T) + h(z:S,T) \\
&= \delta(S,T) + 2 - e_G(z,T-\{z\}) - e_G(z,U) + h(z:S,T) \\
&\leqslant -e_G(z,T-\{z\}) - e_G(z,U) + h(z:S,T), \quad \text{since } \delta(S,T) \leqslant -2.
\end{aligned}$$

This implies that

$$e_G(z,T-\{z\}) + e_G(z,U) - h(z:S,T) \leqslant 0.$$

Because $e_G(z,U) - h(z:S,T) \geqslant 0$ always holds, the above inequality particularly implies that

$$e_G(z,T-\{z\}) = 0 \quad \text{for any } z \in T \quad \text{and} \qquad e_G(z,U) - h(z:S,T) = 0.$$

This proves statements (1)-(3).

To show (4), let D be an odd component with respect to (S,T) and let $x \in V(D)$ be any vertex. Then by the assumption that $|S|$ is maximum, we know that $\delta(S \cup \{x\}, T) \geqslant 0$. So,

$$\begin{aligned}
0 &\leqslant \delta(S\cup\{x\},T) = 2|S| - 2|T| + 2 + \sum_{y\in T} d_{G-(S\cup\{x\})}(y) - h(S\cup\{x\},T) \\
&= 2|S| - 2|T| + 2 + \sum_{y\in T} d_{G-S}(y) - e_G(x,T) - h(S\cup\{x\},T) \\
&\leqslant 2|S| - 2|T| + 2 + \sum_{y\in T} d_{G-S}(y) - e_G(x,T) - (h(S,T)-1) \\
&= \delta(S,T) + 2 - e_G(x,T) + 1 \\
&\leqslant -e_G(x,T) + 1, \quad \text{since } \delta(S,T) \leqslant -2.
\end{aligned}$$

Hence, $e_G(x,T) \leqslant 1$. □

Let G be a graph without a 2-factor and let (S,T) be a biased barrier of G. We call (S,T) a *good biased barrier* of G if $h(S,T)$ is smallest among all biased barriers of G.

Lemma 3.3. *Let G be a graph without a 2-factor, and let (S,T) be a good biased barrier of G. For any $y \in T$, if $h(y:S,T) \geqslant 2$, then for any $D \in \mathcal{H}(y:S,T)$, $|V(D)| \geqslant 3$.*

Proof. Let $D \in \mathcal{H}(y:S,T)$ be an odd component of $G-(S\cup T)$. By (4) of Lemma 3.2, $|V(D)| \geqslant 3$ if $e_G(D,T) \geqslant 3$. So we assume that $e_G(D,T) = 1$ and assume to the contrary that $|V(D)| \leqslant 2$. Let x be the vertex in D if $|V(D)| = 1$, and be a vertex in D which is not adjacent to any vertex in T if $|V(D)| = 2$. Let $z \in T$ be the vertex such that $e_G(D, z) = 1$, and let $T' = (T - \{z\}) \cup \{x\}$ and $U = V(G) - (S \cup T)$. Let D_z be the component of $G - (S \cup T')$ which contains the vertex z. Then since $e_G(z, D') = 1$ for any $D' \in \mathcal{H}(z : S, T)$ by (3) of Lemma 3.2, we have that

$$e_G(D_z, T) = \begin{cases} \displaystyle\sum_{D' \in \mathcal{H}(z:S,T)-\{D\}} (e_G(D', T) - 1) + e_G(x, z), & \text{if } |V(D)| = 1; \\ \displaystyle\sum_{D' \in \mathcal{H}(z:S,T)} (e_G(D', T) - 1) + e_G(x, V(D) - \{x\}), & \text{if } |V(D)| = 2. \end{cases}$$

Since $e_G(D', T)$ is odd for any $D' \in \mathcal{H}(z : S, T)$, and $e_G(x, z) = e_G(x, V(D) - \{x\}) = 1$, we know that $D_z \in \mathcal{H}(S, T')$ is an odd component of $G - (S \cup T')$. Hence, $h(S, T') = h(S, T) - h(z : S, T) + 1$. So

$$\begin{aligned} &\delta(S, T') = 2|S| - 2|T| + \sum_{y \in T'} d_{G-S}(y) - h(S, T') \\ = \;& 2|S| - 2|T| + \sum_{y \in T} d_{G-S}(y) + e_G(x, V(D - x) \cup \{z\}) - \\ & e_G(z, U) - h(S, T) + h(z : S, T) - 1 \\ = \;& \delta(S, T) + e_G(x, V(D - x) \cup \{z\}) - e_G(z, U) + h(z : S, T) - 1 \\ \leqslant \;& \delta(S, T) \leqslant -2, \quad \text{since } e_G(x, V(D - x) \cup \{z\}) = 1, \text{and } e_G(z, U) \geqslant h(z : S, T). \end{aligned}$$

Thus, (S, T') is a biased barrier. However, $h(S, T') = h(S, T) - h(y : S, T) + 1 \leqslant h(S, T) - 1$, showing a contradiction to the assumption that (S, T) is a good biased barrier. □

4 Proof of Theorem 1.3

Let G be an n-vertex Δ-critical graph such that $\tau(G) \geqslant 3/2$ and $\Delta \geqslant n/3$. We show that G has a 2-factor.

Since G is 3/2-tough, $\Delta(G) \geqslant \delta(G) \geqslant 3$. Assume to the contrary that G does not have a 2-factor. Then by Tutte's 2-factor Theorem (Lemma 3.1), G has a barrier. Let (S, T) be a good biased barrier of G. Since S and T are already fixed, we simply denote $\mathcal{H}(S, T)$ by $\mathcal{H}$. Let $U = V(G) - (S \cup T)$ and let $\mathcal{H}_k$ be the set of components D of $G - (S \cup T)$ with $e_G(D, T) = k$. Then we have $\mathcal{H} = \bigcup_{k \geqslant 0} \mathcal{H}_{2k+1}$. For any $y \in T$, let

$$\begin{aligned} \mathcal{H}(y) &= \{ D \in \mathcal{H} \mid e_G(y, D) = 1 \}, \\ \mathcal{H}_1(y) &= \{ D \in \mathcal{H}_1 \mid e_G(y, D) = 1 \}. \end{aligned}$$

It is clear that $\mathcal{H}_1(y) \subseteq \mathcal{H}(y)$. Note also that $\mathcal{H}(y) = \mathcal{H}(y : S, T)$. We use this notation $\mathcal{H}(y)$ for simplicity since S and T are already fixed.

