Toric Mutations in the dP_2 Quiver and Subgraphs of the dP_2 Brane Tiling

Yibo Gao*
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA, U.S.A.
gaoyibo@mit.edu

Zhaoqi Li†
Department of Statistics
University of Washington
Seattle, WA, U.S.A.
zli9@uw.edu

Thuy-Duong Vuong‡
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA, U.S.A.
dvuong@mit.edu

Lisa Yang§
Department of Mathematics
Massachusetts Institute of Technology
Cambridge, MA, U.S.A.
lisayang@mit.edu

Submitted: Feb 20, 2017; Accepted: Mar 6, 2019; Published: May 3, 2019

Abstract

Brane tilings are infinite, bipartite, periodic, planar graphs that are dual to quivers. In this paper, we study the del Pezzo 2 (dP_2) quiver and its associated brane tiling which arise in theoretical physics. Specifically, we prove explicit formulas for all cluster variables generated by toric mutation sequences of the dP_2 quiver. Moreover, we associate a subgraph of the dP_2 brane tiling to each toric cluster variable such that the sum of weighted perfect matchings of the subgraph equals the Laurent polynomial of the cluster variable.

Mathematics Subject Classifications: 13F60, 05B45, 05C20

1 Introduction

Cluster algebras are a class of commutative rings generated by cluster variables, which are partitioned into sets called clusters. Given an initial seed, an operation known as seed mutation can be applied iteratively to generate all cluster variables. The concept

*Supported by NSF grant DMS-1351590.
†Supported by NSF grant DMS-1351590.
‡Supported by NSF grant DMS-1351590.
§Supported by NSF RTG grant DMS-1148634.

of cluster algebras was first introduced by Fomin and Zelevinsky [4] as a tool to study total positivity and dual canonical bases in Lie theory. They have rich applications in different branches of mathematics including algebraic combinatorics, tropical geometry, Teichmuller theory, and representation theory.

It is common to picture a cluster as a quiver with a cluster variable on each vertex. Some special quivers have planar duals known as brane tilings which are doubly-periodic, bipartite, planar graphs. The notion of brane tilings was first introduced in theoretical physics [3]. They are important in physics since perfect matchings of brane tilings carry information on the geometry of certain toric varieties which are Calabi-Yau 3-folds. For such quivers, combinatorial interpretations of the cluster variables have been obtained by associating a subgraph of the brane tiling to each cluster variable such that the Laurent polynomial of the cluster variable is recoverable from a weighting scheme applied to the subgraph ([13], [14], [12]). In particular, the quiver and brane tiling of the third del Pezzo (dP_3) surface ([5]) has been studied widely in [2], [16], [11], and [10]. In this paper, we generalize the techniques utilized in these papers and focus on the second del Pezzo (dP_2) surface. Specifically, we classify all cluster variables generated by toric mutations and give combinatorial interpretations for their Laurent polynomials.

In Section 2, we start with background material on quivers and cluster mutations, and then we introduce our main objects of study: the dP_2 quiver and its corresponding brane tiling. In Section 3, we define ρ-mutations (Definition 5) and show that all toric mutation sequences can be obtained as a specific form of ρ-mutation sequences (Theorem 7). From there, we are able to explicitly write down the cluster variables arising from these ρ-mutation sequences (Theorem 13) and classify all of them in a simple form (Corollary 14). In the second half of the paper, we give combinatorial interpretations for these toric cluster variables by associating subgraphs of the dP_2 brane tiling to each variable. We prove that the weighted perfect matchings of a subgraph are terms in the Laurent polynomial of the cluster variable. Our main theorem (Theorem 24) specifies a contour for each cluster variable. The weighting scheme is described in Section 5 and the procedure for obtaining a subgraph from a contour is given in Section 6. The proof of our main theorem is shown in Section 7 and it proceeds by induction and case work. We provide a detailed description of our proof techniques and sample proofs for a few cases. The rest of the cases are proved analogously and we provide the necessary data in Appendix 9. We finish with a discussion regarding a similar quiver which also generates the Laurent polynomials of the Somos-5 sequence and is studied by David Speyer [15] in Section 8.

2 Preliminaries

2.1 Quiver and Cluster Mutations

Definition 1 (Quiver and Cluster)**.** A *quiver* is a finite directed graph Q with a set of vertices V and a set of edges E. We can associate a cluster variable x_i to the vertex labeled i. The *cluster* is the ordered set of cluster variables $\{x_1, \dots, x_n\}$ at each vertex where $n = |V|$. For a cluster $S = \{x_1, \dots, x_n\}$, let $S[i]$ refer to the ith cluster variable.

In this paper, we allow quivers to have multiple edges connecting two vertices but there can be no 2-cycles or 1-cycles (loops).

Definition 2 (Quiver Mutation)**.** Mutation at vertex i in Q is denoted by μ_i. Mutating performs the following actions on the quiver:

- For every 2-path through i (e.g. $j \to i \to k$), add an edge from j to k.
- Reverse the directions of the arrows incident to i.
- Delete any 2-cycles created by the previous two steps.

Correspondingly, the cluster variable at vertex i is updated and all other cluster variables stay the same. The update follows the binomial exchange relation

$$x_i' x_i = \prod_{i \to j \text{ in } Q} x_j^{a_{i\to j}} + \prod_{j \to i \text{ in } Q} x_j^{a_{j \to i}},$$

where x_i' is the new cluster variable at vertex i and $a_{i \to j}$ is the number of edges from i to j.

The binomial exchange relation replaces $S[i]$ by the new cluster variable x_i'. We denote this replacement by

$$S[i] \leftarrow \frac{\prod_{i \to j \text{ in } Q} x_j^{a_{i\to j}} + \prod_{j \to i \text{ in } Q} x_j^{a_{j \to i}}}{x_i}.$$

2.2 The Del Pezzo 2 Quiver and its Brane Tiling

In this paper, we study a special quiver associated to the second del Pezzo surface (dP_2) [1] and its brane tiling, as shown in Figure 1.

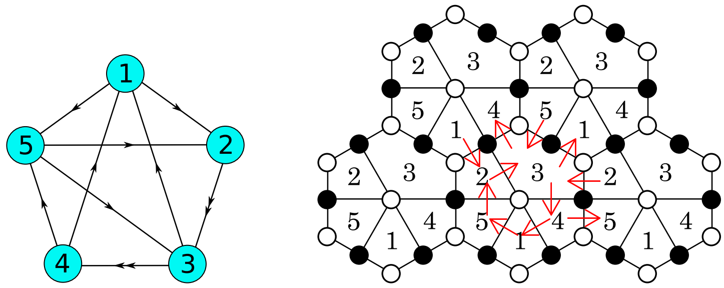

Figure 1: dP_2 quiver Q, where the number of arrowheads stands for the number of arrows, and its associated brane tiling $\mathcal{T}$ (Figure 30 of [5])

To obtain a quiver from its corresponding brane tiling, we look at each edge e up to translation, noting that any brane tiling is periodic, bipartite, and planar. Assume that e borders block i and j such that as we go across from block i to block j, the black end point of e is on the left and the white end point of e is on the right. For this edge e, we add an edge in the quiver that goes from i to j. The red arrows in Figure 1 show this process.

We use Q to denote the dP_2 quiver and $\mathcal{T}$ to denote its associated brane tiling.

2.3 Toric Mutation and Two Models of Quivers

Definition 3 (Toric Vertex and Toric Mutation)**.** We say that a vertex in a quiver is *toric* if it has in-degree 2 and out-degree 2. A *toric mutation* is a cluster mutation at a toric vertex.

Definition 4 (Model)**.** We say that two quivers Q_1 and Q_2 are of the same *model* if they are either isomorphic as directed graphs (i.e. if there exists a bijection between their vertices that preserves edges) or if Q_1 is isomorphic to Q_2 with all edges in Q_2 reversed.

It is easy to show that the dP_2 quiver Q has two models that can be reached from the original quiver by toric mutations. We use **model 1** to refer to the original quiver Q and **model 2** to refer to the quiver obtained from Q by mutating at vertex 2. Figure 2 shows these two models. As a side note, the word "model" sometimes appears as "phase" in physics literature [5].

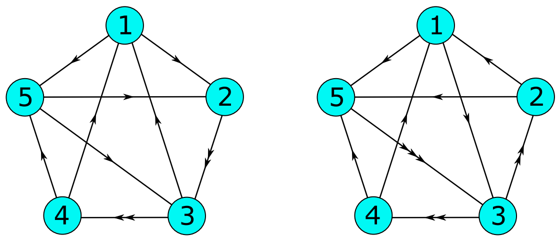

Figure 2: model 1 and model 2 of the dP_2 quiver (Figure 30 and 31 of [5])

Transitions between these two models are shown in Figure 3.

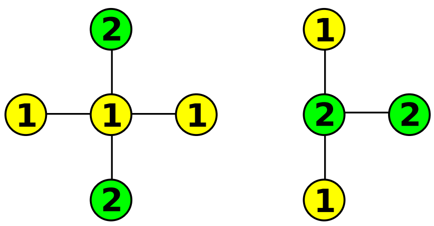

Figure 3: Adjacency between different models (Figure 18 of [2])

3 Classification of Toric Mutation Sequences

Definition 5 ($\boldsymbol{\rho}$-mutation sequences)**.** We define the following operation sequences for model 1 dP_2 quivers, consisting of mutations and permutations, where operations are performed from left to right. A permutation permutes the vertices and their associated cluster variables accordingly.

$$\rho_1 = \mu_1 \circ (54321), \quad \rho_2 = \mu_5 \circ (12345), \quad \rho_3 = \mu_2 \circ \mu_4 \circ (24),$$

$$\rho_4 = \mu_2 \circ \mu_1 \circ \mu_4 \circ (531), \quad \rho_5 = \mu_4 \circ \mu_5 \circ \mu_2 \circ (351),$$
$$\rho_6 = \mu_2 \circ \mu_1 \circ \mu_2 \circ (531)(24), \quad \rho_7 = \mu_4 \circ \mu_5 \circ \mu_4 \circ (135)(24).$$

We call each ρ_i a *ρ-mutation* and any concatenation of ρ_i's a *ρ-mutation sequence.*

As a side note, it is technically more accurate to refer to our "ρ-mutations" as "ρ-operations". However, we choose to follow the conventions set in [11] and [10].

These ρ-mutations all fix the quiver but not the cluster variables so that $\rho_i(Q) = Q$ for $i = 1, \ldots, 7$. Notice that in the original quiver Q, there are no edges connecting vertex 2 and 4. This means a mutation at 2 and a mutation at 4 commute, so ρ_3 can also be written as $\rho_3 = \mu_4 \circ \mu_2 \circ (24)$.

From Figure 3, we construct Figure 4 which shows all possible toric mutation sequences that start from the original dP_2 quiver and return to model 1. From this figure, it is clear that combinations of the seven ρ-mutations give us all possible toric mutation sequences that start in model 1 and end in model 1 up to a permutation of vertices.

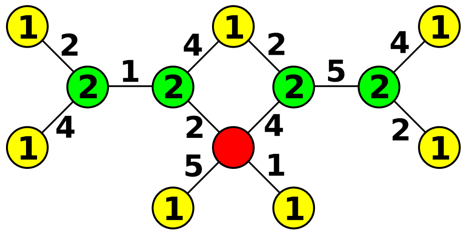

Figure 4: All possible toric mutation sequences that start from model 1 and return to model 1. The red circle represents the initial quiver Q. The number on each edge specifies which vertex is mutated.

Proposition 6 (Relations between $\boldsymbol{\rho}$-mutations)**.**

$$\begin{aligned}
\rho_4\{x_1, x_2, x_3, x_4, x_5\} &= \rho_1^2\rho_3\{x_1, x_2, x_3, x_4, x_5\},\\
\rho_5\{x_1, x_2, x_3, x_4, x_5\} &= \rho_2^2\rho_3\{x_1, x_2, x_3, x_4, x_5\},\\
\rho_6\{x_1, x_2, x_3, x_4, x_5\} &= \rho_1^2\{x_1, x_2, x_3, x_4, x_5\},\\
\rho_7\{x_1, x_2, x_3, x_4, x_5\} &= \rho_2^2\{x_1, x_2, x_3, x_4, x_5\}.
\end{aligned}$$

$$\rho_1\rho_2\{x_1, \ldots, x_5\} = \rho_2\rho_1\{x_1, \ldots, x_5\} = \rho_3^2\{x_1, \ldots, x_5\} = \{x_1, x_2, x_3, x_4, x_5\},$$
$$\rho_1^2\rho_3\{x_1, \ldots, x_5\} = \rho_3\rho_1^2\{x_1, \ldots, x_5\}, \quad \rho_2^2\rho_3\{x_1, \ldots, x_5\} = \rho_3\rho_2^2\{x_1, \ldots, x_5\},$$
$$\rho_1\rho_3\rho_2\{x_1, \ldots, x_5\} = \rho_2\rho_3\rho_1\{x_1, \ldots, x_5\}.$$

Note that it suffices to define ρ_1, ρ_2, ρ_3 because ρ_4, ρ_5, ρ_6, ρ_7 can be written in terms of the previous three.

Theorem 7. *Any toric mutation sequence in the dP_2 quiver that starts and ends at model 1 can be written, up to a permutation of cluster variables, as $\rho_t^k(\rho_3\rho_1)^m\rho_3^w$ where $k, m \in \mathbb{Z}_{\geqslant 0}$, $t \in \{1, 2\}$, and $w \in \{0, 1\}$.*

Proof. This essentially states that all ρ-mutation sequences can be written in a certain form. Fix a generic ρ-mutation sequence.

Since $\rho_1\rho_2 = \rho_2\rho_1 = \rho_3^2 = 1$, we can assume that this sequence does not contain consecutive ρ_3's and does not contain adjacent ρ_1 and ρ_2. Therefore, we can write it as $\rho_{j_1}^{\alpha_1}\rho_3\rho_{j_2}^{\alpha_2}\rho_3\cdots\rho_{j_N}^{\alpha_N}$ where $j_i \in \{1,2\}$ and $\alpha_i \in \mathbb{Z}_{>0}$ and with possibly a ρ_3 at the beginning and a ρ_3 at the end.

Notice that by Proposition 6, ρ_1^2 and ρ_2^2 commute with every ρ-mutation. So we move any two consecutive ρ_1's or consecutive ρ_2's to the front to get $\rho_t^n\rho_3\rho_{\ell_1}\rho_3\rho_{\ell_2}\cdots\rho_{\ell_s}$ where $t, \ell_1, \ldots, \ell_s \in \{1,2\}$ and $n \in \mathbb{Z}_{\geqslant 0}$ and with possibly a ρ_3 at the end.

Proposition 6 gives $\rho_1\rho_3\rho_2 = \rho_2\rho_3\rho_1$, which means ρ_1 and ρ_2 "commute" with a ρ_3 in between. Therefore, we are able to put all ρ_1's in front of ρ_2's in $\rho_3\rho_{\ell_1}\rho_3\rho_{\ell_2}\cdots\rho_{\ell_s}$ with possibly a ρ_3 at the end. The sequence now has the form $\rho_t^n(\rho_3\rho_1)^r(\rho_3\rho_2)^s$ with possibly a ρ_3 at the end.

For a sufficiently large M, we can write the sequence as $\rho_t^n\rho_2^M\rho_1^M(\rho_3\rho_1)^r(\rho_3\rho_2)^s$. Since ρ_1^2 commutes with everything, we can use ρ_1^2 in the term ρ_1^M to cancel with the ρ_2's in $(\rho_3\rho_2)^s$. Finally, we naturally merge the remaining ρ_1's in the previous ρ_1^M with ρ_t^n and ρ_2^M to get $\rho_t^k(\rho_3\rho_1)^m$ with $t \in \{1,2\}$ and $m, k \in \mathbb{Z}_{\geqslant 0}$ and with possibly a ρ_3 at the end, as desired. □

Remark 8. Figure 5 gives a visualization of the ρ-mutation sequences as analogs of the alcove walks discussed for the dP3 case in [10]. In Figure 5, each vertex corresponds to a cluster with a model 1 quiver. We can arbitrarily select one as the initial cluster. A horizontal step to the right is ρ_1, a horizontal step to the left is ρ_2, and a vertical step is ρ_3.

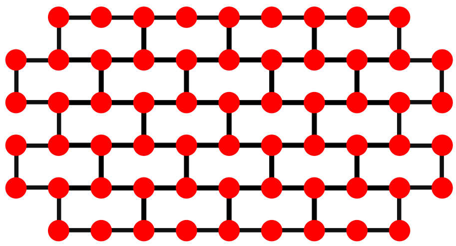

Figure 5: Visualization of ρ-mutation sequences

4 Explicit Formulas for Cluster Variables

In this section, we give explicit formulas for all cluster variables that can be generated through toric mutations on the dP_2 quiver.

Let the initial cluster variables be $\{x_1, x_2, x_3, x_4, x_5\}$.

Definition 9 (Laurent Polynomial for Somos-5 Sequence)**.**

For $n \geqslant 6$, recursively define

$$x_n := \frac{x_{n-1}x_{n-4} + x_{n-2}x_{n-3}}{x_{n-5}}.$$

For $n \leqslant 0$, recursively define

$$x_n := \frac{x_{n+1}x_{n+4} + x_{n+2}x_{n+3}}{x_{n+5}}.$$

Remark 10. For each $n \in \mathbb{Z}$, Definition 9 gives us a way to define x_n as a rational function in x_1, x_2, x_3, x_4, x_5. Moreover, the equation

$$x_n x_{n+5} = x_{n+1}x_{n+4} + x_{n+2}x_{n+3} \tag{1}$$

is satisfied for each $n \in \mathbb{Z}$. Therefore, it is clear that if we assign 1 to $x_1, \ldots, x_5$, then both $\{x_n\}_{n=1,2,\ldots}$ and $\{x_{6-n}\}_{n=1,2,\ldots}$ are the Somos-5 sequence.

