

On the maximum degree of path-pairable planar graphs

António Girão

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Cambridge, U.K.

a.girao@dpmms.cam.ac.uk

Gábor Mészáros Kamil Popielarz Richard Snyder

Department of Mathematical Sciences
The University of Memphis
Memphis, Tennessee, U.S.A.

gmeszaros@purestorage.com, {kamil.popielarz,rjsnyder23}@gmail.com

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Abstract

A graph is *path-pairable* if for any pairing of its vertices there exist edge-disjoint paths joining the vertices in each pair. We investigate the behaviour of the maximum degree in path-pairable planar graphs. We show that any n -vertex path-pairable planar graph must contain a vertex of degree linear in n . Our result generalizes to graphs embeddable on a surface of finite genus.

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1 Introduction

We are interested in *path-pairability*, a graph theoretical notion that emerged from a practical networking problem. This notion was introduced by Csaba, Faudree, Gyárfás, Lehel, and Schelp [3], and further studied by Faudree, Gyárfás, and Lehel [4, 5, 6], and by Kubicka, Kubicki and Lehel [14]. Given a fixed integer k and a simple undirected graph G on at least $2k$ vertices, we say that G is *k -path-pairable* if, for any pair of disjoint sets of distinct vertices $\{x_1, \dots, x_k\}$ and $\{y_1, \dots, y_k\}$ of G , there exist k edge-disjoint paths P_1, P_2, \dots, P_k , such that P_i is a path from x_i to y_i , $1 \leq i \leq k$. The problems of finding k edge(vertex)-disjoint paths routing some prescribed pairs of vertices in a graph is a well-known problem in algorithmic graph theory and combinatorial optimization (see the

surveys[7, 8, 20]). Recently, for a fixed integer k , Kawarabayashi, Kobayashi and Reed [13] constructed a $O(n^2)$ time algorithm which for any graph G on n vertices either finds such k vertex-disjoint paths or concludes no such paths exist. As a corollary they obtained a $O(n^2)$ time algorithm for the edge-disjoint path problem. This improved upon the seminal work of Robertson and Seymour [18], which initially gave a $O(n^3)$ time algorithm for the vertex-disjoint path problem. Note that the problem of finding edge(vertex)-disjoint paths between an unbounded number of prescribed pairs of vertices is known to be NP-complete, even when restricted to planar graphs [17].

The concept of k -path-pairability is closely related to the well-studied notions of k -linkedness and k -weak-linkedness. A graph is said to be k -(weakly)linked if for any choice $\{s_1, \dots, s_k, t_1, \dots, t_k\}$ of $2k$ vertices (not necessarily distinct) there are vertex(edge) internally disjoint paths P_1, \dots, P_k with P_i joining s_i to t_i , $1 \leq i \leq k$. While any k -(weakly)linked graph is $(2k - 1)$ -vertex connected (k -edge connected), the same need not hold for k -path-pairable graphs. Observe that the stars S_{2k} ($k \geq 1$) are k -path-pairable and yet have very low edge density and edge connectivity. On the other hand, a result of Bollobás and Thomason [2] shows that if G is a $2k$ -connected graph with average degree at least $22k$ then G is k -linked. This was later improved by Thomas and Wollan [19] who showed that a $2k$ -connected graph with average degree at least $10k$ is necessarily k -linked. In the context of weakly-linked graphs, a theorem of Hirata, Kubota and Saito [11] states that a $(2k + 1)$ -edge connected graph is $(k + 2)$ -weakly-linked for $k \geq 2$. A few years later, Huck [12] showed that any $(k + 2)$ -edge-connected graph is k -weakly-linked.

A k -path-pairable graph on $2k$ vertices is simply said to be *path-pairable*. Some of the most central questions in the study of path-pairable graphs concern determining the behaviour of their maximum degree. It is fairly easy to construct path-pairable graphs on n vertices (n even) with maximum degree linear in n . For example, complete graphs K_{2n} and complete bipartite graphs $K_{m,n}$ are path-pairable for all choices of $m, n \in \mathbb{N}$ with $m + n$ even, $m \neq 2, n \neq 2$.

It is slightly more challenging to construct an infinite family of path-pairable graphs where the maximum degree grows sublinearly. We shall now describe such a family. Let K_t be the complete graph on t vertices and let K_t^q be constructed from K_t by attaching $q - 1$ leaves to each of the original vertices of K_t . This family was introduced by Csaba, Faudree, Gyárfás, and Lehel [3], who also proved that K_t^q is path-pairable as long as $t \cdot q$ is even and $q \leq \left\lfloor \frac{t}{3+2\sqrt{2}} \right\rfloor$. The bound on q has been recently improved to $\approx \frac{1}{3}t$ [9]. Observe that $n = |V(K_t^q)| = t \cdot q$ and $\Delta(K_t^q) = t + q - 2 = O(\sqrt{n})$ when $q = \Omega(t)$. Additional path-pairable constructions with maximum degree $c\sqrt{n}$ can be found in [14] and [16].