Claim 4.1. $|T| \geqslant |S| + \sum_{k \geqslant 1} k|\mathcal{H}_{2k+1}| + 2$. *In particular,* $|T| \geqslant |S| + 2$.

Proof. Since (S,T) is a barrier,

$$\begin{aligned}\delta(S,T) &= 2|S| - 2|T| + \sum_{y\in T} d_{G-S}(y) - h(S,T)\\ &= 2|S| - 2|T| + \sum_{y\in T} d_{G-S}(y) - \sum_{k\geqslant 0}|\mathcal{H}_{2k+1}| \leqslant -2.\end{aligned}$$

By Lemma 3.2 (1) and (2),

$$\sum_{y\in T} d_{G-S}(y) = \sum_{y\in T} e_G(y,U) = e_G(T,U) = \sum_{k\geqslant 0}(2k+1)|\mathcal{H}_{2k+1}|.$$

Therefore, we have

$$-2 \geqslant 2|S| - 2|T| + \sum_{k\geqslant 0}(2k+1)|\mathcal{H}_{2k+1}| - \sum_{k\geqslant 0}|\mathcal{H}_{2k+1}|,$$

which yields $|T| \geqslant |S| + \sum_{k\geqslant 1} k|\mathcal{H}_{2k+1}| + 2$. □

We perform the following operations to G.

(1) Remove all even components, and remove all components in $\mathcal{H}_1$.

(2) Remove all edges in $G[S]$.

(3) For a component $D \in \mathcal{H}_{2k+1}$ with $k \geqslant 1$ introduce a set of k independent vertices $U^D = \{u_1^D, u_2^D, \dots, u_k^D\}$ and replace D with U^D. By Lemma 3.2 (3), $|N_G(D) \cap T| = e_G(T,D) = 2k+1$. Let $N_G(D) \cap T = \{v_0, v_1, \dots, v_{2k}\}$. Add two new edges $u_i^D v_{2i-1}$ and $u_i^D v_{2i}$ for each i with $1 \leqslant i \leqslant k$. Moreover, add one extra edge $u_1^D v_0$.

Let H be the resulting graph, and let

$$U^{\mathcal{H}} = \bigcup_{k\geqslant 1}\left(\bigcup_{D\in\mathcal{H}_{2k+1}} U^D\right), \quad X = S \cup U^{\mathcal{H}}.$$

By the construction, the graph H satisfies the following properties.

(1) H is a bipartite graph with partite sets X and T,

(2) $|U^{\mathcal{H}}| = \sum_{k\geqslant 1} k|\mathcal{H}_{2k+1}|$, $|X| = |S| + |U^{\mathcal{H}}| = |S| + \sum_{k\geqslant 1} k|\mathcal{H}_{2k+1}|$, and

(3) For each $k \geqslant 1$ and each $D \in \mathcal{H}_{2k+1}$, $d_H(u_1^D) = 3$ and $d_H(u_i^D) = 2$ for each i with $2 \leqslant i \leqslant k$.

We will show that there is a matching in H which saturates T, which gives that $|X| = |S| + \sum_{k\geqslant 1} k|\mathcal{H}_{2k+1}| \geqslant |T|$, giving a contradiction to Claim 4.1.

For notational simplicity, for a set $\mathcal{D} \subseteq \mathcal{H}$, let

$$V(\mathcal{D}) = \bigcup_{D\in\mathcal{D}} V(D).$$

Claim 4.2. *$|S| < |U^{\mathcal{H}}|$.*

Proof. Assume to the contrary that $|S| \geqslant |U^{\mathcal{H}}| = \sum_{k\geqslant 1} k|\mathcal{H}_{2k+1}|$. We may assume that $|U^{\mathcal{H}}| \geqslant 1$. For otherwise, since there is no edge between even component of $G-(S\cup T)$ and T, and each component in $\mathcal{H}_1$ is connected to a single vertex in T, $c(G-S) \geqslant |T|$. Since $|T| \geqslant 2$ by Claim 4.1, S is a cutset of G. This implies that $\tau(G) \leqslant \frac{|S|}{|T|} < 1$, giving a contradiction.

For each $D \in \mathcal{H}_{2k+1}$ with $k \geqslant 1$, let W_D be a set of any $2k$ vertices in D such that for each $x \in W_D$, $e_G(x,T) = 1$. Thus, $D - W_D$ is only connected to a single vertex in T. Let

$$W = S \cup \left(\bigcup_{D\in\mathcal{H}_{2k+1},k\geqslant 1} W_D \right).$$

Since T is an independent set in G, and each component in $G-W$ is connected to S or only a single vertex in T, we have that $c(G-W) \geqslant |T|$. Again, W is a cutset of G as $|T| \geqslant 2$ by Claim 4.1. So

$$\begin{aligned}\tau(G) &\leqslant \frac{|W|}{|T|} \leqslant \frac{|S| + \sum_{k\geqslant 1} 2k|\mathcal{H}_{2k+1}|}{|S| + |U^{\mathcal{H}}| + 2} \\ &\leqslant \frac{\sum_{k\geqslant 1} k|\mathcal{H}_{2k+1}| + \sum_{k\geqslant 1} 2k|\mathcal{H}_{2k+2}|}{\sum_{k\geqslant 1} k|\mathcal{H}_{2k+1}| + \sum_{k\geqslant 1} k|\mathcal{H}_{2k+1}| + 2} < \frac{3}{2},\end{aligned}$$

showing a contradiction to the assumption that $\tau(G) \geqslant 3/2$. □

Because of $|T| > |S| + |U^{\mathcal{H}}|$ and $|U^{\mathcal{H}}| > |S|$, we get the following Claim.

Claim 4.3. *$|T| \geqslant 2|S| + 2$.*

Claim 4.4. *T contains no Δ-vertex of G.*

Proof. Suppose to the contrary that there exists $z \in T$ such that $d_G(z) = \Delta$. We may assume that $|\mathcal{H}(z)| \geqslant 2$. Otherwise, $e_G(z,S) \geqslant \Delta - 1$ and so $|S| \geqslant \Delta - 1$. Hence by Claims 4.2 and 4.3,

$$\begin{aligned} n &= |S| + |T| + |U| \\ &\geqslant 3|S| + 2 + |U^{\mathcal{H}}| \quad (|U| \geqslant |U^{\mathcal{H}}| \text{ by Lemma 3.2 (4)}) \\ &\geqslant 4|S| + 3 \geqslant 4\Delta - 1 \geqslant 4n/3 - 1, \end{aligned}$$

implying that $n \leqslant 3$. This gives a contradiction to the fact that $\Delta \geqslant 3$.