Definition 11. Define the following constants

$$A := \frac{x_1x_5 + x_3^2}{x_2x_4}, \qquad B := \frac{x_2x_6 + x_4^2}{x_3x_5}\Big(= \frac{x_1x_4^2 + x_2x_3x_4 + x_2^2x_5}{x_1x_3x_5}\Big).$$

Lemma 12. *For each $n \in \mathbb{Z}$,*

$$A = \frac{x_{2n-1}x_{2n+3} + x_{2n+1}^2}{x_{2n}x_{2n+2}}, \qquad B = \frac{x_{2n}x_{2n+4} + x_{2n+2}^2}{x_{2n+1}x_{2n+3}}.$$

Proof. The lemma holds for $n = 1$ by definition. By an inductive argument, it suffices to show that for each $m \in \mathbb{Z}$,

$$\frac{x_m x_{m+4} + x_{m+2}^2}{x_{m+1}x_{m+3}} = \frac{x_{m+2}x_{m+6} + x_{m+4}^2}{x_{m+3}x_{m+5}}.$$

According to Equation (1), we have

$$\begin{aligned}
\frac{x_{m+2}x_{m+6} + x_{m+4}^2}{x_{m+3}x_{m+5}} &= \frac{x_{m+2}\dfrac{x_{m+2}x_{m+5} + x_{m+3}x_{m+4}}{x_{m+1}} + x_{m+4}^2}{x_{m+3}x_{m+5}} \\
&= \frac{x_{m+2}^2}{x_{m+1}x_{m+3}} + \frac{x_{m+4}\big(x_{m+2}x_{m+3} + x_{m+1}x_{m+4}\big)}{x_{m+1}x_{m+3}x_{m+5}} \\
&= \frac{x_{m+2}^2}{x_{m+1}x_{m+3}} + \frac{x_{m+4}x_m x_{m+5}}{x_{m+1}x_{m+3}x_{m+5}} \\
&= \frac{x_m x_{m+4} + x_{m+2}^2}{x_{m+1}x_{m+3}}.
\end{aligned}$$

□

Theorem 13. *Define $\rho_1^k := \rho_2^{-k}$ for $k < 0$. Define $g(s,k) := \left\lfloor \frac{s}{2} \right\rfloor \left\lfloor \frac{s+1}{2} \right\rfloor$ if k is even and $g(s,k) := \left\lfloor \frac{s-1}{2} \right\rfloor \left\lfloor \frac{s}{2} \right\rfloor$ if k is odd. Then for $k \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geqslant 0}$,*

$$\begin{aligned}
\rho_1^k(\rho_3\rho_1)^s\{x_1, x_2, x_3, x_4, x_5\} = \{&A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+1}, \\
&A^{g(s,k)}B^{g(s,k+1)}x_{k+s+2}, \\
&A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+3}, \\
&A^{g(s,k)}B^{g(s,k+1)}x_{k+s+4}, \\
&A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+5}\}.
\end{aligned}$$

Proof. We divide our toric mutation sequence into two steps: ρ_1^k and $(\rho_3\rho_1)^s$. Then we proceed by a straightforward induction.

Step 1: $\rho_1^k\{x_1, x_2, x_3, x_4, x_5\} = \{x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}$ for $k \in \mathbb{Z}$.

This holds for $k = 0$. Assume that this holds for some $k \geqslant 0$. Then

$$\begin{aligned}\rho_1^{k+1}\{x_1, x_2, x_3, x_4, x_5\} =& \rho_1\{x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}\\ =&(54321)\big(\mu_1\{x_{k+1}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}\big)\\ =&(54321)\{\frac{x_{k+2}x_{k+5} + x_{k+3}x_{k+4}}{x_{k+1}}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}\\ =&(54321)\{x_{k+6}, x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}\}\\ =&\{x_{k+2}, x_{k+3}, x_{k+4}, x_{k+5}, x_{k+6}\}.\end{aligned}$$

By induction, this proves the claim for $k \geqslant 0$. The proof for $k \leqslant 0$ is analogous.

Note that if $k + s$ is odd, then

$$\begin{aligned}g(s+1, k) =& 2g(s, k) - g(s-1, k) + 1\\ g(s+1, k+1) =& 2g(s, k+1) - g(s-1, k+1).\end{aligned} \tag{2}$$

If k is even, then s is odd and these two equations become

$$\begin{aligned}&\left\lfloor\frac{s+1}{2}\right\rfloor \left\lfloor\frac{s+2}{2}\right\rfloor = 2\left\lfloor\frac{s}{2}\right\rfloor \left\lfloor\frac{s+1}{2}\right\rfloor - \left\lfloor\frac{s-1}{2}\right\rfloor \left\lfloor\frac{s}{2}\right\rfloor + 1\\ \Leftrightarrow &\left(\frac{s+1}{2}\right)^2 = 2\left(\frac{s-1}{2}\right)\left(\frac{s+1}{2}\right) - \left(\frac{s-1}{2}\right)^2 + 1\end{aligned}$$

$$\begin{aligned}&\left\lfloor\frac{s}{2}\right\rfloor \left\lfloor\frac{s+1}{2}\right\rfloor = 2\left\lfloor\frac{s-1}{2}\right\rfloor \left\lfloor\frac{s}{2}\right\rfloor - \left\lfloor\frac{s-2}{2}\right\rfloor \left\lfloor\frac{s-1}{2}\right\rfloor\\ \Leftrightarrow &\left(\frac{s-1}{2}\right)\left(\frac{s+1}{2}\right) = 2\left(\frac{s-1}{2}\right)^2 - \left(\frac{s-3}{2}\right)\left(\frac{s-1}{2}\right)\end{aligned}$$

which clearly hold.

If k is odd, then s is even and these two equations become

$$\begin{aligned}&\left\lfloor\frac{s}{2}\right\rfloor \left\lfloor\frac{s+1}{2}\right\rfloor = 2\left\lfloor\frac{s-1}{2}\right\rfloor \left\lfloor\frac{s}{2}\right\rfloor - \left\lfloor\frac{s-1}{2}\right\rfloor \left\lfloor\frac{s-2}{2}\right\rfloor + 1\\ \Leftrightarrow &\left(\frac{s}{2}\right)^2 = 2\left(\frac{s}{2}\right)\left(\frac{s-2}{2}\right) - \left(\frac{s-2}{2}\right)^2 + 1\end{aligned}$$

$$\begin{aligned}&\left\lfloor\frac{s+1}{2}\right\rfloor \left\lfloor\frac{s+2}{2}\right\rfloor = 2\left\lfloor\frac{s}{2}\right\rfloor \left\lfloor\frac{s+1}{2}\right\rfloor - \left\lfloor\frac{s-1}{2}\right\rfloor \left\lfloor\frac{s}{2}\right\rfloor\\ \Leftrightarrow &\left(\frac{s}{2}\right)\left(\frac{s+2}{2}\right) = 2\left(\frac{s}{2}\right)^2 - \left(\frac{s-2}{2}\right)\left(\frac{s}{2}\right)\end{aligned}$$

which clearly hold.

By the same argument, we can show that if $k+s$ is even, then

$$\begin{aligned} g(s+1,k) =& 2g(s,k)-g(s-1,k) \\ g(s+1,k+1) =& 2g(s,k+1)-g(s-1,k+1)+1. \end{aligned} \tag{3}$$

Step 2: Calculate $\rho_1^k(\rho_3\rho_1)^s\{x_1,x_2,x_3,x_4,x_5\}$.

By step 1, $\rho_1^k(\rho_3\rho_1)^s\{x_1,x_2,x_3,x_4,x_5\} = (\rho_3\rho_1)^s\{x_{k+1},x_{k+2},x_{k+3},x_{k+4},x_{k+5}\}$.

Since $g(0,k) = g(1,k) = 0$, regardless of the parity of k, the theorem holds when $s=0$. Now assume that the theorem holds for some $s-1 \geqslant 0$. It suffices to show

$$\begin{aligned} &(\rho_3\rho_1)\{A^{g(s,k)}B^{g(s,k+1)}x_{k+s}, A^{g(s-1,k)}B^{g(s-1,k+1)}x_{k+s+1}, A^{g(s,k)}B^{g(s,k+1)}x_{k+s+2}, \\ &\quad A^{g(s-1,k)}B^{g(s-1,k+1)}x_{k+s+3}, A^{g(s,k)}B^{g(s,k+1)}x_{k+s+4}\} \\ &\quad = \{A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+1}, A^{g(s,k)}B^{g(s,k+1)}x_{k+s+2}, A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+3}, \\ &\quad A^{g(s,k)}B^{g(s,k+1)}x_{k+s+4}, A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+5}\}. \end{aligned}$$

Denote $\rho_1^k(\rho_3\rho_1)^{s-1}\{x_1,\ldots,x_5\}$ as S, and let $S[i]$ be the i^{th} element of S. Recall that to apply $\rho_3\rho_1$ to S, we first perform $\rho_3 = \mu_2 \circ \mu_4 \circ (24)$. As we mutate vertex 2, the new cluster variable at vertex 2 is updated to

$$\begin{aligned} S[2] \leftarrow & \frac{S[1]S[5]+S[3]^2}{S[2]} \\ =& \frac{A^{2g(s,k)}B^{2g(s,k+1)}(x_{k+s}x_{k+s+4}+x_{k+s+2}^2)}{A^{g(s-1,k)}B^{g(s-1,k+1)}x_{k+s+1}} \end{aligned}$$

According to Lemma 12 and Equation (2), if $k+s$ is odd, the above expression becomes

$$\frac{A^{2g(s,k)}B^{2g(s,k+1)}Ax_{k+s+3}}{A^{g(s-1,k)}B^{g(s-1,k+1)}} = A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+3}.$$

Similarly by Lemma 12 and Equation (3), if $k+s$ is even, we have

$$\frac{A^{2g(s,k)}B^{2g(s,k+1)}Bx_{k+s+3}}{A^{g(s-1,k)}B^{g(s-1,k+1)}} = A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+3}.$$

By the same argument, we can show that as we mutate vertex 4,

$$S[4] \leftarrow A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+1}.$$

So if we let $S' = \rho_3 S$, then S' and S differ only in the 2^{nd} and the 4^{th} coordinate. Specifically,

$$S'[2] = A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+1}, \quad S'[4] = A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+3}.$$

Finally, we mutate at vertex 1 in S' and get

$$S'[1] \leftarrow \frac{S'[2]S'[5]+S'[3]S'[4]}{S'[1]}$$

$$
\begin{aligned}
&= \frac{A^{g(s+1,k)+g(s,k)}B^{g(s+1,k+1)+g(s,k+1)}(x_{k+s+1}x_{k+s+4}+x_{k+s+2}x_{k+s+3})}{A^{g(s,k)}B^{g(s,k+1)}x_{k+s}} \\
&= A^{g(s+1,k)}B^{g(s+1,k+1)}x_{k+s+5}.
\end{aligned}
$$

After applying a permutation (54321), we obtain the desired identity, which completes the inductive step. $\square$

Corollary 14. *All cluster variables that may appear in toric mutation sequences can be written in the form*

$$A^{n^2}B^{n(n-1)}x_{2m} \quad \text{or} \quad A^{n(n-1)}B^{n^2}x_{2m-1} \qquad \textit{where } m, n \in \mathbb{Z}.$$

And conversely, all such expressions can be obtained as cluster variables from some toric mutation sequences.

Proof. We first show that all cluster variables that arise from toric mutations can be achieved by ρ-mutation sequences of the form $\rho_1^k(\rho_3\rho_1)^s$ for some $k \in \mathbb{Z}$ and $s \in \mathbb{Z}\geqslant 0$. According to Theorem 7, every toric mutation sequence starting and ending at model 1 can be written as $\rho_1^k(\rho_3\rho_1)^s$ or $\rho_1^k(\rho_3\rho_1)^s\rho_3$ for some $k \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geqslant 0}$. The proof for Theorem 13 shows that cluster variables of $\rho_1^k(\rho_3\rho_1)^s\rho_3\{x_1,\ldots,x_5\}$ are included in $\rho_1^k(\rho_3\rho_1)^s\{x_1,\ldots,x_5\}$ and $\rho_1^k(\rho_3\rho_1)^{s+1}\{x_1,\ldots,x_5\}$. Now we consider any toric mutation sequence that takes the original model 1 quiver to some model 2 quiver. According to Figure 4, this model 2 quiver can reach two different model 1 quivers in one step of toric mutation. So the cluster variables corresponding to this specific toric mutation sequence that ends on a model 2 quiver are included amongst the cluster variables that are generated by these two model 1 quivers.

We take a closer look at the cluster variables in Theorem 13. Since $g(s,k)$ only depends on the value of s and the parity of k, it is easy to see that all cluster variables that arise can be written as $A^{g(s,k)}B^{g(s,k+1)}x_{k+s}$ for some $k \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geqslant 0}$. Conversely, for any $k \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geqslant 0}$, $A^{g(s,k)}B^{g(s,k+1)}x_{k+s}$ can be generated by a toric mutation sequence according to Theorem 13. We consider the following four cases according to the parity of s and k.

Case 1: s is even and k is even. Let $s = 2n$ and $k+s = 2m$. We have $n \geqslant 0$. Then

$$A^{g(s,k)}B^{g(s,k+1)}x_{k+s} = A^{\lfloor \frac{s}{2} \rfloor \lfloor \frac{s+1}{2} \rfloor}B^{\lfloor \frac{s-1}{2} \rfloor \lfloor \frac{s}{2} \rfloor}x_{2m} = A^{n^2}B^{n(n-1)}x_{2m}.$$

Case 2: s is odd and k is odd. Let $s = 2n+1$ and $k+s = 2m$. We have $n \geqslant 0$. Then

$$A^{g(s,k)}B^{g(s,k+1)}x_{k+s} = A^{\lfloor \frac{s-1}{2} \rfloor \lfloor \frac{s}{2} \rfloor}B^{\lfloor \frac{s}{2} \rfloor \lfloor \frac{s+1}{2} \rfloor}x_{2m} = A^{n^2}B^{n(n+1)}x_{2m}.$$

Case 3: s is even and k is odd. Let $s = 2n$ and $k+s = 2m-1$. We have $n \geqslant 0$. Then

$$A^{g(s,k)}B^{g(s,k+1)}x_{k+s} = A^{\lfloor \frac{s-1}{2} \rfloor \lfloor \frac{s}{2} \rfloor}B^{\lfloor \frac{s}{2} \rfloor \lfloor \frac{s+1}{2} \rfloor}x_{2m-1} = A^{n(n-1)}B^{n^2}x_{2m-1}.$$

Case 4: s is odd and k is even. let $s = 2n+1$ and $k+s = 2m-1$. We have $n \geqslant 0$. Then

$$A^{g(s,k)}B^{g(s,k+1)}x_{k+s} = A^{\lfloor \frac{s}{2} \rfloor \lfloor \frac{s+1}{2} \rfloor}B^{\lfloor \frac{s-1}{2} \rfloor \lfloor \frac{s}{2} \rfloor}x_{2m-1} = A^{n(n+1)}B^{n^2}x_{2m-1}.$$

Cases 1 and 2 can be combined by considering $n \in \mathbb{Z}$ instead of just $n \in \mathbb{Z}_{\geqslant 0}$. Similarly cases 3 and 4 can be combined. Finally, we conclude that all cluster variables generated by toric mutations can be written as either

$$A^{n^2} B^{n(n-1)} x_{2m} \quad \text{or} \quad A^{n(n-1)} B^{n^2} x_{2m-1} \qquad \text{where } m, n \in \mathbb{Z}.$$

The conversely holds by simply noticing that all expressions appeared in the proof of Theorem 13 can be obtained from some ρ-mutation sequences. □

5 Subgraphs of the Brane Tiling

In this paper, every graph we consider is a subgraph of the dP_2 brane tiling so it is bipartite, planar, and weighted. For such a bipartite graph G, let V_1 and V_2 be a partition of G's vertices such that there are no edges within V_1 or V_2. For any vertex set $V_0 \subset V_1 \cup V_2$, define $G - V_0$ to be the graph obtained by removing each vertex in V_0 and its incident edges from G.

We want to find a subgraph for each cluster variable that appears through toric mutations such that the subgraph's weight equals the cluster variable. We use the weighting scheme utilized in [10], [11], [15], [16], and elsewhere in literature.

Definition 15 (Weight of Subgraphs)**.** Associate a weight of $\frac{1}{x_i x_j}$ to each edge bordering block labeled i and j. For a set of edges M, define its weight $w(M)$ to be the product of the weights of the edges. Recall that a *perfect matching* of a graph is a subset of its edges such that every vertex is incident to exactly one of these edges. For a subgraph G of the brane tiling, let $\mathcal{M}(G)$ be the collection of G's perfect matchings. We define the weight of G as

$$w(G) = \sum_{M \in \mathcal{M}(G)} w(M).$$

In order to get recursive relations on the variables which correspond to subgraphs, we use Kuo's condensation theorems ([9], [8]) which represent the weight of a large graph in terms of the weights of its smaller subgraphs.

Lemma 16 (Balanced Kuo Condensation; Theorem 5.1 in [8])**.** *Let G be a weighted planar bipartite graph with $|V_1| = |V_2|$. Assume that p_1, p_2, p_3, p_4 are four vertices appearing in cyclic order on a face of G with $p_1, p_3 \in V_1$ and $p_2, p_4 \in V_2$. Then*

$$\begin{aligned} w(G)w(G - \{p_1,p_2,p_3,p_4\}) =& w(G - \{p_1,p_2\})w(G - \{p_3,p_4\}) \\ &+ w(G - \{p_1,p_4\})w(G - \{p_2,p_3\}). \end{aligned}$$

Lemma 17 (Unbalanced Kuo Condensation; Theorem 5.2 in [8])**.** *Let G be a weighted planar bipartite graph with $|V_1| = |V_2| + 1$. Assume that p_1, p_2, p_3, p_4 are four vertices appearing in cyclic order on a face of G with $p_1, p_2, p_3 \in V_1$ and $p_4 \in V_2$. Then*

$$\begin{aligned} w(G - \{p_2\})w(G - \{p_1,p_3,p_4\}) =& w(G - \{p_1\})w(G - \{p_2,p_3,p_4\}) \\ &+ w(G - \{p_3\})w(G - \{p_1,p_2,p_4\}). \end{aligned}$$

Lemma 18 (Non-alternating Kuo Condensation; Theorem 5.3 in [8]). *Let G be a weighted planar bipartite graph with $|V_1| = |V_2|$. Assume that p_1, p_2, p_3, p_4 are four vertices appearing in cyclic order on a face of G with $p_1, p_2 \in V_1$ and $p_3, p_4 \in V_2$. Then*

$$\begin{aligned} w(G - \{p_1, p_4\})w(G - \{p_2, p_3\}) =& w(G)w(G - \{p_1, p_2, p_3, p_4\}) \\ &+ w(G - \{p_1, p_3\})w(G - \{p_2, p_4\}). \end{aligned}$$

6 Contours for Cluster Variables

In this section, we describe a method to get the subgraph corresponding to any cluster variable obtained by toric mutations of the dP$_2$ quiver. Specifically, we will use 5-sided contours to cut our brane tiling and we define rules to obtain subgraphs from these contours. In Theorem 24, we state the explicit formulas of the contours corresponding to toric cluster variables.

6.1 Graphs from Contours

Given a 5-tuple $(a, b, c, d, e) \in \mathbb{Z}^5$ with $a + b = d$ and $a + e = c$ (these relations are shown in see Figure 6 right), we can define a 5-sided **contour** whose side-lengths are a, b, c, d, e in clockwise order starting from the upper right corner. Figure 6 (left) shows the fundamental shape of the contour where each side length, or integer in the tuple, is positive. In the case of negative side lengths, we draw the corresponding side in the opposite direction.

See Figure 7 (left) for an example of a 5-tuple and its contour. We will abuse notation and denote a geometric contour by its corresponding 5-tuple.

