The perimeter and the site-perimeter of set partitions

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Abstract

In this paper, we study the generating function for the number of set partitions of $[n]$ represented as bargraphs according to the perimeter/site-perimeter. In particular, we find explicit formulas for the total perimeter and the total site-perimeter over all set partitions of $[n]$.

Mathematics Subject Classifications: 05A18

1 Introduction

A *partition* of a positive integer n is a non-increasing sequence of positive integers, called parts, whose sum is n. A *composition* is a partition in which the parts may come in any order, as originally defined by MacMahon [11]. Compositions can be represented as bargraphs. A *bargraph* is a column convex polyomino such that the lower edge lies on a horizontal axis, when it is drawn on a regular planar lattice grid and is made up of square cells. Clearly, the number parts and the size of a composition is the number columns and the total number of cells in the representing bargraph, respectively. For instance, Figure 1 presents the bargraph 122341411. Recently, statistics on bargraphs have been

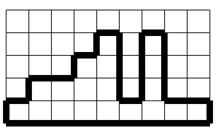

Figure 1: The bargraph 122341411

received a lot of attention. For instance, in [10, 17] it is found the generating function for the number of bargraphs according to the number of horizontal and up steps. In [14] it is studied the generating function for the number of bargraphs according to the number of interior vertices and edges. Moreover, Blecher et al. counted bargraphs according to statistics: descents [3], levels [1], peaks [2] and walls [4].

The enumeration of polyominoes according to their area and perimeter is an interesting problem in combinatorics [7, 9]. When one studies combinatorial families presented geometrically, the perimeter and the area are the most natural and most important statistics to be considered. The *perimeter* of a bargraph B, denoted by $per(B)$, is the number of edges on the boundary of B. The *site-perimeter* of a bargraph B, denoted by $sper(B)$, is the number of nearest-neighbor cells outside the boundary of B. The perimeter and the site perimeter of words were studied in [5] and [6], respectively. Motivated by these results, we extend the study of perimeter/site-perimeter to set partitions.

A *set partition* of $[n]$ is any collection of nonempty, pairwise disjoint subsets, called *blocks*, whose union is $[n]$ (there is a single empty set partition of $[0]$ which has no blocks). We denote the set of all set partitions of $[n]$ by $P(n)$ and the set of all set partitions of $[n]$ with k blocks by $P(n,k)$. We will write a set partition $\pi \in P(n,k)$ as $\pi = B_1/B_2/\cdots/B_k$, where $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$. Equivalently, we will write a set partition by the *canonical sequential form* $\pi = \pi_1\pi_2\cdots\pi_n$, wherein $i \in B_{\pi_i}$ for $i \in [n]$ (see, e.g., [12, 19]). For example, the set partition $\pi = \{1,4,8,9\}/\{2,3\}/\{5\}/\{6,7\}$ has the canonical sequential form $\pi = 122134411$. We will represent each set partition π by the corresponding bargraph of the canonical sequential form of π. For instance, Figure 1 represents the bargraph of $\pi = \{1,6,8,9\}/\{2,3\}/\{4\}/\{5,7\}$. Recently, statistics on bargraphs of set partitions have been investigated by several authors (see, e.g., [13, 15, 16]).

The aim of this paper is to study the perimeter (see Section 2) and the site-perimeter (see Section 3) of set partitions. For example, the perimeter and the site-perimeter of the set partition 122341411 are 32 and 24, respectively. In particular, we show that the total of the half of the perimeter over all set partitions of $[n]$ is given by (see Corollary 5)

$$a_n = \frac{n+3}{6}B_{n+1} - \frac{5}{36}B_{n+2} + \frac{18n+13}{18}B_n - \frac{6n+7}{36}B_{n-1}$$

and the total of the site-perimeter over all set partitions of $[n]$ is given by (see Corollary 7)

$$b_n = \left(\frac{n}{4}+\frac{73}{72}\right)B_{n+1} - \frac{11}{48}B_{n+2} + \left(\frac{4n}{3}+\frac{91}{48}\right)B_n + \left(\frac{n}{4}-\frac{67}{72}\right)B_{n-1} + \left(\frac{n}{6}-\frac{23}{144}\right)B_{n-2},$$

where B_n is the nth Bell number.

