Extensions of set partitions and permutations

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Abstract

Extensions of a set partition obtained by imposing bounds on the size of the parts is examined. Arithmetical and combinatorial properties of these sequences are established.

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1 Introduction

A *partition* of a set $[n] := \{1, 2, \ldots, n\}$ is a collection of nonempty disjoint subsets, called *blocks*, whose union is $[n]$. The *Stirling numbers of the second kind* ${n \brace k}$ count the number of partitions of $[n]$ into k non-empty blocks. The total number of partitions of $[n]$ is given by the *Bell number*

$$B_n = \sum_{\ell=0}^{n} {n \brace \ell} \text{ for } n > 0 \text{ and } B_0 = 1, \tag{1}$$

starting with 1, 1, 2, 5, 15, 52, 203. In this count, the order of the blocks is not relevant. On the other hand, if the order of the blocks is important, then the total number of partitions of $[n]$ is known as the *Fubini number* F_n. The expression

$$F_n = \sum_{\ell=0}^{n} \ell! \left\{ {n \atop \ell} \right\} \text{ for } n > 0 \text{ and } F_0 = 1, \tag{2}$$

is the analogue of (1), starting at 1, 1, 3, 13, 75, 541, 4683. Additional information about set partitions may be found in [34].

Similar sequences of numbers are obtained by enumerating permutations on n elements with k cycles. For example, the *(unsigned) Stirling numbers of the first kind*, denoted by $\left[{n \atop k} \right]$. The Stirling numbers of both kinds are related by the relation

$$\sum_{\ell=k}^{n} (-1)^{\ell-k} \left\{ {n \atop \ell} \right\} \left[{\ell \atop k} \right] = \delta_{n,k}, \tag{3}$$

where $\delta_{n,k}$ is the Kronecker delta symbol.

The literature contains (at least) two generalizations of these combinatorial sequences. As an example, for $r \in \mathbb{N}$, an *r-partition* of n is one in which the first r elements are in distinct blocks. The *r-Stirling numbers*, denoted by $\left\{ {n \atop k} \right\}_r$, count the r-partitions of $[n+r]$ into $k+r$ blocks and satisfy the recurrence

$$\begin{aligned} \left\{ {n \atop k} \right\}_r &= \left\{ {n \atop k} \right\}_{r-1} - (r-1) \left\{ {n-1 \atop k} \right\}_{r-1}, \quad \text{for } n \in \mathbb{N},\ 0 \leqslant k \leqslant n \text{ and } 1 \leqslant r \leqslant k, \\ \left\{ {n \atop k} \right\}_0 &= \left\{ {n \atop k} \right\}. \end{aligned} \tag{4}$$

These numbers were introduced by Broder [7]. The r-Stirling numbers may be expressed in terms of the classical Stirling numbers by

$$\left\{ {n \atop k} \right\}_r = \sum_{i=0}^{n-r} \binom{n-r}{i} \left\{ {i \atop k-r} \right\} r^{n-r-i}. \tag{5}$$

Mező [37] introduced the r-Bell numbers by

$$B_{n,r} = \sum_{k=0}^{n} \left\{ {n+r \atop k+r} \right\}_r, \tag{6}$$

with $B_{n,0} = B_n$, the Bell numbers in (1). These numbers satisfy the recurrence

$$\begin{aligned} B_{n,r} &= B_{n+1,r-1} - (r-1) B_{n,r-1}, \\ B_{n,0} &= B_n. \end{aligned} \tag{7}$$

Similarly, an *r-permutation*, is one in which the first r elements are in distinct cycles. The number of all r-permutations of $[n+r]$ into $k+r$ cycles are counted by the *r-Stirling numbers of the first kind*, denoted by $\left[{n \atop k} \right]_r$. Other combinatorial objects introduced in this manner include the *r-derangement numbers* [60, 59], the *r-Bell numbers* [37], the *r-Whitney numbers* and their q-analogues [8, 35, 36, 40, 42, 51], the *r-Lah* and *r-Lah-Whitney numbers* [46, 52, 54], the *r-Fubini* and *r-Whitney-Fubini numbers* [14] and the *r-Whitney-Eulerian numbers* [50, 53]. The extension of the results presented here for these other classes is the subject of current work.

These generalizations are known as *incomplete combinatorial structures.* They come from imposing a restriction on the size of the blocks and cycles. If the size of the sub-structure (block, cycles, etc.) is required to be bounded from above, then one speaks of a *restricted combinatorial structure*; the case of a lower bound is named an *associated combinatorial structure.* In the situation where the notion of special elements are included, the letter r is added to the name.

For the convenience of the reader, this section contains the list of the numbers discussed in the present work.

1. The symbol $[n]$ denotes the set $\{1, 2, \dots, n\}$.

2. A *partition* of $[n]$ is a collection of non-empty subsets, called *blocks*, whose union is $[n]$.

3. An *r-partition* of n is a partition of $[n]$ in which the first r elements are placed in distinct blocks. The numbers $1, 2, \dots, r$ are called *special.* A *special block* is one containing a special element.

4. The *Stirling number of the second kind* $\left\{ {n \atop k} \right\}$ counts the number of partitions of $[n]$ into k non-empty subsets (or blocks).

5. The *Bell number* B_n counts the total number of partitions of $[n]$ into non-empty subsets.

6. The *Fubini number* F_n counts the total number of partitions of $[n]$ into non-empty blocks, where the order in which the blocks appear is taken into consideration.

7. The *r-Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}_r$ count the number of r-partitions of $[n+r]$ into $k+r$ blocks.

8. The *r-Bell numbers* $B_{n,r}$ count the total number of r-partitions of $[n+r]$.

9. The *r-Fubini numbers* $F_{n,r}$ count the total number of ordered r-partitions of $[n+r]$.

10. The *restricted Stirling numbers* $\left\{ {n \atop k} \right\}_{\leqslant m}$ count the number of partitions of $[n]$ into k subsets with blocks of size at most m.

11. The *associated Stirling numbers* $\left\{ {n \atop k} \right\}_{\geqslant m}$ count the number of partitions of $[n]$ into k subsets with blocks of size at least m.

12. The *restricted r-Stirling numbers* $\left\{ {n \atop k} \right\}_{\leqslant m,r}$ count the number of r-partitions of $[n+r]$ into $k+r$ subsets with blocks of size at most m.

13. The *associated r-Stirling numbers* $\left\{ {n \atop k} \right\}_{\geqslant m,r}$ count the number of r-partitions of $[n+r]$ into $k+r$ subsets with blocks of size at least m.

14. The *restricted Bell numbers* $B_{n,\leqslant m}$ count the total number of partitions of $[n]$ into non-empty subsets with blocks of size at most m.

15. The *restricted Fubini numbers* $F_{n,\leqslant m}$ count the total number of ordered partitions of $[n]$ into blocks of length at most m.

16. The *associated Fubini numbers* $F_{n,\geqslant m}$ count the total number of ordered partitions of $[n]$ into blocks of length at least m.

17. The *restricted r-Bell number* $B_{n,\leqslant m,r}$ is the number of r-partitions of n, with block size at most m.

18. The *associated r-Bell number* $B_{n,\geqslant m,r}$ is the number of r-partitions of n, with block size at least m.

The work presented here contains combinatorial and arithmetical information on these sequences of numbers. The arithmetical part includes congruences as well as valuations. Recall that, for a prime p and $n \in \mathbb{N}$, the p-adic valuation of n is the highest power of p that divides n. An important tool in the analysis of valuations is Legendre's formula [30] for the valuation of factorials:

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1}, \tag{8}$$

where $s_p(n)$ is the sum of digits of n in its base p representation.

Remark 1. It is often the case that an analytic expression for $\nu_p(a_n)$ is hard to find. In many situations one finds that the valuations are given by a *valuation tree*. This concept is illustrated with $\nu_2(n)$. The vertices of the tree have associated to them a subset of indices, some of these vertices have descendants one level down. The rules are these: start with a root vertex and associate to it the set $\mathbb{N}$. Since the valuation $\{\nu_2(n) : n \in \mathbb{N}\}$ is not constant, this vertex is split onto two new vertices (one per residue class modulo 2), which form the next level. To the first vertex one associates the indices $\{n \in \mathbb{N} : n \equiv 0 \pmod 2\}$ and to the second one $\{n \in \mathbb{N} : n \equiv 1 \pmod 2\}$. Since the valuation $\nu_2(n)$ of indices associated to the second vertex is constant ($\equiv 1$), this vertex is declared *terminal*. The constant value 0 is then attached to this vertex. The first vertex has non-constant valuation, so its indices are split according to its residues modulo 4, into $\{n \equiv 2 \pmod 4\}$ and $\{n \equiv 0 \pmod 4\}$. The process continues: the first vertex has constant valuation 1 and the second is then split modulo 2^3 to continue the process. In the situation where this process does not terminate in a finite number of steps, one says that the valuation admits a tree structure.

Another question of interest in the current work is to examine the periodicity of sequences modulo a number $m \in \mathbb{N}$.

Definition 2. An integer sequence $A = (a_n)_{n\geqslant 0}$ is a *periodic sequence modulo* m, with period t, if there exists $s \geqslant 0$ such that $a_{n+t} \equiv a_n \pmod{m}$, for $n \geqslant s$. The smallest t is called the *minimum period* of A.

Remark 3. Let p be a prime and $\{a_n\}$ an integer sequence of period p. If $a_j \not\equiv 0 \pmod{p}$ for $1 \leqslant j \leqslant p$, then $\nu_p(a_n) \equiv 0$.

Example 4. A direct application of the pigeon-hole principle shows that the *Fibonacci numbers*, defined by the recurrence $f_n = f_{n-1} + f_{n-2}$ and initial conditions $f_1 = f_2 = 1$, is a periodic sequence modulo m, for any $m \in \mathbb{N}$. The minimal period for $m = 5$ is 20.

Among the other questions considered here is the distribution of the last digit of a sequence. The periodicity of the last digit has been studied for several combinatorial sequences. For example, the last digit of the Fibonacci numbers is a periodic sequence of period 60; see [58, 61] for more information.

The plan of the paper is as follows: Section 2 presents properties of the *Bell numbers* B_n which will be extended to other families of numbers. Section 3 considers the *restricted Bell numbers* $B_{n,\leqslant m}$ which count the number of partition of $[n]$ with blocks of size at most m. Section 4 extends the results of the previous section to r-partitions of $[n+r]$ with blocks of length at most m. Recurrences, congruences and divisibility issues are discussed. The distribution of the last digit of $B_{n,\leqslant 2,r}$ and $B_{n,\leqslant 3,r}$ is settled in Section 5. The general problem for $r \geqslant 4$ remains to be determined. The valuations $\nu_2(B_{n,\leqslant m,r})$ are completely determined for $m = 2, 3$ in Section 6. The general case remains open. Similar results for the *associated Bell numbers* $B_{n,\geqslant m,r}$ are presented in Section 7. The question of divisibility of these numbers remains an open question. An extension of these numbers to polynomials is discussed in Section 10. Exponential generating functions for these polynomials are established. Section 8 discusses the *restricted Stirling numbers of the second kind* and Section 9 presents a combinatorial proof of an identity involving these numbers. The *Fubini numbers*, counting partitions of $[n]$ taking into account the order of the participating blocks are discussed in Section 11. Recursions are established as well as the periodicity of the last digit. The divisibility question is presented in detail for the primes 2 and 3. For primes $p \geqslant 5$ experimental results are discussed. The restricted and associated Fubini numbers $F_{n,\leqslant m}$ and $F_{n,\geqslant m}$ are discussed in Section 12. Their arithmetic properties appear in Section 13. These results are extended to the r-Fubini numbers in Section 14. Finally a generalization of factorials is discussed in Section 15.

2 The Bell numbers

The Bell numbers B_n, defined in (1), satisfy the well-known recurrence

$$B_n = \sum_{k=0}^{n-1} \binom{n-1}{k} B_k, \quad \text{for} \quad n \geqslant 1, \tag{9}$$

with initial condition $B_0 = 1$; see [62]. Spivey [55] combined (1) and (9) into

$$B_{n+m} = \sum_{k=0}^{n} \sum_{j=0}^{m} \left\{ {m \atop j} \right\} \binom{n}{k} j^{n-k} B_k, \tag{10}$$

taking 0^0 to be 1.

Congruences for the Bell numbers B_n include one by Touchard [56]:

$$B_{n+p} \equiv B_{n+1} + B_n \pmod{p}.$$

A stronger form of Touchard's congruence is given by Junod [27]:

$$B_{n+sp^m} \equiv \sum_{k=0}^{s} \binom{s}{k} m^{s-k} B_{n+k} \pmod{p}. \tag{11}$$

for all positive integer m. The congruence $B_{np} \equiv B_{n+1} \pmod{p}$ appears in Comtet [12]. This has been generalized in [19] to $B_{np} \equiv B_{n+1} \pmod{p^{\nu+1}}$, where $\nu = \nu_p(n)$.

Hall [23] showed that $\{B_n \pmod{p}\}$ has period $N_p = \frac{p^p-1}{p-1}$. This result was rediscovered by Williams [63], who showed this is the minimum period when $p = 2, 3$ and 5. Radoux [49] conjectured that N_p is always the minimum period. Several authors have established special cases. For example, Montgomery et al. [45] proved it for most primes $p < 180$.

The valuation of B_n are discussed in Amdeberhan et al. [1]. For the prime $p = 2$, it is shown that

$$\nu_2(B_n) = \begin{cases} 0 & \text{if } j \equiv 0,\ 1,\ 3,\ 4,\ 6,\ 7,\ 9,\ 10 \pmod{12}, \\ 1 & \text{if } j \equiv 2,\ 11 \pmod{12}, \\ 2 & \text{if } j \equiv 5,\ 8 \pmod{12}. \end{cases} \tag{12}$$

In the case $p = 3$, experimental data shows that $\nu_3(B_n) = 0$ unless

$$n \equiv \{4,\ 8,\ 9,\ 11,\ 17,\ 21,\ 22,\ 24\} \pmod{26}.$$

Also $\nu_3(B_n) = 1$ if $n \equiv 9,\ 11,\ 22,\ 24 \pmod{26}$ and that $\nu_3(B_n)$ has a *valuation tree structure* for $n \equiv 4,\ 8,\ 17,\ 21 \pmod{26}$. This concept is described in Remark 1.

Remark 5. The last digit is the value $B_n \pmod{10}$. Since $\{B_n \pmod{p}\}$ is a periodic sequence of period $N_p = (p^p - 1)/(p - 1)$, the sequence of last digits is periodic of period $\mathrm{lcm}\{N_2, N_5\} = \mathrm{lcm}\{3, 781\} = 2343$.