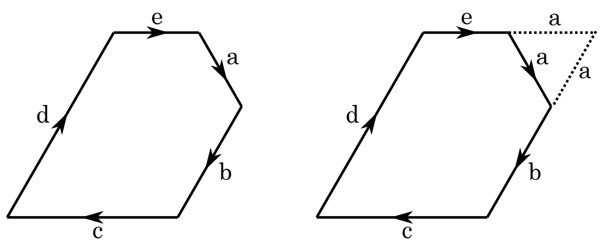

Figure 6: **Left**: 5-sided fundamental shape; **Right**: relations between side lengths.

Now we define the rules to obtain a subgraph from a contour.

Definition 19 (Rules to Get Subgraph).

Suppose that we are given a 5-sided contour $C = (a, b, c, d, e) \in \mathbb{Z}^5$ as above. Call the white vertex between edge c and edge d the **special vertex**.

Step 1: Superimpose the contour on the brane tiling $\mathcal{T}$, where the unit length equals the horizontal distance between two nearest white vertices with the same vertical coordinates, such that the vertex between side a and e sits on any white vertex of degree 5.

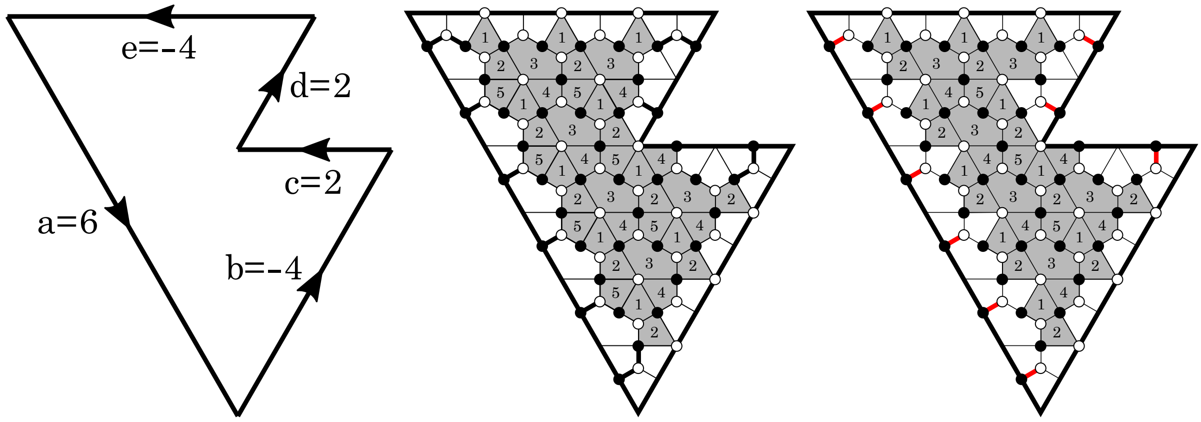

Figure 7: Example of a contour $C = (6, -4, 2, 2, -4)$ and its subgraphs $\mathcal{G}(C)$ (shaded region and darkened edges) and $\widehat{\mathcal{G}}(C)$ (shaded region).

Step 2: On each side of positive length, we keep the black points while removing the white points; on each side of negative length, we keep the white points while removing the black points; on each side of zero length, we remove the single white point if it is not the special vertex.

Step 3: Each corner vertex is white. If the two adjacent sides of a corner vertex are both non-positive, then we keep the vertex; otherwise, we remove it. If a is even, then we keep the special vertex; if a is odd, then we remove the special vertex. We refer to the graph that remains inside the contour as $\mathcal{G}(C)$.

Step 4: In the resulting graph, we connect any vertex of valence 1 to its adjacent vertex. We call this edge a **forced matching**. Then delete these two vertices from the graph. Repeat this step until every vertex in the subgraph has valence at least 2.

Step 5: We refer to the resulting graph as $\widehat{\mathcal{G}}(C)$ or the **subgraph** of contour C. Often we may refer to $\widehat{\mathcal{G}}(C)$ as either $\widehat{\mathcal{G}}(a, b, c, d, e)$ or simply $\widehat{\mathcal{G}}$.

Definition 20. For any graph G, let $\widehat{G}$ denote the graph obtained by removing all forced matchings.

Remark 21. Our notation refers to graphs and contours by $\mathcal{G}$ and $\widehat{\mathcal{G}}$ respectively which is the opposite of the notation used in [10].

To fully recover the cluster variables, we define the covering monomial of a contour and combine this with our definition of the weight of a graph obtained from a contour (Definition 15). For a more general definition of the covering monomial, see [7] and [6].

Definition 22 (Covering Monomial)**.** For this definition, we think of every block labeled 3 as two separate blocks labeled 3. Given a contour C, let a_j be the number of blocks labeled j enclosed in C. Let b_j be the number of blocks labeled j adjacent to a forced matching in C. The contour will pass through the middle of a 3-block near the special vertex (see Figure 7 for an example). If the special vertex is kept (i.e. if a is even), let $c_3 = 1$. Otherwise, let $c_3 = 0$. The covering monomial of graph $\mathcal{G}(C)$ is the product

$m(\mathcal{G}(C)) := x_1^{a_1}x_2^{a_2}x_3^{a_3+c_3}x_4^{a_4}x_5^{a_5}$. The covering monomial of graph $\widehat{\mathcal{G}}(C)$ is the product $m(\widehat{\mathcal{G}}(C)) := x_1^{a_1-b_1}x_2^{a_2-b_2}x_3^{a_3-b_3+c_3}x_4^{a_4-b_4}x_5^{a_5-b_5} = \frac{m(\mathcal{G}(C))}{x_1^{b_1}x_2^{b_2}x_3^{b_3}x_4^{b_4}x_5^{b_5}}$.

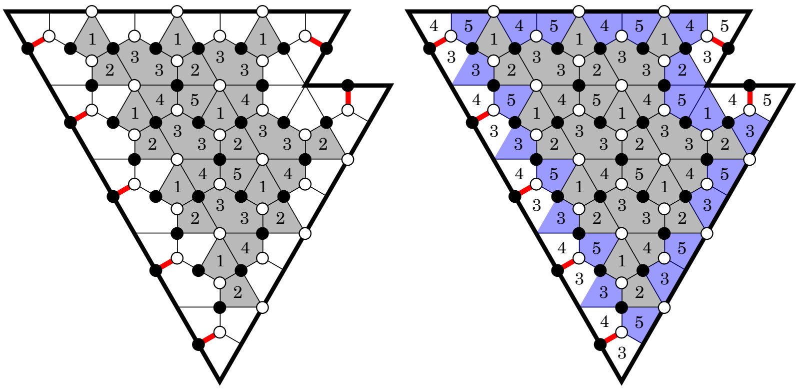

Figure 8: Example of a subgraph $\widehat{\mathcal{G}}(5,-4,1,1,-4)$ and its covering monomial $m(\widehat{\mathcal{G}}(5,-4,1,1,-4))$. The gray and purple blocks are included in the covering monomial.

Remark 23. Our definitions of weight and covering monomial remain unchanged if we think of each six sided 3-block as two four sided blocks without an edge between them. Each 3-block will be drawn as two separate 3-blocks if they appear on the boundary of our contour for the sake of visualizing weight and covering monomial.

For any graph G with an associated contour, denote the product of its weight and its covering monomial by

$$c(G) := w(G)m(G).$$

6.2 Contours of Cluster Variables

By Corollary 4.5, all toric cluster variables are of the form $A^{n^2}B^{n^2-n}x_{2k}$ or $A^{n^2+n}B^{n^2}x_{2k-1}$ where $n,k \in \mathbb{Z}$. Now we state Theorem 24, our main result which gives a formula for the contours of these two families of cluster variables. Theorem 24 is proved in Section 7.

Theorem 24. *For $k \geqslant 2$, we associate the following contours to the toric cluster variables such that if C is the contour associated to a cluster variable, then $c(\widehat{\mathcal{G}}(C))$ equals the Laurent polynomial of that cluster variable.*

$$A^{n^2}B^{n^2-n}x_{2k} = c\left(\widehat{\mathcal{G}}\left(k-2+n, -\left\lceil\frac{k-4+5n}{2}\right\rceil, 2n-1, \left\lfloor\frac{k-3n}{2}\right\rfloor, 1+n-k\right)\right),$$
$$A^{n^2+n}B^{n^2}x_{2k-1} = c\left(\widehat{\mathcal{G}}\left(k-2+n, -\left\lceil\frac{k-2+5n}{2}\right\rceil, 2n, \left\lfloor\frac{k-2-3n}{2}\right\rfloor, 2+n-k\right)\right).$$

Remark 25. Notice that when $n \leqslant 2$, we can reflect the subgraph of $A^pB^qx_{6-n}$ where $p,q \in \mathbb{Z}$ along x_3 which interchanges x_2 with x_4 and x_1 with x_5 in the Laurent polynomial of $A^pB^qx_{6-n}$ since block 2 and block 4, and block 1 and block 5 are symmetric with respect to block 3 in the brane tiling. By Definition 9, interchanging x_2 with x_4 and x_1 with x_5 in the Laurent polynomial (in variables $(x_i)_{i=1}^5$) of x_{6-n} gives x_n but fixes A, B. Hence, interchanging variables, or equivalently, reflecting the subgraph, of $A^pB^qx_{6-n}$ gives the subgraph $A^pB^qx_n$. Therefore, we only need to consider the situation where $k \geqslant 2$.

Figure 9 shows the six possible contour shapes depending on the relationship between n and k.

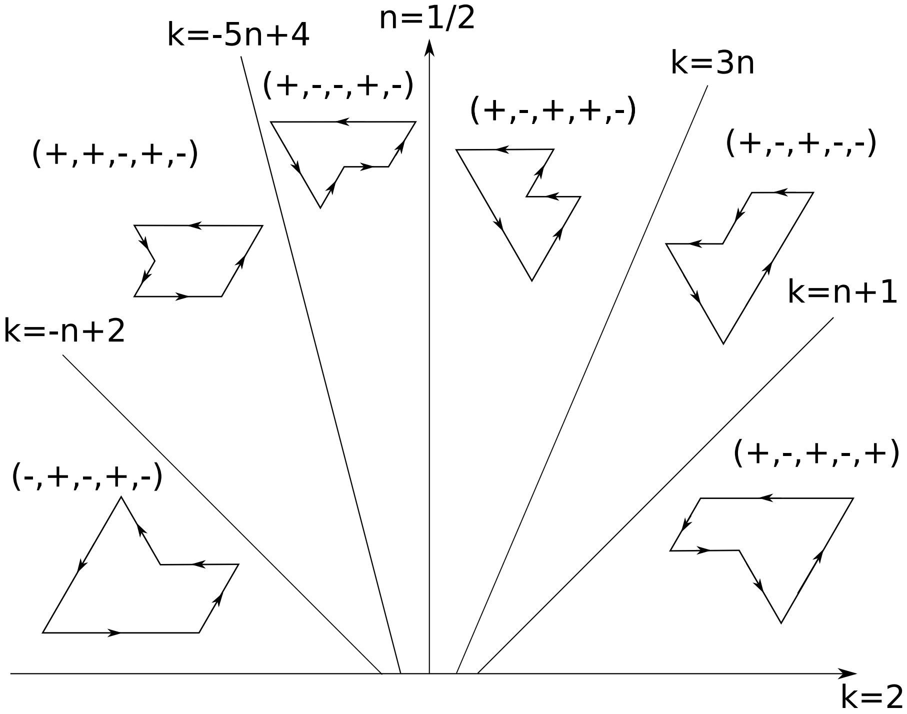

Figure 9: Possible shapes of 5-sided contour

7 Proof of Main Theorem (Theorem 24)

7.1 Overview of induction procedure

We use Kuo's condensation to inductively prove that multiplying the weight and covering monomial of the contours in Theorem 24 yields the Laurent polynomials of the toric cluster variables. First we show that the weights satisfy the desired recurrence. Then we show that for any recurrence of this form, multiplying the weight and covering monomial will give the desired Laurent polynomial. We abuse notation by saying a graph G equals a cluster variable when we mean that the weight $w(G)$, when multiplied by an appropriate covering monomial, will give us the cluster variable's Laurent polynomial.

The base case is $n = 0$ which is proved in Section 7.3. Notice that when $n = 0$, our formula for the contour in the main theorem contains two families: $\{x_{2k-1}\}_{k\geqslant 2}$ and $\{x_{2k}\}_{k\geqslant 2}$.

After proving the base case, we consider the families of variables with $n > 0$ and the families of variables with $n < 0$ separately.

To prove the theorem for $n \geqslant 1$, we assume that the contours for the variables $A^{m^2} B^{m(m-1)} x_{2k}$ and $A^{m(m+1)} B^{m^2} x_{2k-1}$ are correct for all $k \geqslant 2$ and $0 \leqslant m \leqslant n-1$. Then for each $k \geqslant 2$, we consider the following recurrence identity, which is obtained via straightforward algebraic manipulation, as are all other recurrence identities in this section:

$$\begin{aligned}&(A^{n^2} B^{n(n-1)} x_{2k})(A^{(n-1)^2} B^{(n-1)(n-2)} x_{2k+2})\\ =&(A^{(n-1)n} B^{(n-1)^2} x_{2k-1})(A^{(n-1)n} B^{(n-1)^2} x_{2k+3}) + (A^{(n-1)n} B^{(n-1)^2} x_{2k+1})^2.\end{aligned}$$

Among all five terms above, $A^{n^2} B^{n(n-1)} x_{2k}$ is the only term for which we have not proved the theorem. We present a graph G and points p_1, p_2, p_3, p_4 and use a version of Kuo's condensation theorem to prove that the subgraph described in Theorem 24 corresponds to this $A^{n^2} B^{n(n-1)} x_{2k}$.

Once we have the terms $\{A^{n^2} B^{n(n-1)} x_{2k-1}\}$ for $k \geqslant 2$, we consider the following recurrence identity for each $k \geqslant 3$:

$$\begin{aligned}&(A^{n(n+1)} B^{n^2} x_{2k-1})(A^{(n-1)^2+(n-1)} B^{(n-1)^2} x_{2k+1})\\ =&(A^{n^2} B^{n(n-1)} x_{2k-2})(A^{n^2} B^{n(n-1)} x_{2k+2}) + (A^{n^2} B^{n(n-1)} x_{2k})^2.\end{aligned}$$

Similarly, $A^{n(n+1)} B^{n^2} x_{2k-1}$ is the only term for which we have not proved the theorem and we will use a version of Kuo's condensation theorem to do so. Note that the above recurrence cannot be applied to $k = 2$ since we do not associate a contour with $A^{n^2} B^{n(n-1)} x_2$. To solve this problem and to prove the contour for $A^{n(n+1)} B^{n^2} x_3$, we use the following recurrence:

$$\begin{aligned}&(A^{n(n+1)} B^{n^2} x_3)(A^{n^2} B^{n(n-1)} x_8)\\ =&(A^{n(n+1)} B^{n^2} x_5)(A^{n^2} B^{n(n-1)} x_6) + (A^{n(n+1)} B^{n^2} x_7)(A^{n^2} B^{n(n-1)} x_4).\end{aligned}$$

This completes the inductive step.

The prove the theorem for $n \leqslant -1$, the argument is very similar. We assume that we have already proved the contours for variables $A^{m^2} B^{m(m-1)} x_{2k}$ and $A^{m(m+1)} B^{m^2} x_{2k-1}$ for all $n+1 \leqslant m \leqslant 0$. The recurrence

$$\begin{aligned}&(A^{n^2} B^{n(n-1)} x_{2k})(A^{(n+1)^2} B^{n(n+1)} x_{2k+2})\\ =&(A^{n(n+1)} B^{n^2} x_{2k-1})(A^{n(n+1)} B^{n^2} x_{2k+3}) + (A^{n(n+1)} B^{n^2} x_{2k+1})^2\end{aligned}$$

allows us to prove the contours for $A^{n^2} B^{n(n-1)} x_{2k}$ for all $k \geqslant 2$. Then the recurrence

$$\begin{aligned}&(A^{n(n+1)} B^{n^2} x_{2k-1})(A^{(n+1)^2+(n+1)} B^{(n+1)^2} x_{2k+1})\\ =&(A^{(n+1)^2} B^{(n+1)n} x_{2k-2})(A^{(n+1)^2} B^{(n+1)n} x_{2k+2}) + (A^{(n+1)^2} B^{(n+1)n} x_{2k})^2\end{aligned}$$

allows us to prove the contours for $A^{n(n+1)}B^{n^2}x_{2k-1}$ for all $k \geqslant 3$. For variables of the form $A^{n(n+1)}B^{n^2}x_3$, we use the recurrence

$$\begin{aligned}&(A^{n(n+1)}B^{n^2}x_3)(A^{(n+1)^2}B^{n(n+1)}x_8)\\=&(A^{n(n+1)}B^{n^2}x_5)(A^{(n+1)^2}B^{n(n+1)}x_6)+(A^{n(n+1)}B^{n^2}x_7)(A^{(n+1)^2}B^{n(n+1)}x_4).\end{aligned}$$

This completes the inductive step.

Section 7.3 proves the base case and Section 7.4 proves one case of the inductive step. Notice that for the inductive step, we have 28 cases in total and will not present explicit proofs for all cases. The cases are divided by whether side lengths of the contour are greater or smaller than 0 and by some parity conditions on n and k. Section 7.2 gives a summary of the techniques used to prove the remaining cases. The proofs of the remaining cases are analogous to the proof in Section 7.4, and we provide the necessary data for readers to verify the remaining cases in Appendix 9.

7.2 Overview of Proof Techniques

We divide our proof into cases depending on the relations between n and k which lead to different contour shapes. In this section, we give an overview of the proof.

Step 1. Consider a contour $C = (a, b, c, d, e)$ with the special vertex kept or removed and 4 points p_1, p_2, p_3, p_4 inside the contour. The version of Kuo's condensation theorem that we use depends on whether the graph $\mathcal{G}(C)$ is balanced or not, i.e. whether the graph has the same number of black vertices as white vertices, and on the colors and positions of p_1, p_2, p_3, p_4. Kuo's condensation theorem is of the form

$$w\left(\mathcal{G}(C) - S_1\right) w(\mathcal{G}(C)-S_2) = w\left(\mathcal{G}(C) - S_3\right) w(\mathcal{G}(C)-S_4)+w\left(\mathcal{G}(C) - S_5\right) w(\mathcal{G}(C)-S_6),$$

where each S_i is a subset of $\{p_1, p_2, p_3, p_4\}$. Notice that $\mathcal{G}(C) - S_i$ may include many forced matchings. We multiply both sides of the equation by $m(\mathcal{G}(C))^2$, the square of the covering monomial of the graph G. Each term in the equation is then of the form $m(\mathcal{G}(C))w(\mathcal{G}(C) - S_i)$.