The following result due to Faudree [5] shows that the maximum degree of a path-pairable graph has to grow with the order of the graph.

Theorem 1. *If G is path-pairable on n vertices with maximum degree Δ , then $n \leq 2\Delta^\Delta$.*

Letting $\Delta_{\min}(n) := \min\{\Delta(G) : G \text{ is a path-pairable graph on } n \text{ vertices}\}$, this result implies that

$$\Delta_{\min}(n) \geq c_1 \frac{\log n}{\log \log n},$$

for some constant c_1 . To date, the best known upper bound on $\Delta_{\min}(n)$ is due to Győri, Mezei, and Mészáros, exhibiting a path-pairable graph with maximum degree $\Delta \approx 5.5 \cdot \log n$ [10]. In summary, we have the following general asymptotic bounds on $\Delta_{\min}(n)$:

$$c_1 \frac{\log n}{\log \log n} \leq \Delta_{\min}(n) \leq c_2 \log n.$$

Recall that the star $K_{1,n-1}$ is path-pairable. This is simply due to the presence of a vertex of large degree. Are there properties we may impose on a general path-pairable graph to force a vertex of large degree, say, linear in n ? Along these lines, Faudree, Gyárfás and Lehel [6] proved that an n -vertex path-pairable graph with maximum degree at most $n - 2$ must have at least $3n/2 - \log n - c$ edges, for some absolute constant c . Instead of simply imposing a condition on the number of edges, we wished to determine whether or not a structural property like *planarity* would be enough to force a vertex of linear degree in a path-pairable graph. To formulate this precisely, let us define $\Delta_{\min}^p(n)$ to be

$$\min\{\Delta(G) : G \text{ is a path-pairable planar graph on } n \text{ vertices}\}.$$

Our problem, then, is to determine whether or not $\Delta_{\min}^p(n) = \Theta(n)$. We first note that a simple application of the Planar Separator Theorem of Lipton and Tarjan [15] shows that every path-pairable planar graph on n vertices must contain a vertex of degree at least $c\sqrt{n}$. Indeed, if G is such a graph, then the Separator Theorem allows us to partition $V(G)$ into three sets S, A, B , where $|S| = O(\sqrt{n})$, $|A| \leq |B| \leq 2n/3$, and there are no edges between A and B . Now, while path-pairable graphs G need not be highly connected or edge connected, they must satisfy certain connectivity-like conditions. More precisely, they must satisfy the *cut-condition*: for every subset $X \subset V(G)$ of size at most $n/2$, there are at least $|X|$ edges between X and $V(G) \setminus X$. Note that the cut-condition is not sufficient to guarantee path-pairability; see [16] for additional details. Accordingly, since $n/4 < |A| < n/2$ and there are no edges between A and B , the cut-condition implies that there are at least $|A|$ edges between A and S . We therefore obtain a vertex in S of degree $\Omega(\sqrt{n})$.

Our main theorem, which we state below, shows that we can do much better than this. Namely, every path-pairable planar graph must have a vertex of *linear* degree.

Theorem 2. *There exists $c \geq 10^{-10^{10}}$ such that if G is a path-pairable planar graph on n vertices then $\Delta(G) \geq cn$.*

We have not made an attempt to optimize the constant c obtained in the proof. The value we give is surely far from the truth.

In the other direction, there are easy examples of path-pairable planar graphs with very large maximum degree; for example, consider the star $K_{1,n-1}$. Our second result finds an infinite family of path-pairable planar graphs with smaller (but of course still linear) degree.

Theorem 3. *There exist path-pairable planar graphs G on n vertices with $\Delta(G) = \frac{2}{3}n$.*

Combining Theorems 2 and 3, we have that

$$10^{-10^{10}} n \leq \Delta_{\min}^p(n) \leq \frac{2}{3}n.$$

However, there is currently a significant gap between the constants in the upper and lower bounds. Closing this gap and finding the truth is an interesting open problem.

We remark that using the same techniques it is possible to generalize these results to path-pairable graphs which can be embedded on a surface of a finite genus.

1.1 Organization

The remainder of the paper is organized as follows. In the next short section, we shall describe our construction establishing Theorem 3. The third section of this paper contains a proof of our main theorem, Theorem 2. This proof relies on three preparatory lemmas and on some common facts concerning planar graphs. In particular, we use heavily the fact that any subset X of the vertices of a planar graph induces less than $3|X|$ edges, and any bipartite planar graph on n vertices has less than $2n$ edges. Finally, we close with some remarks and open problems.

1.2 Notation

Our notation is standard. Thus, for a graph G and two subsets $X, Y \subset V(G)$ we say that a path in G is an $X - Y$ path if it begins in X and ends in Y . If $X = \{x\}$ and $Y = \{y\}$ are singletons, we shall simply say that the path is an $x - y$ path. For subsets $X, Y \subset V(G)$, $e(X, Y)$ is the number of edges with one endpoint in X and the other in Y . As usual, $G[X]$ denotes the graph induced in G with vertex set X .

2 The Construction

Our aim in this section is to prove Theorem 3, which we restate here for convenience.