3 Restricted Bell Numbers

The Bell numbers B_n considered in Section 2 were extended by Miksa et al. [43] to the family $\{B_{n,\leqslant m}\}$, counting the number of partitions of $[n]$ with blocks of size at most m. These are named the *restricted Bell numbers* (denoted by $G_{n,m}$ in [43]). The value

$B_{n,\leqslant 1} = 1$ is clear: there is a single way to partition $[n]$ into non-empty blocks with at most one element. On the other hand, if $m > n$, then $B_{n,\leqslant m} = B_n$. Miksa et al. established (as Theorem 2) the recurrence

$$B_{n,\leqslant m} = \sum_{j=0}^{m-1} \binom{n-1}{j} B_{n-1-j,\leqslant m}, \tag{13}$$

with initial condition $B_{0,\leqslant m} = 1$ in analogy to (9).

Some congruences appear in [43]. For p prime and $0 \leqslant m < p$,

$$B_{n+p,\leqslant m} \equiv B_{n,\leqslant m} \pmod{p}. \tag{14}$$

This shows that $B_{n,\leqslant p} \pmod{p}$ is a periodic sequence. The particular value $B_{p,\leqslant m} \equiv 1 \pmod{p}$ follows from $B_{0,\leqslant m} = 1$. The explicit expression presented in [43] shows that

$$B_{n,\leqslant m} = \sum_{c_1+2c_2+\cdots+mc_m=n} \frac{n!}{c_1!c_2!\cdots c_m!1!^{c_1}2!^{c_2}\cdots m!^{c_m}},$$

which leads to

$$B_{p^s,\leqslant m} \equiv 1 \pmod{p^s} \quad \text{for } m < p \quad \text{and } s \geqslant 1, \tag{15}$$

generalizing (14).

Remark 6. The periodicity of $B_{n,\leqslant m} \pmod{p}$ shows that the last digit of $B_{n,\leqslant m}$ is also a periodic sequence. For example, for $m = 5$, the sequence $B_{n,\leqslant 5} \pmod{2}$ has period 8 and $B_{n,\leqslant 5} \pmod{5}$ has period 20. Therefore the last digit of $B_{n,\leqslant 5}$ has period $\operatorname{lcm}\{8, 20\} = 40$.

Some analysis of the p-adic valuation of the restricted Bell numbers appears in the literature. Amdeberhan et al. [4] established an expression for the 2-adic valuation of the restricted Bell numbers $B_{n,\leqslant 2}$:

$$\nu_2(B_{n,\leqslant 2}) = \left\lfloor \frac{n}{2} \right\rfloor - 2\left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor = \begin{cases} k, & \text{if } n = 4k; \\ k, & \text{if } n = 4k+1; \\ k+1, & \text{if } n = 4k+2; \\ k+2, & \text{if } n = 4k+3. \end{cases}$$

The sequence $B_{n,\leqslant 2}$ coincides with the number of *involutions* of n elements, denoted in [4] by $\text{Inv}_1(n)$. This sequence is also called *Bessel numbers of the second kind*, see [10] for further information.

The valuation for the prime $p = 3$ is easy to determine using Remark 3.

Proposition 7. *For $n \geqslant 2$, the prime 3 does not divide $B_{n,\leqslant 2}$. Therefore $\nu_3(B_{n,\leqslant 2}) = 0$.*

Proof. Formula (14) shows that $B_{n,\leqslant 2}$ is periodic modulo 3. The result follows from the values $B_{2,\leqslant 2} = 1$, $B_{3,\leqslant 2} = 1$, and $B_{4,\leqslant 2} = 4 \neq 0 \pmod{3}$. □

Remark 8. The discussion of $\nu_5(B_{n,\leqslant 2})$ is used now to introduce the concept of a *valuation tree*. This notion has been mentioned in the context of the valuation $\nu_3(B_n)$ for $n \equiv 4,\ 8,\ 17,\ 21 \pmod{26}$, in Section 2. The statements made here are based on computer experiments. The reader is encouraged to use congruences above to provide rigorous proofs.

Start with a root node representing all positive integers. Then observe that $\nu_5(B_{n,\leqslant 2}) = 0$ if $n \not\equiv 4 \pmod 5$. Construct a new level, called the *first level*, with 5 nodes connected to the root and label them by the residues classes modulo 5. Each node corresponds to a collection of indices. It is denoted by $V_{1,j}$ and it corresponds to the indices

$$V_{1,j} = \{5n + j : \ n \in \mathbb{N}\}, \quad 0 \leqslant j \leqslant 4.$$

The vertex is called *terminal* if the valuation $\nu_5(B_{m,\leqslant 2})$ for every $m \in V_{1,j}$ is independent of m. For example, since $\nu_5(B_{5n,\leqslant 2}) = 0$, the vertex $V_{1,0}$ is terminal. The constant valuation of a terminal vertex V is called *the valuation of the vertex* and is denoted by $\nu_p(V)$, or $\nu_p(V; B_{n,\leqslant 2})$, to mention the sequence under study. For example, $\nu_5(V_{1,0}; B_{n,\leqslant 2}) = 0$. In this example, there are four terminal vertices $V_{1,j} : j = 0,\ 1,\ 2,\ 3$. Each of them has valuation 0. Now construct the *second level* by splitting the indices in $V_{1,4}$ modulo 5^2. This gives the vertices

$$V_{2,j} = \{5(5n + j) + 4 = 5^2 n + 5j + 4 : \ n \in \mathbb{N}\}, \quad 0 \leqslant j \leqslant 4.$$

On this level, there are four terminal vertices, with valuation

$$\nu_5(V_{2,0}; B_{n,\leqslant 2}) = \nu_5(V_{2,1}; B_{n,\leqslant 2}) = \nu_5(V_{2,2}; B_{n,\leqslant 2}) = \nu_5(V_{2,3}; B_{n,\leqslant 2}) = 1. \tag{16}$$

This represents the fact that $\nu_5(B_{5n+4,\leqslant 2}) = 1$ for $n \not\equiv 4 \pmod 5$. Repeating this process and now forming the *third level* gives

$$\nu_5(V_{3,0}; B_{n,\leqslant 2}) = \nu_5(V_{3,1}; B_{n,\leqslant 2}) = \nu_5(V_{3,2}; B_{n,\leqslant 2}) = \nu_5(V_{3,4}; B_{n,\leqslant 2}) = 2. \tag{17}$$

It is conjectured that this process can be continued indefinitely. The resulting tree is called *the valuation tree for the prime* 5 *and the sequence* $B_{n,\leqslant 2}$; or simply, that $\nu_5(B_{n,\leqslant 2})$ has a *valuation tree structure*.

4 Restricted r-Bell Numbers

This section introduces a new extension of the restricted Bell numbers $B_{n,\leqslant m}$ studied in Section 3. Some basic properties of arithmetical and combinatorial properties are presented. The definition employs the notion of r-partition given in (6) of Section 1.

Definition 9. For $n,\ m,\ r \in \mathbb{N}$, the numbers $B_{n,\leqslant m,r}$ count all r-partitions of $[n+r]$ such that each block has size at most m. These numbers are called the *restricted r-numbers*. The elements $1, 2, \ldots, r$ will be called *special elements* and a block of a partition is called *special* if it contains a special element.

Remark 10. The case $r = 0$ yields the restricted Bell numbers

$$B_{n,\leqslant m,0} = B_{n,\leqslant m}, \tag{18}$$

of Section 3 and the limiting case

$$B_{n,\leqslant\infty,r} = B_{n,r}, \tag{19}$$

gives the *r-Bell numbers* in (6). A second special case is $B_{n,\leqslant 1,r} = B_{n,\leqslant 1} = 1$, since the size of each block must be exactly 1 and then the condition on special block is vacuous.

Example 11. $B_{2,\leqslant 2,2} = 8$, with the corresponding partitions being

$$\begin{array}{llll} \{\{\overline{1}\},\{\overline{2}\},\{3\},\{4\}\}, & \{\{\overline{1}\},\{\overline{2}\},\{3,4\}\}, & \{\{\overline{1},3\},\{\overline{2}\},\{4\}\}, & \{\{\overline{1},4\},\{\overline{2}\},\{3\}\}, \\ \{\{\overline{1}\},\{\overline{2},3\},\{4\}\}, & \{\{\overline{1}\},\{\overline{2},4\},\{3\}\}, & \{\{\overline{1},3\},\{\overline{2},4\}\}, & \{\{\overline{1},4\},\{\overline{2},3\}\}. \end{array}$$

The special elements are overlined.

The first result gives a recurrence similar to (13). Observe that in (20), the index r on the left-hand side is reduced by 1 on the right. Iteration of this recurrence takes $B_{n,\leqslant m,r}$ to $B_{n,\leqslant m,0} = B_{n,\leqslant m}$. This is computable using (13).

Theorem 12. *The restricted r-Bell numbers, $B_{n,\leqslant m,r}$, satisfy the recurrence*

$$B_{n,\leqslant m,r} = \sum_{\ell=0}^{m-1} \binom{n}{\ell} B_{n-\ell,\leqslant m,r-1}, \tag{20}$$

for $n \geqslant 1$, $r \geqslant 1$ and $m \geqslant 1$. The initial values are $B_{n,\leqslant m,0} = B_{n,\leqslant m}$ from (13) and $B_{0,\leqslant m,r} = 1$ and $B_{n,\leqslant m,r} = 0$ for $r < 0$ or $n < 0$.

Proof. Suppose the first special block is of size ℓ, where $1 \leqslant \ell \leqslant m$. This block contains the minimal element $\overline{1}$, and the rest of the block is formed by choosing $\ell - 1$ elements, with $0 \leqslant \ell - 1 \leqslant m - 1$. Therefore, the number of r-partitions of $[n + r]$ with exactly ℓ elements in the first block is $\binom{n}{\ell} B_{n-\ell,\leqslant m,r-1}$ for $0 \leqslant \ell \leqslant m-1$. Summing over ℓ completes the proof. ☐

A second recurrence is presented next.

Theorem 13. *The restricted r-Bell numbers, $B_{n,\leqslant m,r}$, satisfy the recurrence*

$$B_{n,\leqslant m,r} = r \sum_{\ell=0}^{m-2} \binom{n-1}{\ell} B_{n-1-\ell,\leqslant m,r-1} + \sum_{\ell=0}^{m-1} \binom{n-1}{\ell} B_{n-1-\ell,\leqslant m,r}, \tag{21}$$

for $n \geqslant 1$, $r \geqslant 0$ and $m \geqslant 1$. The initial values are the same as in Theorem 12.

Proof. Let i be the size of the block containing the last element, namely $n+r$. Then $1 \leqslant i \leqslant m$. If this block is special, there are r ways to choose the special element in this block. The remaining ℓ non-special elements, with $0 \leqslant \ell \leqslant m-2$, can be chosen in $\binom{n-1}{\ell}$ ways. This corresponds to the first sum in (21). The second sum appears for the non-special elements. □

The recurrence relations are now used to establish some congruences of restricted r-Bell numbers. The proof uses an elementary congruence established below.

Lemma 14. *Let n, k, p, s be non-negative integers and p a prime with $0 \leqslant k < p$. Then, for each $s \in \mathbb{N}$,*

$$\binom{n+p^s}{k} \equiv \binom{n}{k} \pmod{p^s}.$$

Proof. The binomial theorem gives $(1+x)^{n+p^s} = \sum_{k=0}^{n+p^s} \binom{n+p^s}{k} x^k$. On the other hand, if $p \nmid \ell$ then $\binom{p^s}{\ell} \equiv 0 \pmod{p^s}$, so $(1+x)^{p^s} \equiv \sum_{\substack{\ell=0 \\ p|\ell}}^{p^s} \binom{p^s}{\ell} x^\ell \pmod{p^s}$. Therefore

$$\begin{aligned}(1+x)^{n+p^s} = (1+x)^n(1+x)^{p^s} &\equiv (1+x)^n \sum_{\substack{\ell=0 \\ p|\ell}}^{p^s} \binom{p^s}{\ell} x^\ell \pmod{p^s} \\ &= \sum_{k=0}^{n} \binom{n}{k} x^k + (1+x)^n \sum_{\substack{\ell=1 \\ p|\ell}}^{p^s} \binom{p^s}{\ell} x^\ell \pmod{p^s}.\end{aligned}$$

The congruence now comes by matching the corresponding coefficients. □

The next result is preliminary for a further generalization of (14) stated in Theorem 16.

Lemma 15. *Let p be a prime, $s \in \mathbb{N}$ and $m < p$. Then $B_{p^s, \leqslant m, r} \equiv 1 \pmod{p^s}$.*

Proof. Theorem 12 gives

$$B_{p^s,\leqslant m,r} = \sum_{\ell=0}^{m-1} \binom{p^s}{\ell} B_{p^s-\ell,\leqslant m,r-1} \equiv B_{p^s,\leqslant m,r-1} \pmod{p^s}.$$

Iteration and (15) produce the result. □

The next statement establishes the periodicity of the restricted Bell numbers modulo a power of a prime.

Theorem 16. *Let $n, r, s \in \mathbb{N}$, p a prime and $1 \leqslant m < p$. As a function of $n \in \mathbb{N}$, the restricted r-Bell numbers, $B_{n,\leqslant m,r}$, is a periodic sequence modulo p^s, with period p^s. That is*

$$B_{n+p^s,\leqslant m,r} \equiv B_{n,\leqslant m,r} \pmod{p^s}.$$

Proof. Proceed by induction on n. Lemma 15 gives $n = 0$. Theorem 13 and Lemma 14 give

$$\begin{aligned} B_{n+1+p^s,\leqslant m,r} &= r \sum_{\ell=0}^{m-2} \binom{n+p^s}{\ell} B_{n+p^s-\ell,\leqslant m,r-1} + \sum_{\ell=0}^{m-1} \binom{n+p^s}{\ell} B_{n+p^s-\ell,\leqslant m,r} \\ &\equiv r \sum_{\ell=0}^{m-2} \binom{n}{\ell} B_{n-\ell,\leqslant m,r-1} + \sum_{\ell=0}^{m-1} \binom{n}{\ell} B_{n-\ell,\leqslant m,r} \pmod{p^s} \\ &= B_{n+1,\leqslant m,r}. \end{aligned}$$

The proof is complete. ☐

Remark 17. The sequence $\{B_{n,\leqslant m,r}\} \pmod p$ is periodic modulo p. The result of Hall [23] for the periods of the Bell numbers modulo p cited in Section 2 has been extended by Mező and Ramírez [41] to the r-Bell numbers.

5 The last digit of the restricted r-Bell numbers.

Given $x \in \mathbb{N}$, the value $x \pmod{10}$ is the *last digit* of x. This section discusses properties of the last digit of the restricted r-Bell numbers, $B_{n,\leqslant r,m}$. The proofs use the congruence in Theorem 16.

The discussion starts with $\{B_{n,\leqslant 2,r}\}$ for $r = 1, 2, 3$. Figure 1 shows the first 100 values of the last digit in $\{B_{n,\leqslant 2,1}\}$. The data suggests that this is a periodic sequence of period 5.