Step 2. For each $i = 1, \ldots, 6$, we describe a contour C_i inside C such that $\mathcal{G}\widehat{(C) - S_i} = \widehat{\mathcal{G}}(C_i)$. Recall that $\widehat{G}$ is graph G with all forced matchings removed. We find C_i by first describing points p_1, p_2, p_3, p_4 and how removing each point individually will change the contour C. Then we can aggregate these changes to get the total effect of removing S_i. In general, the additivity of such effects is nontrivial, but it is straightforward to verify for each of our cases.

This is the core step of our proof. The effects of removing each point p_i from $\widehat{\mathcal{G}}(C)$ will be stated and justified through diagrams.

Step 3. Now we want to relate $m(\mathcal{G}(C))w(\mathcal{G}(C) - S_i)$ to $c(\widehat{\mathcal{G}}(C_i))$. By definition, we know that $\mathcal{G}(C_i)$ and $\mathcal{G}(C) - S_i$ only differ by a set of forced matchings of $\mathcal{G}(C) - S_i$ inside contour C and outside contour C_i. Meanwhile, $m(\mathcal{G}(C))$ and $m(\mathcal{G}(C_i))$ differ by a factor of the product of all the blocks (the product of variables corresponding to the blocks) inside C but outside C_i. Each block can be in only one forced matching since otherwise the matching would not be forced. Therefore, the quotient

$$\frac{m(\mathcal{G}(C))w(\mathcal{G}(C)-S_i)}{m(\mathcal{G}(C_i))w(\mathcal{G}(C_i))}$$

is the product of all the blocks inside C and outside C_i that are not adjacent to any forced matchings inside C and outside C_i. Let these blocks form set T_i. We will use $T(S_i)$ and T_i interchangeably. For each case, we explicitly provide $T_1,\ldots,T_6$ for a choice of points p_1,p_2,p_3,p_4 and verify that

$$\left(\prod_{j\in T_1} x_j\right)\left(\prod_{j\in T_2} x_j\right)=\left(\prod_{j\in T_3} x_j\right)\left(\prod_{j\in T_4} x_j\right)=\left(\prod_{j\in T_5} x_j\right)\left(\prod_{j\in T_6} x_j\right). \tag{4}$$

Notice that

$$m(\mathcal{G}(C_i))w(\mathcal{G}(C_i))=m(\widehat{\mathcal{G}}(C_i))w(\widehat{\mathcal{G}}(C_i))=:c(\widehat{\mathcal{G}}(C_i))$$

since by definition, both $m(\mathcal{G}(C_i))/m(\widehat{\mathcal{G}}(C_i))$ and $w(\widehat{\mathcal{G}}(C_i))/w(\mathcal{G}(C_i))$ equal the product of blocks adjacent to the forced matchings of $\mathcal{G}(C_i)$. Combining these arguments, we conclude that

$$c(G_1)c(G_2)=c(G_3)c(G_4)+c(G_5)c(G_6)$$

where $G_i=\mathcal{G}(C_i)$.

This step essentially verifies that the covering monomials match up with the weights used in Kuo's condensation theorems to give the desired Laurent polynomials.

Step 4. By the induction hypothesis, we can identify five of the expressions $c(G_i)$ as the Laurent polynomials of toric cluster variables. Therefore, the sixth expression is the Laurent polynomial of the next cluster variable in the sequence.

We provide the details of these steps in Section 7.3 and Section 7.4.

Definition 26. We establish the following notation used in our proof.

Let $(a,b,c,d,e)-K$ be the contour of side lengths a,b,c,d,e with the special vertex *kept* and $(a,b,c,d,e)-R$ be the contour of side lengths a,b,c,d,e with the special vertex *removed*. We write $\mathcal{G}(a,b,c,d,e,)-K$ (resp. $-R$) to denote the subgraph obtained from contour $(a,b,c,d,e)-K$ (resp. $-R$). The notation is similar for $\widehat{\mathcal{G}}$.

Point p_i is a white (or black) point *on edge* a (or b,c,d,e) if it is one of the white (or black) points on the boundary of $\widehat{\mathcal{G}}(C)$ facing edge a, where C is some contour. This notation follows from [10] and it does not necessarily mean that p_i is on edge a (or b,c,d,e) of the contour.

7.3 Base case ($n = 0$)

When $n = 0$, the cluster variables $A^{n^2}B^{n^2-n}x_{2k}$ and $A^{n^2+n}B^{n^2}x_{2k-1}$ where $n \in \mathbb{Z}, k \in \mathbb{Z}_{\geqslant 0}$ are the terms $\{x_m\}_{m\in\mathbb{Z}}$ of the Somos-5 sequence.

For $1 \leqslant i \leqslant 5$, let C_i be the contour defined in Theorem 24 (and Remark 25) for the initial cluster variable x_i. As examples, the contour for x_1 is $C_1 = (-1, 0, 0, -1, 1)$ and the contour for x_2 is $C_2 = (-1, 1, -1, 0, 0)$, as are shown in Figure 10. We verify the weights and covering monomials of these contours. As shown in Figure 10, the subgraphs for these cluster variables are empty so they have weight 1. Recall that by definition, the covering monomials for C_3 and C_4 have an additional x_3 term. Notice that $c(\widehat{\mathcal{G}}(C_i)) = x_i$ for $1 \leqslant i \leqslant 5$.

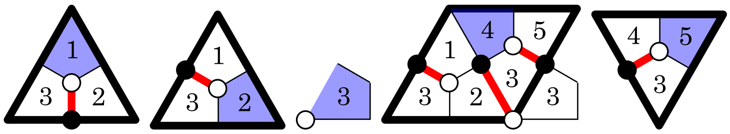

Figure 10: For $1 \leqslant i \leqslant 5$, we give contours C_i for terms x_i of the Somos-5 sequence. The purple blocks remain after multiplying the weights and covering monomials of these graphs.

Now assume the contours for x_i for all $i \leqslant m-1$ give the correct Laurent polynomials for the cluster variables. We show the contour defined in Theorem 24 for x_m is correct.

Case 1: $m = 2k - 1$. Take the following contour

$$C = (a, b, c, d, e) = \left(k-2, -\left\lceil \frac{k-2}{2} \right\rceil, 0, \left\lfloor \frac{k-2}{2} \right\rfloor, 2-k\right).$$

Since $k > 3$, we have $a > 0, b < 0, d > 0$ and $e < 0$.

Let $G = \widehat{\mathcal{G}}(C)$. Then we follow the steps described in Section 7.2.

Step 1. By balanced Kuo's condensation theorem (Lemma 16),

$$\begin{aligned} w(\mathcal{G}(C))w(\mathcal{G}(C) - \{p_1, p_2, p_3, p_4\}) =& w(\mathcal{G}(C) - \{p_1, p_2\})w(\mathcal{G}(C) - \{p_3, p_4\}) \\ &+ w(\mathcal{G}(C) - \{p_1, p_4\})w(\mathcal{G}(C) - \{p_2, p_3\}). \end{aligned}$$

where we let $S_1 = \emptyset$, $S_2 = \{p_1, p_2, p_3, p_4\}$, $S_3 = \{p_1, p_2\}$, $S_4 = \{p_3, p_4\}$, $S_5 = \{p_1, p_4\}$, $S_6 = \{p_2, p_3\}$. Then we multiply both sides by $m(\mathcal{G}(C))^2$.

Step 2. We define the black points p_1, p_3 and white points p_2, p_4 as follows.

- Let p_1 be any black point on edge e.
- Let p_2 be any white point on edge a.
- Let p_3 be any black point on edge b.

- Let p_4 be a white point near edge c defined as follows:
  - If $k \equiv 0 \pmod 2$, then $a \equiv 0 \pmod 2$ so the special vertex is kept. Let p_4 be the kept special white point between edges c and d.
  - If $k \equiv 1 \pmod 2$, then $a \not\equiv 0 \pmod 2$ so the special vertex is removed. Let p_4 be the other white point on the 5-block which contains the removed white point between edges c and d.

We also give the effects of removing each point separately:

- The effect of removing p_1 is $(a, b, c, d, e) \to (a-1, b, c, d-1, e+1)$. We may also write this succinctly as $-\{p_1\} = (-1, 0, 0, -1, 1)$. This effect is equivalent to deleting a trapezoid along edge e of the original contour.
- The effect of removing p_2 is $(a, b, c, d, e) \to (a-1, b+1, c, d, e+1)$. It is equivalent to deleting a trapezoid along edge a.
- The effect of removing p_3 is $(a, b, c, d, e) \to (a-1, b+1, c-1, d, e)$.
- The effect of removing p_4 is $(a, b, c, d, e) - K \to (a, b, c, d, e) - R$ and $(a, b, c, d, e) - R \to (a, b+1, c, d+1, e) - K$ depending on the parity of k.

The position of each point and the effect of removing each point can be seen in Figure 11 where the special point kept and in Figure 12 where special point removed. In the figures, the enlarged red dots indicate point p_i and the red edges indicate forced matchings. The shaded region indicates what is removed from the original contour after deleting the corresponding point. The black letters K/R indicate whether the special point is kept or removed in the original contour and the blue letters indicate this for the new contour.

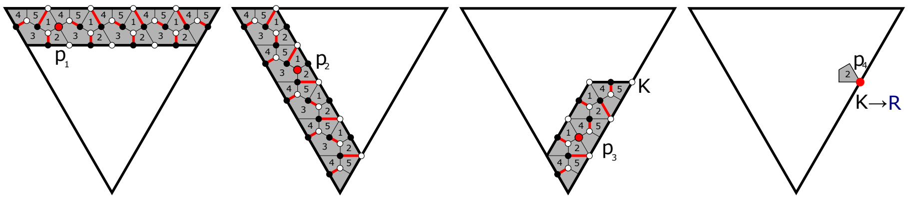

Figure 11: Effects of removing points for x_{2k-1}, k even.

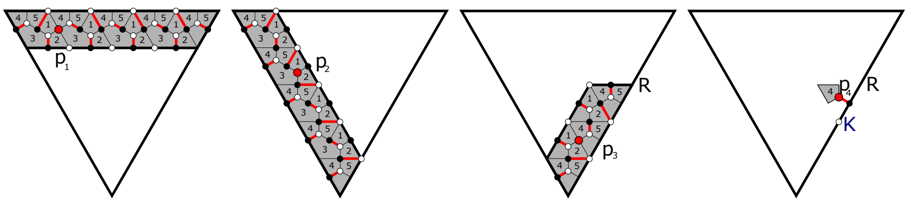

Figure 12: Effects of removing points for x_{2k-1}, k odd.

Below, we explicitly write down the five contours C_i satisfying $\mathcal{G}\widehat{(C) - S_i} = \widehat{\mathcal{G}}(C_i)$ and their corresponding cluster variables. These equivalences follow from the induction hypothesis.

Subcase 1: k is even, i.e. $m \equiv 3 \pmod 4$. We have $C = (a, b, c, d, e) - K$.

$$\begin{aligned}
G - \{\widehat{p_1, p_2,} p_3, p_4\} - K &= \widehat{\mathcal{G}}(a-3, b+2, c-1, d-1, e+2) - R \\
&= \widehat{\mathcal{G}}(C_2), \text{ graph of } x_{2k-6} \\
G - \widehat{\{p_1, p_2\}} - K &= \widehat{\mathcal{G}}(a-2, b+1, c, d-1, e+2) - K \\
&= \widehat{\mathcal{G}}(C_3), \text{ graph of } x_{2k-5} \\
G - \widehat{\{p_3, p_4\}} - K &= \widehat{\mathcal{G}}(a-1, b+1, c-1, d, e) - R \\
&= \widehat{\mathcal{G}}(C_4), \text{ graph of } x_{2k-2} \\
G - \widehat{\{p_2, p_3\}} - K &= \widehat{\mathcal{G}}(a-1, b, c, d-1, e+1) - R \\
&= \widehat{\mathcal{G}}(C_5), \text{ graph of } x_{2k-3} \\
G - \widehat{\{p_1, p_4\}} - K &= \widehat{\mathcal{G}}(a-2, b+2, c-1, d, e+1) - K \\
&= \widehat{\mathcal{G}}(C_6), \text{ graph of } x_{2k-4}
\end{aligned}$$

Subcase 2: k is odd, i.e. $m \equiv 1 \pmod 4$. We have $C = (a, b, c, d, e) - R$.

$$\begin{aligned}
G - \{\widehat{p_1, p_2,} p_3, p_4\} - R &= \widehat{\mathcal{G}}(a-3, b+3, c-1, d, e+2) - K \\
&= \widehat{\mathcal{G}}(C_2), \text{ graph of } x_{2k-6} \\
G - \widehat{\{p_1, p_2\}} - R &= \widehat{\mathcal{G}}(a-2, b+1, c, d-1, e+2) - R \\
&= \widehat{\mathcal{G}}(C_3), \text{ graph of } x_{2k-5} \\
G - \widehat{\{p_3, p_4\}} - R &= \widehat{\mathcal{G}}(a-1, b+2, c-1, d+1, e) - K \\
&= \widehat{\mathcal{G}}(C_4), \text{ graph of } x_{2k-2} \\
G - \widehat{\{p_2, p_3\}} - R &= \widehat{\mathcal{G}}(a-1, b+1, c, d, e+1) - K \\
&= \widehat{\mathcal{G}}(C_5), \text{ graph of } x_{2k-3} \\
G - \widehat{\{p_1, p_4\}} - R &= \widehat{\mathcal{G}}(a-2, b+2, c-1, d, e+1) - R \\
&= \widehat{\mathcal{G}}(C_6), \text{ graph of } x_{2k-4}
\end{aligned}$$

Step 3. Now we specify the sets T_i (defined in Section 7.2) for a specific choice of p_1, p_2, p_3, p_4. We specify the choices of p_i's to explicitly compute T_i.

$G - K$ (Special vertex kept): let p_1 be the rightmost black (denoted by B) point on edge e (not in a forced matching), p_2 be the topmost white (denoted by W) point on edge a (not in a forced matching), p_3 be the bottommost (B) point on edge b (in a forced matching), p_4 be the special vertex. See Figure 13.

$$T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_3x_3x_3x_4, \quad T(\{p_1, p_2\}) = x_3x_4,$$

$$T(\{p_3, p_4\}) = x_3 x_3, \quad T(\{p_2, p_3\}) = x_3 x_3, \quad T(\{p_1, p_4\}) = x_3 x_4.$$

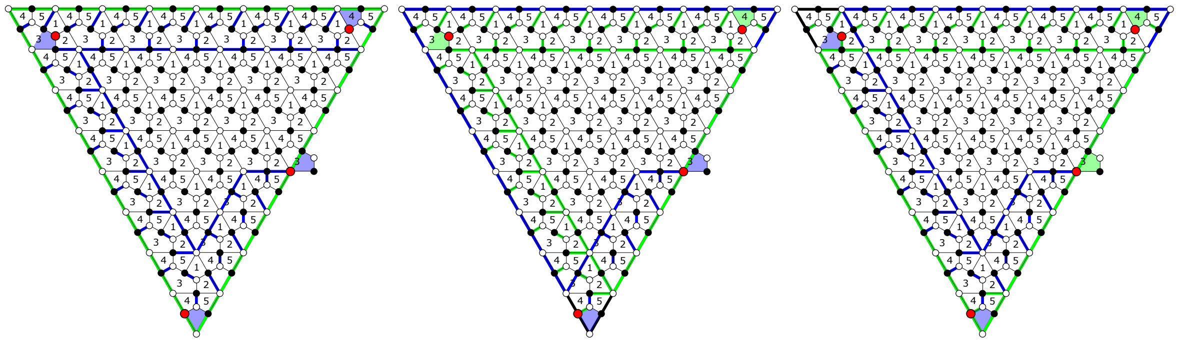

Figure 13: Covering monomial for x_{2k-1}, k even. Left: $T(\emptyset)$ and $T(\{p_1, p_2, p_3, p_4\})$. Middle: $T(\{p_1, p_2\})$ and $T(\{p_3, p_4\})$. Right $T(\{p_2, p_3\})$ and $T(\{p_1, p_4\})$.

$G - R$ (Special vertex removed): let p_1 be the rightmost (B) point on edge e (not in a forced matching), p_2 be the topmost (W) point on edge a (not in a forced matching), p_3 be the bottommost (B) point on edge b (in a forced matching), p_4 be the other white vertex on the 5-block below the special vertex. See Figure 14.

$$T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_3 x_3 x_3 x_4, \quad T(\{p_1, p_2\}) = x_3 x_4,$$
$$T(\{p_3, p_4\}) = x_3 x_5, \quad T(\{p_2, p_3\}) = x_3 x_3, \quad T(\{p_1, p_4\}) = x_4 x_4.$$

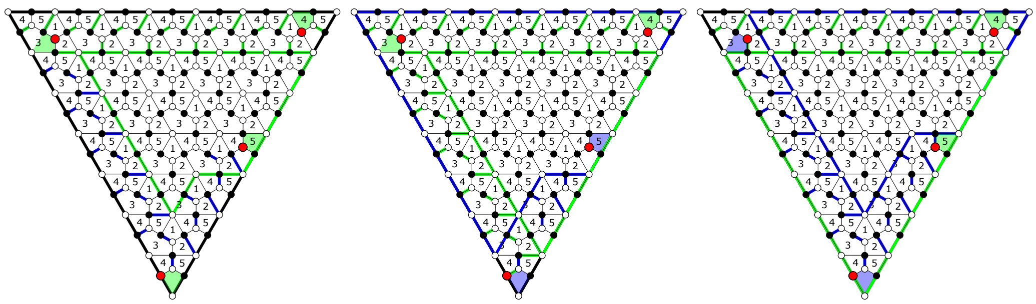

Figure 14: Covering monomial for x_{2k-1}, k even. Left: $T(\emptyset)$ and $T(\{p_1, p_2, p_3, p_4\})$. Middle: $T(\{p_1, p_2\})$ and $T(\{p_3, p_4\})$. Right $T(\{p_2, p_3\})$ and $T(\{p_1, p_4\})$.

We see that equation 4 is satisfied:

$$\left(\prod_{j \in T_1} x_j\right)\left(\prod_{j \in T_2} x_j\right) = \left(\prod_{j \in T_3} x_j\right)\left(\prod_{j \in T_4} x_j\right) = \left(\prod_{j \in T_5} x_j\right)\left(\prod_{j \in T_6} x_j\right).$$

By the Somos-5 recurrence, $x_{2k-1}x_{2k-6} = x_{2k-5}x_{2k-2} + x_{2k-3}x_{2k-4}$. So we conclude that $c(\widehat{\mathcal{G}}(C_1))$ is the Laurent polynomial of x_{2k-1}, as desired.

Case 2: $m = 2k$. Consider the following contour

$$C = (a, b, c, d, e) = \left(k - 2, -\left\lceil \frac{k-4}{2} \right\rceil, -1, \left\lfloor \frac{k}{2} \right\rfloor, 1 - k\right).$$

Since $k \geqslant 3$, we have $a > 0, b \leqslant 0, d > 0$ and $e < 0$. The proof is similar to the proof for Case 1. In Step 1 we again use balanced Kuo's condensation on $\mathcal{G}(C)$ with the same notation for each S_i. In Step 2 we define the four points as follows.