Theorem 3. *There exist path-pairable planar graphs G on n vertices with $\Delta(G) = \frac{2}{3}n$.*

Proof. Let G be a graph on $n = 6k$ vertices with vertex set $V(G) = A \cup B \cup C \cup \{x_{AB}, x_{BC}, x_{CA}\}$ where $|A| = |B| = |C| = 2k - 1$, and x_{AB}, x_{BC}, x_{CA} denote three additional vertices forming a triangle such that x_{AB}, x_{BC}, x_{CA} are joined to every vertex in $A \cup B$, $B \cup C$, and $C \cup A$, respectively, and A, B, C are independent sets. This graph is clearly planar. Let \mathcal{P} be a pairing of the vertices and let $\{u, v\} \in \mathcal{P}$. We describe how to join u and v by a path in all possible cases.

1. If there is an edge between u and v , join them by this edge.
2. If $u \in \{x_{AB}, x_{BC}, x_{CA}\}$ and $v \in A \cup B \cup C$ such that there is no edge between them, join them by the path uvw where the edge uw is consistent with the cyclic ordering x_{AB}, x_{BC}, x_{CA} . For example, if $u = x_{AB}$ and $v \in C$, we join u and v by the path $ux_{BC}v$. The remaining cases can be dealt using the same pattern.

3. If $u, v \in A \cup B \cup C$ and they are in the same class, join them by the path uvw where w is an arbitrary common neighbour (out of the two available).
4. If $u, v \in A \cup B \cup C$ and they are in different classes, join them by the path uvw where w is the unique common neighbour.

It is straightforward to check that the above instructions find edge-disjoint paths joining terminals, regardless of the choice of \mathcal{P} . \square

3 The Proof of Theorem 2

The aim of this section is to prove our main theorem, Theorem 2. Our proof is based on three preparatory lemmas. First, we shall introduce some terminology. Let G be a multigraph. We say that two multiedges e, f of G are at distance d if the shortest path in G joining an endpoint of e and an endpoint of f has length d . If two multiedges are at distance 0, we shall simply say they are *incident*. Further, we shall refer to a matching of size k as a *k-matching*. We say that a *k-matching* is *good* if every pair of edges in the matching is at distance exactly 1. Notice that contracting all the edges of a good *k-matching* results in the complete graph K_k (with potential multiple edges and loops).

If we consider the complete graph whose vertices are the multiedges of some graph, and 3-colour its edges according to certain notions of ‘density’, then by Ramsey’s theorem one of the colours must contain a ‘dense’ subgraph. Our first lemma is a Ramsey-type result in a similar vein (although we do not need to appeal to Ramsey’s theorem itself). More precisely, our lemma says that in any multigraph with enough multiedges, either some multiedges cluster together, or many pairs of multiedges are far apart, or one can find a good *k-matching*. We first need the following inequality.

Fact 4. *If $k \geq 2$ then $2^{-k} \left(\frac{1+2^{-k-1}}{(1-2^{-k})^2} \right) \leq 2^{-k+1}$.*

The above inequality is easily seen to be equivalent to $(2^{-k+2} - 1)(2^{-k-1} - 1) \geq 0$.

Lemma 5. *Let k be a natural number and $\varepsilon_1, \varepsilon_2$ be positive reals such that $\varepsilon_1 + \varepsilon_2 \leq 2^{-k}$. Then, for sufficiently large $M = M(k)$, if G is a multigraph on M multiedges, then at least one of the following conditions is satisfied.*

1. *There is a multiedge in G which is incident with at least $\varepsilon_1 M$ multiedges;*
2. *There are at least $\varepsilon_2 \binom{M}{2}$ pairs of multiedges which are at distance greater than 1;*
3. *G contains a good k -matching.*

Proof. We shall use induction on k . The base case when $k = 1$ is trivial - Condition 3 is always satisfied. Assume then that $k \geq 2$ and the lemma is true for $k - 1$.

Suppose every multiedge is incident with at most $\varepsilon_1 M$ multiedges and at most $\varepsilon_2 \binom{M}{2}$ pairs of multiedges are at distance greater than 1. We shall show that G contains a good *k-matching*. By an averaging argument there is a multiedge e which is at distance at

most 1 from at least $(1 - \varepsilon_2)M - 1$ multiedges. Let E' be the set of those multiedges which are at distance exactly 1 from e . It follows from our assumptions that $M' = |E'| \geq (1 - \varepsilon_1 - \varepsilon_2)M - 1 \geq (1 - 2^{-k})M - 1$. Let G' be the multigraph spanned by E' . By assumption, at most $\varepsilon_2 \binom{M}{2}$ of the multiedges in G' are at distance greater than 1. Therefore, since $M \leq \frac{M'+1}{1-2^{-k}}$, for large enough M (and hence large enough M') we have that at most