In order to obtain asymptotic estimates for the moments as well as the limiting distribution, we need

$$B_{n+h} = B_n\frac{(n+h)!}{n!r^h}\left(1+O\left(\frac{\log n}{n}\right)\right) \tag{1}$$

uniformly for $h = O(\log n)$, where B_n is the nth Bell number and r is the positive root of $re^r = n+1$. See [8] for an even stronger form that includes further terms in the asymptotic expansion. We say that the sequence f_n is asymptotically equivalent to the sequence g_n, denoted by $f_n \sim g_n$, if $\lim_{n\to\infty} \frac{f_n}{g_n} = 1$. So,

$$\begin{aligned}\frac{a_n}{B_n} &\sim \frac{n+3}{6}\frac{B_{n+1}}{B_n} - \frac{5}{36}\frac{B_{n+2}}{B_n} + \frac{18n+13}{18} - \frac{6n+7}{36}\frac{B_{n-1}}{B_n}\\ &\sim \frac{n+3}{6}\frac{B_{n+1}}{B_n} - \frac{5}{36}\frac{B_{n+2}}{B_n}\end{aligned}$$

and

$$\begin{aligned}\frac{b_n}{B_n} &\sim \frac{36n+146}{144}\frac{B_{n+1}}{B_n} - \frac{33}{144}\frac{B_{n+2}}{B_n} + \frac{192n+273}{144} - \frac{36n-134}{144}\frac{B_{n-1}}{B_n} + \frac{24n-23}{144}\frac{B_{n-2}}{B_n}\\ &\sim \frac{36n+146}{144}\frac{B_{n+1}}{B_n} - \frac{33}{144}\frac{B_{n+2}}{B_n},\end{aligned}$$

which, by (1), leads to the following corollary.

Corollary 1. *Asymptotically,*

$$a_n = \frac{n^2 B_n}{6(\log(n) - \log\log n)}\left(1 - \frac{5}{6(\log n - \log\log n)}\right)\left(1 + O\left(\frac{\log n}{n}\right)\right)$$

and

$$b_n = \frac{n^2 B_n}{48(\log(n) - \log\log n)}\left(12 - \frac{11}{\log n - \log\log n}\right)\left(1 + O\left(\frac{\log n}{n}\right)\right).$$

2 The perimeter of set partitions

Let $P_k(x,q) = \sum_{n\geqslant k}\sum_{\pi\in P_{n,k}} x^n q^{\frac{1}{2}per(\pi)}$ be the generating function for the number of set partitions of n with exactly k blocks, according to the half of the perimeter. Generally, let $P_k(x,q|a_1a_2\cdots a_s) = \sum_{n\geqslant k}\sum_{\pi=\pi' a_1a_2\cdots a_s\in P_{n,k}} x^n q^{\frac{1}{2}per(\pi)}$ be the generating function for the number of set partitions $\pi = \pi' a_1a_2\cdots a_s$ of n with exactly k blocks, according to the half of the perimeter. We define $P_0(x,q) = 1$. Since each set partition with one block has the form $11\cdots 1$, we have $P_1(x,q) = \frac{xq^2}{1-xq}$. By definitions we have

$$P_k(x,q|a) = \sum_{b=1}^{k} P_k(x,q|ba) = x\sum_{b=1}^{a} q^{a-b+1}P_k(x,q|b) + xq\sum_{b=a+1}^{k} P_k(x,q|b), \qquad (2)$$

for all $1 \leqslant a \leqslant k-1$. Moreover,

$$P_k(x,q|k) = x\sum_{b=1}^{k} q^{k-b+1}P_k(x,q|b) + x\sum_{b=1}^{k-1} q^{k-b+1}P_{k-1}(x,q|b). \qquad (3)$$

Define $P_k(x,q,v)=\sum_{a=1}^{k}P_k(x,q|a)v^{a-1}$. Then (2)-(3) can be written as

$$\begin{aligned}P_k(x,q,v)&=P_k(x,q|k)v^{k-1}+\frac{x}{1-qv}(qP_k(x,q,v)-q^kv^{k-1}P_k(x,q,1/q))\\&\quad+\frac{xq}{1-v}(P_k(x,q,1)-P_k(x,q,v))\end{aligned}$$

with

$$P_k(x,q|k)=xq^kP_k(x,q,1/q)+xq^kP_{k-1}(x,q,1/q).$$

Thus, we can state the following result.