- Let p_1 be any white point on edge a.
- Let p_2 be any black point on edge e.
- Let p_3 be any white point on edge d.
- Let p_4 be a black point near edge c on edge d defined as follows:
  - If $k \equiv 0 \pmod 2$, then $a \equiv 0 \pmod 2$ so the special point is kept. Let p_4 be the black point on the edge between the 4-block and 5-block above the special point.
  - If $k \equiv 1 \pmod 2$, then $a \not\equiv 0 \pmod 2$ so the special point is removed. Let p_4 be the lowest black point on edge d.

We give the effects of removing each point separately:

- The effect of removing p_1 is $(a, b, c, d, e) \to (a - 1, b + 1, c, d, e + 1)$.
- The effect of removing p_2 is $(a, b, c, d, e) \to (a - 1, b, c, d - 1, e + 1)$.
- The effect of removing p_3 is $(a, b, c, d, e) - K \to (a, b - 1, c + 1, d - 1, e + 1) - R$ and $(a, b, c, d, e) - R \to (a, b, c + 1, d, e + 1) - K$ depending on the parity of k.
- The effect of removing p_4 is $(a, b, c, d, e) - K \to (a, b - 1, c, d - 1, e) - R$ and $(a, b, c, d, e) - R \to (a, b, c, d, e) - K$ depending on the parity of k.

The position of each point and the effect of removing each point is shown in Figure 15 (special point kept) and Figure 16 (special point removed).

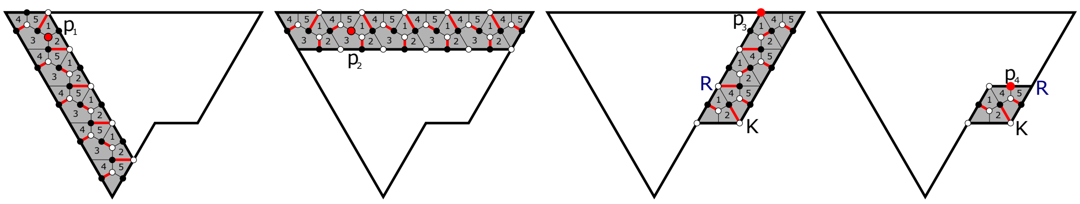

Figure 15: Effects of removing points for x_{2k}, k even.

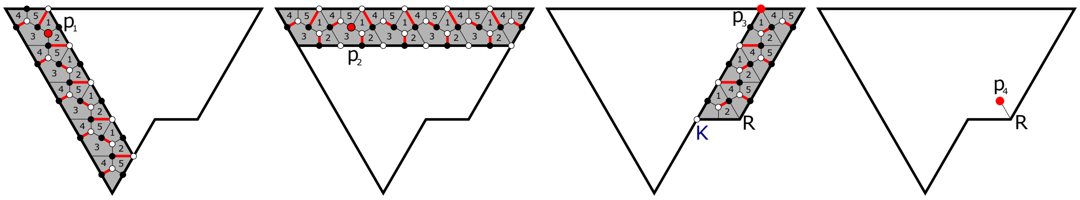

Figure 16: Effects of removing points for x_{2k}, k odd.

Below, we explicitly write down the five contours C_i satisfying $\widehat{\mathcal{G}(C) - S_i} = \widehat{\mathcal{G}}(C_i)$ and their corresponding cluster variables. These equivalences follow from the induction hypothesis.

Subcase 1: k is even, i.e. $m \equiv 0 \pmod 4$. We have $C = (a, b, c, d, e) - K$.

$$\begin{aligned} G - \widehat{\{p_1, p_2, p_3, p_4\}} - K &= \widehat{\mathcal{G}}(a-2, b, c+1, d-2, e+3) - K \\ &= \widehat{\mathcal{G}}(C_2),\ \text{graph of } x_{2k-5} \\ G - \widehat{\{p_1, p_2\}} - K &= \widehat{\mathcal{G}}(a-2, b+1, c, d-1, e+2) - K \\ &= \widehat{\mathcal{G}}(C_3),\ \text{graph of } x_{2k-4} \\ G - \widehat{\{p_3, p_4\}} - K &= \widehat{\mathcal{G}}(a, b-1, c+1, d-1, e+1) - K \\ &= \widehat{\mathcal{G}}(C_4),\ \text{graph of } x_{2k-1} \\ G - \widehat{\{p_2, p_3\}} - K &= \widehat{\mathcal{G}}(a-1, b-1, c+1, d-2, e+2) - R \\ &= \widehat{\mathcal{G}}(C_5),\ \text{graph of } x_{2k-3} \\ G - \widehat{\{p_1, p_4\}} - K &= \widehat{\mathcal{G}}(a-1, b, c, d-1, e+1) - R \\ &= \widehat{\mathcal{G}}(C_6),\ \text{graph of } x_{2k-2} \end{aligned}$$

Subcase 2: k is odd, i.e. $m \equiv 2 \pmod 4$. We have $C = (a, b, c, d, e) - R$.

$$\begin{aligned} G - \widehat{\{p_1, p_2, p_3, p_4\}} - R &= \widehat{\mathcal{G}}(a-2, b, c+1, d-2, e+3) - R \\ &= \widehat{\mathcal{G}}(C_2),\ \text{graph of } x_{2k-5} \\ G - \widehat{\{p_1, p_2\}} - R &= \widehat{\mathcal{G}}(a-2, b+1, c, d-1, e+2) - R \\ &= \widehat{\mathcal{G}}(C_3),\ \text{graph of } x_{2k-4} \\ G - \widehat{\{p_3, p_4\}} - R &= \widehat{\mathcal{G}}(a, b-1, c+1, d-1, e+1) - R \\ &= \widehat{\mathcal{G}}(C_4),\ \text{graph of } x_{2k-1} \\ G - \widehat{\{p_2, p_3\}} - R &= \widehat{\mathcal{G}}(a-1, b, c+1, d-1, e+2) - K \\ &= \widehat{\mathcal{G}}(C_5),\ \text{graph of } x_{2k-3} \\ G - \widehat{\{p_1, p_4\}} - R &= \widehat{\mathcal{G}}(a-1, b+1, c, d, e+1) - K \\ &= \widehat{\mathcal{G}}(C_6),\ \text{graph of } x_{2k-2} \end{aligned}$$

In Step 3 we specify the sets T_i and verify equation 4.

$G-K$ (Special vertex kept): let p_1 be the bottommost (W) point on edge a (not in a forced matching), p_2 be the leftmost (B) point on edge e (not in a forced matching), p_3 be the topmost (W) point on edge d, p_4 be the (B) point on the edge between the 4-block and 5-block above the special vertex. See Figure 17.

$$T(\emptyset)=1, \quad T(\{p_1,p_2,p_3,p_4\})=x_3x_4x_4x_5, \quad T(\{p_1,p_2\})=x_3x_5,$$
$$T(\{p_3,p_4\})=x_4x_4, \quad T(\{p_2,p_3\})=x_4x_5, \quad T(\{p_1,p_4\})=x_3x_4.$$

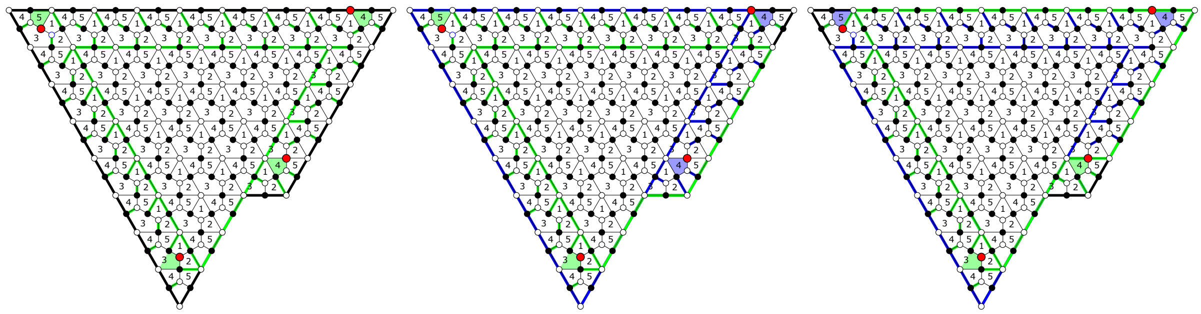

Figure 17: Covering monomial for x_{2k}, k even. Left: $T(\emptyset)$ and $T(\{p_1,p_2,p_3,p_4\})$. Middle: $T(\{p_1,p_2\})$ and $T(\{p_3,p_4\})$. Right $T(\{p_2,p_3\})$ and $T(\{p_1,p_4\})$.

$G-R$ (Special vertex removed): let p_1 be the bottommost (W) point on edge a (not in a forced matching), p_2 be the leftmost (B) point on edge e (not in a forced matching), p_3 be the topmost (W) point on edge d, p_4 be the (B) point on the edge between the 2-block and 3-block above the special vertex.

$$T(\emptyset)=1, \quad T(\{p_1,p_2,p_3,p_4\})=x_2x_3x_4x_5, \quad T(\{p_1,p_2\})=x_3x_5,$$
$$T(\{p_3,p_4\})=x_2x_4, \quad T(\{p_2,p_3\})=x_4x_5, \quad T(\{p_1,p_4\})=x_2x_3.$$

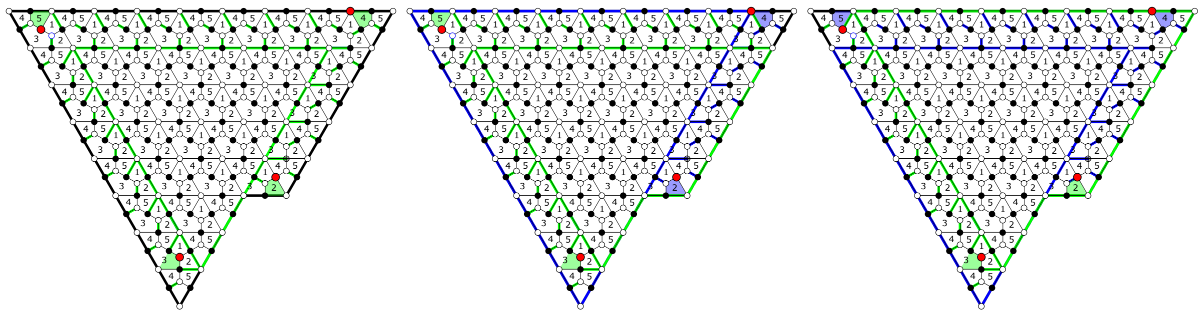

Figure 18: Covering monomial for x_{2k}, k odd. Left: $T(\emptyset)$ and $T(\{p_1,p_2,p_3,p_4\})$. Middle: $T(\{p_1,p_2\})$ and $T(\{p_3,p_4\})$. Right $T(\{p_2,p_3\})$ and $T(\{p_1,p_4\})$.

By the Somos-5 recurrence, $x_{2k-1}x_{2k-6}=x_{2k-5}x_{2k-2}+x_{2k-3}x_{2k-4}$. So we conclude that $c(\widehat{\mathcal{G}}(C_1))$ is the Laurent polynomial of x_{2k-1}, as desired.

7.4 Inductive Step for $A^{n^2}B^{n(n-1)}x_{2k}$, $n \geqslant 1$, $k \geqslant 3n-1$

As explained in Section 7.1, we will only show the inductive step for toric cluster variables of the form $A^{n^2}B^{n(n-1)}x_{2k}$. The proofs for the remaining cases are analogous and we provide the data for their verification in Appendix 9.

Assume the contours of $A^{m^2}B^{m(m-1)}x_{2k}$ and $A^{m(m+1)}B^{m^2}x_{2k+1}$, as defined in Theorem 24, give the correct cluster variables for any $m \leqslant n-1$ and $k \geqslant 3n-1$. Now we want to show that the contour of $A^{m^2}B^{m(m-1)}x_{2k}$ is correct for any $k \geqslant 3n-1$ and $m = n$.

For this case, we use the following recurrence:

$$\begin{aligned} &(A^{n^2}B^{n(n-1)}x_{2k})(A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2}) && (5)\\ =&(A^{(n-1)n}B^{(n-1)^2}x_{2k-1})(A^{(n-1)n}B^{(n-1)^2}x_{2k+3}) + (A^{(n-1)n}B^{(n-1)^2}x_{2k+1})^2 && (6)\end{aligned}$$

By the induction hypothesis, we have the correctness of the contours for each of the cluster variables $A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2}$, $A^{(n-1)n}B^{(n-1)^2}x_{2k-1}$, $A^{(n-1)n}B^{(n-1)^2}x_{2k+1}$ and $A^{(n-1)n}B^{(n-1)^2}x_{2k+3}$.

Let contour C be the following:

$$C = (a,b,c,d,e) = \left(k-1+n, -\left\lceil \frac{k+5n-5}{2} \right\rceil, 2n-2, \left\lfloor \frac{k-3n+3}{2} \right\rfloor, n-k-1\right).$$

Since $k \geqslant 3n-1$, we have $a > 0, b < 0, c \geqslant 0, d > 0, e < 0$. Again, we follow the steps described in Section 7.2. Let $G = \widehat{\mathcal{G}}(C)$.

Step 1: By non-alternating Kuo Condensation theorem (Lemma 18),

$$\begin{aligned} w(G-\{p_1,p_2\})w(G-\{p_3,p_4\}) =&w(G)w(G-\{p_1,p_2,p_3,p_4\})\\ &+ w(G-\{p_1,p_3\})w(G-\{p_2,p_4\}).\end{aligned}$$

where we let $S_1 = \{p_1,p_2\}$, $S_2 = \{p_3,p_4\}$, $S_3 = \emptyset$, $S_4 = \{p_1,p_2,p_3,p_4\}$, $S_5 = \{p_1,p_3\}$, $S_6 = \{p_2,p_4\}$. Then we multiply both sides by $m(\mathcal{G}(C))^2$.

Step 2. We define the four points p_1,p_2,p_3,p_4 on edge d,e,b,c respectively, where p_1 and p_4 are white while p_2 and p_3 are black.

$$\begin{aligned} -\{p_1\} &= \begin{cases} (0,-1,1,-1,1) - R, & \text{if } G = (a,b,c,d,e) - K\\ (0,0,1,0,1) - K, & \text{if } G = (a,b,c,d,e) - R\end{cases}\\ -\{p_2\} &= (-1,0,0,-1,1)\\ -\{p_3\} &= (-1,1,-1,0,0)\\ -\{p_4\} &= \begin{cases} (0,0,0,0,0) - R, & \text{if } G = (a,b,c,d,e) - K\\ (0,1,0,1,0) - K, & \text{if } G = (a,b,c,d,e) - R\end{cases}\end{aligned}$$

The positions of these points and the effects of removing each point individually is shown in Figure 19 (p_1) and Figure 20 (p_2,p_3,p_4) in the case where the special point is kept.

As shown in Figure 19, after we remove p_1, some blocks are removed (grey) while some blocks are appended (pink). This transformation looks unusual but it actually follows the same procedure as the previous ones. After the removal of p_1, some matchings are no longer forced and the forced matchings become different, which results in the addition of pink blocks. Note that the matchings in the pink blocks are all forced, so the subgraph still becomes smaller after the removal. Figure 21 shows the case where the special point is removed.

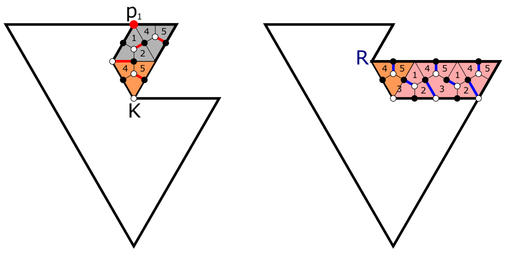

Figure 19: The effect of removing p_1 for $A^{n^2}B^{n(n-1)}$ with $n \geqslant 1$, $k \geqslant 3n-1$ when the special point is kept. Left: before removal. Right: after removal.

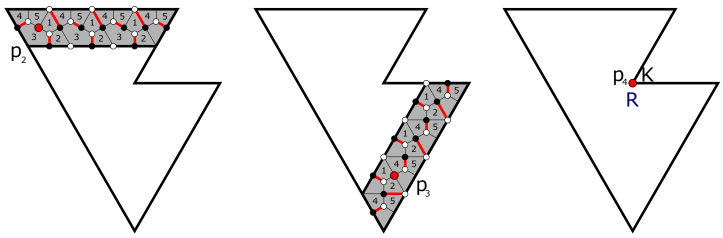

Figure 20: The effects of removing p_2, p_3, p_4 for $A^{n^2}B^{n(n-1)}$ with $n \geqslant 1$, $k \geqslant 3n-1$ when the special point is kept.

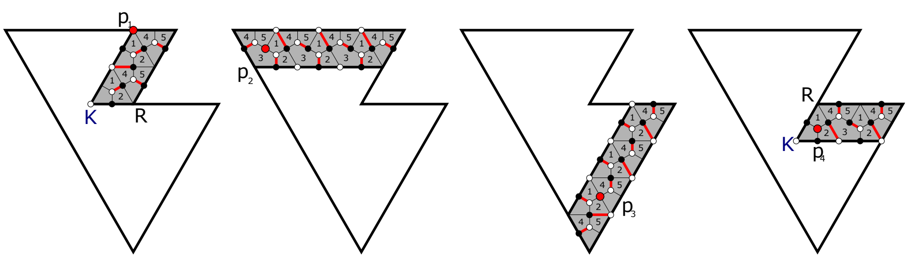

Figure 21: The effects of removing p_i's for $A^{n^2}B^{n(n-1)}$ with $n \geqslant 1$, $k \geqslant 3n-1$ when the special point is removed.