$$\begin{aligned} \varepsilon_2 \binom{M}{2} &\leq \varepsilon_2 \binom{\frac{M'+1}{1-2^{-k}}}{2} = \frac{\varepsilon_2}{(1-2^{-k})^2} \left(1 + \frac{1}{M'}\right) \left(1 + \frac{1+2^{-k}}{M'-1}\right) \binom{M'}{2} \\ &\leq \frac{\varepsilon_2(1+2^{-k-1})}{(1-2^{-k})^2} \binom{M'}{2}, \end{aligned}$$

pairs of multiedges in G' are at distance greater than 1. Also, for M' large enough, each multiedge in G' is incident with at most $\varepsilon_1 M \leq \varepsilon_1 \frac{M'+1}{1-2^{-k}} = \frac{\varepsilon_1}{1-2^{-k}} \left(1 + \frac{1}{M'}\right) M' \leq \frac{\varepsilon_1(1+2^{-k-1})}{1-2^{-k}} M'$ multiedges. Note that for $k \geq 2$ one has

$$\begin{aligned} \varepsilon_1 \frac{1+2^{-k-1}}{1-2^{-k}} + \varepsilon_2 \frac{1+2^{-k-1}}{(1-2^{-k})^2} &\leq \varepsilon_1 \frac{1+2^{-k-1}}{(1-2^{-k})^2} + \varepsilon_2 \frac{1+2^{-k-1}}{(1-2^{-k})^2} \\ &\leq 2^{-k} \frac{1+2^{-k-1}}{(1-2^{-k})^2} \leq 2^{-(k-1)}, \end{aligned}$$

where the last inequality is precisely Fact 4. Therefore, by the induction hypothesis, G' contains a good $(k-1)$ -matching. But since e is at distance 1 from any multiedge in G' , we also have a good k -matching in G . \square

Since we shall be operating with planar graphs, we single out the following corollary.

Corollary 6. *Let M be a sufficiently large integer and let $\varepsilon_1, \varepsilon_2$ be positive reals such that $\varepsilon_1 + \varepsilon_2 \leq \frac{1}{32}$. If G is a planar multigraph with M multiedges then either G has a multiedge which is incident with at least $\varepsilon_1 M$ multiedges or there are at least $\varepsilon_2 \binom{M}{2}$ pairs of multiedges at distance greater than 1.*

Proof. If G contained a good 5-matching then it would contain a K_5 minor. \square

We say that that a pairing of vertices of a graph *contributes* to k edges if any system of edge-disjoint paths joining the vertices of each pair uses at least k edges. One strategy in the proof of our main theorem is to consider a suitable bipartition of our path-pairable planar graph, and to exploit the fact that any bipartite planar graph on n vertices has at most $2n - 4$ edges. To exploit this last property we shall need ways of finding pairings of the vertices which contribute to ‘many’ edges to the bipartition. This is formalized in the following lemma.

Lemma 7. *Let D be an integer and $0 < \varepsilon \leq 1/2$. Then there exists $c > 0$ such that the following is true. Suppose G is a path-pairable planar graph on n vertices with $\Delta = \Delta(G) \leq cn$. Let $A, U \subset V(G)$ be given with $U \subset A$ such that every vertex in A has degree at most D , $|A| \geq (1 - \varepsilon)n$ and $|U| \geq \varepsilon n$. Let $B = V(G) \setminus A$. Then there is a pairing of the vertices in U which contributes to at least $2|U| - 16\varepsilon n$ edges between A and B .*

Proof. We say that a path in G is *weak* if it begins and ends in A , uses no edges inside B , and uses at most 2 edges between A and B . Now, let $C := \lceil 4\varepsilon^{-1} \rceil$ and note that since $\varepsilon \leq 1/2$ we have that $\frac{3}{C-2} \leq \varepsilon$. For every $x \in U$, let $X(x) = \{u \in U : \text{dist}_G(u, x) \leq C\}$ and $Y_x = \{u \in U : \exists \text{ a weak } x-u \text{ path in } G\}$. Finally, consider the set $U_x = X(x) \cap Y_x$. We claim that U_x is small for every $x \in U$. So fix $x \in U$ and consider the vertices reachable from x at distance at most C using only weak paths. Such a path can either stay entirely in A , or use precisely two edges between A and B . In the first case, since vertices in A have degree at most D , the number of such vertices in U_x is at most D^C . To bound the number of vertices reachable from paths of the second type, note that such a path can first move around inside A , then leave A and use precisely two edges between A and B (and no edges inside B), then finally move around inside A again. Recalling that the maximum degree of G is Δ and the maximum degree of vertices in A is D , we obtain the bound $D^C \cdot D\Delta \cdot D^C$. It follows that

$$|U_x| \leq D^C + D^C D\Delta D^C = D^C(1 + D^{C+1}\Delta).$$

Choose $c = c(D, \varepsilon) = \frac{\varepsilon}{4D^{2C+1}}$ so that $\Delta \leq cn$. Then $|U_x| \leq D^C(1 + D^{C+1}\Delta) \leq (D^C + D^{2C+1})\Delta$, and using the upper bound on Δ , we find that