Hence, by Lemma 3.3, we have that

$$\begin{aligned} n &= |S| + |T| + |U| \geqslant e_G(z,S) + 3|\mathcal{H}(z)| + |T| \\ &\geqslant e_G(z,S) + 3|\mathcal{H}(z)| + 2e_G(z,S) + 2 \quad (|T| \geqslant 2|S| + 2 \geqslant 2e_G(z,S) + 2) \\ &= 3(e_G(z,S) + |\mathcal{H}(z)|) + 2 = 3\Delta + 2 \geqslant n + 2, \end{aligned}$$

showing a contradiction. □

Claim 4.5. $|\mathcal{H}| \geqslant 2$.

Proof. Assume to the contrary that $|\mathcal{H}| \leqslant 1$. By Claim 4.2, we then know that $|\mathcal{H}| = 1$ and the component is adjacent to at least three vertices in T. Let D be the only odd component in $G - (S \cup T)$ and assume that $e_G(D, T) = 2k + 1$ for some $k \geqslant 1$. By Lemma 3.2 (3), $|T| \geqslant 2k + 1$. Again, by Claim 4.2, $|S| \leqslant k - 1$. Let W be the union of the set S and the set of neighbors of vertices in T from the component D. Then $|W| \leqslant k - 1 + 2k + 1 = 3k$ and $c(G - W) \geqslant |T| \geqslant 2k + 1$. This gives a contradiction to the toughness of G. □

Let $D_1 \in \mathcal{H}$ be a component such that

$$|V(D_1)| = \max\{\, |V(D)| \;\big|\; D \in \mathcal{H} \,\},$$

and $D_2 \in \mathcal{H} - \{D_1\}$ such that

$$|V(D_2)| = \max\{\, |V(D)| \;\big|\; D \in \mathcal{H} - \{D_1\} \,\}.$$

Claim 4.6. *Let $D \in \mathcal{H} - \{D_1, D_2\}$. Then D contains no Δ-vertex of G. Furthermore, if D_1 contains a Δ-vertex of G, then $|V(D)| \leqslant |V(D_1)| - 1$; and if D_2 contains a Δ-vertex of G, then $|V(D)| \leqslant |V(D_2)| - 2 \leqslant |V(D_1)| - 2$, and for any $x \in V(D)$, $d_G(x) \leqslant \Delta - 2$.*

Proof. Note that by the choice of D_1 and D_2, $|V(D)| \leqslant |V(\mathcal{H})|/3$, recall here that $V(\mathcal{H})$ is the union of vertex sets of components in $\mathcal{H}$. Since $|T| \geqslant 2|S| + 2$ by Claim 4.3, we have that $n \geqslant |S| + |T| + |V(\mathcal{H})| \geqslant 3|S| + 2 + |V(\mathcal{H})|$. Consequently, $|V(\mathcal{H})|/3 \leqslant (n-2)/3 - |S|$. Thus, for any $x \in V(D)$,

$$d_G(x) \;\leqslant\; |V(D)| - 1 + 1 + |S| \leqslant |V(\mathcal{H})|/3 + |S| \leqslant (n-2)/3 < \Delta.$$

Suppose that D_1 contains a Δ-vertex of G, and there exists $D \in \mathcal{H} - \{D_1, D_2\}$ such that $|V(D)| = |V(D_1)|$. This implies that $|V(D_1)| = |V(D_2)| = |V(D)|$, so $|V(D_1)| \leqslant |V(\mathcal{H})|/3$. Then by exactly the same argument above, we have that for any $x \in V(D_1)$,

$$d_G(x) \;\leqslant\; |V(D)| - 1 + 1 + |S| \leqslant |V(\mathcal{H})|/3 + |S| \leqslant (n-2)/3 < \Delta.$$

Hence, $|V(D)| \leqslant |V(D_1)| - 1$.

Suppose now that D_2 contains a Δ-vertex of G. Since $|V(D_1)| \geqslant |V(D_2)|$, we then have that $|V(D_i)| + |S| \geqslant \Delta$ for $i = 1, 2$. So for any $D \in \mathcal{H} - \{D_1, D_2\}$,

$$\begin{aligned} n &\geqslant |S| + |T| + |V(D_1)| + |V(D_2)| + |V(D)| \\ &\geqslant |S| + 2|S| + 2 + |V(D_1)| + |V(D_2)| + |V(D)| \\ &= |S| + |V(D_1)| + |S| + |V(D_2)| + |S| + |V(D)| + 2. \end{aligned}$$

Because of $|V(D_i)| + |S| \geqslant \Delta$ for $i = 1, 2$, it follows that

$$|S| + |V(D)| \leqslant n - 2\Delta - 2 \leqslant n/3 - 2 \leqslant \Delta - 2.$$

Consequently, $|V(D)| \leqslant |V(D_2)| - 2 \leqslant |V(D_1)| - 2$, and for any $x \in V(D)$, $d_G(x) \leqslant \Delta - 2$. □

We introduce some further notation here. Let

$$T_1 = \{\, y \in T \mid |\mathcal{H}_1(y)| = 1 \,\}, \quad \text{and} \quad T_2 = \{\, y \in T \mid |\mathcal{H}_1(y)| \geqslant 2 \,\}.$$

For each component $D \in \mathcal{H}_1$, let $y_D \in T$ be the vertex such that $e_G(D,T) = e_G(D, y_D) = 1$. Let

$$\mathcal{H}_{11} = \{\, D \in \mathcal{H}_1 \mid y_D \in T_1 \,\}, \quad \text{and} \quad \mathcal{H}_{12} = \{\, D \in \mathcal{H}_1 \mid y_D \in T_2 \,\}(= \mathcal{H}_1 - \mathcal{H}_{11}).$$

Claim 4.7. *For each component $D \in \mathcal{H}_{12}$, $|\mathcal{H}(y_D)| \geqslant 2$. Consequently, $|V(D)| \geqslant 3$.*

Proof. Since $D \in \mathcal{H}_{12}$, we have that $|\mathcal{H}(y_D)| \geqslant |\mathcal{H}_1(y_D)| \geqslant 2$. Then $|V(D)| \geqslant 3$ by Lemma 3.3. □

Denote

$$\begin{aligned}
m_1 &= |\mathcal{H}_{11}|, \quad m_2 = |\mathcal{H}_{12}|, \quad \text{and} \quad m_3 = |\mathcal{H} - \mathcal{H}_1|,\\
S_1 &= \{x \in S \mid x \,\text{has a non}\ \Delta\text{-degree neighbor in}\ V(G) - T\}, \quad \text{and} \quad S_0 = S - S_1,\\
p_y &= |\mathcal{H}_1(y)| \quad \text{for any}\ y \in T.
\end{aligned}$$

Note that by the definition, if $m_2 \neq 0$, then $m_2 \geqslant 2$.