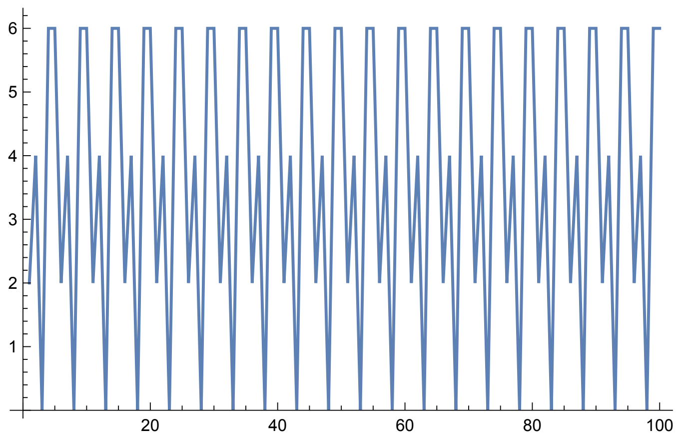

Figure 1: The last digit of the sequences $\{B_{n,\leqslant 2,r}\}$, for $r = 1$.

Theorem 18. *Fix $r \in \mathbb{N}$. The last digit of the sequence $\{B_{n,\leqslant 2,r}\}$ is a periodic sequence of period 5; that is, for any $n \in \mathbb{N}$*

$$B_{n+5,\leqslant 2,r} \equiv B_{n,\leqslant 2,r} \pmod{10} \quad \textit{for } n \geqslant 2.$$

The same statement holds for $\{B_{n,\leqslant 3,r}\}$, for $n \geqslant 4$.

Proof. Theorem 16 with $p = 5$ gives $B_{n+5,\leqslant 2,r} \equiv B_{n,\leqslant 2,r} \pmod 5$. Theorem 13 and the corresponding initial values, show that $B_{n,\leqslant 2,r}$ is an even number. Therefore the congruence extends from modulo 5 to modulo 10. The proof for $B_{n,\leqslant 3,r}$ is similar. □

Table 1 shows the fundamental period for the last digit of the sequences $\{B_{n,\leqslant m,r}\}$ for $m = 2, 3$ and $r = 0, 1, \ldots, 7$.

r	$B_{n,\leqslant 2,r}$	$B_{n,\leqslant 3,r}$
0	$2,4,0,6,6$	4, 6, 6, 2, 0
1	$2,4,0,6,6$	4, 6, 6, 2, 0
2	$8,2,6,6,8$	2, 6, 8, 0, 0
3	$4,4,6,4,6$	4, 2, 6, 4, 2
4	$2,8,8,6,0$	0, 6, 0, 6, 2
5	$6,2,4,0,6$	0, 4, 6, 6, 2
6	$4,0,6,6,2$	6, 6, 2, 0, 4
7	$8,8,2,6,6$	0, 2, 6, 8, 0

Table 1: The fundamental period for the last digit.

The congruences for $B_{n+\ell,\leqslant 2,r}$ for $0 \leqslant \ell \leqslant 4$ are presented next.

Corollary 19. *Let $r \geqslant 0$. Then*

$$B_{n+\ell,\leqslant 2,r} \equiv B_{n,\leqslant 2,r} + a_2(n,\ell) \pmod{10} \quad \textit{for } n \geqslant 2,$$
$$B_{n+\ell,\leqslant 3,r} \equiv B_{n,\leqslant 3,r} + a_3(n,\ell) \pmod{10} \quad \textit{for } n \geqslant 4,$$

where $a_s(5n+i,\ell) \equiv B_{i+\ell,\leqslant s,r} - B_{i,\leqslant s,r} \pmod{10}$, for $i \in \{0,1,2,3,4\}$ and $s = 2, 3$.

For example, if $r = 0$ and $\ell = 1, 3$, the congruences modulo 10 are

$$B_{n+1,\leqslant 2} \equiv B_{n,\leqslant 2} + \begin{cases} 0, & n \equiv 0 \pmod{10}; \\ 6, & n \equiv 1 \pmod{10}; \\ 2, & n \equiv 2 \pmod{10}; \\ 6, & n \equiv 3 \pmod{10}; \\ 6, & n \equiv 4 \pmod{10}. \end{cases} \quad B_{n+3,\leqslant 2} \equiv B_{n,\leqslant 2} + \begin{cases} 8, & n \equiv 0 \pmod{10}; \\ 4, & n \equiv 1 \pmod{10}; \\ 4, & n \equiv 2 \pmod{10}; \\ 2, & n \equiv 3 \pmod{10}; \\ 2, & n \equiv 4 \pmod{10}. \end{cases}$$

Experimental evidence leads to the next conjectures.

Conjecture 20. For any $r \geqslant 0$

$$B_{n+20,\leqslant 4,r} \equiv B_{n,\leqslant 4,r} \pmod{10} \quad (n \geqslant 0);$$
$$B_{n+40,\leqslant 5,r} \equiv B_{n,\leqslant 5,r} \pmod{10} \quad (n \geqslant 0);$$
$$B_{n+200,\leqslant 6,r} \equiv B_{n,\leqslant 6,r} \pmod{10} \quad (n \geqslant 0).$$

Figure 2 shows the first 200 values of the last digit of the sequences $\{B_{n,\leqslant 4,0}\} = \{B_{n,\leqslant 4}\}$. The data suggests that this sequence has period 20.

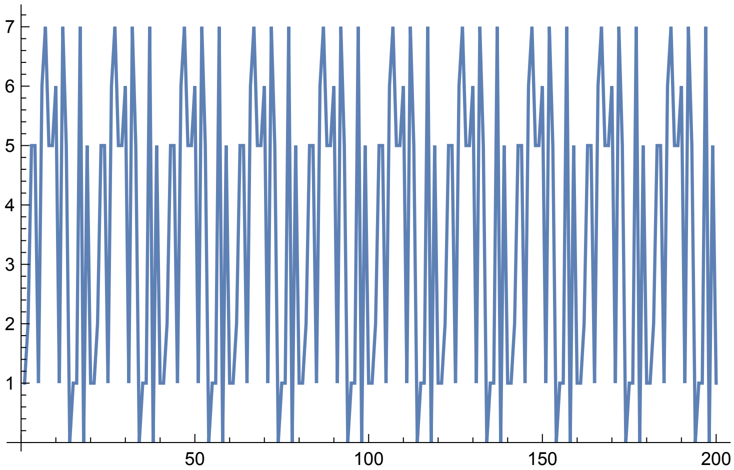

Figure 2: The last digit of the sequences $\{B_{n,\leqslant 4}\}$.

Conjecture 21. The sequence of the last digit of the restricted r-Bell numbers is a periodic sequence.

6 p-adic valuations of restricted r-Bell numbers

Given a sequence $\{a_n\}$ of positive integers and a prime p, the sequence $\{\nu_p(a_n)\}$ of p-adic valuations offers interesting challenges. Interesting examples include the 2-adic valuation of Stirling numbers of the second kind [3, 15, 24, 31], the 2-adic valuation of a sequence of integers appearing in the evaluation of a definite integral [33] and also the 2-adic valuation of domino tilings [11].

Amdeberhan et al. [4] established an expression for the 2-adic valuation of the restricted Bell numbers $B_{n,\leqslant 2}$:

$$\nu_2(B_{n,\leqslant 2}) = \left\lfloor \frac{n}{2} \right\rfloor - 2\left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n+1}{4} \right\rfloor = \begin{cases} k, & \text{if } n = 4k; \\ k, & \text{if } n = 4k+1; \\ k+1, & \text{if } n = 4k+2; \\ k+2, & \text{if } n = 4k+3. \end{cases}$$

The sequence $B_{n,\leqslant 2}$ coincides with the number of *involutions* of n elements, denoted in [4] by $\mathrm{Inv}_1(n)$. This sequence is also called *Bessel numbers of the second kind*, see [10] for further information.

The 2-adic valuation of the restricted r-Bell numbers $B_{n,\leqslant 2,r}$ follows a similar pattern. Figure 3 shows the first few values of $\nu_2(B_{n,\leqslant 2,6})$. Jung et al. [26] described $\nu_2(B_{n,\leqslant 2,r})$. The general formula is divided into many cases. For example, if $r \equiv 0 \pmod 4$, then

$$\nu_2(B_{n,\leqslant 2,r}) = \begin{cases} k, & \text{if } n = 4k; \\ k, & \text{if } n = 4k+1; \\ k+1, & \text{if } n = 4k+2; \\ \alpha, & \text{if } n = 4k+3; \end{cases}$$

where $\alpha \geqslant k+2$.

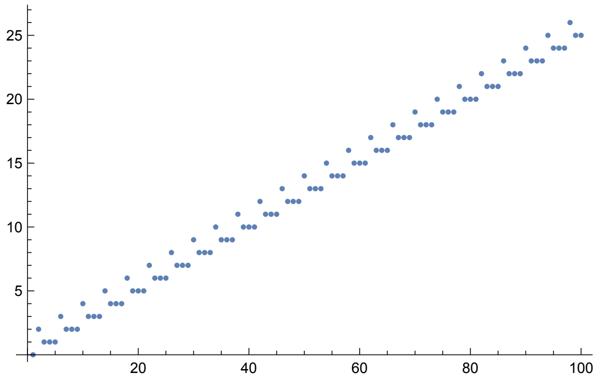

Figure 3: The 2-adic valuation of $B_{n,\leqslant 2,r}$.

The reader is invited to verify that, in the case $n \equiv 3 \pmod 4$, the valuation $\nu_2(B_{n,\leqslant 2,r})$ admits a simple formula for $r \equiv 0 \pmod 8$ and it is more complicated if $r \equiv 4 \pmod 8$.

The next goal is to discuss the 3-adic valuation $\nu_3(B_{n,\leqslant 2,r})$. The first results give some congruences for the restricted r-Bell numbers modulo 3.

Lemma 22. *The sequence of residues $B_{n,\leqslant 2,r}$ modulo 3, as a function of n, is a periodic sequence of period 3. The fundamental period is $\{1,1,2\}, \{1,2,1\}$ and $\{1,0,2\}$ for $r \equiv 0,1,2 \pmod 3$, respectively.*

Proof. Assume $n \equiv 0 \pmod 3$ and $r \equiv 1 \pmod 3$. Write $n = 3k$ and $r = 3t+1$. Theorem 13 gives

$$\begin{aligned} B_{n,\leqslant 2,r} = B_{3k,\leqslant 2,3t+1} &= (3t+1)B_{3k-1,\leqslant 2,3t} + B_{3k-1,\leqslant 2,3t+1} + (3k-1)B_{3k-2,\leqslant 2,3t+1} \\ &\equiv 2+1-2 = 1 \pmod 3. \end{aligned}$$

The other cases are similar. □

Corollary 23. *The 3-adic valuation* $\nu_3(B_{n,\leqslant 2,r})$ *is zero unless* $r \equiv 2 \pmod 3$ *and* $n \equiv 1 \pmod 3$*.*

The next lemma extends the analysis of Lemma 22 and its corollary to indices modulo 9. The proof is left to the reader.

Lemma 24. *The 3-adic valuation* $\nu_3(B_{n,\leqslant 2,r})$ *is* 1 *if* $r \equiv 2,\ 5 \pmod 9$*.*

It remains to analyze the sequence $\nu_3(B_{n,\leqslant 2,r})$ for $n \equiv 8 \pmod 9$. The description below, for $r = 8$, describes the valuation tree, as introduced in Remark 1.

Remark 25. The description of the valuation $\nu_3(B_{n,\leqslant 2,8})$ is given by a *valuation tree.* The root of the tree corresponds to all indices $n \in \mathbb{N}$. A sequence of nodes is constructed as follow: each node has attached a collection of indices. In this construction, one asks the following question: *is the valuation* $\nu_3(B_{n,\leqslant 2,8})$ `independent` of n. If the answer is positive, then the vertex is declared *terminal* and the constant value is assigned to it. If the answer is negative, the index set is split according to the residue modulo a power of the prime $p = 3$.

In the present case, since $\nu_3(B_{n,\leqslant 2,8})$ is not constant (it starts as $\{0,\ 2,\ 0\}$). Then the vertex is split into three vertices, corresponding to the residue of n modulo 3. The sequence $\nu_3(B_{3n,\leqslant 2,8})$ has constant value 0 and $\nu_3(B_{3n+2,\leqslant 2,8})$ also has constant value 0. The third class does not have constant value; it starts as $\{2,\ 3,\ 2,\ 2,\ 4\}$. This class of indices $\equiv 1 \pmod 3$ is now split into $n \equiv 1,\ 4,\ 7 \pmod 9$. Then $\nu_3(B_{9n+2,\leqslant 2,8}) \equiv 2$ and $\nu_3(B_{9n+7,\leqslant 2,8}) \equiv 2$ and the sequence $\equiv 4 \pmod 9$ is split modulo 27. It is conjectured that this process extends indefinitely.

Remark 26. Using this type of analysis, it is possible to establish similar experimental results. For example,

- The sequence of residues $B_{n,\leqslant 2,r}$ modulo 5 is a periodic sequence of period 5. In particular, if $r \equiv 2 \pmod 5$, the fundamental period is $\{1, 3, 3, 2, 1\}$. The sequence $B_{n,\leqslant 2,r}$ modulo 5 contains 0 precisely when $r \equiv 0,\ 1,\ 4 \pmod 5$. Therefore

  $$\nu_5(B_{n,\leqslant 2,r}) = 0 \text{ for } r \equiv 2,\ 3 \pmod 5.$$

  In the cases $r \equiv 2,\ 3 \pmod 5$, the sequence $\nu_5(B_{n,\leqslant 2,r})$ has a valuation tree structure.

- The sequence of residues $B_{n,\leqslant 5,r}$ modulo 13 is a periodic sequence of period 13. For example, if $r \equiv 2 \pmod{13}$, the fundamental period is

  $$\{1, 3, 10, 11, 8, 9, 12, 1, 8, 2, 11, 6, 6\}.$$

  The data shows that if $r \not\equiv 3,\ 4,\ 6,\ 7 \pmod{13}$, then $\nu_{13}(B_{n,\leqslant 5,r}) = 0$ and in the remaining cases there is a valuation tree structure.

The complete analysis of the valuations $\nu_p(B_{n,\leqslant m,r})$ cannot (*up to now*) be derived from this type of arguments.

7 Associated r-Bell Numbers

This section discusses some properties for the *associated r-Bell numbers* $B_{n,\geqslant m,r}$ which count the number of r-partitions of $[n+r]$ with blocks of length at least m. The results are similar to those presented in the previous sections for the restricted r-Bell numbers $B_{n,\leqslant m,r}$. The first statement is the analog of Theorems 12 and 13. The proofs are similar, so they are not presented here.