$$|U_x| \leq \varepsilon n/2.$$

Let us define an auxiliary graph G_U with vertex set U where we join two vertices x, y provided $y \notin U_x$ (equivalently, $x \notin U_y$). It is easy to see that G_U has a perfect matching (or ‘almost’ perfect, if $|U|$ is odd; this makes no difference for us). Indeed, the degree of every vertex in G_U is at least $|U| - \frac{\varepsilon}{2}n \geq |U|/2$, and therefore G_U has a Hamilton cycle. Fix a perfect matching \mathcal{M} in G_U according to this Hamilton cycle and fix a pairing \mathcal{P} of the vertices of G where each edge of \mathcal{M} forms a pair. Finally, since G is path-pairable, choose a collection of edge-disjoint paths \mathcal{R} that connect the vertices of each pair in \mathcal{P} . Observe that any path from \mathcal{R} must use an even number of edges between A and B . We single out two types of edges $e = xy$ in \mathcal{M} with respect to \mathcal{R} : either the $x - y$ path in \mathcal{R} is weak but $\text{dist}(x, y) > C$, or this $x - y$ path uses at least 4 edges between A and B . Let $\mathcal{M} = \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$, where \mathcal{E}_0 denotes the edges satisfying the former condition, \mathcal{E}_1 the latter, and \mathcal{E}_2 denotes the remaining edges. We claim that most edges are in \mathcal{E}_1 . Indeed, observe that if $e = xy \in \mathcal{E}_2$, then the $x - y$ path must use edges from B . By planarity we have $e(B) < 3|B|$, and therefore $|\mathcal{E}_2| < 3\varepsilon n$. Using planarity again we have that $e(A) < 3|A|$. On the other hand, for each edge in \mathcal{E}_0 its path in \mathcal{R} uses more than C edges, at most 2 of which are in the cut $\{A, B\}$, and none of which belong to B . Accordingly, since these paths are edge-disjoint, we have that $e(A) \geq (C - 2)|\mathcal{E}_0|$ and so

$$|\mathcal{E}_0| < \frac{3}{C-2}|A| \leq \varepsilon|A|.$$

Therefore, $|\mathcal{E}_1| \geq \frac{1}{2}|U| - \varepsilon|A| - 3\varepsilon n \geq \frac{1}{2}|U| - 4\varepsilon n$. It follows that since every path in \mathcal{R} connecting an edge in \mathcal{E}_1 contributes at least 4 edges between A and B , and these paths must be edge-disjoint, we have

$$e(A, B) > 2|U| - 16\varepsilon n.$$

This completes the proof of Lemma 7. \square

Our final lemma allows us to quantify more precisely the degree distribution in any bipartite planar graph.

Lemma 8. *Let G be a bipartite planar graph on n vertices with parts A, B , and let $A' \subset A$ be the set of vertices in A with degree at least 3. Then the following are true.*

1. *The number of vertices in A with degree exactly 2 is at least $e(A, B) - n - 3|B|$;*
2. *$|A'| < 2|B|$;*
3. *$e(A', B) < 6|B|$.*

Proof. For each $i \geq 0$ let $A_i, A_{\leq i}$, and $A_{\geq i}$ denote the number of vertices in A that have degree i in G , degree at most i , and degree at least i , respectively. Because of planarity we have that $e(A', B) < 2(|A'| + |B|)$. Alternatively, $e(A', B) \geq 3|A'|$ so it follows that $A_{\geq 3} = |A'| < 2|B|$, and so $e(A', B) \leq 2(|A'| + |B|) < 6|B|$, establishing the second and third items. Further, we can bound the number of edges between A and B as

$$\begin{aligned} e(A, B) &\leq A_{\leq 1} + 2(|A| - A_{\leq 1} - A_{\geq 3}) + e(A', B) \\ &\leq A_{\leq 1} + 2(|A| - A_{\leq 1} - |A'|) + 2(|A'| + |B|) \\ &\leq 2|A| - A_{\leq 1} + 2|B|. \end{aligned}$$

It follows that $A_{\leq 1} \leq 2|A| + 2|B| - e(A, B)$. Finally, we see that $A_2 = |A| - |A'| - A_{\leq 1} > e(A, B) - |A| - 4|B| = e(A, B) - n - 3|B|$, as required. \square

We are now in a position to prove our main theorem. First, let us give a rough sketch of the proof. Let G be a path-pairable planar graph. We first partition the vertex set of G into the set A of vertices of small degree and the set B of vertices of large degree. We can apply Lemma 7 to find that there are many edges in this cut. We shall then show that most vertices in A have degree 2 in this bipartite graph. If $Y \subset A$ denotes the vertices of degree 2, then we define a planar multigraph with vertex set B where we join $x, y \in B$ whenever there is a $v \in Y$ joined to precisely x and y . Now, using Corollary 6, we are able to either find a vertex of linear degree in B , or we can find many pairs of multiedges in our multigraph that are far apart. This, however, allows us to find a pairing which contributes to more than $2n$ edges between A and B , a contradiction to planarity.

We restate Theorem 2 for convenience.