Claim 4.8. *Let $y \in T$ be a vertex. Then*

$$|N_G(y) \cap S| \geqslant \begin{cases} |S_0| + m_1/3 + m_2 - 1, & \text{if } \emptyset \neq \mathcal{H}(y) \not\subseteq \{D_1, D_2\};\\ 2, & \text{if } \mathcal{H}(y) = \{D_1\} \text{ or } \{D_2\};\\ 1, & \text{if } \mathcal{H}(y) = \{D_1, D_2\}. \end{cases}$$

Moreover, $N_G(y) \cap S \neq \emptyset$.

Proof. Since G is 3/2-tough, $\delta(G) \geqslant 3$. As each $y \in T_1$ satisfies $d_G(y) = e_G(y,S) + e_G(y, V(\mathcal{H}))$ and $e_G(y, D_i) \leqslant 1$ for $i = 1,2$, we get $e_G(y,S) \geqslant 2$ if $\mathcal{H}(y) = \{D_1\}$ or $\{D_2\}$. If $\mathcal{H}(y) = \{D_1, D_2\}$, then $|N_G(y) \cap S| \geqslant 1$.

Thus we assume that there exists $D \in \mathcal{H}(y) - \{D_1, D_2\}$. Let x_D be the neighbor of y in D. By Claim 4.6, x_D is not a Δ-vertex of G. Moreover, y is adjacent to at least $\Delta - d_G(x_D) + 1$ Δ-vertices of G by Vizing's Adjacency Lemma.

Note that each component in $\mathcal{H} - \mathcal{H}_1$ contains at least three vertices by Lemma 3.2 (4). So $n \geqslant$

$$\begin{cases} |S| + |T| + |V(D_1)| + |V(D_2)| + |V(D)| + m_1 + 3(m_2 + m_3 - 3), & \text{if } D_1, D_2, D \\ & \in \mathcal{H} - \mathcal{H}_{11};\\ |S| + |T| + |V(D_1)| + |V(D_2)| + |V(D)| + m_1 - 1 + 3(m_2 + m_3 - 2), & \text{otherwise.} \end{cases}$$

Thus, because $|T| \geqslant 2|S| + 2$ by Lemma 4.3, and $|U^{\mathcal{H}}| \geqslant |S| + 1$ implying that $m_3 \geqslant 1$, we get that

$$n \geqslant |S| + |T| + |V(D_1)| + |V(D_2)| + |V(D)| + m_1 + 3(m_2 + m_3 - 3)$$

$$\geqslant 3|S| + 2 + |V(D_1)| + |V(D_2)| + |V(D)| + m_1 + 3(m_2 - 2)$$
$$\geqslant \begin{cases} 3|S| + 3|V(D)| + m_1 + 3m_2, & \text{if } D_2 \text{ contains a } \Delta\text{-vertex;} \\ 3|S| + 3|V(D)| + m_1 + 3m_2 - 3, & \text{if } D_1 \text{ contains a } \Delta\text{-vertex;} \\ 3|S| + 3|V(D)| + m_1 + 3m_2 - 4, & \text{if neither } D_1 \text{ nor } D_2 \text{ contains a } \Delta\text{-vertex.} \end{cases}$$

The above bounds were obtained because of $|V(D_1)| \geqslant |V(D_2)| \geqslant |V(D)|$ and Claim 4.6. Thus since no component in $\mathcal{H} - \{D_1, D_2\}$ containing a Δ-vertex of G by Claim 4.6, we have that

$$|N_G(y) \cap S \cap V_\Delta| \geqslant \begin{cases} \Delta - d_G(x_D) - 1, & \text{if } D_2 \text{ contains a } \Delta\text{-vertex;} \\ \Delta - d_G(x_D), & \text{if } D_1 \text{ contains a } \Delta\text{-vertex but } D_2 \text{ has no } \Delta\text{-vertex;} \\ \Delta - d_G(x_D) + 1, & \text{if neither } D_1 \text{ nor } D_2 \text{ contains a } \Delta\text{-vertex.} \end{cases}$$

Because x_D is not a Δ-vertex of G, by the definitions of S_0 and S_1, we have that $N_G(x_D) \cap S = N_G(x_D) \cap S_1$. So $d_G(x_D) \leqslant |S_1| + |V(D)|$. Replacing Δ by $\frac{n}{3}$ in the above bounds on $|N_G(y) \cap S|$, and combining the bounds on n, we get that

$$|N_G(y) \cap S| \geqslant \begin{cases} |S_0| + \frac{m_1}{3} + m_2 - 1, & \text{if } D_2 \text{ contains a } \Delta\text{-vertex;} \\ |S_0| + \frac{m_1}{3} + m_2 - 1, & \text{if } D_1 \text{ contains a } \Delta\text{-vertex but } D_2 \text{ has no } \Delta\text{-vertex;} \\ |S_0| + \frac{m_1}{3} + m_2 - \frac{1}{3}, & \text{if neither } D_1 \text{ nor } D_2 \text{ contains a } \Delta\text{-vertex.} \end{cases}$$

For the second part of the statement, if $\mathcal{H}(y) = \emptyset$, then $N_G(y) = N_G(y) \cap S$. So assume that $\mathcal{H}(y) \neq \emptyset$. By the first part of the statement, it easily follows that $|N_G(y) \cap S| \geqslant 1$ unless $\mathcal{H}(y) \not\subseteq \{D_1, D_2\}$. Let $D \in \mathcal{H}(y) - \{D_1, D_2\}$, and let x_D be the neighbor of y in D. By Claim 4.6, x_D is not a Δ-vertex of G. Moreover, y is adjacent to at least $\Delta - d_G(x_D) + 1$ Δ-vertices of G by Vizing's Adjacency Lemma. Note that no component in $\mathcal{H} - \{D_1, D_2\}$ contains a Δ-vertex of G by Claim 4.6. If D_2 does not contain a Δ-vertex of G, then y is adjacent to at least $\Delta - d_G(x_D) \geqslant 1$ Δ-vertices which are contained in S. If D_2 contains a Δ-vertex of G, then by the second part of Claim 4.6, $d_G(x_D) \leqslant \Delta - 2$. So y is adjacent to at least $\Delta - d_G(x_D) - 1 \geqslant 1$ Δ-vertices which are contained in S.