Theorem 27. *For $n \geqslant 1$, $r \geqslant 0$ and $m \geqslant 1$ the associated r-Bell numbers $B_{n,\geqslant m,r}$ satisfy the recurrences*

$$B_{n,\geqslant m,r} = r\sum_{\ell=m-2}^{n-1}\binom{n-1}{\ell}B_{n-1-\ell,\geqslant m,r-1} + \sum_{\ell=m-1}^{n-1}\binom{n-1}{\ell}B_{n-1-\ell,\geqslant m,r},$$

with the initial values $B_{0,\geqslant m,0} = 1$ and $B_{n,\geqslant m,0} = 0$ for $0 < n \leqslant m-1$, and for $r > 0$ and $0 \leqslant n < (m-1)r$ the values are $B_{0,\geqslant m,0} = 0$. Additionally, for $r \geqslant 1$,

$$B_{n,\geqslant m,r} = \sum_{\ell=m-1}^{n}\binom{n}{\ell}B_{n-\ell,\geqslant m,r-1}.$$

The next statement offers a relation between the restricted r-Bell numbers $B_{n,\leqslant m,r}$ and the associated r-Bell numbers $B_{n,\geqslant m,r}$.

Theorem 28. *The associated r-Bell numbers $B_{n,\geqslant m,r}$ can be calculated from the r-Bell numbers $B_{n,r}$ and the restricted r-Bell numbers $B_{n,\leqslant m,r}$, via*

$$B_{n,\geqslant m,r} = B_{n,r} - \sum_{\substack{i+j>0\\ 0\leqslant i\leqslant n,\, 0\leqslant j\leqslant r}}\binom{n}{i}\binom{r}{j}B_{i,\leqslant m-1,j}B_{n-i,\geqslant m,r-j}. \tag{22}$$

Proof. Let $\mathbb{B}_{n,r}$ denote the set of all r-partitions of $[n+r]$, with cardinality $B_{n,r}$. Suppose $\pi \in \mathbb{B}_{n,r}$ is represented as $\pi = \pi_1/\pi_2/\cdots/\pi_\ell$. Given a partition $\pi \in \mathbb{B}_{n,r}$, consider the set

$$A_\pi = \{i \in [n+r] \setminus [r] : \text{if } i \in \pi_j \text{ then } |\pi_j| < m\}.$$

This is the set of all non-special elements appearing in blocks of length less than m. Let B_π the complement of A_π, i.e., $B_\pi = \{i \in [n+r]\setminus[r] : \text{if } i \in \pi_j \text{ then } |\pi_j| \geqslant m\}$. Construct the partition $S_i = \{\pi \in \mathbb{B}_{n,r} : |A_\pi| = i\}$. Then $\mathbb{B}_{n,r} = \bigcup_{i=0}^{n} S_i$ and

$$|S_i| = \binom{n}{i}\sum_{j=\max\{0,1-i\}}^{r}\binom{r}{j}B_{i,\leqslant m-1,j}\, B_{n-i,\geqslant m,r-j}.$$

Summing over i yields the desired result. □

The previous result is now considered modulo a prime p.

Corollary 29. *Let p be a prime. Then*

$$\sum_{j=0}^{r} \binom{r}{j} B_{p,\geqslant m,r-j} \equiv r+2-B_{p,\leqslant m-1,r} \pmod{p}.$$

If $2 \leqslant m \leqslant p$ then

$$\sum_{j=0}^{r} \binom{r}{j} B_{p,\geqslant m,r-j} \equiv r+1 \pmod{p}.$$

Moreover,

$$B_{p,\geqslant m} \equiv 1 \pmod{p}.$$

Proof. Theorem 28 gives

$$B_{p,\geqslant m,r} \equiv B_{p,r} - B_{p,\leqslant m-1,r} - \sum_{j=1}^{r} \binom{r}{j} B_{p,\geqslant m,r-j} \pmod{p}.$$

Theorem 3 of [41] gives $B_{p,r} \equiv r+2 \pmod{p}$. This implies the first equality. The second identity follows from Theorem 16. The last congruence is the special case $r=0$. □

Remark 30. The congruence in Corollary 29 may be written as

$$B_{p,\geqslant m,r} \equiv r+2-B_{p,\leqslant m-1,r} - \sum_{j=0}^{r-1} \binom{r}{j} B_{p,\geqslant m,j} \pmod{p}. \tag{23}$$

This form is useful in an inductive argument (in r) of modular properties of $B_{p,\geqslant m,r}$. For example, $r=1$ yields

$$B_{p,\geqslant m,1} \equiv 3 - B_{p,\leqslant m-1,1} - B_{p,\geqslant m,0} \pmod{p}. \tag{24}$$

Remark 31. Unlike the restricted r-Bell numbers, the associated r-Bell numbers do not have a predictable behavior for their last digit. This unpredictability extends to their valuations. Moll et al. [44] studied the function $\nu_2(B_{n,\geqslant 2})$, proving that

$$\nu_2(B_{n,\geqslant 2}) = 0 \text{ if } n \equiv 0,\, 2 \pmod{3}.$$

For $n \equiv 1 \pmod{3}$, the valuation satisfies $\nu_2(B_{n,\geqslant 2}) \geqslant 1$. A more detailed study of this function is in progress.

8 A combinatorial identity

The Stirling numbers $\left\{ {n \atop k} \right\}$ count the number of partitions of $[n]$ into k non-empty blocks. It is natural to consider the extensions $\left\{ {n \atop k} \right\}_r$ of r-partitions of $[n+r]$ into $k+r$ non-empty blocks as well as $\left\{ {n \atop k} \right\}_{\leqslant m,r}$, the number of r-partitions of $[n+r]$ into $k+r$ non-empty blocks of size at most m. There are also families of corresponding associated numbers. These sequences are discussed in [44]. Further information about these numbers may be found in [6, 9, 28].

The goal of this section is to present a combinatorial proof of an interesting identity for $\left\{ {n \atop k} \right\}_{\leqslant 2,r}$ given in [26]. This sequence was studied by Cheon et al. [9] by means of Riordan arrays.

Theorem 32. *The restricted r-Stirling numbers and the associated r-Stirling numbers satisfy the following recurrences*

(a)

$$\left\{ {n+1 \atop k} \right\}_{\leqslant m,r} = r\sum_{i=0}^{m-2}\binom{n}{i}\left\{ {n-i \atop k} \right\}_{\leqslant m,r-1} + \sum_{i=0}^{m-1}\binom{n}{i}\left\{ {n-i \atop k-1} \right\}_{\leqslant m,r},$$

with initial conditions $\left\{ {n \atop k} \right\}_{\leqslant m,0} = \left\{ {n \atop k} \right\}_{\leqslant m}$, $\left\{ {0 \atop 0} \right\}_{\leqslant m,r} = 1$, $\left\{ {n \atop 0} \right\}_{\leqslant m,r} = r!\left\{ {n \atop r} \right\}_{\leqslant m-1}$, $\left\{ {0 \atop k} \right\}_{\leqslant m,r} = 0$.

(b)

$$\left\{ {n+1 \atop k} \right\}_{\geqslant m,r} = r\sum_{i=m-2}^{n}\binom{n}{i}\left\{ {n-i \atop k} \right\}_{\geqslant m,r-1} + \sum_{i=m-1}^{n}\binom{n}{i}\left\{ {n-i \atop k-1} \right\}_{\geqslant m,r},$$

with initial conditions $\left\{ {n \atop k} \right\}_{\geqslant m,0} = \left\{ {n \atop k} \right\}_{\geqslant m}$, $\left\{ {0 \atop k} \right\}_{\geqslant m,r} = 0$, $\left\{ {n \atop 0} \right\}_{\geqslant m,r} = r!\left\{ {n \atop r} \right\}_{\geqslant m-1}$.

The notation $\left\{ {[n] \atop k} \right\}$ is used for the set of partitions of $[n]$ into k non-empty blocks with cardinality $\left|\left\{ {[n] \atop k} \right\}\right| = \left\{ {n \atop k} \right\}$.

Theorem 33. *For $n, r \geqslant 0$, the identity*

$$\sum_{j=0}^{n}\left\{ {n \atop j} \right\}\left\{ {j \atop k} \right\}_{\leqslant 2,r} = \sum_{j=0}^{n}\binom{n}{j}2^{j-k}r^{n-j}\left\{ {j \atop k} \right\}$$

holds.

Proof. Consider the partition $\pi = \pi_1/\cdots/\pi_r/\pi_{r+1}/\cdots/\pi_{r+k}$ defined by

- $\pi_i \subseteq [\pm(n+r)] = \{\pm 1, \pm 2, \cdots, \pm(n+r)\}$ for every $i \in [k+r]$.
- $i \in \pi_i$ for $i \in [r]$, i.e., π_i is a signed special block containing their indices.
- $|\pi| = \{|\pi_1|, \ldots, |\pi_{k+r}|\}$, where $|\pi_i| = \{|\ell| : \ell \in \pi_i\}$ is a partition of $[n+r]$.

- For every $i \in [k+r]$, the minimal element of each block is positive.
- For $i \leqslant r$, the blocks contain only positive numbers.

The set of these signed $r-$partitions with the property that the minimal element of each block is positive is denoted by $\mathbb{A}_{n,k,r}$. For example,

$$\lambda = \{\underbrace{\{1,8,9\},\{2\},\{3,5\}}_{\text{Signed blocks}},\{4,-6,10\},\{7,-11,-12\}\} \in \mathbb{A}_{9,2,3}.$$

Now consider $S_i^{n,k,r} = \{\pi \in \mathbb{A}_{n,k,r} : |R_\pi| = i\}$, where

$$R_\pi = \{i \in [n+r] \setminus [r] : \text{ if } i \in B \in \pi, \text{ then } |B \cap [r]| = 1\},$$

i.e., the elements on the signed blocks, giving the disjoint union

$$\mathbb{A}_{n,k,r} = \bigcup_{i=0}^{n} S_i^{n,k,r}.$$

In the set R_π we have put the elements for the first r blocks, and so counting elements in the k remaining blocks is equivalent to partition $[n-i]$ into k blocks and attaching a sign to them, except for the minimal ones. This yields

$$|S_i^{n,k,r}| = \underbrace{\binom{n}{i} r^i}_{\text{defines } R_\pi} \left\{ {n-i \atop k} \right\} 2^{n-i-k}.$$

The right-hand side of the required identity appears as the cardinality of $\mathbb{A}_{n,k,r}$.

In order to complete the proof, consider the map

$$\varphi : \bigcup_{j=0}^{n} \left\{ {[n] \atop j} \right\} \times \left\{ {[j] \atop k} \right\}_{\leqslant 2,r} \longrightarrow \mathbb{A}_{n,k,r},$$

given by

$$\varphi(\pi,\lambda) = \gamma_1/\cdots/\gamma_r/\gamma_{r+1}/\cdots/\gamma_{r+k} \in \mathbb{A}_{n,k,r},$$

where $\pi = \pi_1/\cdots/\pi_j$, $\lambda = \lambda_1/\cdots/\lambda_k/\cdots/\lambda_{k+r}$ with $\lambda_i = \{\lambda_{i,1},\lambda_{i,2}\}$ or $\lambda_i = \{\lambda_{i,1}\}$ and

$$\gamma_i = \begin{cases} \{i\}, & i \leqslant r, \ |\lambda_i| = 1, \\ \{i\} \cup \{s+r : s \in \pi_{\lambda_{i,2}-r}\}, & i \leqslant r, \ |\lambda_i| = 2. \\ \{s+r : s \in \pi_{\lambda_{i,1}-r}\}, & i > r, \ |\lambda_i| = 1, \\ \{s+r : s \in \pi_{\lambda_{i,1}-r}\} \cup \{-(s+r) : s \in \pi_{\lambda_{i,2}-r}\}, & i > r, \ |\lambda_i| = 2. \end{cases}$$

For example, take $n = 10$, $j = 6$, $r = 2$, $k = 2$ and

$$\pi = \underbrace{\{1,3\}}_{\pi_1}/\underbrace{\{2,6,8\}}_{\pi_2}/\underbrace{\{4\}}_{\pi_3}/\underbrace{\{5,10\}}_{\pi_4}/\underbrace{\{7\}}_{\pi_5}/\underbrace{\{9\}}_{\pi_6} \text{ and } \lambda = \underbrace{\{1,6\}/\{2,8\}}_{\text{signed blocks}}/\{3,4\}/\{5,7\}.$$

Then

$$\begin{aligned}\varphi(\pi,\lambda) &= \underbrace{\{1,\underbrace{5+r,10+r}_{\pi_4}\}/\{2,\underbrace{9+r}_{\pi_6}\}}_{i\leqslant r}/\{\underbrace{1+r,3+r}_{\pi_1},\underbrace{-(2+r),-(6+r),-(8+r)}_{\pi_2}\}/\{\underbrace{4+r}_{\pi_3},\underbrace{-(7+r)}_{\pi_5}\}\\ &= \underbrace{\{1,7,12\}/\{2,11\}}_{i\leqslant r}/\{3,5,-4,-8,-10\}/\{6,-9\}\end{aligned}$$

This map is a bijection and the identity follows. Details are left to the reader. □

Remark 34. In the case $r = 0$, the statement above gives a relation for the Bessel numbers of the second kind (see [64]):

$$\sum_{j=0}^{n} \left\{ {n \atop j} \right\} \left\{ {j \atop k} \right\}_{\leqslant 2} = 2^{n-k} \left\{ {n \atop k} \right\}.$$

Remark 35. Cheon et al. [9] studied a related sequence $b_r(n,k)$ orthogonal to $\left\{ {n \atop k} \right\}_{\leqslant 2,r}$, i.e.,

$$\sum_{i=k}^{n} \left\{ {n \atop i} \right\}_{\leqslant 2,r} b_r(i,k) = \sum_{i=k}^{n} b_r(n,i) \left\{ {i \atop k} \right\}_{\leqslant 2,r} = \delta_{k,n}.$$

The numbers $\widehat{b}_r(n,k) := (-1)^{n-k} b_r(n,k)$ are called *the unsigned r-Bessel numbers of the first kind*, with exponential generating function

$$\sum_{n=k}^{2k+r} \widehat{b}_r(n,k) \frac{x^n}{n!} = \frac{1}{k!} \frac{(1-\sqrt{1-2x})^k}{(1-2x)^{r/2}}.$$

An interesting combinatorial interpretation of this sequence, using the concept of G-partitions, is given in [9].

9 Generalized Howard's Identities

The *restricted Stirling numbers of the second kind*, $\left\{ {n \atop k} \right\}_{\leqslant m}$ and *associated Stirling numbers of the second kind*, $\left\{ {n \atop k} \right\}_{\geqslant m}$ were introduced in Section 8. The goal here is to present a combinatorial proof of some identities given by Howard in [25]. The symbol $\binom{[n]}{k}$ denotes the set of n-combinations of k-elements, with cardinality $\binom{n}{k}$.