Theorem 2. *There exists $c \geq 10^{-10^{10}}$ such that if G is a path-pairable planar graph on n vertices then $\Delta(G) \geq cn$.*

Proof. Suppose G is a path-pairable planar graph and fix some large constant D so that $D^{-1} \leq 8.5 \cdot 10^{-6}$. Partition the vertex set of G into sets A and B , where $B = \{v \in V(G) : d(v) \geq D\}$ and $A = V(G) \setminus B$. Since $e(G) < 3n$ it easily follows that $|B| \leq 6D^{-1}n := \varepsilon n$. Suppose that $\Delta(G) < cn$, where c is sufficiently small (depending only on D) given by Lemma 7. More precisely, we may take $c = \frac{\varepsilon}{4D^{2\lceil 4/\varepsilon \rceil + 1}}$.

Our aim is to obtain a contradiction by showing such graph G can not exist.

Of course, this is true if $cn < 1$, so we shall assume throughout that $n \geq 1/c$. By Lemma 7 (with $U = A$) we have that there are at least $2|A| - 16\epsilon n \geq 2n - 18\epsilon n$ edges between A and B .

Next, we shall show that there is a large subset of A which induces a graph with maximum degree at most 2. To see this, let $A_0 = A, B_0 = B$. Suppose A_i, B_i have been defined already. If there is a vertex $v \in A_i$ such that $d_{A_i}(v) > d_{B_i}(v)$, then let $A_{i+1} = A_i \setminus \{v\}$ and $B_{i+1} = B_i \cup \{v\}$. Notice that $e(A_{i+1}, B_{i+1}) \geq e(A_i, B_i) + 1$, and so $e(A_{i+1}, B_{i+1}) \geq e(A, B) + i \geq 2n - 18\epsilon n + i$. Let $t \geq 0$ be such that there is no $v \in A_t$ with more neighbours in A_t than in B_t . Observe that $t \leq 18\epsilon n$ (otherwise $e(A_t, B_t) \geq 2n$), and accordingly $|B_t| = |B| + t \leq \epsilon n + 18\epsilon n = 19\epsilon n$.

Let $X \subset A_t$ be the set of vertices in A_t with at least 3 neighbours in A_t . Since every vertex in A_t has more neighbours in B_t than in A_t , we have that every vertex in X has at least 3 neighbours in B_t . Therefore, by Lemma 8, $|X| \leq 2|B_t|$, $e(X, B_t) \leq 6|B_t|$, and there are at least $e(A_t, B_t) - n - 3|B_t| \geq e(A, B) - n - 3|B_t|$ vertices in A_t with exactly two neighbours in B_t . Let $A^* = A_t \setminus X$ and $B^* = B_t \cup X$. Now we have that every vertex in A^* has at most 2 neighbours in A^* and $|B^*| \leq 3|B_t| \leq 57\epsilon n$, so $|A^*| \geq n - 57\epsilon n$. We have to make sure we still have many vertices in A^* with exactly two neighbours in B^* . Notice that if a vertex $v \in A_t$ had two neighbours in B_t and was not adjacent to any vertex in X then $v \in A^*$ and v still has exactly two neighbours in B^* . Therefore we only have to worry about the vertices in A^* which are adjacent to some vertices in X . Observe that $e(X, A^*) \leq e(X, B_t) \leq 6|B_t|$, and so there are at least $e(A, B) - n - 9|B_t| \geq (2n - 18\epsilon n) - n - 9 \cdot 19\epsilon n = n - 189\epsilon n$ vertices in A^* with exactly 2 neighbours in B^* . Hence there are at most $189\epsilon n$ vertices in A^* which do not have degree 2 in B^* .

In what follows, we shall deem certain edges of our graph as ‘bad’. It will turn out that these edges are few in number, and, moreover, that paths connecting vertices in certain pairings, and which do not contain bad edges, must contribute to many edges between A^* and B^* . Formally, we say that an edge $uv \in G$ is *bad* if one of the following holds:

1. (Type I) $uv \in G[B^*]$.
2. (Type II) $uv \in G[A^*]$ and u (or v) has degree not equal to 2 in B^* .
3. (Type III) $uv \in G[A^*]$, $d_{B^*}(u) = d_{B^*}(v) = 2$, and $N_{B^*}(u) \neq N_{B^*}(v)$.
4. (Type IV) $uv \in G$, such that $u \in A^*, v \in B^*$, and $d_{B^*}(u) \geq 3$.

We have the following bound on the number of bad edges.

Claim 9. *There are at most $1233\epsilon n$ bad edges.*

Proof. We are going to bound the number of bad edges of each type.

Note that by planarity, there are at most $3|B^*|$ edges in B^* so there are at most $3|B^*| \leq 171\epsilon n$ edges of Type I.

Now, since every vertex in A^* has at most two neighbours in A^* , each vertex in A^* with degree not equal to 2 in B^* contributes to at most two bad edges of Type II. As there are at most $189\epsilon n$ vertices in A^* which do not have degree 2 in B^* , it follows that there are at most $378\epsilon n$ bad edges of Type II.