The proof is finished. □

If $\{D_1, D_2\} \cap \mathcal{H}_1 \neq \emptyset$, say $D_1 \in \mathcal{H}_1$, then there exists a unique vertex $y \in T$ such that $e_G(y, D_1) = 1$. If $\{D_1, D_2\} \cap \mathcal{H}_1 \neq \emptyset$ and there exists $y \in T$ such that $\mathcal{H}(y) = \{D_1, D_2\}$,

we denote y by y_ω.

Claim 4.9. *Let $y \in T$ be a vertex such that $|\mathcal{H}(y)| \geqslant 2$ and $y \neq y_\omega$. Then $d_G(y) \geqslant 4$.*

Proof. Assume to the contrary that $d_G(y) = 3$. Let $D \in \mathcal{H}(y) - \{D_1, D_2\}$, and let x_D be the neighbor of y in D. Then x_D is adjacent to at least $\Delta - 3 + 1$ Δ-vertices of G by Vizing's Adjacency Lemma. Since $V(D)$ contains no Δ-vertex of G by Claim 4.6, and T contains no Δ-vertex of G by Claim 4.4, we conclude that $|S| \geqslant |N_G(x_D) \cap S \cap V_\Delta| \geqslant \Delta - 2$. Since each $D \in \mathcal{H}_{2k+1}$ contains at least $2k+1$ vertices by Lemma 3.2 (4), $|V(\mathcal{H})| \geqslant 2|U^{\mathcal{H}}|$. Thus

$$n \geqslant |S| + 2|U^{\mathcal{H}}| + |T| \geqslant |S| + 2(|S| + 1) + 2|S| + 2$$

$$= \quad 5|S| + 4 \geqslant 5(\Delta - 2) + 4 \geqslant 5\left(\frac{n}{3} - 2\right) + 4 = \frac{5n}{3} - 6,$$

implying that $n \leqslant 9$.

By Claim 4.8, $N_G(y) \cap S \neq \emptyset$. Since $|\mathcal{H}(y)| \geqslant 2$, by Lemma 3.3, $|U| \geqslant |V(\mathcal{H}(y))| \geqslant 6$. Since $|S| \geqslant 1$, $|T| \geqslant 2|S| + 2 \geqslant 4$. Hence, $n \geqslant |S| + |T| + |U| \geqslant 1 + 4 + 6 \geqslant 11$, a contradiction. □

Claim 4.10. *Let $xy \in E(H)$ be an edge with $x \in X$ and $y \in T$. Then each of the following holds.*

(1) If $x \in S_0$, then $d_H(y) + p_y \geqslant d_H(x) + 1$.

(2) If $x \in S_1$, then $d_H(y) + p_y \geqslant d_H(x) + 2$.

(3) If $x \in U^{\mathcal{H}}$ and $p_y = 0$, then $d_H(y) \geqslant d_H(x)$.

(4) If $x \in U^{\mathcal{H}}$, $p_y \geqslant 1$, and $y \neq y_\omega$, then $d_H(y) + p_y \geqslant d_H(x) + 1$.

(5) If $x \in U^{\mathcal{H}}$ and $y = y_\omega$, then $d_H(y) + p_y \geqslant d_H(x)$.

Proof. Statements (1) and (2) follow from Lemma 2.2 by taking $\sigma_x = 0$ and 1, respectively. The statements (3) and (5) are clear, since $d_H(y) + p_y = d_G(y) \geqslant \delta(G) \geqslant 3$, and $d_H(x) \leqslant 3$ for any $x \in U^{\mathcal{H}}$. Now we show statement (4). By the assumption that $x \in U^{\mathcal{H}}$ and $p_y \geqslant 1$, we have that $|\mathcal{H}(y)| \geqslant 2$. Then the statement follows by Claim 4.9, since $d_H(y) + p_y = d_G(y) \geqslant 4$, while $d_H(x) \leqslant 3$. □

Claim 4.11. *H has a matching which saturates T.*

Proof. Suppose to the contrary that H has no matching saturating T. By Hall's Theorem, there is a nonempty subset $B \subseteq T$ such that $|N_H(B)| < |B|$. Among all such subsets with this property, we choose B with smallest cardinality. Let $A = N_H(B)$ and $H' = H[A \cup B]$. Then we claim that in H', there is a matching which saturates A. Suppose this is not the case. By Hall's Theorem again, there is a nonempty subset $A' \subseteq A$ such that $|N_{H'}(A')| < |A'|$. Since $A' \subseteq A = N_H(B) \neq \emptyset$ (T contains no isolated vertex of H), $N_{H'}(A') \neq \emptyset$. Let $B' = B - N_{H'}(A')$. As $|B| > |A| \geqslant |N_{H'}(A')| > 0$, $0 < |B'| < |B|$. On the other hand, we have $N_{H'}(B') = N_H(B') \subseteq A - A'$. However, $|B'| = |B| - |N_{H'}(A')| > |A| - |N_{H'}(A')| > |A| - |A'| = |A - A'| \geqslant |N_H(B')|$, showing a contradiction to the choice of B.

Let M be a matching of $H' = H[A \cup B]$ which saturates A. We consider two cases below.

Case 1. $B \cap T_2 \subseteq \{y_\omega\}$.

For any $y \in B$ with $y \neq y_\omega$, $p_y \leqslant 1$. Since $|B| > |A|$, there exists $y_0 \in B - V(M)$. Since $p_{y_0} \leqslant 1$ if $y_0 \neq y_\omega$, we have that $d_H(y_0) \geqslant 2$ if $y_0 \neq y_\omega$. Otherwise, $d_H(y_0) \geqslant 1$.

Assume first that $y_\omega \notin V(M)$. So applying Claim 4.10, we have that

$$e_H(A, B) \quad \leqslant \quad \sum_{\substack{xy \in M \\ x \in A, y \in B}} d_H(x)$$

$$\leqslant \sum_{\substack{xy\in M\\ x\in S_0\cup S_1 \text{ or } p_y=1}} (d_H(y)+p_y-1) + \sum_{\substack{xy\in M\\ x\in U^{\mathcal{H}}, y\in B, p_y=0}} d_H(y)$$

$$< \sum_{\substack{xy\in M\\ x\in A, y\in B}} d_H(y) + d_H(y_0) \leqslant e_H(A,B),$$

showing a contradiction.