Below, we explicitly write down the contours C_i satisfying $\widehat{\mathcal{G}(C) - S_i} = \widehat{\mathcal{G}}(C_i)$. For the five contours $C_2, \ldots, C_6$, we specify their corresponding cluster variables, which are $A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2}$, $A^{n(n-1)}B^{(n-1)^2}x_{2k+3}$, $A^{n^2-n}B^{(n-1)^2}x_{2k-1}$, $A^{n^2-n}B^{(n-1)^2}x_{2k+1}$

and $A^{n^2-n}B^{(n-1)^2}x_{2k+1}$. These equivalences follow from the induction hypothesis. By recurrence 5, we conclude that $\widehat{\mathcal{G}}(C_1)$ is the graph corresponding to $A^{n^2}B^{n^2-n}x_{2k}$.
Case 1: $n+k$ is odd. The special vertex is kept and $C=(a,b,c,d,e)-K$.

$$\begin{aligned}
G-\widehat{\{p_1,p_2\}} &= \widehat{\mathcal{G}}(a-1,b-1,c+1,d-2,e+2)-R\\
&= \widehat{\mathcal{G}}(C_1)\\
G-\widehat{\{p_3,p_4\}} &= \widehat{\mathcal{G}}(a-1,b+1,c-1,d,e)-R\\
&= \widehat{\mathcal{G}}(C_2),\ \text{graph of } A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2}\\
\widehat{G} &= \widehat{\mathcal{G}}(a,b,c,d,e)-K\\
&= \widehat{\mathcal{G}}(C_3),\ \text{graph of } A^{n(n-1)}B^{(n-1)^2}x_{2k+3}\\
G-\{\widehat{p_1,p_2},p_3,p_4\} &= \widehat{\mathcal{G}}(a-2,b+1,c,d-1,e+2)-K\\
&= \widehat{\mathcal{G}}(C_4),\ \text{graph of } A^{n^2-n}B^{(n-1)^2}x_{2k-1}\\
G-\widehat{\{p_1,p_3\}} &= \widehat{\mathcal{G}}(a-1,b,c,d-1,e+1)-R\\
&= \widehat{\mathcal{G}}(C_5),\ \text{graph of } A^{n^2-n}B^{(n-1)^2}x_{2k+1}\\
G-\widehat{\{p_2,p_4\}} &= \widehat{\mathcal{G}}(a-1,b,c,d-1,e+1)-R\\
&= \widehat{\mathcal{G}}(C_6),\ \text{graph of } A^{n^2-n}B^{(n-1)^2}x_{2k+1}
\end{aligned}$$

Case 2: $n+k$ is even. The special vertex is removed and $C=(a,b,c,d,e)-R$.

$$\begin{aligned}
G-\widehat{\{p_1,p_2\}} &= \widehat{\mathcal{G}}(a-1,b,c+1,d-1,e+2)-K\\
&= \widehat{\mathcal{G}}(C_1)\\
G-\widehat{\{p_3,p_4\}} &= \widehat{\mathcal{G}}(a-1,b+2,c-1,d+1,e)-K\\
&= \widehat{\mathcal{G}}(C_2),\ \text{graph of } A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2}\\
\widehat{G} &= \widehat{\mathcal{G}}(a,b,c,d,e)-R\\
&= \widehat{\mathcal{G}}(C_3),\ \text{graph of } A^{n(n-1)}B^{(n-1)^2}x_{2k+3}\\
G-\{\widehat{p_1,p_2},p_3,p_4\} &= \widehat{\mathcal{G}}(a-2,b+1,c,d-1,e+2)-R\\
&= \widehat{\mathcal{G}}(C_4),\ \text{graph of } A^{n^2-n}B^{(n-1)^2}x_{2k-1}\\
G-\widehat{\{p_1,p_3\}} &= \widehat{\mathcal{G}}(a-1,b+1,c,d,e+1)-K\\
&= \widehat{\mathcal{G}}(C_5),\ \text{graph of } A^{n^2-n}B^{(n-1)^2}x_{2k+1}\\
G-\widehat{\{p_2,p_4\}} &= \widehat{\mathcal{G}}(a-1,b+1,c,d,e+1)-K\\
&= \widehat{\mathcal{G}}(C_6),\ \text{graph of } A^{n^2-n}B^{(n-1)^2}x_{2k+1}
\end{aligned}$$

We now verify the correctness of the covering monomial.

Step 3. We specify the sets T_i for a choice of p_1, p_2, p_3, p_4 and verify equation 4. In each of the diagrams in Figures 22 and 23, p_1, p_2, p_3, p_4 are the red points. Each figure shows the new contours C_i and C_{i+1} in green and in blue. Note that the choices of p_i's are the same as in Step 2.

There is a bijection between perfect matchings of $\mathcal{G}(C_i)$ and perfect matchings of $\mathcal{G}(C) - S_i$. Let M be any perfect matching of $\mathcal{G}(C_i)$. Essentially, the weight of the blocks in T_i is exactly what we need to multiply $m(\mathcal{G}(C_i))w(M)$ by so that it corresponds to a term of $m(\mathcal{G}(C))w(\mathcal{G}(C) - S_i)$.

We explain how the leftmost diagram of Figure 22 shows that the weight of T_1 is $x_3x_3x_4x_4$. Let us start with the green contour C_1. A perfect matching of $\mathcal{G}(C_1)$ corresponds to a perfect matching of $G - \{p_1, p_2\}$ if we remove the red matchings and add in the green matchings. Algebraically, this corresponds to multiplying by the weight of these matchings. The covering monomial of $\mathcal{G}(C_1)$ must be multiplied by the weight of all blocks that are outside the green contour C_1 and within the largest contour C. Note that the weight of these blocks is divided out by many of the green matchings and only the two 4-blocks (green) along edge e and the single 3-block (cyan) near the special vertex (as is defined in Definition 19) remain.

In this particular case, the contour C_1 is not completely contained in C, so we must also divide by the weight of all blocks within C_1 and outside C. Again, note that these weight of these blocks divide out all but one of the red matchings. So overall, the weight of T_1 includes $x_3x_4x_4$ from the covering monomial of C, part of the weight $\frac{1}{x_3x_4x_5}$ of two green matchings (shaded), and the weight x_4x_5 of the single red matching within C_1 (green). So the weight of T_1 is x_4x_4. Similarly, we find the weight of T_2 is x_3x_3 since we simply need to multiply by the weight of blocks outside C_2 within C and the only blocks that are not divided out by forced matchings are the 3-block near the special vertex (cyan) and the 3-block near p_3 (blue).

Case 1: Special vertex kept. See Figure 22.

For this choice of p_1, p_2, p_3, p_4, we have

$$\prod_{j \in T_1} x_j = x_3x_4x_4 \frac{1}{x_3x_4x_5} x_4x_5 = x_4x_4, \qquad \prod_{j \in T_2} x_j = x_3x_3,$$
$$\prod_{j \in T_3} x_j = 1, \qquad \prod_{j \in T_4} x_j = x_3x_3x_4x_4,$$
$$\prod_{j \in T_5} x_j = x_3x_3x_4 \frac{1}{x_3x_4x_5} x_4x_5 = x_3x_4, \qquad \prod_{j \in T_6} x_j = x_3x_4.$$

Case 2: Special vertex removed. See Figure 23.

For this choice of p_1, p_2, p_3, p_4, we have

$$\prod_{j \in T_1} x_j = x_4x_4, \qquad \prod_{j \in T_2} x_j = x_1x_3,$$

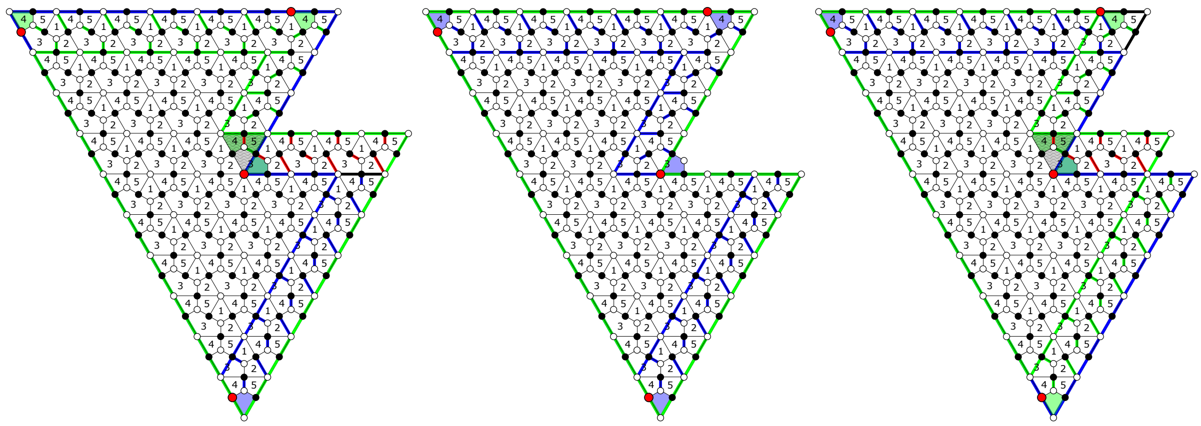

Figure 22: Covering monomial for the case of $A^{n^2}B^{n(n-1)}$ with $n \geqslant 1$, $k \geqslant 3n-1$ and the special point kept. Left: $T(\{p_1,p_2\})$ and $T(\{p_3,p_4\})$. Middle: $T(\{p_1,p_2,p_3,p_4\})$ and $T(\emptyset)$. Right: $T(\{p_1,p_3\})$ and $T(\{p_2,p_4\})$.

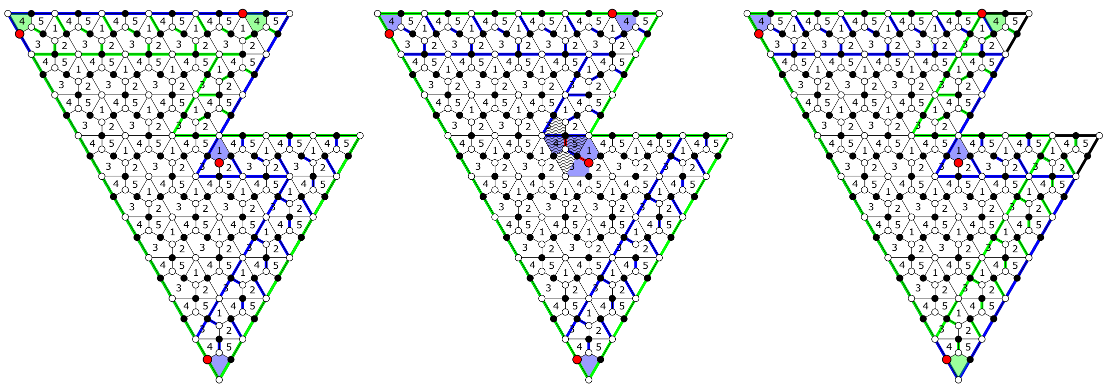

Figure 23: Covering monomial for the case of $A^{n^2}B^{n(n-1)}$ with $n \geqslant 1$, $k \geqslant 3n-1$ and the special point removed. Left: $T(\{p_1,p_2\})$ and $T(\{p_3,p_4\})$. Middle: $T(\{p_1,p_2,p_3,p_4\})$ and $T(\emptyset)$. Right: $T(\{p_1,p_3\})$ and $T(\{p_2,p_4\})$.

$$\prod_{j\in T_3} x_j = 1, \qquad \prod_{j\in T_4} x_j = x_3x_3x_4\frac{x_3\cdot x_1x_3x_4x_5}{x_3x_3x_4x_5} = x_1x_3x_3x_4,$$
$$\prod_{j\in T_5} x_j = x_3x_4, \qquad \prod_{j\in T_6} x_j = x_1x_4.$$

As desired, equation 4 holds in both cases.

Remark 27. As long as we fix the side and the color of a point p_i, the effect of removing p_i is the same regardless of the shape of the contour, i.e. regardless of the signs of the other side lengths. For instance, as shown in Figure 24, the effects of removing p_4 in shapes $(+,-,+,+,-)$ and $(+,-,+,-,+)$ are the same.

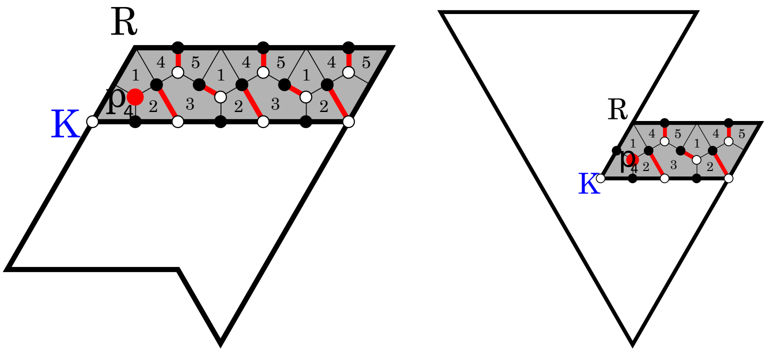

Figure 24: The effects of removing p_4 in shapes $(+,-,+,-,+)$ and $(+,-,+,+,-)$.

8 Comparison with other results

Our paper provides a complete description of the cluster variables generated by toric mutations of the dP_2 quiver through an algebraic formula and a combinatorial interpretation. In this section, we discuss the relation between our results for the dP_2 quiver and two other results: a similar result for the dP_3 quiver and the octahedron recurrence.

8.1 The dP_3 Quiver

The dP_3 quiver and its brane tiling have been studied widely in [2], [16], [11], and [10] using similar techniques.

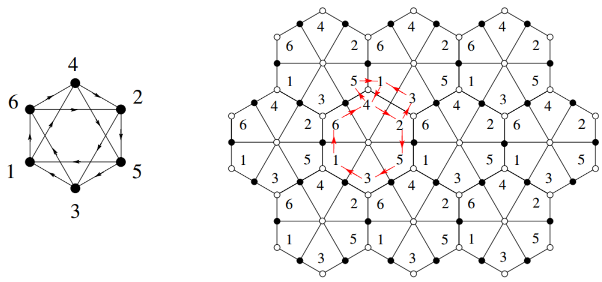

Figure 25: dP_3 quiver and brane tiling [10]

The dP_3 quiver is shown in Figure 25. If we contract the edge between vertex 4 and vertex 2 and remove 2-cycles, we form the dP_2 quiver. Correspondingly, if we merge block 4 and block 2 in the dP_3 brane tiling, we obtain the dP_2 brane tiling. However, this observation does not transfer the association of contours with cluster variables in the dP_3 case to the dP_2 case. In particular, the subgraphs we consider in dP_2 brane tiling significantly differ from those in dP_3 brane tiling. For instance, when side c is long, then there are many forced edges resulting in shapes which do not occur in the dP_3 case. See Figure 26.

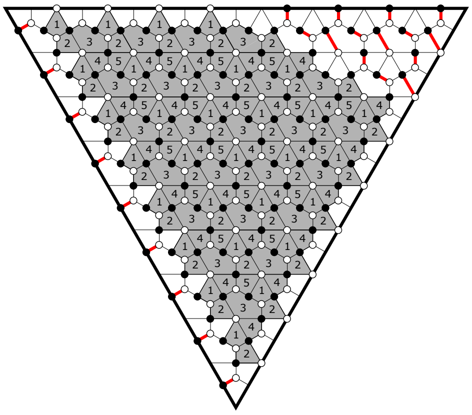

Figure 26: Graph for $A^6B^4x_{17}$. A long edge c results in many forced edges.

8.2 The Octahedron Recurrence

In [15], David Speyer gives a combinatorial interpretation for the Laurent polynomials of the Somos-5 sequence in terms of the weight of subgraphs of a different brane tiling. See Figure 27 for this brane tiling and its corresponding quiver.

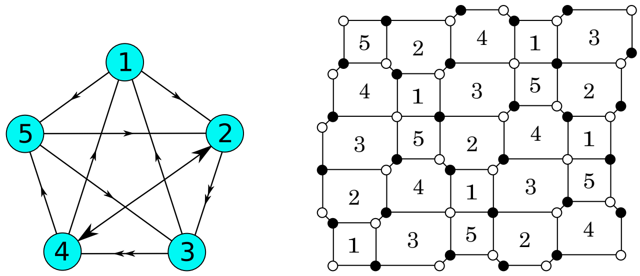

Figure 27: The quiver and the brane tiling studied in [15].

Notice that by adding a 2-cycle between vertex 2 and vertex 4 in our dP_2 quiver, we obtain the quiver studied in Figure 27. However, the two resulting brane tilings are quite different and it is hard to describe a transformation between these tilings in a simple way.

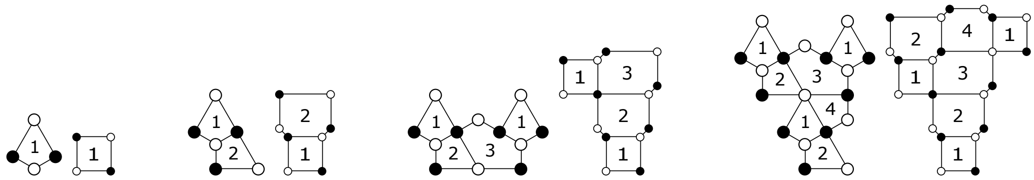

Figure 28: The subgraphs corresponding to terms x_6, x_7, x_8, x_9 in the two different brane tilings.

We provide a few terms of the Laurent polynomial of the Somos-5 sequence written as subgraphs of these two different brane tilings in Figure 28 and Figure 29. The blocks in each pair of subgraphs are similar but not identical. Moreover, the subgraphs corresponding to x_n in the dP_2 brane tiling grow in two different directions (upper right and lower right) depending on the parity of n. On the other hand, the subgraphs considered by Speyer grow in a way that appears to be unrelated to the parity of n. Therefore, we believe that the problem of finding contour families in these two brane tilings are inherently different. There should exist some bijection between these subgraph families (and some bijection between their perfect matchings) since we know how to generate them given x_n. We leave the proving such bijections as an open problem for future research.

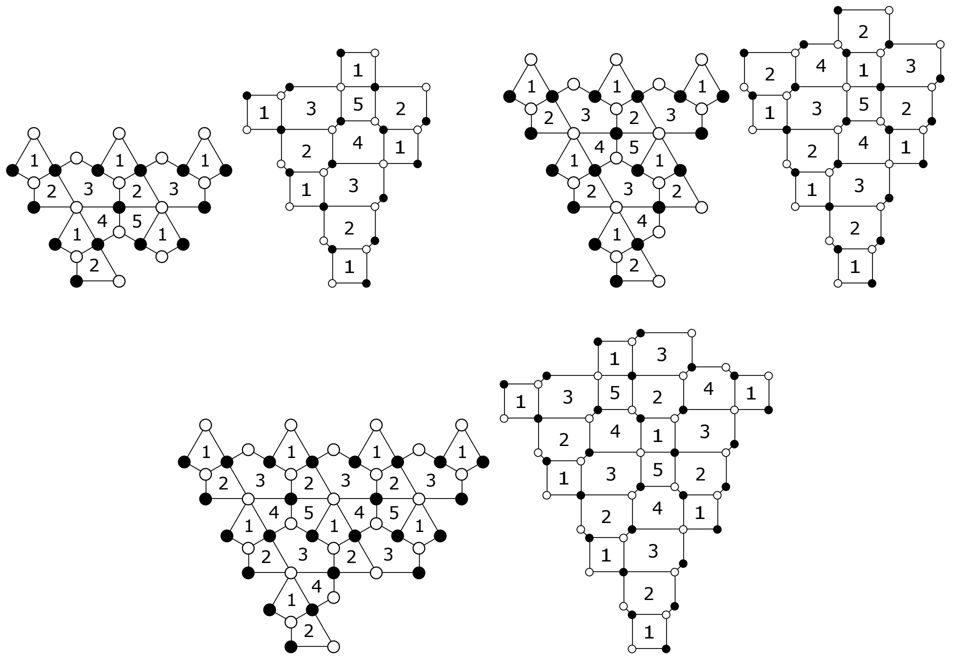

Figure 29: The subgraphs corresponding to terms x_{10}, x_{11}, x_{12} in the two different brane tilings.