Let us consider bad edges of Type III. Since $G[A^*]$ has maximum degree 2, we can partition the edges of $G[A^*]$ into at most 3 matchings, M_1, M_2, M_3 . It is well known (and easy to see) that contracting an edge in a planar graph preserves planarity. It follows that, for $i \in \{1, 2, 3\}$, we can contract the edges of M_i while still preserving planarity. Denote this new graph by \tilde{G}_i with vertex set $\tilde{A}_i \cup B^*$. Since \tilde{G}_i is planar, from Lemma 8 we have that there are at most $2|B^*|$ vertices in \tilde{G}_i with at least 3 neighbours in B^* . Therefore, at most $2|B^*|$ edges in M_i can be bad of Type III. Hence, there are at most $6|B^*| \leq 342\epsilon n$ bad edges of Type III.

Finally, by Lemma 8 there can be at most $6|B^*| \leq 342\epsilon n$ bad edges of Type IV.

So in total there are at most $1233\epsilon n$ bad edges of any type. \square

Let $Y \subseteq A^*$ be the set of vertices with degree exactly 2 in B^* . We now define an auxiliary *multigraph* G_{B^*} in the following way. The vertex set of G_{B^*} is B^* and for any two vertices $x, y \in B^*$, join x to y by an edge for every $v \in Y$ that is joined precisely to x and y .

Claim 10. G_{B^*} is planar.

Proof. This is clear since the bipartite graph $G[Y, B^*]$ between Y and B^* is planar, and contracting edges preserves planarity. \square

Apply Corollary 6 to the multigraph G_{B^*} with $\epsilon_1 = \epsilon_2 = 1/100$. Notice that if an edge in G_{B^*} is incident with more than $\frac{1}{100}|Y|$ vertices then one of its endpoints has degree at least $\frac{1}{200}|Y|$. However, recall that we initially assumed $\Delta(G) < cn$, and certainly $c \leq 1/400$ by our choice of D . Accordingly, since $|Y| \geq n - 189\epsilon n \geq n/2$, we obtain a vertex of degree at least

$$2c|Y| \geq cn,$$

a contradiction.

So we may assume that there are at least $\frac{1}{100} \binom{|Y|}{2}$ pairs of edges in G_{B^*} which are at distance greater than 1. Note that if H is any graph on n vertices with edge density at least δ , then it is easy to greedily find a matching of size at least $\frac{\delta}{10}n$. It follows that we may select a collection of pairwise disjoint pairs \mathcal{P} in Y , such that $|\mathcal{P}| \geq \frac{1}{1000}|Y| \geq \frac{1}{2000}n$, and such that for every $\{u, v\} \in \mathcal{P}$, their corresponding edges in G_{B^*} are at distance greater than 1.

We need the following two claims.

Claim 11. *Let P be a path contained in A^* which has at least two vertices and does not contain any bad edges. Then every vertex $v \in P$ has the same neighbourhood (of size 2) in B^* .*

Proof. This is immediate from the definition of a bad edge. \square

Claim 12. *Let $u, v \in Y$ be two vertices whose corresponding edges in G_{B^*} are at distance greater than 1. Then any path in G joining u and v either contains some bad edges, or uses at least 6 edges between A^* and B^* .*

Proof. Suppose P is a path joining u and v which does not use any bad edges. By definition and using claim 11, all vertices of $V(P) \cap A^*$ are in Y , it can not have an edge inside B^* and it must use 2 or 4 edges between A^* and B^* . We may assume P uses 4 edges as the other case follows from the same argument. Let $P = P_1e_1e_2P_2e_3e_4P_3$, where $\{e_1, e_2, e_3, e_4\}$ are edges between A^* and B^* and P_1, P_2, P_3 are paths inside Y . From claim 11 applied to P_1, P_2 and P_3 we deduce that the edge of u in G_{B^*} is at distance at most 1 to the edge of v in G_{B^*} . \square

The proof of Theorem 2 is nearly complete. Indeed, since G is path-pairable, there are edge-disjoint paths joining every pair of \mathcal{P} , and hence the pairs in \mathcal{P} contribute to at least $6(|\mathcal{P}| - 1233\epsilon n)$ edges between A^* and B^* , by Claims 9 and 12.

Let P be the union of the vertices in \mathcal{P} and let $U = A^* \setminus P$. Recall that $|A^*| \geq n - 57\epsilon n$. Suppose first that $|U| < 57\epsilon n$. It follows that

$$2|\mathcal{P}| > (n - 57\epsilon n) - 57\epsilon n,$$

so $|\mathcal{P}| > n/2 - 57\epsilon n$. Then the above pairing contributes at least $6(n/2 - 1290\epsilon n) = 3n - 7740\epsilon n$ edges between A^* and B^* . But this is at least $2n$ whenever $\epsilon \leq 7740^{-1}$ which is guaranteed by our choice of D , a contradiction. Therefore, we may assume that $|U| \geq 57\epsilon n$. By Lemma 7 (since c is small enough) there is a pairing of the vertices in U which contributes to at least $2|U| - 16 \cdot 57\epsilon n = 2|U| - 912\epsilon n$ edges between A^* and B^* . Hence in total the number of edges between A^* and B^* is

$$\begin{aligned} &\geq 6(|\mathcal{P}| - 1233\epsilon n) + 2|U| - 912\epsilon n \\ &= 6(|\mathcal{P}| - 1233\epsilon n) + 2|A^*| - 4|\mathcal{P}| - 912\epsilon n. \end{aligned}$$