Theorem 36. *Let $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n$. Then*

$$\left\{ {n \atop n-k} \right\} = \sum_{j=0}^{k} \binom{n}{2k-j} \left\{ {2k-j \atop k-j} \right\}_{\geqslant 2}.$$

Proof. Elementary manipulations transform the desired identity to

$$\left\{ {n \atop k} \right\} = \sum_{j=0}^{n-k} \binom{n}{2(n-k)-j} \left\{ {2(n-k)-j \atop n-k-j} \right\}_{\geqslant 2} = \sum_{j=0}^{n-k} \binom{n}{k-j} \left\{ {n-(k-j) \atop j} \right\}_{\geqslant 2}.$$

Define the function

$$\varphi : \left\{ {[n] \atop k} \right\} \longrightarrow \bigcup_{j=0}^{n-k} \binom{[n]}{k-j} \times \left\{ {[n-(k-j)] \atop j} \right\}_{\geqslant 2},$$

by $\varphi(\pi) = (\mathrm{Fix}(\pi), \pi \setminus \mathrm{Fix}(\pi))$, where $\mathrm{Fix}(\pi) = \{B \in \pi : |B| = 1\}$ are the singletons of π. The map

$$\psi(X, \pi) = \pi \cup \{\{x\} : x \in X\}$$

is the inverse of φ. It follows that φ is a bijection, establishing the result. □

A similar argument gives the next generalization.

Theorem 37. *Let n, m, $k \in \mathbb{N}$ with $0 \leqslant k \leqslant n$. Then*

$$\left\{ {n \atop k} \right\}_{\geqslant m} = \sum_{j=0}^{n} \binom{n}{mj} \frac{(mj)!}{m!^j j!} \left\{ {n-mj \atop k-j} \right\}_{\geqslant m+1}.$$

10 Restricted and associated r-Bell Polynomials

The classical *Bell polynomials* $B_n(x)$ are defined by

$$B_n(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^k.$$

These polynomials extend the Bell numbers B_n, since $B_n(1) = B_n$. Mező [37] introduced the r-Bell polynomials by

$$B_{n,r}(x) := \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r x^k,$$

where $\left\{ {n \atop k} \right\}_r$ are the r-Stirling numbers of the second kind. Further generalizations of these polynomials appear in Corcino et al. [14]. The restricted (associated) r-Bell numbers appear as

$$B_{n,\leqslant m,r} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\leqslant m,r} \quad \text{and} \quad B_{n,\geqslant m,r} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\geqslant m,r}. \tag{25}$$

The objects of interest in this section are two families of polynomials: the *restricted r-Bell polynomials* $B_{n,\leqslant m,r}(x)$ and the *associated r-Bell polynomials* $B_{n,\geqslant m,r}(x)$ defined by

$$B_{n,\leqslant m,r}(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\leqslant m,r} x^k \quad \text{and} \quad B_{n,\geqslant m,r}(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\geqslant m,r} x^k.$$

The exponential generating function of these families appeared in [6, 9].

Theorem 38. *The exponential generating function of the restricted/associated r-Bell polynomials are given by*

$$\sum_{n=0}^{\infty} B_{n,\leqslant m,r}(x)\frac{z^n}{n!} = \left(1+z+\frac{z^2}{2!}+\cdots+\frac{z^{m-1}}{(m-1)!}\right)^r e^{x\left(z+\frac{z^2}{2!}+\cdots+\frac{z^m}{m!}\right)} \quad \textit{and}$$
$$\sum_{n=0}^{\infty} B_{n,\geqslant m,r}(x)\frac{z^n}{n!} = \left(\frac{z^{m-1}}{(m-1)!}+\frac{z^m}{m!}+\cdots\right)^r e^{x\left(\frac{z^m}{m!}+\frac{z^{m+1}}{(m+1)!}+\cdots\right)}.$$

The classical Stirling numbers of the second kind satisfy the identity

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}},$$

where $x^{\underline{k}}$ is the falling factorial defined by $x^{\underline{k}} := x(x-1)\cdots(x-(k-1))$ for $k \geqslant 1$ and $x^{\underline{0}} = 1$. A generalization for the incomplete r-Stirling numbers is presented next, the relation to r-Bell polynomials is stated in Theorem 40. The proof is a direct application of Theorem 8 of [42].

Theorem 39. *For t, $r \in \mathbb{N}$, define*

$$f_{t,r}(x) = \left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^m}{m!}\right)^t \left(1+x+\frac{x^2}{2!}+\cdots+\frac{x^{m-1}}{(m-1)!}\right)^r.$$

Then

$$\frac{d^n}{dx^n} f_{t,r}(x)\bigg|_{x=0} = f_{t,r}^{(n)}(0) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\leqslant m,r} t^{\underline{k}}. \tag{26}$$

Similarly, for t, $r \in \mathbb{N}$ define $g_{t,r}(x)$ by

$$g_{t,r}(x) = \left(1+\frac{x^m}{m!}+\frac{x^{m+1}}{(m+1)!}+\cdots\right)^t \left(\frac{x^{m-1}}{(m-1)!}+\frac{x^m}{m!}+\cdots\right)^r.$$

Then

$$\frac{d^n}{dx^n} g_{t,r}(x)\bigg|_{x=0} = g_{t,r}^{(n)}(0) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\geqslant m,r} t^{\underline{k}}. \tag{27}$$

The next statements are analogues of Dobinski's formula (cf. [17, 47]) for Bell numbers

$$B_n = \frac{1}{e}\sum_{\ell=0}^{\infty}\frac{\ell^n}{\ell!}.$$

Theorem 40. *The restricted r-Bell polynomials satisfy the identity*

$$B_{n,\leqslant m,r}(x) = \frac{1}{e^x}\sum_{\ell=0}^{\infty} f_{\ell,r}^{(n)}(0)\frac{x^\ell}{\ell!}.$$

In particular,

$$B_{n,\leqslant m,r} = \frac{1}{e}\sum_{\ell=0}^{\infty}\frac{1}{\ell!}f_{\ell,r}^{(n)}(0).$$

Proof. Theorem 39 implies

$$\begin{aligned}\sum_{\ell=0}^{\infty} f_{\ell,r}^{(n)}(0)\frac{x^\ell}{\ell!} &= \sum_{\ell=0}^{\infty}\sum_{k=0}^{n}\left\{ {n \atop k} \right\}_{\leqslant m,r} \ell^{\underline{k}}\frac{x^\ell}{\ell!} = \sum_{\ell=0}^{\infty}\sum_{k=0}^{n}\left\{ {n \atop k} \right\}_{\leqslant m,r}\binom{\ell}{k}\frac{k!}{\ell!}x^\ell \\ &= \sum_{s=0}^{\infty}\sum_{k=0}^{n}\left\{ {n \atop k} \right\}_{\leqslant m,r}\binom{s+k}{k}\frac{k!}{(s+k)!}x^{s+k} = \sum_{s=0}^{\infty}\frac{x^s}{s!}\sum_{k=0}^{n}\left\{ {n \atop k} \right\}_{\leqslant m,r} x^k \\ &= e^x B_{n,\leqslant m,r}(x).\end{aligned}$$

The proof is complete. □

Example 41. The identity

$$\left\{ {n \atop k} \right\}_{\leqslant 2} = \frac{k!}{2^{n-k}(2k-n)!}\binom{n}{k},$$

produces the expression

$$B_{n,\leqslant 2} = \frac{1}{e}\sum_{\ell=0}^{\infty}\sum_{k=0}^{n}\frac{n^{\underline{k}}}{2^{n-k}(\ell-k)!(2k-n)!}.$$

The analogous result the associated type is stated next.

Theorem 42. *The associated r-Bell polynomials satisfy the identity*

$$B_{n,\geqslant m,r}(x) = \frac{1}{e^x}\sum_{\ell=0}^{\infty} g_{\ell,r}^{(n)}(0)\frac{x^\ell}{\ell!}.$$

In particular,

$$B_{n,\geqslant m,r} = \frac{1}{e}\sum_{\ell=0}^{\infty}\frac{1}{\ell!}g_{\ell,r}^{(n)}(0).$$

11 Fubini Numbers

The Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ count the number of partitions of $[n]$ into k non-empty blocks. The total number of partitions is given by the Bell number B_n in (1). The corresponding counting situation, where now the k blocks are ordered, is given by the *Fubini numbers* F_n, also called the *ordered Bell numbers.* They are given by

$$F_n = \sum_{\ell=0}^{n} \ell! \left\{ {n \atop \ell} \right\} \text{ for } n > 0 \text{ and } F_0 = 1, \tag{28}$$

as stated in (2). This section discusses some of their properties.

Remark 43. The explicit formula for the Stirling number of the second kind

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} (k-\ell)^n,$$

gives

$$F_n = \sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^{\ell} \binom{k}{\ell} (k-\ell)^n. \tag{29}$$

Remark 44. The Fubini numbers satisfy the recurrence

$$F_n = \sum_{\ell=1}^{n} \binom{n}{\ell} F_{n-\ell}, \tag{30}$$

with initial condition $F_0 = 1$. A proof is given in Corollary 72.

Remark 45. The exponential generating function for $\{F_n\}$ is given by

$$\sum_{n=0}^{\infty} F_n \frac{x^n}{n!} = \frac{1}{2 - e^x}. \tag{31}$$

The next statements deal with modular properties of the Fubini numbers.

Theorem 46. *Let p be a prime. Then $\{F_n \pmod p\}$ is a periodic sequence of period $p-1$.*

Proof. Fermat's little theorem gives $a^p \equiv a \pmod p$ and the identity

$$\sum_{i=0}^{\ell} (-1)^i \binom{\ell}{i} (\ell - i)^n = 0 \quad (\ell > n)$$

imply that

$$\begin{aligned}
F_{n+p-1} &= \sum_{k=0}^{n+p-1} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (k-\ell)^{n+p-1} \\
&= \sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (k-\ell)^{n+p-1} + \sum_{k=n+1}^{n+p-1} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (k-\ell)^{n+p-1} \\
&\equiv \sum_{k=0}^{n} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (k-\ell)^{n} \pmod{p} \\
&= F_n.
\end{aligned}$$

□

A similar argument shows that $\{F_n\}$ is periodic modulo p^s, with period $p^{s-1}(p-1)$, i.e.,

$$F_{n+p^{s-1}(p-1)} \equiv F_n \pmod{p^s}, \tag{32}$$

see Barsky [5]. Diagana and Maïga [16] established the generalization

$$F_{n+tp^s} \equiv F_{n+tp^{s-1}} \pmod{p^s} \tag{33}$$

with $\gcd(t, p) = 1$. For p prime, Good [20] conjectured that $p-1$ is the minimal period of the Fubini numbers modulo p. This conjecture was verified in [20], for $2 \leqslant p \leqslant 73$. The general case was established in Poonen [48].

Theorem 47. *Let p be a prime and $s \in \mathbb{N}$. Then $\{F_n\}$ modulo p^s is a periodic sequence, with minimal period $p^{s-1}(p-1)$.*

The expression (29) yields the following result.

Corollary 48. *Let p be a prime. Then*

$$F_{p-1} \equiv 0 \pmod{p} \quad \text{and} \quad F_p \equiv 1 \pmod{p}.$$

The next result establishes the structure of the last digit of F_n. The proof uses the periodicity of the F_n for $p = 2$, 5, first established by Gross [22].

Corollary 49. *For $n \geqslant 1$ the congruence $F_{n+4} \equiv F_n \pmod{10}$ holds. The fundamental period for the last digit is $\{1, 3, 3, 5\}$.*

The recurrences stated below were proven first by Poonen [48] using induction. Diagana and Maïga [16] used the Laplace transform of a p-adic measure to present a new proof. Two different proofs are presented below: one using combinatorial identities and then a bijective proof.

Theorem 50. *Let $n, q \in \mathbb{N}$. Then*

$$(2^q - 1)F_n = \sum_{\ell=0}^{n-1} q^{n-\ell}\binom{n}{\ell}F_\ell + \sum_{\ell=1}^{q-1} 2^{q-\ell-1}\ell^n.$$

$$(2^q - 1)F_n = -2^q\sum_{\ell=0}^{n-1}(-q)^{n-\ell}\binom{n}{\ell}F_\ell + (-1)^n\sum_{\ell=1}^{q} 2^{\ell-1}\ell^n.$$

Proof. The identity (see [13, pp. 228])

$$F_n = \frac{1}{2}\sum_{k=0}^{\infty}\frac{k^n}{2^k}, \tag{34}$$

produces

$$\begin{aligned}F_n &= \frac{1}{2}\sum_{\ell=0}^{q-1}\frac{\ell^n}{2^\ell} + \frac{1}{2^{q+1}}\sum_{\ell=0}^{\infty}\frac{(\ell+q)^n}{2^{\ell+q}} = \frac{1}{2}\sum_{\ell=0}^{q-1}\frac{\ell^n}{2^\ell} + \frac{1}{2^{q+1}}\sum_{\ell=0}^{\infty}\frac{1}{2^\ell}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\ell^k \\ &= \frac{1}{2}\sum_{\ell=0}^{q-1}\frac{\ell^n}{2^\ell} + \frac{1}{2^q}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{2}\sum_{\ell=0}^{\infty}\frac{1}{2^\ell}\ell^k = \frac{1}{2}\sum_{\ell=0}^{q-1}\frac{\ell^n}{2^\ell} + \frac{1}{2^q}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}F_k.\end{aligned}$$

Multiplying by 2^q to obtain the result. The second identity is proven in a similar manner. □

Combinatorial proof. Let $L_{q,n} := 2^{[q]} \times \mathbb{F}_n$, where $2^{[q]}$ is the power set of $[q]$ and

$$\mathbb{F}_n := \{(\pi_1, \ldots, \pi_k) : \bigcup_{i=1}^{k}\pi_i = [n] \text{ and } \pi_i \cap \pi_j = \varnothing \text{ if } i \neq j\}$$

the set of ordered partitions of $[n]$. Then $|L_{q,n}| = 2^q F_n$.

Denote by $\|\pi\|$ the number of blocks of the set partition π. Partition the set $L_{q,n}$ in the form: $L_{q,n} = L^+_{q,n} \cup L^-_{q,n}$, where $L^+_{q,n} = \{(X,\pi) : \|\pi\| \geqslant |X|\}$ and $L^-_{q,n} = \{(X,\pi) : \|\pi\| < |X|\}$. Let $[q]^{[n]}$ be the set of functions from $[n]$ to $[q]$ and define

$$\varphi : L^+_{q,n} \longrightarrow \bigcup_{\ell=0}^{n}\left([q]^{[n-\ell]} \times \binom{[n]}{\ell} \times \mathbb{F}_\ell\right),$$

by

$$\varphi(X,\pi) = \left(f_{X,\pi}, \bigcup_{i=1}^{\|\pi\|-|X|}\pi_i\ , (\pi_1, \ldots, \pi_{\|\pi\|-|X|})\right),$$

where

$$f_{X,\pi} : \bigcup_{i=\|\pi\|-|X|+1}^{\|\pi\|}\pi_i \longrightarrow X,$$

is a function mapping the tail of the partition to $X = \{X_1, \ldots, X_s\}$ with $X_1 < \cdots < X_s$ by $f_{X,\pi}(a) = X_{i-\|\pi\|-|X|}$ for $a \in \pi_i$.