Proposition 2. *We have*

$$\begin{aligned}P_k(x,q,v)&=xq^kv^{k-1}P_{k-1}(x,q,1/q)+\frac{xq}{1-qv}(P_k(x,q,v)-q^kv^kP_k(x,q,1/q))\\&\quad+\frac{xq}{1-v}(P_k(x,q,1)-P_k(x,q,v))\end{aligned}$$

with $P_1(x,q,v)=\frac{xq^2}{1-xq}$.

Proposition 2 with $q=1$ gives $P_k(x,1,v)=xv^{k-1}P_{k-1}(x,1,1)+\frac{x(1-v^k)}{1-v}P_k(x,1,1)$ with $P_1(x,1,v)=\frac{x}{1-x}$. If we take $v\to1$, then we obtain $P_k(x,1,1)=\frac{x}{1-kx}P_{k-1}(x,1,1)$ with $P_1(x,1,1)=\frac{x}{1-x}$, which leads to $P_k(x,1,1)=\frac{x^k}{(1-x)\cdots(1-kx)}$, as expected. Thus, for all $k\geqslant1$,

$$P_k(x,1,v)=\frac{x^k(v^{k-1}(1-v)(1-kx)+x(1-v^k))}{(1-v)(1-x)\cdots(1-kx)}. \tag{4}$$

Define $F_k(x,v)=\frac{\partial}{\partial q}P_k(x,q,v)\mid_{q=1}$ and $G_k(x,v)=\frac{\partial}{\partial v}P_k(x,1,v)$. By Proposition 2, we have

$$\begin{aligned}F_k(x,v)&=kxv^{k-1}P_{k-1}(x,1,1)+xv^{k-1}F_{k-1}(x,1)-xv^{k-1}G_{k-1}(x,1)\\&\quad+\frac{x}{(1-v)^2}(P_k(x,1,v)-v^kP_k(x,1,1))\\&\quad+\frac{x}{1-v}(F_k(x,v)-kv^kP_k(x,1,1)-v^kF_k(x,1)+v^kG_k(x,1))\\&\quad+\frac{x}{1-v}(P_k(x,1,1)-P_k(x,1,v))+\frac{x}{1-v}(F_k(x,1)-F_k(x,v))\end{aligned}$$

with $F_1(x,v)=\frac{x(2-x)}{(1-x)^2}$. Thus, by using (4) and taking $v\to1$, we obtain

$$F_k(x,1)=\frac{x}{1-kx}F_{k-1}(x,1)+\frac{x^k(12+3(k^2-5k+2)x-k(2k-7)(k-1)x^2)}{6(1-kx)\prod_{j=1}^{k}(1-jx)} \tag{5}$$

with $F_0(x,1)=0$. Thus by induction, we can state the following result.

Theorem 3. *The generating function $F_k(x,1)$ for the total of the half of the perimeter over all set partitions of $[n]$ with exactly k blocks is given by*

$$\frac{x^k}{6\prod_{j=1}^{k}(1-jx)}\sum_{i=1}^{k}\frac{12+3(i^2-5i+2)x-i(i-1)(2i-7)x^2}{1-ix}.$$

In order to study further the total of the half of the perimeter over all set partitions of $[n]$, we consider the exponential generating function $E_k(x)=\sum_{n\geqslant 0}[x^n](F_k(x,1))\frac{x^n}{n!}$, where $[x^n]f(x)$ denotes the coefficient of x^n in the generating function $f(x)$. Note that $\frac{x^k}{\prod_{j=1}^{k}(1-jx)}=\sum_{n\geqslant 0}S_{n,k}x^n$ and $\frac{1}{k!}(e^x-1)^k=\sum_{n\geqslant 0}S_{n,k}\frac{x^n}{n!}$, where $S_{n,k}$ denotes the Stirling number of the second kind (for example, see [12]). So, by (5), we have

$$(1-kx)F_k(x,1)=xF_{k-1}(x,1)+\frac{12+3(k^2-5k+2)x-k(2k-7)(k-1)x^2}{6}\sum_{n\geqslant 0}S_{n,k}x^n$$

with $F_0(x,1)=0$. Thus, the exponential generating function $E_k(x)$ satisfies,

$$\begin{aligned}E_k(x)&=k\int_0^x E_k(t)dt+\int_0^x E_{k-1}(t)dt+\frac{2(e^x-1)^k}{k!}\\&\quad+\frac{k^2-5k+2}{2}\int_0^x\frac{(e^t-1)^k}{k!}dt-\frac{k(2k-7)(k-1)}{6}\int_0^x\int_0^t\frac{(e^r-1)^k}{k!}drdt\end{aligned}$$