Acknowledgements

This research was carried out as part of the 2016 REU program at the School of Mathematics at University of Minnesota, Twin Cities, and was supported by NSF RTG grant DMS-1148634 and by NSF grant DMS-1351590. The authors would like to thank Sunita Chepuri and Elise DelMas for their comments and suggestions. The authors are especially grateful to Gregg Musiker for his mentorship, support, and valuable advice.

References

[1] Chris E. Beasley and M. Ronen Plesser. Toric duality is Seiberg duality. *J. High Energy Phys.*, (12):Paper 1, 38, 2001.

[2] Richard Eager and Sebastián Franco. Colored BPS pyramid partition functions, quivers and cluster transformations. *J. High Energy Phys.*, (9):038, front matter+42, 2012.

[3] Sebastián Franco, Amihay Hanany, David Vegh, Brian Wecht, and Kristian D. Kennaway. Brane dimers and quiver gauge theories. *J. High Energy Phys.*, (1):096, 48, 2006.

[4] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

[5] Amihay Hanany and Rak-Kyeong Seong. Brane tilings and reflexive polygons. *Fortschr. Phys.*, 60(6):695–803, 2012.

[6] In-jee Jeong. Bipartite graphs, quivers, and cluster variables. http://www.math.umn.edu/~reiner/REU/Jeong2011.pdf, 2011.

[7] I Jeong, G Musiker, and S Zhang. Brane tilings and cluster algebras i, 2000.

[8] Eric H. Kuo. Applications of graphical condensation for enumerating matchings and tilings. *Theoret. Comput. Sci.*, 319(1-3):29–57, 2004.

[9] Eric Kuo. Graphical condensation generalizations involving pfaffians and determinants. arXiv:math/0605154, 2006.

[10] Tri Lai and Gregg Musiker. Beyond Aztec castles: toric cascades in the dP_3 quiver. *Comm. Math. Phys.*, 356(3):823–881, 2017.

[11] Megan Leoni, Gregg Musiker, Seth Neel, and Paxton Turner. Aztec castles and the dP3 quiver. *J. Phys. A*, 47(47):474011, 32, 2014.

[12] Kyungyong Lee and Ralf Schiffler. A combinatorial formula for rank 2 cluster variables. *J. Algebraic Combin.*, 37(1):67–85, 2013.

[13] Gregg Musiker and Ralf Schiffler. Cluster expansion formulas and perfect matchings. *J. Algebraic Combin.*, 32(2):187–209, 2010.

[14] Gregg Musiker. A graph theoretic expansion formula for cluster algebras of classical type. *Ann. Comb.*, 15(1):147–184, 2011.

[15] David E. Speyer. Perfect matchings and the octahedron recurrence. *J. Algebraic Combin.*, 25(3):309–348, 2007.

[16] S Zhang. Cluster variables and perfect matchings of subgraphs of the dp3 lattice (2012). http://www.math.umn.edu/~reiner/REU/Zhang2012.pdf, 2012.

9 Appendix

For the remaining cases of Theorem 24, we provide the data necessary to verify their correctness when following the steps in Section 7.1 and Section 7.2. These cases are grouped by the form of their cluster variable.

9.1 $A^{n^2}B^{n(n-1)}x_{2k}$, $n \geqslant 1$, $k \geqslant 2$

The recurrence we use:

$$\begin{aligned}&(A^{n^2}B^{n(n-1)}x_{2k})(A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2})\\=&(A^{(n-1)n}B^{(n-1)^2}x_{2k-1})(A^{(n-1)n}B^{(n-1)^2}x_{2k+3})+(A^{(n-1)n}B^{(n-1)^2}x_{2k+1})^2\end{aligned}$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge d, e, b, c respectively.

9.1.1 Case 1

$k \geqslant 3n-1$. Non-alternating Kuo. Shape $(+,-,+,+,-)$.

$$G := \widehat{\mathcal{G}}\left(k-1+n, -\left\lceil\frac{k+5n-5}{2}\right\rceil, 2n-2, \left\lfloor\frac{k-3n+3}{2}\right\rfloor, n-k-1\right)$$

$$\begin{aligned}-\{p_1\}(W) :&K \to (0,-1,+1,-1,+1)-R\\&R \to (0,0,+1,0,+1)-K\\-\{p_2\}(B) :&\to (-1,0,0,-1,+1)\\-\{p_3\}(B) :&\to (-1,+1,-1,0,0)\\-\{p_4\}(W) :&K \to (0,0,0,0,0)-R\\&R \to (0,+1,0,+1,0)-K\end{aligned}$$

$$\begin{aligned}G =&A^{(n-1)n}B^{(n-1)^2}x_{2k+3}\\G-\{p_1,p_2,p_3,p_4\} =&A^{(n-1)n}B^{(n-1)^2}x_{2k-1}\\G-\{p_1,p_3\} =&A^{(n-1)n}B^{(n-1)^2}x_{2k+1}\\G-\{p_2,p_4\} =&A^{(n-1)n}B^{(n-1)^2}x_{2k+1}\\G-\{p_1,p_2\} =&A^{n^2}B^{n^2-n}x_{2k}\\G-\{p_3,p_4\} =&A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2}\end{aligned}$$

$G-K$ (Special vertex kept): let p_1 be the topmost (W) point on edge d, p_2 be the leftmost (B) point on edge e, p_3 be the bottommost (B) point on edge b, p_4 be the special vertex.

$$\begin{aligned}&T(\emptyset)=1, \quad T(\{p_1,p_2,p_3,p_4\})=x_3x_3x_4x_4, \quad T(\{p_1,p_3\})=x_3x_4,\\&T(\{p_2,p_4\})=x_3x_4, \quad T(\{p_1,p_2\})=x_4x_4, \quad T(\{p_3,p_4\})=x_3x_3.\end{aligned}$$

$G-R$ (Special vertex removed): let p_1 be the topmost (W) point on edge d, p_2 be the leftmost (B) point on edge e, p_3 be the bottommost (B) point on edge b, p_4 be the (W) vertex bordering the 1-block below the special vertex.

$$\begin{aligned}&T(\emptyset)=1, \quad T(\{p_1,p_2,p_3,p_4\})=x_1x_3x_4x_4, \quad T(\{p_1,p_3\})=x_3x_4,\\&T(\{p_2,p_4\})=x_1x_4, \quad T(\{p_1,p_2\})=x_4x_4, \quad T(\{p_3,p_4\})=x_1x_3.\end{aligned}$$

9.1.2 Case 2

$n+1 \leqslant k \leqslant 3n-2$. Unbalanced Kuo. Shape $(+,-,+,-,-)$.

When $(k+2)+(n-1)$ is odd, let

$$G := \widehat{\mathcal{G}}\left(k-1+n, -\left\lceil \frac{k+5n-5}{2} \right\rceil, 2n-2+1, \left\lfloor \frac{k-3n+3}{2} \right\rfloor, n-k-1+1\right) - K$$

When $(k+2)+(n-1)$ is even, let

$$G := \widehat{\mathcal{G}}\left(k-1+n, -\left\lceil \frac{k+5n-5}{2} \right\rceil - 1, 2n-2+1, \left\lfloor \frac{k-3n+3}{2} \right\rfloor - 1, n-k-1+1\right) - R$$

$$\begin{aligned}
-\{p_1\}(B) :& K \to (0,0,-1,0,-1) - R \\
& R \to (0,+1,-1,+1,-1) - K \\
-\{p_2\}(B) :& \to (-1,0,0,-1,+1) \\
-\{p_3\}(B) :& \to (-1,+1,-1,0,0) \\
-\{p_4\}(W) :& K \to (0,0,0,0,0) - R \\
& R \to (0,+1,0,+1,0) - K
\end{aligned}$$

$$\begin{aligned}
G - \{p_1\} =& A^{(n-1)n} B^{(n-1)^2} x_{2k+3} \\
G - \{p_2, p_3, p_4\} =& A^{(n-1)n} B^{(n-1)^2} x_{2k-1} \\
G - \{p_3\} =& A^{(n-1)n} B^{(n-1)^2} x_{2k+1} \\
G - \{p_1, p_2, p_4\} =& A^{(n-1)n} B^{(n-1)^2} x_{2k+1} \\
G - \{p_2\} =& A^{n^2} B^{n^2-n} x_{2k} \\
G - \{p_1, p_3, p_4\} =& A^{(n-1)^2} B^{(n-1)(n-2)} x_{2k+2}
\end{aligned}$$

$G-K$ (Special vertex kept): let p_1 be the bottommost (B) point on edge d, p_2 be the leftmost (B) point on edge e, p_3 be the bottommost (B) point on edge b, p_4 be the special vertex.

$$T(\{p_1\}) = x_3, \quad T(\{p_2,p_3,p_4\}) = x_3x_3x_4, \quad T(\{p_3\}) = x_3,$$
$$T(\{p_1,p_2,p_4\}) = x_3x_3x_4, \quad T(\{p_2\}) = x_4, \quad T(\{p_1,p_3,p_4\}) = x_3x_3x_3.$$

$G-R$ (Special vertex removed): let p_1 be the bottommost (B) point on edge d, p_2 be the leftmost (B) point on edge e, p_3 be the bottommost (B) point on edge b, p_4 be the (W) vertex bordering the 1-block below the special vertex.

$$T(\{p_1\}) = x_4, \quad T(\{p_2,p_3,p_4\}) = x_1x_3x_4, \quad T(\{p_3\}) = x_3x_3,$$
$$T(\{p_1,p_2,p_4\}) = x_3x_3x_3x_4, \quad T(\{p_2\}) = x_4, \quad T(\{p_1,p_3,p_4\}) = x_1x_3x_3.$$

9.1.3 Case 3

$k \leqslant n$. Balanced Kuo. Shape $(+,-,+,-,+)$.

When $(k+2)+(n-1)$ is odd, let

$$G := \widehat{\mathcal{G}}\left(k-1+n-1, -\left\lceil\frac{k+5n-5}{2}\right\rceil, 2n-2+1, \left\lfloor\frac{k-3n+3}{2}\right\rfloor-1, n-k-1+2\right)-K$$

When $(k+2)+(n-1)$ is even, let

$$G := \widehat{\mathcal{G}}\left(k-1+n-1, -\left\lceil\frac{k+5n-5}{2}\right\rceil-1, 2n-2+1, \left\lfloor\frac{k-3n+3}{2}\right\rfloor-2, n-k-1+2\right)-R$$

$$\begin{aligned}
-\{p_1\}(B) :& K \to (0,0,-1,0,-1)-R\\
& R \to (0,+1,-1,+1,-1)-K\\
-\{p_2\}(W) :& \to (+1,0,0,+1,-1)\\
-\{p_3\}(B) :& \to (-1,+1,-1,0,0)\\
-\{p_4\}(W) :& K \to (0,0,0,0,0)-R\\
& R \to (0,+1,0,+1,0)-K
\end{aligned}$$

$$\begin{aligned}
G-\{p_1,p_2\} =& A^{(n-1)n}B^{(n-1)^2}x_{2k+3}\\
G-\{p_3,p_4\} =& A^{(n-1)n}B^{(n-1)^2}x_{2k-1}\\
G-\{p_2,p_3\} =& A^{(n-1)n}B^{(n-1)^2}x_{2k+1}\\
G-\{p_1,p_4\} =& A^{(n-1)n}B^{(n-1)^2}x_{2k+1}\\
G =& A^{n^2}B^{n^2-n}x_{2k}\\
G-\{p_1,p_2,p_3,p_4\} =& A^{(n-1)^2}B^{(n-1)(n-2)}x_{2k+2}
\end{aligned}$$

$G-K$ (Special vertex kept): let p_1 be the second from top (B) point on edge d, p_2 be the second from left (W) point on edge e, p_3 be the second from bottom (B) point on edge b, p_4 be the special vertex.

$$T(\{p_1,p_2\}) = x_1x_3, \quad T(\{p_3,p_4\}) = x_3x_5, \quad T(\{p_2,p_3\}) = x_3x_5,$$
$$T(\{p_1,p_4\}) = x_1x_3, \quad T(\emptyset) = 1, \quad T(\{p_1,p_2,p_3,p_4\}) = x_1x_3x_3x_5.$$

$G-R$ (Special vertex removed): let p_1 be the second from bottom (B) point on edge d, p_2 be the second from right (W) point on edge e, p_3 be the second from bottom (B) point on edge b, p_4 be the (W) vertex bordering the 1-block below the special vertex.

$$T(\{p_1,p_2\}) = x_1x_3, \quad T(\{p_3,p_4\}) = x_1x_5, \quad T(\{p_2,p_3\}) = x_3x_5,$$
$$T(\{p_1,p_4\}) = x_1x_1, \quad T(\emptyset) = 1, \quad T(\{p_1,p_2,p_3,p_4\}) = x_1x_1x_3x_5.$$

9.2 $A^{n^2+n}B^{n^2}x_{2k-1}$, $n \geqslant 1$, $k \geqslant 3$

The recurrence we use:

$$\begin{aligned}
&(A^{n^2+n}B^{n^2}x_{2k-1})(A^{(n-1)^2+(n-1)}B^{(n-1)^2}x_{2k+1})\\
=&(A^{n^2}B^{n(n-1)}x_{2k-2})(A^{n^2}B^{n(n-1)}x_{2k+2}) + (A^{n^2}B^{n(n-1)}x_{2k})^2
\end{aligned}$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge d, e, b, c respectively.

The effect of removing p_1, p_2, p_3, p_4 and the sets used in the proof of covering monomial are the same as in Section 9.1.

9.2.1 Case 1

$k \geqslant 3n+1$. Non-alternating Kuo. Shape $(+,-,+,+,-)$.

$$G := \widehat{\mathcal{G}}\left(k-1+n, -\left\lceil \frac{k+5n-3}{2} \right\rceil, 2n-1, \left\lfloor \frac{k-3n+1}{2} \right\rfloor, n-k\right).$$

$$\begin{aligned}
G =&A^{n^2}B^{n^2-n}x_{2k+2}\\
G-\{p_1,p_2,p_3,p_4\} =&A^{n^2}B^{n^2-n}x_{2k-2}\\
G-\{p_1,p_3\} =&A^{n^2}B^{n^2-n}x_{2k}\\
G-\{p_2,p_4\} =&A^{n^2}B^{n^2-n}x_{2k}\\
G-\{p_1,p_2\} =&A^{n^2+n}B^{n^2}x_{2k-1}\\
G-\{p_3,p_4\} =&A^{(n-1)^2+(n-1)}B^{(n-1)^2}x_{2k+1}
\end{aligned}$$

9.2.2 Case 2

$n+2 \leqslant k \leqslant 3n$. Unbalanced Kuo. Shape $(+,-,+,-,-)$.

When $(k+1)+n$ is odd, let

$$G := \widehat{\mathcal{G}}\left(k-1+n, -\left\lceil \frac{k+5n-3}{2} \right\rceil, 2n-1+1, \left\lfloor \frac{k-3n+1}{2} \right\rfloor, n-k+1\right) - K$$

When $(k+1)+n$ is even, let

$$G := \widehat{\mathcal{G}}\left(k-1+n, -\left\lceil \frac{k+5n-3}{2} \right\rceil - 1, 2n-1+1, \left\lfloor \frac{k-3n+1}{2} \right\rfloor - 1, n-k-1+1\right) - R$$

$$\begin{aligned}
G-\{p_1\} =&A^{n^2}B^{n^2-n}x_{2k+2}\\
G-\{p_2,p_3,p_4\} =&A^{n^2}B^{n^2-n}x_{2k-2}\\
G-\{p_3\} =&A^{n^2}B^{n^2-n}x_{2k}
\end{aligned}$$

$$
\begin{aligned}
G - \{p_1, p_2, p_4\} =& A^{n^2} B^{n^2-n} x_{2k} \\
G - \{p_2\} =& A^{n^2+n} B^{n^2} x_{2k-1} \\
G - \{p_1, p_3, p_4\} =& A^{(n-1)^2+(n-1)} B^{(n-1)^2} x_{2k+1}
\end{aligned}
$$

9.2.3 Case 3

$3 \leqslant k \leqslant n+1$. Balanced Kuo. Shape $(+,-,+,-,+)$.

When $(k+1)+n$ is odd, let

$$G := \widehat{\mathcal{G}}\left(k-1+n-1, -\left\lceil \frac{k+5n-3}{2} \right\rceil, 2n-1+1, \left\lfloor \frac{k-3n+1}{2} \right\rfloor - 1, n-k+2\right) - K$$

When $(k+1)+n$ is even, let

$$G := \widehat{\mathcal{G}}\left(k-1+n-1, -\left\lceil \frac{k+5n-3}{2} \right\rceil - 1, 2n-1+1, \left\lfloor \frac{k-3n+1}{2} \right\rfloor - 2, n-k+2\right) - R$$

$$
\begin{aligned}
G - \{p_1, p_2\} =& A^{n^2} B^{n^2-n} x_{2k+2} \\
G - \{p_3, p_4\} =& A^{n^2} B^{n^2-n} x_{2k-2} \\
G - \{p_2, p_3\} =& A^{n^2} B^{n^2-n} x_{2k} \\
G - \{p_1, p_4\} =& A^{n^2} B^{n^2-n} x_{2k} \\
G =& A^{n^2+n} B^{n^2} x_{2k-1} \\
G - \{p_1, p_2, p_3, p_4\} =& A^{(n-1)^2+(n-1)} B^{(n-1)^2} x_{2k+1}
\end{aligned}
$$

9.3 $A^{n^2} B^{n(n-1)} x_{2k}$, $n \leqslant -1$, $k \geqslant 2$

The recurrence we use:

$$
\begin{aligned}
&(A^{n^2} B^{n(n-1)} x_{2k})(A^{(n+1)^2} B^{n(n+1)} x_{2k+2}) \\
=&(A^{n(n+1)} B^{n^2} x_{2k-1})(A^{n(n+1)} B^{n^2} x_{2k+3}) + (A^{n(n+1)} B^{n^2} x_{2k+1})^2
\end{aligned}
$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge d, a, b, c respectively.