The index ℓ on the partition of the range is given by

$$\ell = \sum_{i=1}^{\|\pi\|-|X|} |\pi_i| = \left| \bigcup_{i=1}^{\|\pi\|-|X|} \pi_i \right|.$$

For example, if $X = \{2, 4, 5\}$ and $\pi = (\{4, 6, 9\}, \{1, 3\}, \{5, 7\}, \{2\}, \{8, 10\})$, then

$$f_{X,\pi} : \bigcup_{i=3}^{5} \pi_i \longrightarrow X,$$

with $f_{X,\pi}(5) = 2$, $f_{X,\pi}(7) = 2$, $f_{X,\pi}(2) = 4$, $f_{X,\pi}(8) = 5$ and $f_{X,\pi}(10) = 5$. Therefore,

$$\varphi(\{2, 4, 5\}, (\{4, 6, 9\}, \{1, 3\}, \{5, 7\}, \{2\}, \{8, 10\})) = (f_{X,\pi}, \{4, 6, 9, 1, 3\}, (\{4, 6, 9\}, \{1, 3\})).$$

`Claim`: φ is a bijection. To show this, define

$$\psi : \bigcup_{\ell=0}^{n} \left([q]^{[n-\ell]} \times \binom{[n]}{\ell} \times \mathbb{F}_\ell \right) \longrightarrow L^+_{q,n},$$

by

$$\psi(g, Y, \lambda) = (g([n] \setminus Y), (\lambda_1, \lambda_2, \ldots, \lambda_{|\lambda|}, \underbrace{g^{-1}(1), \ldots, g^{-1}(q)}_{\text{discard } j \in [q] \text{ such that } g^{-1}(j) = \varnothing})).$$

It is shown that φ and ψ are inverses of each other. Note that, applying this definition to the example and calling $\lambda = (\{4, 6, 9\}, \{1, 3\}, \{5, 7\}, \{2\}, \{8, 10\})$, we get that

$$\begin{aligned}
\psi(\varphi(\{2, 4, 5\}, \lambda) &= \psi(f_{X,\pi}, \{4, 6, 9, 1, 3\}, (\{4, 6, 9\}, \{1, 3\})) \\
&= (f_{X,\pi}(\{2, 5, 7, 8, 10\}), (\{4, 6, 9\}, \{1, 3\}, f^{-1}_{X,\pi}(1), \cdots, f^{-1}_{X,\pi}(10))) \\
&= (\{4, 2, 5\}, (\{4, 6, 9\}, \{1, 3\}, \varnothing, \{5, 7\}, \varnothing, \{2\}, \{8, 10\}, \varnothing, \cdots, \varnothing)) \\
&= (\{4, 2, 5\}, (\{4, 6, 9\}, \{1, 3\}, \{5, 7\}, \{2\}, \{8, 10\})) \\
&= (\{4, 2, 5\}, \lambda).
\end{aligned}$$

`Step 1`. $\psi \circ \varphi = Id$. Indeed, the composition is

$$\begin{aligned}
\psi(\varphi(X, \pi)) &= \psi(f_{X,\pi}, \bigcup_{i=1}^{\|\pi\|-|X|} \pi_i, (\pi_1, \ldots, \pi_{|X|})) \\
&= (f_{X,\pi}([n] \setminus \bigcup_{i=1}^{\|\pi\|-|X|} \pi_i), (\pi_1, \ldots, \pi_X, f^{-1}_{X,\pi}(1), \ldots, f^{-1}_{X,\pi}(q))).
\end{aligned}$$

The definition of $f_{X,\pi}$ implies

$$f_{X,\pi}([n] \setminus \bigcup_{i=1}^{\|\pi\|-|X|} \pi_i) = f_{X,\pi}(\bigcup_{i=\|\pi\|-|X|+1}^{\|\pi\|} \pi_i) = X$$

and $\pi = (\pi_1, \dots, \pi_{|X|}, \pi_{|X|+1}, \dots, \pi_{\|\pi\|})$.

`Step 2`. $\varphi \circ \psi = Id$. This time, the composition is

$$\varphi(\psi(g, Y, \lambda)) = \varphi(g([n] \setminus Y), (\lambda_1, \dots, \lambda_{|\lambda|}, g^{-1}(1), \dots, g^{-1}(q))),$$

$f_{g([n]\setminus Y),(\lambda_1,\dots,g^{-1}(q))} = g$ by definition of $f_{X,\pi}$. Also, $Y = \bigcup_{i=1}^{|\lambda|} \lambda_i$ by definition of partition and the projection sends λ to λ.

This shows that φ is a bijection.

It follows that

$$\begin{aligned} |L^+_{q,n}| &= \left| \bigcup_{\ell=0}^{n} \left([q]^{[n-\ell]} \times \binom{[n]}{\ell} \times \mathbb{F}_\ell \right) \right| \\ &= \sum_{\ell=0}^{n} q^{n-\ell} \binom{n}{\ell} F_\ell. \end{aligned} \tag{35}$$

For the second part of the identity, consider the function

$$\varphi : \bigcup_{\ell=1}^{q-1} \left(2^{[q-\ell-1]} \times [\ell]^{[n]} \right) \longrightarrow L^-_{q,n},$$

given by $\varphi(X, f) = (X \cup \{i \in [\ell] : f^{-1}(i) \neq \varnothing\}, (f^{-1}(1), \dots, f^{-1}(\ell)))$. The set $L^-_{q,n}$ contains the range of the function, defined as the number of parts of the resulting partition. The map φ is a bijection and its inverse is given by

$$\psi : L^-_{q,n} \longrightarrow \bigcup_{\ell=1}^{q-1} \left(2^{[q-\ell-1]} \times [\ell]^{[n]} \right).$$

Let $X = \{X_1, \dots, X_s\} \subseteq [q]$, such that $X_1 < \dots < X_s$, then

$$\psi(X, \lambda) = (\{X_{|\lambda|+1}, X_{|\lambda|+2}, \dots, X_s\}, f_{X,\lambda}),$$

where $f_{X,\lambda} : [n] \longrightarrow [\ell]$ is defined by $f_{X,\lambda}(a) = X_i$ if $a \in \lambda_i$.

As an example, consider $X = \{2, 4, 5, 7, 8, 9, 11\} \subseteq [20]$ and

$$\lambda = (\{4, 6, 9\}, \{1, 3\}, \{5, 7\}, \{2\}, \{8, 10\}).$$

Then,

$$\psi(X, \lambda) = (\{X_6, X_7\}, f_{X,\lambda}) = (\{9, 11\}, f_{X,\lambda}),$$

and $f_{X,\lambda}(\{4,6,9\}) = 2, f_{X,\lambda}(\{1,3\}) = 4, f_{X,\lambda}(\{5,7\}) = 5, f_{X,\lambda}(\{2\}) = 7, f_{X,\lambda}(\{8,10\}) = 8$. Now, computing φ of the resulting pair

$$\begin{aligned}\varphi(\psi(X,\lambda)) &= \varphi((\{9,11\}, f_{X,\lambda}))\\ &= (\{9,11\} \cup \{i \in [\ell] : f_{X,\lambda}^{-1}(i) \neq \varnothing\}, (f_{X,\lambda}^{-1}(1), \dots, f_{X,\lambda}^{-1}(\ell)))\\ &= (\{9,11\} \cup \{2,4,5,7,8\}, (\{4,6,9\}, \{1,3\}, \{5,7\}, \{2\}, \{8,10\}))\\ &= (X,\lambda)\end{aligned}$$

It follows that

$$|L_{q,n}^{-}| = |\bigcup_{\ell=1}^{q-1} [2]^{[q-\ell-1]} \times [\ell]^{[n]}| = \sum_{\ell=1}^{q-1} 2^{q-\ell-1}\ell^n. \tag{36}$$

Combining (35) and (36), gives the stated identity. □

Diagana and Maïga [16] used Theorem 50 to establish some interesting congruences for the Fubini numbers.

Corollary 51. *Let q, $n \in \mathbb{N}$. Then*

$$(2^q - 1)F_n \equiv nqF_{n-1} + \sum_{j=1}^{q-1} 2^{q-1-j} j^n \pmod{q^2}.$$

In particular, if $n = q = p$ is a prime number, then

$$(2^p - 1)F_p \equiv \sum_{j=1}^{p-1} 2^{p-1-j} j^p \pmod{p^2}.$$

Corollary 52. *Let $q \in \mathbb{N}$. Then*

$$(2^q - 1)F_n \equiv \sum_{j=1}^{q-1} 2^{q-1-j} j^n \pmod{q}.$$

Velleman and Call [57] gave a combinatorial identity for F_n, similar to (29). An alternative proof is presented next.

Theorem 53. *For all $n \geqslant 1$,*

$$F_n = \sum_{k=1}^{n} \sum_{i=0}^{k-1} (-1)^i \binom{n+1}{i} (k-i)^n 2^{n-k}.$$

Proof. The Eulerian numbers $A_{n,k}$, given by

$$A_{n,k} = \sum_{i=0}^{k-1} (-1)^i \binom{n+1}{i} (k-i)^n, \tag{37}$$

count the number of permutations containing exactly k runs. Let $\mathbb{A}_{n,k}$ be the set of these permutations. Information about these permutation appear in [21].

The identity (37) and

$$F_n = \sum_{k=1}^{n} A_{n,k} 2^{n-k}, \tag{38}$$

prove the result. The proof of (38) is presented next.

In an ordered partition $\pi = (\pi_1, \dots, \pi_\ell)$, each block π_i can be written as $\pi_{i,1}\pi_{i,2}\cdots\pi_{i,|\pi_i|}$ so that $\pi_{i,1} < \pi_{i,2} < \cdots < \pi_{i,|\pi_i|}$. Define the function

$$\varphi : \mathbb{F}_n \longrightarrow \bigcup_{k=1}^{n} \mathbb{A}_{n,k} \times 2^{[n-k]},$$

by

$$\varphi(\pi) := (\underbrace{\pi_1\pi_2\cdots\pi_\ell}_{Concatenation}, \{i \in [\ell-1] : \pi_{i,|\pi_i|} < \pi_{i+1,1}\}).$$

For example,

$$\varphi(\{3,6\},\{1,4\},\{2\},\{5,7,8\}) = (36142578, \{3\}).$$

Therefore the partition is being encoded as a permutation where the blocks start with a descent in the permutation or where we indicate with the resulting set which has index of ascents in the permutation. This function has the natural inverse: given a permutation with $k-1$ descents and given a subset of the ascents, we can generate the partition in the following way, consider $\pi = 24\underline{3}58\underline{1}7\underline{6}9 \in \mathbb{A}_{9,4}$ where the descents are underlined. As the first element of the permutation does not count as ascent, then we have $5 = |\{2,4,5,7,9\}|$, where the set is the indices of the ascents in π, places to choose for creating a new block. If we choose $X = \{4\}$ then we can construct the partition $(\{2,4\},\{3\},\{5,8\},\{1,7\},\{6,9\})$. □

12 Restricted and associated Fubini Numbers

The Fubini numbers F_n count the number of ordered set partitions. It is natural to generalize them by restricting the size of the blocks used in the partitions. This gives the *restricted Fubini numbers*, $F_{n,\leqslant m}$, where the blocks are of size at most m and the *associated Fubini numbers* $F_{n,\geqslant m}$, with blocks of size at least m. In terms of the restricted/associated Stirling numbers, the identities

$$F_{n,\leqslant m} = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}_{\leqslant m} \quad \text{and} \quad F_{n,\geqslant m} = \sum_{k=0}^{n} k! \left\{ {n \atop k} \right\}_{\geqslant m},$$

are clear.

Theorem 39 provides the following relations:

$$\frac{d^n}{dx^n} f_t(x)\bigg|_{x=0} = f_t^{(n)}(0) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\leqslant m} t^{\underline{k}}, \tag{39}$$

$$\frac{d^n}{dx^n} g_t(x)\bigg|_{x=0} = g_t^{(n)}(0) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_{\geqslant m} t^{\underline{k}}, \tag{40}$$

where $f_t(x) = \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^m}{m!}\right)^t$ and $g_t(x) = \left(1 + \frac{x^m}{m!} + \frac{x^{m+1}}{(m+1)!} + \cdots\right)^t$. In the case $m \geqslant n$, this identity yields the classical formula

$$t^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} t^{\underline{k}} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} \binom{t}{k} k!.$$

An elementary argument shows that this counts the total number of functions from $[n]$ to the set $[t]$. The equations (39) and (40) admit the following combinatorial interpretation: the expression $f_t^{(n)}(0)$ counts the functions w from $[n]$ to $[t]$, such that $|w^{-1}(i)| \leqslant m$ for every $i \in [t]$. A similar interpretation exists for $g_t^{(n)}(0)$.

Remark 54. Komatsu and Ramírez [29] found the exponential generating functions for the restricted/associated Fubini numbers:

$$\sum_{n=0}^{\infty} F_{n,\leqslant m} \frac{x^n}{n!} = \frac{1}{1 - x - \frac{x^2}{2!} - \cdots - \frac{x^m}{m!}}, \tag{41}$$

$$\sum_{n=0}^{\infty} F_{n,\geqslant m} \frac{x^n}{n!} = \frac{1}{1 - \frac{x^m}{m!} - \frac{x^{m+1}}{(m+1)!} - \cdots}. \tag{42}$$

The next statement is analogous to the identity (34).

Theorem 55. *For $n \geqslant 0$ we have*

$$F_{n,\leqslant m} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\leqslant m} k^{\underline{\ell}} \quad \textit{and} \quad F_{n,\geqslant m} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\geqslant m} k^{\underline{\ell}}.$$

Proof. The expressions in (39) and (41) give

$$\sum_{n=0}^{\infty} F_{n,\leqslant m} \frac{x^n}{n!} = \frac{1}{1 - x - \frac{x^2}{2!} - \cdots - \frac{x^m}{m!}} = \frac{1}{2\left(1 - \frac{1}{2}\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^m}{m!}\right)\right)}$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} \left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^m}{m!}\right)^k = \frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{n=0}^{\infty} \left(\sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\leqslant m} k^{\underline{\ell}} \right) \frac{x^n}{n!}.$$

The result follows by comparing coefficients. The proof of the second identity is similar. □

A recurrence for the restricted/associated Fubini numbers is presented next.