Assume now that $y_\omega \in V(M)$. By the definition of y_ω, $1 \leqslant p_{y_\omega} \leqslant 2$. If $p_{y_\omega} = 2$, then for any edge $xy_\omega \in E(H)$, we have that $x \in S$ and so $d_H(y_\omega) + p_{y_\omega} \geqslant d_H(x) + 1$; and if $p_{y_\omega} = 1$, then for any edge $xy_\omega \in E(H)$, $d_H(y_\omega) + p_{y_\omega} \geqslant d_H(x)$. So for any edge $xy_\omega \in E(H)$, $d_H(x) \leqslant d_H(y_\omega) + 1$. Then applying Claim 4.10, we have that

$$e_H(A,B) \leqslant \sum_{\substack{xy\in M\\ x\in A, y\in B}} d_H(x)$$

$$\leqslant \sum_{\substack{xy\in M\\ x\in S_0\cup S_1 \text{ or } p_y=1, y\neq y_\omega}} (d_H(y)+p_y-1) + \left(\sum_{\substack{xy\in M\\ x\in U^{\mathcal{H}}, y\in B, p_y=0}} d_H(y)\right) + d_H(y_\omega) + 1$$

$$< \left(\sum_{\substack{xy\in M\\ x\in A, y\in B}} d_H(y)\right) + d_H(y_0) \leqslant e_H(A,B),$$

showing a contradiction again.

Case 2. $(B \cap T_2) - \{y_\omega\} \neq \emptyset$.

For any $y \in T_2$, $\mathcal{H}_1(y) \subseteq \mathcal{H}_{12}$. Since $|\mathcal{H}_1(y)| \geqslant 2$ if $y \in T_2$, the assumption that $(B \cap T_2) - \{y_\omega\} \neq \emptyset$ implies that $m_2 \geqslant 2$. Furthermore, if $y_\omega \in T_2$, then $m_2 \geqslant 4$.

Since $|B| > |A|$, there exists $y_0 \in B - V(M)$. Since $N_H(y) \cap S \neq \emptyset$ for any $y \in T$ by Claim 4.8, we have $d_H(y_0) \geqslant 1$. We claim that if y_ω exists and $y_0 \neq y_\omega$, then $d_H(y_0) \geqslant 2$. If $|\mathcal{H}(y_0)| \leqslant 1$, then $d_H(y_0) \geqslant d_G(y_0) - 1 \geqslant 2$. So we assume that $|\mathcal{H}(y_0)| \geqslant 2$. If y_ω exists and $y_0 \neq y_\omega$, then by Claim 4.8, $d_H(y_0) \geqslant |N_H(y_0) \cap S| \geqslant |S_0| + m_1/3 + m_2 - 1 \geqslant m_1/3 + m_2 - 1$. Note that if $y_\omega \in T_2$ then $m_2 \geqslant 4$, and if $y_\omega \notin T_2$, then by the definition of y_ω, $m_1 \geqslant 1$. Thus we have that $d_H(y_0) \geqslant 2$.

For any $y \in T_2 - \{y_\omega\}$, $|N_H(y) \cap S| = |N_G(y) \cap S| \geqslant |S_0| + m_1/3 + m_2 - 1$ by Claim 4.8. Thus, $|A \cap S| \geqslant |S_0| + m_1/3 + m_2 - 1$. Let $A_0 = A \cap S$. Then since $m_2 \geqslant 2$, if $m_1 \leqslant 1$

$$2|A_0 - S_0| \geqslant 2m_1/3 + 2m_2 - 2 \geqslant m_1 + m_2 - 1/3, \tag{1}$$

and if $m_1 \geqslant 2$, then

$$|A_0 - S_0| \geqslant m_1/3 + m_2 - 1 \geqslant m_2 - 1/3. \tag{2}$$

If $y_\omega \in V(M)$, then let x_ω be the vertex with $x_\omega y_\omega \in M$.

Assume first that $y_\omega \notin V(M)$ or $x_\omega \in S$. Note that in both cases, Claim 4.10 (5) does not apply. Applying Claim 4.10 (1-4), we have that

$$e_H(A,B) \leqslant \sum_{\substack{xy\in M\\ x\in A, y\in B}} d_H(x) \leqslant \sum_{\substack{xy\in M\\ x\in S_0}} (d_H(y)+p_y-1) + \sum_{\substack{xy\in M\\ x\in S_1}} (d_H(y)+p_y-2)$$

$$
\begin{aligned}
& + \sum_{\substack{xy\in M\\ x\in U^{\mathcal{H}},y\in B,p_y=0}} d_H(y) + \sum_{\substack{xy\in M\\ x\in U^{\mathcal{H}},y\in B,p_y\geqslant 1}} (d_H(y)+p_y-1)\\
\leqslant & \begin{cases} \left(\sum\limits_{\substack{xy\in M\\ x\in A,y\in B}} d_H(y)\right) + \sum\limits_{\substack{xy\in M\\ x\notin S_1}} p_y + \sum\limits_{\substack{xy\in M\\ x\in S_1}} (p_y-2)\\ \left(\sum\limits_{\substack{xy\in M\\ x\in A,y\in B}} d_H(y)\right) + \sum\limits_{\substack{xy\in M\\ p_y=1}} (p_y-1) + \sum\limits_{\substack{xy\in M\\ x\in S_1 \text{ or } p_y\geqslant 2}} (p_y-1)\end{cases}\\
\leqslant & \begin{cases} \left(\sum\limits_{\substack{xy\in M\\ x\in A,y\in B}} d_H(y)\right) + m_1+m_2-2|A_0-S_0|\\ \left(\sum\limits_{\substack{xy\in M\\ x\in A,y\in B}} d_H(y)\right) + m_2-|A_0-S_0|\end{cases}\\
\leqslant & \left(\sum_{\substack{xy\in M\\ x\in A,y\in B}} d_H(y)\right) + 1/3\\
< & \left(\sum_{\substack{xy\in M\\ x\in A,y\in B}} d_H(y)\right) + d_H(y_0) \leqslant e_H(A,B),
\end{aligned}
$$

showing a contradiction.