with $E_0(x)=0$. Hence,

$$\begin{aligned}\frac{d^2}{dx^2}E_k(x)&=k\frac{d}{dx}E_k(x)+\frac{d}{dx}E_{k-1}(x)+\frac{2e^x(e^x-1)^{k-2}(ke^x-1)}{(k-1)!}\\&\quad+\frac{k^2-5k+2}{2}\frac{(e^x-1)^{k-1}e^x}{(k-1)!}-\frac{k(2k-7)(k-1)}{6}\frac{(e^x-1)^k}{k!}\end{aligned}$$

Define $E(x,y)=\sum_{k\geqslant 0}E_k(x)y^k$. By multiplying by y^k, and summing over $k\geqslant 1$, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial x^2}E(x,y)&=y\frac{\partial^2}{\partial x\partial y}E(x,y)+y\frac{\partial}{\partial x}E(x,y)\\&\quad+\frac{1}{6}y(y^2e^{3x}+9ye^{2x}+3(2-y^2)e^x+2y^2+3y)e^{y(e^x-1)}.\end{aligned}$$

Solving this partial differential equation under the condition $\frac{\partial}{\partial x}E(x,y)\mid_{x=0}=2y$, we obtain the following result.

Theorem 4. *The exponential generating function $E(x,y)$ for the total of the half of the perimeter over all set partitions of $[n]$ with exactly k blocks is given by*

$$\frac{y}{36}\int_0^x((6t-5)y^2e^{3t}+9(6t+1)ye^{2t}+9(y^2+4t+8)e^t-y(4y+9))e^{ye^t-y}dt.$$

By Theorem 4 we have

$$\begin{aligned}
&\frac{\partial}{\partial x}E(x,y)\\
&=\frac{y}{36}((6x-5)y^2e^{3x}+9(6x+1)ye^{2x}+9(y^2+4x+8)e^x-y(4y+9))e^{ye^x-y}\\
&=\frac{6x-5}{36}\frac{\partial^3}{\partial x^3}e^{ye^x-y}+\frac{3x+2}{3}\frac{\partial^2}{\partial x^2}e^{ye^x-y}+\frac{9y^2-6x+53}{36}\frac{\partial}{\partial x}e^{ye^x-y}-\frac{4y^3+9y^2}{36}e^{ye^x-y}.
\end{aligned}$$

By comparing the coefficient of $x^ny^k/n!$ in both sides, we get the following formula.

Corollary 5. *The total of the half of the perimeter over all set partitions of* $[n+1]$ *with exactly* k *blocks is given by*

$$\frac{n+4}{6}S_{n+2,k}-\frac{5}{36}S_{n+3,k}+\frac{36n+53}{36}S_{n+1,k}+\frac{1}{4}S_{n+1,k-2}-\frac{n}{6}S_{n,k}-\frac{1}{9}S_{n,k-3}-\frac{1}{4}S_{n,k-2},$$

and the total of the half of the perimeter over all set partitions of $[n+1]$ *is given by*

$$\frac{n+4}{6}B_{n+2}-\frac{5}{36}B_{n+3}+\frac{18n+31}{18}B_{n+1}-\frac{6n+13}{36}B_n,$$

where $S_{n,k}$ *denotes the Stirling number of the second kind and* B_n *denotes the nth Bell number.*

3 The site-perimeter of set partitions

Let $Q_k(x,q)=\sum_{n\geqslant k}\sum_{\pi\in P_{n,k}}x^nq^{sper(\pi)}$ be the generating function for the number of set partitions of n with k blocks according to the site-perimeter. Generally, let

$$Q_k(x,q|a_1a_2\cdots a_s)=\sum_{n\geqslant k}\sum_{\pi=\pi'a_1a_2\cdots a_s\in P_{n,k}}x^nq^{sper(\pi)}$$

be the generating function for the number of set partition $\pi=\pi'a_1a_2\cdots a_s$ of n with k blocks according to the site-perimeter. We define $Q_0(x,q)=1$. Since each set partition with one block has the form $11\cdots1$, we have

$$Q_1(x,q)=\frac{xq^4}{1-xq^2}.$$

Since each set partition with two blocks has either the form $11\cdots12\pi'2$ or the form $11\cdots12\pi'1$, where π' is a word over alphabet $\{1,2\}$, we obtain

$$Q_2(x)=\frac{q^7x^2}{1-2q^2x}+\left(\frac{q^6x^2}{1-2q^2x}-\frac{q^6x^2}{1-q^2x}\right)=\frac{q^7x^2(1+qx-q^2x)}{(1-2q^2x)(1-q^2x)}.$$