Recalling that $|A^*| \geq n - 57\epsilon n$ and $|\mathcal{P}| \geq n/2000$, we have that there are

$$\begin{aligned} &\geq 2|\mathcal{P}| + 2(n - 57\epsilon n) - 6 \cdot 1233\epsilon n - 912\epsilon n \\ &\geq 2n + n/1000 - 8424\epsilon n, \end{aligned}$$

edges between A^* and B^* .

So by our choice of D we get that $8424\epsilon \leq \frac{1}{1000}$, and so there are at least $2n$ edges between A^* and B^* , a contradiction to the planarity of G . It follows that there must exist a vertex of degree at least cn . \square

4 Final Remarks and Open Problems

It is worth observing that our proof relies only on the following three properties of a planar graph G : contracting edges of G preserves planarity, G does not contain a K_5 -minor, and any bipartite subgraph H of G has at most $2|H|$ edges. We remark that it is possible to

generalize our result in the following sense. Given integers t, c , we say that a graph G is (t, c) -good if G is K_t -minor-free and any bipartite subgraph H of G has at most $2|H| + c$ edges. Moreover, define $\mathcal{G}_{t,c}$ to be the family of (t, c) -good graphs G such that contracting edges of G preserves (t, c) -goodness.

Theorem 13. *For any integers t, c there is a positive constant $C = C(t, c)$ such that the following holds. If G is a path-pairable graph on n vertices with $G \in \mathcal{G}_{t,c}$, then $\Delta(G) \geq Cn$.*

We have the following immediate corollary.

Corollary 14. *For every non-negative integer g there is a positive constant $C = C(g)$ such that the following holds. If G is a path-pairable graph on n vertices which has a 2-cell embedding on a surface with genus g , then $\Delta(G) \geq Cn$.*

Proof. We claim that $G \in \mathcal{G}_{3g+5,2g}$. Indeed, it follows from Euler's formula (see, e.g., [1]) that if G is 2-cell embedded on a surface of genus g then $n + m - f = 2 - g$, where m is the number of edges G and f is the number of faces of the embedding. Since $2m \geq 3f$ ($2m \geq 4f$ if G is triangle-free) it follows that $e(G) \leq 3n + 3g - 6$ ($e(G) \leq 2n + 2g - 4$ if G is triangle-free). In particular, if G is bipartite then $e(G) \leq 2n + 2g - 4 \leq 2n + 2g$. Suppose for contradiction that G contains a K_{3g+5} -minor. Then K_{3g+5} could be 2-cell embedded on a surface of genus g , hence $\binom{3g+5}{2} = e(K_{3g+5}) \leq 12g + 9$, which is easily seen to be a contradiction. \square

Sketch of a proof of Theorem 13. The proof is essentially the same as the proof of Theorem 2. Certain changes have to be made in the preparatory lemmas first.

Corollary 6 generalizes trivially to multigraphs with no K_t -minors.

In Lemma 7 we only use the fact that any subgraph H of a planar graph has at most $3|H|$ edges. Observe that if $G \in \mathcal{G}_{t,c}$ then any subgraph of H of G has at most $4|H| + 2c$ edges. One can therefore modify the proof, at the expense of a worse constant in front of εn in the conclusion of the Lemma.

In the proof of Lemma 8 we only use the fact that a bipartite subgraph H of a planar graph does not use more than $2|H|$ edges. The lemma can be therefore modified to work for graphs in $\mathcal{G}_{t,c}$ by introducing some additive constants, depending only on c , to the inequalities in every part of the Lemma.

In the proof of Theorem 2 all the estimates remain correct by taking ε small enough.

Note that we also need that $\mathcal{G}_{t,c}$ is closed under edge contractions in order to estimate the number of "bad edges", as in Claim 9 (there we used that contracting edges preserves planarity). \square

We believe that the condition on the number of edges in bipartite subgraphs can be omitted while still ensuring the existence of a vertex of linear degree. We therefore make the following conjecture.

Conjecture 15. *For any t there exists a constant $c = c(t)$ such that every path-pairable graph on n vertices without a K_t minor must contain a vertex of degree at least cn .*

Finally, recall that we defined $\Delta_{\min}^p(n)$ to be the minimum of $\Delta(G)$ over all n -vertex path-pairable planar graphs G . We have shown that $\Delta_{\min}^p(n)$ grows linearly in n ; however, as mentioned in the Introduction, the constants in the upper and lower bounds are quite far apart. We close with the following problem.

Problem 16. Determine $\Delta_{\min}^p(n)$ for sufficiently large n .

We do not know if our construction yielding the upper bound of $2n/3$ is optimal, and a significant improvement on our lower bound would be very interesting.

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