Assume now that $y_\omega \in V(M)$ and $x_\omega \in U^{\mathcal{H}}$. By the definition of y_ω, $1 \leqslant p_{y_\omega} \leqslant 2$. If $p_{y_\omega} = 2$, then since $d_H(y_\omega) \geqslant p_{y_\omega} = 2$, we have $d_H(y_\omega) + p_{y_\omega} \geqslant 4 \geqslant d_H(x)+1$ for each edge $xy_\omega \in E(H)$ with $x \in U^{\mathcal{H}}$; and if $p_{y_\omega} = 1$, then for any edge $xy_\omega \in E(H)$, $d_H(y_\omega) + p_{y_\omega} \geqslant d_H(x)$. Therefore, combining Claim 4.10, we know that for any edge $xy_\omega \in E(H)$, $d_H(x) \leqslant d_H(y_\omega)+1$. Applying Claim 4.10 again, we have that

$$
\begin{aligned}
e_H(A,B) \leqslant & \sum_{\substack{xy\in M\\ x\in A,y\in B}} d_H(x) \leqslant \sum_{\substack{xy\in M\\ x\in S_0}} (d_H(y)+p_y-1) + \sum_{\substack{xy\in M\\ x\in S_1}} (d_H(y)+p_y-2)\\
& + \sum_{\substack{xy\in M\\ x\in U^{\mathcal{H}},y\in B,p_y=0}} d_H(y) + \sum_{\substack{xy\in M\\ x\in U^{\mathcal{H}},y\in B,p_y\geqslant 1,y\neq y_\omega}} (d_H(y)+p_y-1) + d_H(y_\omega)+1\\
\leqslant & \begin{cases} \left(\sum\limits_{\substack{xy\in M\\ x\in A,y\in B}} d_H(y)\right) + \sum\limits_{\substack{xy\in M\\ x\notin S_1}} p_y + \sum\limits_{\substack{xy\in M\\ x\in S_1}} (p_y-2)+1\\ \left(\sum\limits_{\substack{xy\in M\\ x\in A,y\in B}} d_H(y)\right) + \sum\limits_{\substack{xy\in M\\ p_y=1}} (p_y-1) + \sum\limits_{\substack{xy\in M\\ x\in S_1 \text{ or } p_y\geqslant 2}} (p_y-1)+1\end{cases}
\end{aligned}
$$

$$\leqslant \begin{cases} \left(\sum\limits_{\substack{xy \in M \\ x \in A, y \in B}} d_H(y) \right) + 1 + m_1 + m_2 - 2|A_0 - S_0| \\ \left(\sum\limits_{\substack{xy \in M \\ x \in A, y \in B}} d_H(y) \right) + 1 + m_2 - |A_0 - S_0| \end{cases}$$

$$\leqslant \left(\sum_{\substack{xy \in M \\ x \in A, y \in B}} d_H(y) \right) + 1 + 1/3$$

$$< \left(\sum_{\substack{xy \in M \\ x \in A, y \in B}} d_H(y) \right) + d_H(y_0) \leqslant e_H(A, B), \quad (d_H(y_0) \geqslant 2 \text{ in this case})$$

showing a contradiction again. □

Claim 4.11 gives a contradiction to Claim 4.1. The proof of Theorem 1.3 is now complete. ■

5 Essentiality of Δ-criticality

In this section, we construct a family of n-vertex graphs G with $\Delta(G) = n - 1$, $\tau(G) \geqslant \frac{3}{2}$, but have no 2-factor. This demonstrates that the condition of Δ-criticality cannot be dropped from Theorem 1.3.

Let $k \geqslant 7$ be an integer, S be the vertex set of a complete graph of order k, T be a set of $6k+1$ isolated vertices, and $\mathcal{H}$ be a set of 5 disjoint complete graphs Q_1, Q_2, Q_3, Q_4, Q_5, each with order $2k+1$. Label the vertices in T as $u_1, u_2, \cdots, u_{6k+1}$. Let G be a graph with $V(G) = S \cup T \cup V(\mathcal{H})$. The edges of G are constructed below:

Step 1 Adding all edges between S and $T \cup V(\mathcal{H})$. (Clearly, every vertex in S is adjacent to every other vertex of G.)

Step 2 For each i, $i = 1, 2, 3, 4$, joining each of

$$u_{(i-1)\times(k-1)+1}, u_{(i-1)\times(k-1)+2}, \cdots, u_{(i-1)\times(k-1)+k-1}$$

to one unused vertex in Q_i and one unused vertex in Q_{i+1}; joining each of

$$u_{4k-3}, u_{4k-2}, u_{4k-1}, u_{4k}, u_{4k+1}, u_{4k+2}, u_{4k+3}, u_{4k+4}$$

to one unused vertex in Q_1 and one unused vertex in Q_5. (By "unused," we meant that the vertex in Q_i has not been joined to any vertex in T.)

Step 3 After Step 2, each of Q_2, Q_3, and Q_4 has only 3 unused vertex left, and each of Q_1 and Q_5 has exactly $k-6$ unused vertex left. Also, there are exactly $(6k+1) - (4k+4) = 2k-3$ vertices in T that are not joined to any vertex in $\mathcal{H}$. Joining each vertex of $u_{4k+5}, u_{4k+6}, \cdots, u_{6k+1}$ to exactly one unused vertex in $\mathcal{H}$.

By the construction,

$$n := |V(G)| = k + 5(2k+1) + 6k + 1 = 17k + 6.$$

All vertices in S have degree $n-1$, which are the only maximum degree vertices in G. Each vertex in T is adjacent to a vertex in $\mathcal{H}$ that has degree $2k+1+|S| = 3k+1$, and is adjacent to exactly k vertices of maximum degree. Since $n = 17k+6$, and

$$\Delta(G) - (3k+1) + 1 = n - 1 - 3k = 14k + 5 > k,$$

Lemma 2.1 implies that G is not Δ-critical.

We next claim that G has no 2-factor. This is clear, since every Q_i is an odd component of $G - (S \cup T)$ that is adjacent to exactly $2k+1$ vertices in T by the construction, and we have that

$$\delta(S,T) = 2k - 2 \times (6k+1) + 5 \times (2k+1) - 5 = -2 < 0.$$

Thus, (S,T) is a barrier and G has no 2-factor by Lemma 3.1.

Finally, we show that $\tau(G) \geqslant \frac{3}{2}$. Let $W \subseteq V(G)$ be a toughset of G. That is, W is a cutset of G such that $\frac{|W|}{c(G-W)} = \tau(G)$. Assume to the contrary that $\tau(G) < \frac{3}{2}$. Since G is 3-connected, the assumption that $\tau(G) < \frac{3}{2}$ implies that $c(G-W) \geqslant 3$.

Since every vertex in S is adjacent to every other vertex in G, we get the following conclusion.