9.3.1 Case 1

$k \geqslant 1-5n$. Non-alternating Kuo. Shape $(+,-,-,+,-)$.

$$G := \widehat{\mathcal{G}}\left(k+n, -\left\lceil \frac{k+5n}{2} \right\rceil, 2n, \left\lfloor \frac{k-3n}{2} \right\rfloor, n-k\right)$$

$$-\{p_1\}(W) : K \to (0, -1, +1, -1, +1) - R$$

$$\begin{aligned}
R &\to (0,0,+1,0,+1) - K\\
-\{p_2\}(W) :&\to (-1,+1,0,0,+1)\\
-\{p_3\}(B) :&\to (-1,+1,-1,0,0)\\
-\{p_4\}(B) :K &\to (0,-1,0,-1,0) - R\\
R &\to (0,0,0,0,0) - K
\end{aligned}$$

$$\begin{aligned}
G =& A^{n^2+n} B^{n^2} x_{2k+3}\\
G - \{p_1, p_2, p_3, p_4\} =& A^{n^2+n} B^{n^2} x_{2k-1}\\
G - \{p_1, p_3\} =& A^{n^2+n} B^{n^2} x_{2k+1}\\
G - \{p_2, p_4\} =& A^{n^2+n} B^{n^2} x_{2k+1}\\
G - \{p_1, p_4\} =& A^{(n+1)^2} B^{(n+1)^2-(n+1)} x_{2k+2}\\
G - \{p_2, p_3\} =& A^{n^2} B^{n^2-n} x_{2k}
\end{aligned}$$

$G - K$ (Special vertex kept): let p_1 be the topmost (W) point on edge d, p_2 be the topmost (W) point on edge a (not in a forced matching), p_3 be the bottommost (B) point on edge b (not in a forced matching), p_4 be the (B) point on the edge between the 4-block and 5-block above the special vertex.

$$\begin{gathered}
T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_4x_4x_5x_5, \quad T(\{p_1, p_3\}) = x_4x_5,\\
T(\{p_2, p_4\}) = x_4x_5, \quad T(\{p_1, p_4\}) = x_4x_4, \quad T(\{p_2, p_3\}) = x_5x_5.
\end{gathered}$$

$G - R$ (Special vertex removed): let p_1 be the topmost (W) point on edge d, p_2 be the topmost (W) point on edge a (not in a forced matching), p_3 be the bottommost (B) point on edge b (in a forced matching), p_4 be the (B) point on the edge between the 2-block and 3-block above the special vertex.

$$\begin{gathered}
T(\emptyset) = 1, \quad T(\{p_1, p_2, p_3, p_4\}) = x_2x_3x_4x_5, \quad T(\{p_1, p_3\}) = x_3x_4,\\
T(\{p_2, p_4\}) = x_2x_5, \quad T(\{p_1, p_4\}) = x_2x_4, \quad T(\{p_2, p_3\}) = x_3x_5.
\end{gathered}$$

9.3.2 Case 2

$2 - n \leqslant k \leqslant -5n$. Unbalanced Kuo. Shape $(+, +, -, +, -)$.

$$G := \widehat{\mathcal{G}}\left(k + n - 1, -\left\lceil \frac{k + 5n}{2} \right\rceil + 1, 2n - 1, \left\lfloor \frac{k - 3n}{2} \right\rfloor, n - k\right)$$

$$\begin{aligned}
-\{p_1\}(W) :K &\to (0,-1,+1,-1,+1) - R\\
R &\to (0,0,+1,0,+1) - K\\
-\{p_2\}(W) :&\to (-1,+1,0,0,+1)
\end{aligned}$$

$$\begin{aligned}
-\{p_3\}(W) &: \to (+1,-1,+1,0,0)\\
-\{p_4\}(B) &: K \to (0,-1,0,-1,0) - R\\
&\quad R \to (0,0,0,0,0) - K
\end{aligned}$$

$$\begin{aligned}
G - \{p_3\} &= A^{n^2+n} B^{n^2} x_{2k+3}\\
G - \{p_1,p_2,p_4\} &= A^{n^2+n} B^{n^2} x_{2k-1}\\
G - \{p_1\} &= A^{n^2+n} B^{n^2} x_{2k+1}\\
G - \{p_2,p_3,p_4\} &= A^{n^2+n} B^{n^2} x_{2k+1}\\
G - \{p_2\} &= A^{n^2} B^{n^2-n} x_{2k}\\
G - \{p_1,p_3,p_4\} &= A^{(n+1)^2} B^{(n+1)^2-(n+1)} x_{2k+2}
\end{aligned}$$

$G-K$ (Special vertex kept): let p_1 be the topmost (W) point on edge d, p_2 be the topmost (W) point on edge a, p_3 be the bottommost (W) point on edge b, p_4 be the (B) point on the edge between the 4-block and 5-block above the special vertex.

$$T(\{p_3\}) = 1, \quad T(\{p_1,p_2,p_4\}) = x_4x_4x_5x_5, \quad T(\{p_1\}) = x_4x_5,$$
$$T(\{p_2,p_3,p_4\}) = x_4x_5, \quad T(\{p_2\}) = x_4x_4, \quad T(\{p_1,p_3,p_4\}) = x_5x_5.$$

$G-R$ (Special vertex removed): let p_1 be the topmost (W) point on edge d, p_2 be the topmost (W) point on edge a, p_3 be the bottommost (B) point on edge b, p_4 be the (B) point on the edge between the 2-block and 3-block above the special vertex.

$$T(\{p_3\}) = x_2, \quad T(\{p_1,p_2,p_4\}) = x_4x_4x_5, \quad T(\{p_1\}) = x_4,$$
$$T(\{p_2,p_3,p_4\}) = x_2x_4x_5, \quad T(\{p_2\}) = x_5, \quad T(\{p_1,p_3,p_4\}) = x_2x_4x_4.$$

9.3.3 Case 3

$2 \leqslant k \leqslant 1-n$. Balanced Kuo. Shape $(-,+,-,+,-)$.

$$G := \widehat{\mathcal{G}}\left(k+n-2, -\left\lceil \frac{k+5n}{2} \right\rceil + 2, 2n-1, \left\lfloor \frac{k-3n}{2} \right\rfloor, n-k+1\right)$$

$$\begin{aligned}
-\{p_1\}(W) &: K \to (0,-1,+1,-1,+1) - R\\
&\quad R \to (0,0,+1,0,+1) - K\\
-\{p_2\}(B) &: \to (+1,-1,0,0,-1)\\
-\{p_3\}(W) &: \to (+1,-1,+1,0,0)\\
-\{p_4\}(B) &: K \to (0,-1,0,-1,0) - R\\
&\quad R \to (0,0,0,0,0) - K
\end{aligned}$$

$$\begin{aligned}
G - \{p_2, p_3\} =& A^{n^2+n} B^{n^2} x_{2k+3}\\
G - \{p_1, p_4\} =& A^{n^2+n} B^{n^2} x_{2k-1}\\
G - \{p_1, p_2\} =& A^{n^2+n} B^{n^2} x_{2k+1}\\
G - \{p_3, p_4\} =& A^{n^2+n} B^{n^2} x_{2k+1}\\
G - \{p_1, p_2, p_3, p_4\} =& A^{(n+1)^2} B^{(n+1)^2-(n+1)} x_{2k+2}\\
G =& A^{n^2} B^{n^2-n} x_{2k}
\end{aligned}$$

$G - K$ (Special vertex kept): let p_1 be the topmost (W) point on edge d, p_2 be the bottommost (B) point on edge a, p_3 be the topmost (W) point on edge b (not in a forced matching), p_4 be the (B) point on the edge between the 4-block and 5-block above the special vertex.

$$T(\{p_2, p_3\}) = x_2 x_2, \quad T(\{p_1, p_4\}) = x_4 x_4, \quad T(\{p_1, p_2\}) = x_2 x_4,$$
$$T(\{p_3, p_4\}) = x_2 x_4, \quad T(\{p_1, p_2, p_3, p_4\}) = x_2 x_2 x_4 x_4, \quad T(\emptyset) = 1.$$

$G - R$ (Special vertex removed): let p_1 be the topmost (W) point on edge d, p_2 be the bottommost (B) point on edge a, p_3 be the topmost (W) point on edge b (not in a forced matching), p_4 be the (B) point on the edge between the 2-block and 3-block above the special vertex.

$$T(\{p_2, p_3\}) = x_2 x_2, \quad T(\{p_1, p_4\}) = x_2 x_4, \quad T(\{p_1, p_2\}) = x_2 x_4,$$
$$T(\{p_3, p_4\}) = x_2 x_2, \quad T(\{p_1, p_2, p_3, p_4\}) = x_2 x_2 x_2 x_4, \quad T(\emptyset) = 1.$$

9.4 $A^{n^2+n} B^{n^2} x_{2k-1}$, $n \leqslant -1$, $k \geqslant 3$

The recurrence we use:

$$\begin{aligned}
&(A^{n^2+n} B^{n^2} x_{2k-1})(A^{(n+1)(n+2)} B^{(n+1)^2} x_{2k+1})\\
=&(A^{(n+1)^2} B^{(n+1)n} x_{2k-2})(A^{(n+1)^2} B^{(n+1)n} x_{2k+2}) + (A^{(n+1)^2} B^{(n+1)n} x_{2k})^2
\end{aligned}$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge d, a, b, c respectively.

The effect of removing p_1, p_2, p_3, p_4 and the sets used in the proof of covering monomial are the same as in Section 9.3.

9.4.1 Case 1

$k \geqslant -1 - 5n$. Non-alternating Kuo. Shape $(+, -, -, +, -)$.

$$G := \widehat{\mathcal{G}}\left(k + n, -\left\lceil \frac{k + 5n + 2}{2} \right\rceil, 2(n + 1) - 1, \left\lfloor \frac{k - 3n - 2}{2} \right\rfloor, 1 + n - k\right)$$

$$
\begin{aligned}
G =& A^{(n+1)^2}B^{(n+1)n}x_{2k+2} \\
G - \{p_1, p_2, p_3, p_4\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k-2} \\
G - \{p_1, p_3\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k} \\
G - \{p_2, p_4\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k} \\
G - \{p_1, p_4\} =& A^{(n+1)^2+(n+1)}B^{(n+1)^2}x_{2k+1} \\
G - \{p_2, p_3\} =& A^{n^2+n}B^{n^2}x_{2k-1}
\end{aligned}
$$

9.4.2 Case 2

$2 - n \leqslant k \leqslant -2 - 5n$. Unbalanced Kuo. Shape $(+, +, -, +, -)$.

$$
G := \widehat{\mathcal{G}}\left(k + n - 1, -\left\lceil \frac{k + 5n + 2}{2} \right\rceil + 1, 2(n + 1) - 1 - 1, \left\lfloor \frac{k - 3n - 2}{2} \right\rfloor, 1 + n - k\right)
$$

$$
\begin{aligned}
G - \{p_3\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k+2} \\
G - \{p_1, p_2, p_4\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k-2} \\
G - \{p_1\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k} \\
G - \{p_2, p_3, p_4\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k} \\
G - \{p_1, p_3, p_4\} =& A^{(n+1)^2+(n+1)}B^{(n+1)^2}x_{2k+1} \\
G - \{p_2\} =& A^{n^2+n}B^{n^2}x_{2k-1}
\end{aligned}
$$

9.4.3 Case 3

$2 \leqslant k \leqslant 1 - n$. Balanced Kuo. Shape $(-, +, -, +, -)$.

$$
G := \widehat{\mathcal{G}}\left(k + n, -\left\lceil \frac{k + 5n + 2}{2} \right\rceil, 2(n + 1) - 1, \left\lfloor \frac{k - 3n - 2}{2} \right\rfloor, 1 + n - k\right)
$$

$$
\begin{aligned}
G - \{p_2, p_3\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k+2} \\
G - \{p_1, p_4\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k-2} \\
G - \{p_1, p_3\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k} \\
G - \{p_2, p_4\} =& A^{(n+1)^2}B^{(n+1)n}x_{2k} \\
G - \{p_1, p_2, p_3, p_4\} =& A^{(n+1)^2+(n+1)}B^{(n+1)^2}x_{2k+1} \\
G =& A^{n^2+n}B^{n^2}x_{2k-1}
\end{aligned}
$$

9.5 $A^{n^2+n}B^{n^2}x_3$, $n \geqslant 1$

The recurrence we use:

$$\begin{aligned}&(A^{n(n+1)}B^{n^2}x_3)(A^{n^2}B^{n(n-1)}x_8)\\=&(A^{n(n+1)}B^{n^2}x_5)(A^{n^2}B^{n(n-1)}x_6)+(A^{n(n+1)}B^{n^2}x_7)(A^{n^2}B^{n(n-1)}x_4).\end{aligned}$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge e, a, c, d respectively.

When $n = 1$: can verify the contour match the graph using Balanced Kuo or just directly verify the matching polynomial.

Let $n \geqslant 2$. Unbalanced Kuo. Shape $(+, -, +, -, +)$.

When $n + 3$ is odd, let

$$G := \widehat{\mathcal{G}}\left(3-2+n, -\left\lceil\frac{3-2+5n}{2}\right\rceil, 2n, \left\lfloor\frac{3-3n-2}{2}\right\rfloor, n-3+2\right) - R$$

When $n + 3$ is even, let

$$G := \widehat{\mathcal{G}}\left(3-2+n, -\left\lceil\frac{3-2+5n}{2}\right\rceil - 1, 2n, \left\lfloor\frac{3-3n-2}{2}\right\rfloor - 1, n-3+2\right) - K$$

$$\begin{aligned}-\{p_1\}(W) :&\to (+1, 0, 0, +1, -1)\\-\{p_2\}(W) :&\to (-1, +1, 0, 0, +1)\\-\{p_3\}(W) :&K \to (0, 0, 0, 0, 0) - R\\&R \to (0, +1, 0, +1, 0) - K\\-\{p_4\}(B) :&K \to (0, 0, -1, 0, -1) - R\\&R \to (0, +1, -1, +1, -1) - K\end{aligned}$$

$$\begin{aligned}G - \{p_1\} =&A^{n^2+n}B^{n^2}x_7\\G - \{p_2, p_3, p_4\} =&A^{n^2}B^{n(n-1)}x_4\\G - \{p_3\} =&A^{n^2+n}B^{n^2}x_5\\G - \{p_1, p_2, p_4\} =&A^{n^2}B^{n(n-1)}x_6\\G - \{p_2\} =&A^{n^2+n}B^{n^2}x_3\\G - \{p_1, p_3, p_4\} =&A^{n^2}B^{n(n-1)}x_8\end{aligned}$$

$G - K$ (Special vertex kept): let p_1 be the leftmost (W) point on edge e (the bottommost point of edge d), p_2 be the bottommost (W) point on edge a (not in a forced matching), p_3 be the special vertex, p_4 be the (B) point on the edge between the 1-block and 4-block below the special vertex.

$$T(\{p_1\}) = x_1, \quad T(\{p_2, p_3, p_4\}) = x_1x_3x_5, \quad T(\{p_3\}) = x_3,$$

$$T(\{p_1,p_2,p_4\}) = x_1x_1x_5, \quad T(\{p_2\}) = x_5, \quad T(\{p_1,p_3,p_4\}) = x_1x_1x_3.$$

$G-R$ (Special vertex removed): let p_1 be the leftmost (W) point on edge e (the bottommost point of edge d), p_2 be the bottommost (W) point on edge a (not in a forced matching), p_3 be the (W) point below the special vertex, p_4 be the (B) point below p_3.

$$T(\{p_1\}) = x_1, \quad T(\{p_2,p_3,p_4\}) = x_1x_3x_5, \quad T(\{p_3\}) = x_1,$$
$$T(\{p_1,p_2,p_4\}) = x_1x_3x_5, \quad T(\{p_2\}) = x_5, \quad T(\{p_1,p_3,p_4\}) = x_1x_1x_3.$$

9.6 $A^{n^2+n}B^{n^2}x_3$, $n \leqslant -1$

The recurrence we use:

$$\begin{aligned}&(A^{n(n+1)}B^{n^2}x_3)(A^{(n+1)^2}B^{n(n+1)}x_8)\\ =&(A^{n(n+1)}B^{n^2}x_5)(A^{(n+1)^2}B^{n(n+1)}x_6) + (A^{n(n+1)}B^{n^2}x_7)(A^{(n+1)^2}B^{n(n+1)}x_4)\end{aligned}$$

Kuo's four points: p_1, p_2, p_3, p_4 are on edge a, e, c, b respectively.
When $n=-1$, can check directly to see contour for Bx_3 is correct.
Let $n \leqslant -2$. Unbalanced Kuo. Shape $(-,+,-,+,-)$.
When $n+3$ is odd, let

$$G := \widehat{\mathcal{G}}\left(3-2+n, -\left\lceil \frac{3-2+5n}{2} \right\rceil + 1, 2n, \left\lfloor \frac{3-3n-2}{2} \right\rfloor + 1, n-3+2\right) - K$$

When $n+3$ is even, let

$$G := \widehat{\mathcal{G}}\left(3-2+n, -\left\lceil \frac{3-2+5n}{2} \right\rceil - 1, 2n, \left\lfloor \frac{3-3n-2}{2} \right\rfloor - 1, n-3+2\right) - R$$

$$\begin{aligned}-\{p_1\}(B) :&\to (+1,-1,0,0,-1)\\ -\{p_2\}(B) :&\to (-1,+1,0,0,+1)\\ -\{p_3\}(B) :&K \to (0,-1,0,-1,0) - R\\ &R \to (0,0,0,0,0) - K\\ -\{p_4\}(W) :&\to (+1,-1,+1,0,0)\end{aligned}$$

$$\begin{aligned}G - \{p_1\} =& A^{n^2+n}B^{n^2}x_7\\ G - \{p_2,p_3,p_4\} =& A^{n^2}B^{n(n-1)}x_4\\ G - \{p_3\} =& A^{n^2+n}B^{n^2}x_5\\ G - \{p_1,p_2,p_4\} =& A^{n^2}B^{n(n-1)}x_6\\ G - \{p_2\} =& A^{n^2+n}B^{n^2}x_3\end{aligned}$$

$$G - \{p_1, p_3, p_4\} = A^{n^2} B^{n(n-1)} x_8$$

$G - K$ (Special vertex kept): let p_1 be the topmost (B) point on edge a (in a forced matching), p_2 be the rightmost (B) point on edge e (in a forced matching), p_3 be the (B) point with 3 neighbors on the 3-block above the special vertex, p_4 be the bottommost (W) point on edge b.

$$T(\{p_1\}) = x_1, \quad T(\{p_2, p_3, p_4\}) = x_2 x_3 x_5, \quad T(\{p_3\}) = x_5,$$
$$T(\{p_1, p_2, p_4\}) = x_1 x_2 x_3, \quad T(\{p_2\}) = x_3, \quad T(\{p_1, p_3, p_4\}) = x_1 x_2 x_5.$$

$G - R$ (Special vertex removed): let p_1 be the topmost (B) point on edge a (in a forced matching), p_2 be the rightmost (B) point on edge e (in a forced matching), p_3 be the (B) point on the edge between the 2-block and 3-block above the special vertex, p_4 be the bottommost (W) point on edge b.

$$T(\{p_1\}) = x_1, \quad T(\{p_2, p_3, p_4\}) = x_2 x_2 x_5, \quad T(\{p_3\}) = x_2,$$
$$T(\{p_1, p_2, p_4\}) = x_1 x_2 x_5, \quad T(\{p_2\}) = x_5, \quad T(\{p_1, p_3, p_4\}) = x_1 x_2 x_2.$$