Theorem 56. *Let n, $q \in \mathbb{N}$. Then*

$$2^q F_{n,\leqslant m} = \sum_{\ell=0}^{n} \binom{n}{\ell} F_{\ell,\leqslant m} \left(\sum_{i=0}^{n-\ell} \left\{ {n-\ell \atop i} \right\}_{\leqslant m} q^{\underline{i}} \right) + \sum_{\ell=1}^{q-1} 2^{q-\ell-1} \left(\sum_{i=0}^{n} \left\{ {n \atop i} \right\}_{\leqslant m} \ell^{\underline{i}} \right),$$

$$2^q F_{n,\geqslant m} = \sum_{\ell=0}^{n} \binom{n}{\ell} F_{\ell,\geqslant m} \left(\sum_{i=0}^{n-\ell} \left\{ {n-\ell \atop i} \right\}_{\geqslant m} q^{\underline{i}} \right) + \sum_{\ell=1}^{q-1} 2^{q-\ell-1} \left(\sum_{i=0}^{n} \left\{ {n \atop i} \right\}_{\geqslant m} \ell^{\underline{i}} \right).$$

Proof. Theorem 55 and (39) imply

$$\begin{aligned}
F_{n,\leqslant m} &= \frac{1}{2} \sum_{k=0}^{q-1} \frac{1}{2^k} \sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\leqslant m} k^{\underline{\ell}} + \frac{1}{2} \sum_{k=q}^{\infty} \frac{1}{2^k} \sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\leqslant m} k^{\underline{\ell}} \\
&= \frac{1}{2} \sum_{k=0}^{q-1} \frac{1}{2^k} \sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\leqslant m} k^{\underline{\ell}} + \frac{1}{2^{q+1}} \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\leqslant m} (k+q)^{\underline{\ell}} \\
&= \frac{1}{2} \sum_{k=0}^{q-1} \frac{1}{2^k} \sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\leqslant m} k^{\underline{\ell}} + \frac{1}{2^{q+1}} \sum_{k=0}^{\infty} \frac{1}{2^k} f_{k+q}^{(n)}(0) \\
&= \frac{1}{2} \sum_{k=0}^{q-1} \frac{1}{2^k} \sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\leqslant m} k^{\underline{\ell}} + \frac{1}{2^{q+1}} \sum_{k=0}^{\infty} \frac{1}{2^k} \sum_{j=0}^{n} \binom{n}{j} f_k^{(j)}(0) f_q^{(n-j)}(0) \\
&= \frac{1}{2} \sum_{k=0}^{q-1} \frac{1}{2^k} \sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\leqslant m} k^{\underline{\ell}} + \frac{1}{2^q} \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{2} \sum_{k=0}^{\infty} \frac{1}{2^k} f_k^{(j)}(0) \right) f_q^{(n-j)}(0) \\
&= \frac{1}{2} \sum_{k=0}^{q-1} \frac{1}{2^k} \sum_{\ell=0}^{n} \left\{ {n \atop \ell} \right\}_{\leqslant m} k^{\underline{\ell}} + \frac{1}{2^q} \sum_{j=0}^{n} \binom{n}{j} F_{j,\leqslant m} \left(\sum_{i=0}^{n} \left\{ {n-j \atop i} \right\}_{\leqslant m} q^{\underline{i}} \right).
\end{aligned}$$

The second identity follows in a similar manner. □

The previous theorem is now used to generate some congruences.

Corollary 57. *Let q, $n \in \mathbb{N}$. Then*

$$(2^q - 1) F_{n,\leqslant m} \equiv \sum_{\ell=1}^{q-1} 2^{q-\ell-1} \left(\sum_{i=0}^{n} \left\{ {n \atop i} \right\}_{\leqslant m} \ell^{\underline{i}} \right) \pmod{q},$$

$$(2^q - 1) F_{n,\geqslant m} \equiv \sum_{\ell=1}^{q-1} 2^{q-\ell-1} \left(\sum_{i=0}^{n} \left\{ {n \atop i} \right\}_{\geqslant m} \ell^{\underline{i}} \right) \pmod{q}.$$

The next statement is a generalization of Theorem 53.

Theorem 58. *The restricted/associated Fubini numbers satisfy the identities*

$$F_{n,\leqslant m} = \sum_{k=1}^{n}\sum_{i=0}^{k}(-1)^i\binom{n+1}{i}2^{n-k}\left(\sum_{j=0}^{n}{n \brace j}_{\leqslant m}(k-i)^{\underline{j}}\right),$$
$$F_{n,\geqslant m} = \sum_{k=1}^{n}\sum_{i=0}^{k}(-1)^i\binom{n+1}{i}2^{n-k}\left(\sum_{j=0}^{n}{n \brace j}_{\geqslant m}(k-i)^{\underline{j}}\right).$$

Proof. The proof is analogous to the one given for Theorem 53. Simply use the combinatorial observation (39). □

The final result in this section is an identity relating the associated Fubini numbers $F_{n,\geqslant k}$ with the Fubini numbers F_n and the incomplete Stirling numbers ${i \brace j}_{\leqslant k}$ and ${i \brace j}_{\geqslant k}$.

Theorem 59. *Let $n \in \mathbb{N}$. Then*

$$F_{n,\geqslant k} = F_n - \sum_{i=1}^{n}\sum_{k_1=0}^{i}\sum_{k_2=0}^{n-i}\binom{n}{i}(k_1+k_2)!{i \brace k_1}_{\leqslant k-1}{n-i \brace k_2}_{\geqslant k}. \tag{43}$$

Proof. Let $\pi = (\pi_1, \ldots, \pi_\ell) \in \mathbb{F}_n$ be an ordered partition. Write $[n] = A_\pi \cup B_\pi$ where $A_\pi := \{i \in [n] : \text{if } i \in \pi_j \text{ then } |\pi_j| < k\}$ and $B_\pi := \{i \in [n] : \text{if } i \in \pi_j \text{ then } |\pi_j| \geqslant k\}$. The sets $S_i := \{\pi \in \mathbb{F}_n : |A_\pi| = i\}$ provide the disjoint decomposition $\mathbb{F}_n = \bigcup_{i=0}^{n} S_i$ and it follows that

$$|S_i| = \binom{n}{i}\sum_{k_1=0}^{i}\sum_{k_2=0}^{n-i}\binom{k_1+k_2}{k_1}\underbrace{k_1!{i \brace k_1}_{\leqslant k-1}}_{\text{Blocks of } A_\pi}\underbrace{k_2!{n-i \brace k_2}_{\geqslant k}}_{\text{Blocks of } B_\pi}.$$

The identity follows from here. □

13 Arithmetical properties of the restricted/associated Fubini numbers

This section discusses some arithmetical properties of the Fubini numbers F_n and their generalizations. Particular emphasis is placed on congruences and p-adic valuations.

Proposition 60. *The Fubini numbers F_n are odd; that is, $\nu_2(F_n) = 0$.*

Proof. Proceed by induction and use the recurrence stated in Remark 45, to obtain

$$F_n \equiv \sum_{k=0}^{n-1}\binom{n}{k} = 2^n - 1 \equiv 1 \pmod 2.$$ □

The next results deal with congruences and valuations with respect to the prime $p = 3$.

Proposition 61. *The Fubini numbers satisfy*

$$F_n \equiv \begin{Bmatrix} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{Bmatrix} \pmod{3}. \tag{44}$$

Proof. Take $q = 3$ in Corollary 51 to obtain

$$7F_n \equiv 3nF_{n-1} + \sum_{j=1}^{2} 2^{2-j} j^n \equiv 2 + 2^n \pmod{3}. \tag{45}$$

This simplifies to $F_n \equiv 2^n - 1 \pmod{3}$ and the result follows. □

The next result gives the 3-adic valuation of F_n.

Theorem 62. *The 3-adic valuation of the Fubini number is given by*

$$\nu_3(F_n) = \begin{Bmatrix} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{Bmatrix}. \tag{46}$$

Proof. Proposition 61 shows that $F_n \equiv 1 \pmod{3}$ for n odd. Therefore, $\nu_3(F_n) = 0$ for n odd. Now assume n is even. Corollary 51 with $q = 3$ gives

$$F_n \equiv 3nF_{n-1} + 2^{n+2} - 1 \pmod{9}. \tag{47}$$

A symbolic calculation shows that, starting at $n = 2$, the numbers F_n modulo 9 are periodic with repeating pattern $\{3, 4, 3, 1, 3, 7\}$. This is proved by induction, using (47). To reduce the calculation, write $n = 6t + a$, with $t \in \mathbb{N}$ and $a \in \{0, 2, 4\}$, since n is even. Then

$$2^{n+2} = 2^{6t+a+2} = (2^6 = 64)^t \times 2^{a+2} \equiv 2^{a+2} \pmod{9}. \tag{48}$$

Consider first the case $n \equiv 2 \pmod{6}$. Then, by induction, $F_{n-1} \equiv 7 \pmod{9}$. It is required to show that $F_n \equiv 3 \pmod{9}$. Then (47) gives

$$F_n \equiv 3(6t + 2)F_{n-1} + 2^{2+1} - 1 \equiv 6F_{n-1} + 7 - 1 \equiv 6 \times 7 + 6 = 48 \equiv 3 \pmod{9}, \tag{49}$$

as required. The other two choices for a are dealt with in a similar manner.

Proposition 61 shows that $F_n \equiv 0 \pmod{3}$ and by the argument above $F_n \not\equiv 0 \pmod{9}$. Therefore $\nu_3(F_n) = 1$. The proof is complete. □

Remark 63. The structure of the p-adic valuation for $p \geqslant 5$ prime is described by a *valuation tree*, as introduced in Remark 1. These valuation trees also appeared in [2] in the context of the 2-adic valuation of the Stirling numbers $\left[n \atop k \right]$. The results given next are *all experimental*.

Divide $\mathbb{N}$ into four classes according to the residue of n modulo 4. The numbers F_n corresponding to indices in three of these classes have 5-adic valuation *independent* of n.

For example, if $n \equiv 1 \pmod 4$, it turns out that $\nu_5(F_n) = 0$; that is, $\nu_5(F_{4n+1}) = 0$ for all $n \in \mathbb{N}$. Similarly, $\nu_5(F_{4n+2}) = 0$ and $\nu_5(F_{4n+3}) = 0$ for all $n \in \mathbb{N}$. (It is a coincidence that each of the classes has the same value, namely 0. The important point is that this value is the same for each index in the class). In the case $n \equiv 0 \pmod 4$, the value $\nu_5(F_{4n})$ *does depend* on the index n. Therefore, based on experience acquired with other sequences, these numbers are split modulo 5 to produce five classes of indices: $20n$, $20n+4$, $20n+8$, $20n+12$, and $20n+16$. Now there are four classes for which the valuation of the Fubini number with index in the class, has a valuation independent of the index. For example, $\nu_5(F_{20n}) = 1$ for all $n \in \mathbb{N}$. Similarly

$$\nu_5(F_{20n+8}) = 1, \quad \nu_5(F_{20n+12}) = 1 \quad \text{and} \quad \nu_5(F_{20n+16}) = 1 \quad \text{for all } n \in \mathbb{N}. \tag{50}$$

As before, in the remaining case $n \equiv 4 \pmod{20}$, the value $\nu_5(F_{20n+4})$ *does depend* on the index n. The process is continued by splitting the set of indices of the form $20n+4$ into five classes modulo 100.

$$100n+4, \quad 100n+24, \quad 100n+44, \quad 100n+64, \quad 100n+84. \tag{51}$$

Conjecture 64. The process described above continues ad infinitum. At each step, there is a single class where the valuation is not constant. Moreover, this phenomena happens for every prime $p \geqslant 5$.

Definition 65. The *restricted Fubini numbers* $F_{n,\leqslant m}$ count all the partitions of $[n]$ into blocks of length at most m, where the order in which the blocks appear is taken into consideration. The corresponding *associated Fubini numbers* $F_{n,\geqslant m}$ are defined in a similar form, now with blocks of length at least m.

Remark 66. Mező [38] established the recurrence

$$F_{n,\leqslant m} = \sum_{\ell=1}^{m} \binom{n}{\ell} F_{n-\ell,\leqslant m}, \tag{52}$$

$$F_{n,\leqslant m} = F_n, \ \ n \leqslant m. \tag{53}$$

Similarly, there is a recurrence for the associated Fubini numbers as

$$F_{n,\geqslant m} = \sum_{\ell=m}^{n} \binom{n}{\ell} F_{n-\ell,\geqslant m},$$

$$F_{n,\geqslant m} = 0, \ \ n < m.$$

This section discusses some elementary arithmetic properties of the numbers $F_{n,\leqslant m}$ and $F_{n,\geqslant m}$. Lengyel [32] establish some additional arithmetical properties for this sequence. The first result states that, in the case $p = 2$, these numbers are related to the restricted and associated Stirling numbers.

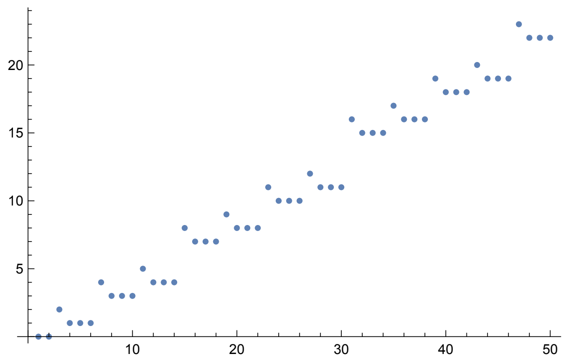

Figure 4: The 2-adic valuation of $F_{n,\leqslant 2}$.

Lemma 67. *Let* n, $m \in \mathbb{N}$. *Then*

$$F_{n,\leqslant m} \equiv \left\{ {n \atop 1} \right\}_{\leqslant m} \pmod 2 \quad \textit{and} \quad F_{n,\geqslant m} \equiv \left\{ {n \atop 1} \right\}_{\geqslant m} \pmod 2.$$

Proof. Use Corollary 57 with $q = 2$ and observe that, since $\left\{ {n \atop 1} \right\}_{\leqslant m} = 1$ for $1 \leqslant n \leqslant m$ and 0 otherwise, then $F_{n,\leqslant m}$ is odd for $1 \leqslant n \leqslant m$ and even for all $n > m$. $\square$

The 2-adic behavior of the restricted Fubini numbers is discusses next. Figure 4 shows the first few values of the sequence $\nu_2\left(\{F_{n,\leqslant 2}\}\right)$.

An analytic expression explaining this figure is presented in the next theorem.