Claim 5.1. *$S \subseteq W$.*

Claim 5.2. *Let $U \subseteq W$ be a proper non-empty subset of W. Then all vertices in U are adjacent to in total at least $\frac{2|U|}{3} + 1$ components of $G - W$. In particular, for any $x \in W$, x is adjacent to at least two components of $G - W$.*

Proof. Assume to the contrary that there exists a proper non-empty subset U of W such that all vertices in U are adjacent to in total at most $\frac{2|U|}{3}$ components of $G-W$. Let $\tau = \tau(G)$. Then $|W| = \tau c(G-W)$ and $G - (W-U)$ has at least $c(G-W) - \frac{2|U|}{3} + 1 > c(G-W) - \frac{|U|}{\tau} + 1$ components of $G-W$. Since $\tau(G) < \frac{3}{2}$, we see that $c(G-(W-U)) \geqslant 2$. Thus $W^* = W - U$ is a cutset of G. However,

$$\frac{|W^*|}{c(G-W^*)} < \frac{|W-U|}{c(G-W) - \frac{1}{\tau}|U|} = \tau,$$

contradicting the fact that W is a toughset of G. □

Claim 5.3. *$W \cap T = \emptyset$.*

Proof. Assume to the contrary that there exists $x \in W \cap T$. Note that each vertex in T, besides being adjacent to all vertices in S, is adjacent to at most two vertices in $\mathcal{H}$. As $S \subseteq W$ by Claim 5.1, we know by the particular part of Claim 5.2 that $x \in$

$\{u_1, u_2, \cdots, u_{4k+4}\}$ and the two neighbors of x in $\mathcal{H}$ are contained in two distinct components of $G-W$. Assume that the neighbors of x in $\mathcal{H}$ are in Q_1 and Q_5. (The argument for the other cases is similar.) Then $x \in \{u_{4k-3}, u_{4k-2}, u_{4k-1}, u_{4k}, u_{4k+1}, u_{4k+2}, u_{4k+3}, u_{4k+4}\}$. Let $W^* = W \cap (\{u_{4k-3}, u_{4k-2}, u_{4k-1}, u_{4k}, u_{4k+1}, u_{4k+2}, u_{4k+3}, u_{4k+4}\} \cup V(Q_1 \cup Q_5))$. It follows that W^* separates the two neighbors of x in $Q_1 \cup Q_5$ into two distinct components in $G - W$. By the construction of the edges of G in Step 2, we know that there are 8 vertex-disjoint paths between Q_1 and Q_5 going through the eight vertices $u_{4k-3}, u_{4k-2}, u_{4k-1}, u_{4k}, u_{4k+1}, u_{4k+2}, u_{4k+3}, u_{4k+4}$. Thus, $|W^*| \geqslant 8$. Note that each of $u_{4k-3}, \cdots, u_{4k+4}$ is only adjacent to Q_1 and Q_5 in $G - S$, and Q_1 and Q_5 are cliques in G. Assume there are exactly t vertices y from

$$\{u_{4k-3}, u_{4k-2}, u_{4k-1}, u_{4k}, u_{4k+1}, u_{4k+2}, u_{4k+3}, u_{4k+4}\}$$

such that W contains both the two neighbors of y in $Q_1 \cup Q_5$. Then $|W^*| \geqslant 8 + t$ and vertices in W^* are adjacent to at most $t + 2$ components of $G - W$. Since $t \leqslant 8$, $t + 2 < \frac{2}{3}(8 + t) + 1$. This gives a contradiction to Claim 5.2. □

Claim 5.4. *Let $x \in V(\mathcal{H})$ such that x is a neighbor of some vertex in $\{u_1, u_2, \cdots, u_{4k+4}\}$. Then $x \notin W$.*

Proof. Assume to the contrary and without loss of generality, that $x \in V(Q_1) \cap W$ is a neighbor of u_1. (The argument for all other cases follows a similar idea, and if x is a neighbor of a vertex in $\{u_{4k-3}, u_{4k-2}, \cdots, u_{4k+4}\}$, we replace $k - 1$ in the following argument by 8.) Since the neighbors of x in Q_1 form a clique in G, x is adjacent to exactly two components of $G - W$ by Claim 5.2, where one of the two components contains some neighbors of x in Q_1, and the other of the two contains the vertex u_1. Let the neighbor of u_i in Q_1 be a_i, and the neighbor of u_i in Q_2 be b_i, for $i = 1, 2, \cdots, k - 1$. By this labeling of the vertices, $x = a_1$. Let $W^* = W \cap \{a_1, a_2, \cdots, a_{k-1}, b_1, b_2, \cdots, b_{k-1}\}$. To separate u_1 from some neighbors of x contained in Q_1, since $W \cap T = \emptyset$ by Claim 5.3, we have that $|W^*| \geqslant 2$. We assume, without loss of generality, that $a_1, a_2, \cdots, a_p \in W$, $b_1, b_2, \cdots, b_p \notin W$, $a_{p+1}, a_{p+2}, \cdots, a_{p+q} \notin W$, $b_{p+1}, b_{p+2}, \cdots, b_{p+q} \in W$, and $a_{p+q+1}, b_{p+q+1}, \cdots, a_{p+q+r}, b_{p+q+r} \in W$ for some integers p, q, r with $1 \leqslant p, q, r \leqslant k-1$ and $p+q+r \leqslant k-1$. Then $|W^*| = p+q+2r$, and vertices in W^* are adjacent to in total at most $r+2$ components in $G-W$, and they are adjacent to in total exactly $r+2$ components in $G-W$ only if $p+q+r = k-1$. If $r \geqslant 3$, then $r + 2 < \frac{2}{3}(2r) + 1 \leqslant \frac{2}{3}(p + q + 2r) + 1$. This gives a contradiction to Claim 5.2. So we assume that $r \leqslant 2$. If $p + q + r \geqslant 3$, then since $r \leqslant 2$, we get $r + 2 < \frac{2}{3}(p + q + 2r) + 1$, showing a contradiction to Claim 5.2 again. Thus, we have that $p + q + r \leqslant 2 < k - 1$, as $k \geqslant 7$. This particularly implies that vertices in W^* are adjacent to at most $r + 1$ components in $G - W$. As $|W^*| \geqslant 2$, Claim 5.2, together with the argument above that vertices in W^* are adjacent to at most $r + 1$ components in $G - W$, implies that $r = 2$. However, $r + 1 = 3 < \frac{8}{3} + 1 \leqslant \frac{2}{3}(p + q + 2r) + 1$. Again, we achieve a contradiction to Claim 5.2. □

Let the neighbors of vertices $u_{4k+5}, u_{4k+6}, \cdots, u_{6k+1}$ in $\mathcal{H}$ be $x_1, x_2, \cdots, x_{2k-3}$, respectively. Now by Claim 5.3 and Claim 5.4, we have that $W \subseteq S \cup \{x_1, x_2, \cdots, x_{2k-3}\}$.

Assume that $|W \cap \{x_1, x_2, \cdots, x_{2k-3}\}| = t$. Then $G - W$ has exactly $t + 1$ components, and since $t \leqslant 2k - 3$, we get

$$\frac{|W|}{c(G-W)} = \frac{k+t}{t+1} \geqslant \frac{3}{2}.$$

This gives a contradiction to the assumption that $\tau(G) < \frac{3}{2}$.

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