Theorem 68. *The 2-adic valuation for the restricted Fubini numbers* $F_{n,\leqslant 2}$ *is*

$$\nu_2(F_{n,\leqslant 2}) = \begin{cases} \frac{n}{2} - s_2\left(\frac{n}{2}\right), & \textit{if } n \equiv 0 \pmod 4, \\ \frac{n-1}{2} - s_2\left(\frac{n-1}{2}\right), & \textit{if } n \equiv 1 \pmod 4, \\ \frac{n-2}{2} - s_2\left(\frac{n-2}{2}\right), & \textit{if } n \equiv 2 \pmod 4, \\ \frac{n-3}{2} - s_2\left(\frac{n+1}{2}\right) + 3, & \textit{if } n \equiv 3 \pmod 4; \end{cases}$$

where $s_2(n)$ *is the sum of the digits of* n *in its binary expansion.*

Proof. The proof is by induction, and is divided into four cases according to the residue of n modulo 4. The symbols O_i denote an odd number. If $n = 4k$ then (52) with $m = 2$ and the induction hypothesis yields

$$\begin{aligned} F_{4k,\leqslant 2} &= 4kF_{4k-1,\leqslant 2} + \frac{(4k-1)(4k)}{2} F_{4k-2,\leqslant 2} \\ &= 4k\left(2^{2k-2-s_2(2k)+3}O_1\right) + 2k(4k-1)\left(2^{2k-2-s_2(2k-2)}O_2\right) \\ &= 2^{2k-s_2(2k)+3}kO_1 + 2^{2k-1-s_2(2k-2)}kO_3 \\ &= 2^{2k-s_2(2k-2)-1}k\left(2^{4-s_2(2k)+s_2(2k-2)}O_1 + O_3\right) \\ &= 2^{2k-s_2(2k-2)-1}kO_4. \end{aligned}$$

If k is odd, then $F_{4k,\leqslant 2} = 2^{2k-1-s_2(2k-2)} O_5$, and it follows that

$$\nu_2(F_{4k,\leqslant 2}) = 2k - s_2(2k-2) - 1 = 2k - s_2(2k),$$

using $s_2(x) - s_2(x-1) = -\nu_2(x)$. This is a direct consequence of Legendre's formula for the p-adic valuation of factorials: $\nu_p(x!) = \dfrac{x - s_p(x)}{p-1}$. On the other hand, if k is even, then

$$\nu_2(F_{4k,\leqslant 2}) = 2k - (s_2(2k-2) - \nu_2(k) + 1) = 2k - s_2(2k).$$

The remaining cases are analyzed in a similar manner. □

Symbolic computations produce the next statement. The reader is invited to produce a proof in the style presented for the previous theorem.

Theorem 69. *The* 3*-adic valuation of* $F_{n,\leqslant 2}$ *has a* 3*-block structure; that is, for* $n \in \mathbb{N}$*:*

$$\nu_3(F_{3n-1,\leqslant 2}) = \nu_3(F_{3n,\leqslant 2}) = \nu_3(F_{3n+1,\leqslant 2}). \tag{54}$$

The common value is given by

$$\nu_3(F_{3n,\leqslant 2}) = n + \nu_3(n!) = \frac{3n - s_3(n)}{2}, \tag{55}$$

where $s_3(n)$ *is the sum of the digits in the expansion of* n *in base* 3*.*

Problem 70. Describe the valuations $\nu_5(F_{n,\leqslant 2})$.

14 *r*-Fubini Numbers

The r-Fubini numbers, $F_{n,r}$, have appeared in [39]. They are defined as the number of ordered r-partitions of $[n+r]$. Thus,

$$F_{n,r} = \sum_{k=0}^{n} (k+r)! \left\{ {n \atop k} \right\}_r .$$

The first statement gives a recurrence for $F_{n,r}$. The initial condition $F_{0,r} = r!$ is clear from the definition.

Theorem 71. *Let* $n \in \mathbb{N}$*. Then the* r*-Fubini numbers satisfy the recurrence*

$$F_{n,r} = \sum_{k=0}^{n-1} \binom{n}{k} F_{k,r} + r \sum_{k=0}^{n} \binom{n}{k} F_{k,r-1}$$

with initial condition $F_{n,0} = F_n$*.*

Proof. Assume the last block in an ordered r-partition of $[n+r]$ is non-special and has k elements, with $1 \leqslant k \leqslant n$. There are $\binom{n}{k}$ ways to choose these elements and the remaining $n-k$ elements can be ordered in $F_{n-k,r}$ ways. If the last block is special then the number of choices is $r\binom{n}{k-1}F_{n-(k-1),r-1}$. Summing over k gives

$$\begin{aligned} F_{n,r} &= \sum_{k=1}^{n}\binom{n}{k}F_{n-k,r} + r\sum_{k=1}^{n+1} n\binom{n}{k-1}F_{n-(k-1),r-1} \\ &= \sum_{k=0}^{n-1}\binom{n}{k}F_{k,r} + r\sum_{k=0}^{n}\binom{n}{k}F_{k,r-1}. \end{aligned}$$

□

The special case $r = 0$ gives the next result, stated in Remark 44.

Corollary 72. *The Fubini numbers satisfy the recurrence*

$$F_n = \sum_{k=0}^{n-1}\binom{n}{k}F_k, \quad n \geqslant 1. \tag{56}$$

Proposition 73. *Let p be a prime. Then*

$$F_{p,r} \equiv 2r! + rF_{p,r-1} \pmod{p}.$$

In particular, since $F_p \equiv 1 \pmod{p}$, then

$$F_{p,r} \equiv r!(2r-1) \pmod{p}.$$

The next result gives an exponential generating function.

Theorem 74. *The exponential generating function for $F_{n,r}$ is*

$$F_r(x) := \sum_{n=0}^{\infty} F_{n,r}\frac{x^n}{n!} = \frac{r!e^{rx}}{(2-e^x)^{r+1}}.$$

Proof. Theorem 71 gives

$$\begin{aligned} 2F_r(x) &= 2\sum_{n=0}^{\infty} F_{n,r}\frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}F_{k,r}\frac{x^n}{n!} + r\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n}{k}F_{k,r-1}\frac{x^n}{n!} \\ &= \sum_{k=0}^{\infty}F_{k,r}\frac{x^k}{k!}\sum_{n=0}^{\infty}\frac{x^n}{n!} + r\sum_{k=0}^{\infty}F_{k,r-1}\frac{x^k}{k!}\sum_{n=0}^{\infty}\frac{x^n}{n!} \\ &= e^xF_r(x) + re^xF_{r-1}(x). \end{aligned}$$

Then

$$F_r(x) = \frac{re^x}{2-e^x}F_{r-1}(x) = \cdots = \frac{r!e^{rx}}{(2-e^x)^r}F_0(x).$$

The value $F_0(x) = 1/(2-e^x)$, given in Remark 45, yields the desired result. □

Theorem 75 below generalizes the identity (34).

Theorem 75. *Let n, $r \in \mathbb{N}$. Then*

$$F_{n,r} = \frac{r!}{2^{r+1}} \sum_{\ell=0}^{\infty} \frac{1}{2^{\ell}} \binom{r+\ell}{\ell} (r+\ell)^n,$$
$$= \frac{1}{2^{r+1}} \sum_{\ell=0}^{\infty} \frac{1}{2^{\ell}} (r+\ell)^{\underline{r}} (r+\ell)^n.$$

Proof. Theorem 74 yields

$$\sum_{n=0}^{\infty} F_{n,r} \frac{x^n}{n!} = \frac{r! e^{rx}}{2^{r+1}\left(1 - \frac{1}{2}e^x\right)^{r+1}} = \frac{r! e^{rx}}{2^{r+1}} \sum_{\ell=0}^{\infty} \binom{r+\ell}{\ell} \left(\frac{1}{2}e^x\right)^{\ell}$$
$$= \frac{r!}{2^{r+1}} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{1}{2^{\ell}} \binom{r+\ell}{\ell} (\ell+r)^n \frac{x^n}{n!}.$$

Comparing the n-th coefficient gives the desired result. □

The next statement is based on an *experimental observation*:

Problem 76. The 2-adic valuation of $F_{n,r}$ is independent of the index n. Moreover

$$\nu_2(F_{n,2r}) = \nu_2(F_{n,2r+1}). \tag{57}$$

with common expression

$$\nu_2(F_{n,2r}) = \nu_2(F_{n,2r+1}) = 2r + 1 - s_2(r+1) - \nu_2(r+1), \tag{58}$$

for every $r \in \mathbb{N}$.

Problem 77. Develop similar results for the associated r-Fubini numbers $F_{n,r}$. The pattern for $\nu_3(F_{n,r})$ is relatively simple. Conduct a similar study for the corresponding restricted and associated families.

15 Generalized Factorial Numbers

Given a permutation π on n symbols, it can be written, uniquely up to order, as a product of disjoint cycles. The *Stirling number of the first kind*, denoted by $\left[{n \atop k}\right]$, counts the number of permutations of n elements containing k cycles. Since

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} = n!, \tag{59}$$

the sum of the left is also called *factorial numbers*. In this section similar numbers, counting permutations on n elements with restrictions on the size of cycles are introduced.

In the case where the restriction is that none of the cycles of a permutation of n contain more than m items, one obtains the *restricted Stirling numbers of the first kind*, denoted by $\left[{n \atop k}\right]_{\leqslant m}$ (cf. [18, 38, 44]). The *associated Stirling numbers of the first kind* $\left[{n \atop k}\right]_{\geqslant m}$ counts the case where the size of the cycles are at least m.

The corresponding r-generalizations, those where the first r elements are in distinct cycles (cf. [6, 9, 28]), produce the *restricted r-Stirling numbers of the first kind* $\left[{n \atop k}\right]_{\leqslant m,r}$. The *associated Stirling numbers of the first kind* $\left[{n \atop k}\right]_{\geqslant m}$ and the *associated r-Stirling numbers of the first kind* $\left[{n \atop k}\right]_{\geqslant m,r}$ are defined similarly.

The first statement is a combinatorial proof of an identity of Howard [25].

Theorem 78. *For k, $n \in \mathbb{N}$, the identity*

$$\left[{n \atop n-k}\right] = \sum_{\ell=0}^{k} \binom{n}{2k-\ell} \left[{2k-\ell \atop k-\ell}\right]_{\geqslant 2}$$

holds.

Proof. The change of indices $j = n - k - \ell$, converts the desired identity into

$$\left[{n \atop k}\right] = \sum_{\ell=0}^{n-k} \binom{n}{k-\ell} \left[{n-(k-\ell) \atop \ell}\right]_{\geqslant 2}.$$

The right-hand side corresponds to the decomposition of the permutation of n with k cycles into its fixed points and the cycles of length $\geqslant 2$. □

The next identity admits a proof similar to the one presented for Theorem 36.

Theorem 79. *Let n, k, $m \in \mathbb{N}$. Then*

$$\left[{n \atop k}\right]_{\geqslant m} = \sum_{\ell=0}^{n} \binom{n}{m\ell} \frac{(ml)!}{m^{\ell}\ell!} \left[{n-m\ell \atop k-\ell}\right]_{\geqslant m+1}.$$

A new sequence of numbers is introduced next.

Definition 80. Let $A_{n,\leqslant m,r}$ the total number of r-permutations of $[n+r]$ with the condition that each cycle has size at most m. This sequence is called *restricted r-factorial numbers*. The *associated r-factorial numbers* $A_{n,\geqslant m,r}$ are defined in a similar manner.

The following identities are immediate:

$$A_{n,\leqslant m,r} = \sum_{k=0}^{n} \left[{n \atop k}\right]_{\leqslant m,r} \quad \text{and} \quad A_{n,\geqslant m,r} = \sum_{k=0}^{n} \left[{n \atop k}\right]_{\geqslant m,r}.$$

Remark 81. In the special case $m = 2$ one has $A_{n,\leqslant 2,r} = B_{n\leqslant 2,r}$. The numbers $A_{n,\geqslant 2,r}$ are the *r-derangements numbers*, discussed in [60, 59].

The next statements are analogous to results discussed in the previous sections. The details of the proofs are left to the reader.

Theorem 82. *For $n \geqslant 1$, $r \geqslant 0$ and $m \geqslant 2$ the restricted r-factorial numbers satisfy the recurrence relation*

$$A_{n,\leqslant m,r} = r\sum_{\ell=0}^{m-2}(\ell+1)(n-1)^{\overline{\ell}}A_{n-1-\ell,\leqslant m,r-1} + \sum_{\ell=0}^{m-1}(n-1)^{\overline{\ell}}A_{n-1-\ell,\leqslant m,r}.$$

Moreover,

$$A_{n,\leqslant m,r} = \sum_{\ell=0}^{m-1} n^{\overline{\ell}}A_{n-\ell,\leqslant m,r-1}.$$

Theorem 83. *For $n \geqslant 1$, $r \geqslant 0$ and $m \geqslant 1$ the associated r-factorial numbers satisfy the recurrence relation*

$$A_{n,\geqslant m,r} = r\sum_{\ell=m-2}^{n-1}(\ell+1)(n-1)^{\overline{\ell}}A_{n-1-\ell,\geqslant m,r-1} + \sum_{\ell=m-1}^{n-1}(n-1)^{\overline{\ell}}A_{n-1-\ell,\geqslant m,r}.$$

Moreover,

$$A_{n,\geqslant m,r} = \sum_{\ell=m-1}^{n} n^{\overline{\ell}}A_{n-\ell,\geqslant m,r-1}.$$

As before, some polynomials are constructed with these families of numbers.

Definition 84. The *restricted r-factorial polynomials*, $A_{n,\leqslant m,r}(x)$, and the *associated r-factorial polynomials*, $A_{n,\geqslant m,r}(x)$ are defined by the expressions

$$A_{n,\leqslant m,r}(x) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{\leqslant m,r}x^k \quad \text{and} \quad A_{n,\geqslant m,r}(x) = \sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{\geqslant m,r}x^k.$$

The exponential generating functions of these polynomials are stated next.

Theorem 85. *The exponential generating function of the restricted/associated r-factorial polynomials are*

$$\sum_{n=0}^{\infty}A_{n,\leqslant m,r}(x)\frac{z^n}{n!} = \left(1+z+z^2+\cdots+z^{m-1}\right)^r e^{x\left(z+\frac{z^2}{2}+\cdots+\frac{z^m}{m}\right)},$$
$$\sum_{n=0}^{\infty}A_{n,\geqslant m,r}(x)\frac{z^n}{n!} = \left(z^{m-1}+z^m+\cdots\right)^r e^{x\left(\frac{z^m}{m}+\frac{z^{m+1}}{m+1}+\cdots\right)}.$$

The next statement is the analogous to the Theorem 39.

Theorem 86. *For* t, $r \in \mathbb{N}$, *define*

$$h_{t,r}(x) = \left(1 + x + \frac{x}{2} + \cdots + \frac{x^m}{m}\right)^t \left(1 + x + x^2 + \cdots + x^{m-1}\right)^r.$$

Then

$$\frac{d^n}{dx^n} h_{t,r}(x)\bigg|_{x=0} = h_{t,r}^{(n)}(0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\leqslant m,r} t^{\underline{k}}. \tag{60}$$

Similarly, for t, $r \in \mathbb{N}$ *define* $w_{t,r}(x)$ *by*

$$w_{t,r}(x) = \left(1 + \frac{x^m}{m} + \frac{x^{m+1}}{m+1} + \cdots\right)^t \left(x^{m-1} + x^m + \cdots\right)^r.$$

Then

$$\frac{d^n}{dx^n} w_{t,r}(x)\bigg|_{x=0} = w_{t,r}^{(n)}(0) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\geqslant m,r} t^{\underline{k}}. \tag{61}$$

16 Conclusions

A variety of numbers of combinatorial origin are discussed. These sequences are obtained by restricting sizes of substructures in set partitions, either from above or below. Arithmetic properties include congruences and structures of their p-adic valuations were discussed.

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