Clearly, $Q_k(x,q) = \sum_{a=1}^{k} Q_k(x,q|a)$ and $Q_k(x,q|a) = \sum_{b=1}^{k} Q_k(x,q|ba)$ for all $1 \leqslant a \leqslant k-1$. By the definitions we have

$$Q_k(x,q|a) = \sum_{b=1}^{a-1} Q_k(x,q|ba) + xq^2 Q_k(x,q|a) + xq \sum_{b=a+1}^{k} Q_k(x,q|b), \tag{6}$$

$$\begin{aligned} Q_k(x,q|ba) &= xq^{2a-2b+1} \sum_{c=1}^{b} Q_k(x,q|cb) + x \sum_{c=b+1}^{a} q^{2a-b-c+2} Q_k(x,q|cb) \\ &\quad + x \sum_{c=a+1}^{k} q^{a-b+2} Q_k(x,q|cb), \end{aligned} \tag{7}$$

where $1 \leqslant b \leqslant a-1$ and $1 \leqslant a \leqslant k-1$. Moreover,

$$Q_k(x,q|k) = \sum_{b=1}^{k-1} Q_k(x,q|bk) + xq^2 Q_k(x,q|k), \tag{8}$$

$$\begin{aligned} Q_k(x,q|bk) &= xq^{2k-2b+1} \sum_{c=1}^{b} (Q_k(x,q|cb) + Q_{k-1}(x,q|cb)) \\ &\quad + x \sum_{c=b+1}^{k-1} q^{2k-b-c+2} (Q_k(x,q|cb) + Q_{k-1}(x,q|cb)) + xq^{k-b+2} Q_k(x,q|kb), \end{aligned} \tag{9}$$

where $1 \leqslant b \leqslant k-1$.

Our goal is to find a recurrence relation for the generating function

$$Q'_k(x) = \frac{\partial}{\partial q} Q_k(x,q) \mid_{q=1} .$$

To do that, let $H_k(x) = \frac{x^k}{\prod_{j=1}^{k}(1-jx)}$ to be the generating function for the number of set partitions of n with exactly k blocks (see [12]). Clearly,

$$Q_k(x,1|a) = xH_k(x), \qquad a = 1,2,\ldots,k-1, \tag{10}$$

$$Q_k(x,1|k) = xH_{k-1}(x) + xH_k(x). \tag{11}$$

Define $Q'_k(x|a) = \frac{\partial}{\partial q} Q_k(x,q|a) \mid_{q=1}$ and $Q'_k(x|ba) = \frac{\partial}{\partial q} Q_k(x,q|ba) \mid_{q=1}$. Differentiating (6)-(7) at $q=1$ gives

$$Q'_k(x|a) = \sum_{b=1}^{a-1} Q'_k(x|ba) + xQ_k(x,1|a) + x\sum_{b=a}^{k} Q_k(x,1|b) + x\sum_{b=a}^{k} Q'_k(x|b),$$

$$\begin{aligned} Q'_k(x|ba) &= x(2a-2b+1) \sum_{c=1}^{b} Q_k(x,1|cb) + x \sum_{c=b+1}^{a} (2a-b-c+2) Q_k(x,1|cb) \\ &\quad + x \sum_{c=a+1}^{k} (a-b+2) Q_k(x,1|cb) + x \sum_{c=1}^{k} Q'_k(x|cb), \end{aligned}$$

where $1 \leqslant b \leqslant a-1$ and $1 \leqslant a \leqslant k-1$. Thus, by (10)-(11), we have

$$Q'_k(x|ba) = \frac{x^2((a^2-b^2-a-b)x+2a-2b+4)}{2}H_k(x) + xQ'_k(x|b),$$

which, by substituting into $Q'_k(x|a)$ and using (10)-(11), implies

$$\begin{aligned}Q'_k(x|a) &= \sum_{b=1}^{a-1}\frac{x^2((a^2-b^2-a-b)x+2a-2b+4)}{2}H_k(x)\\ &\quad + (k-a+2)x^2H_k(x) + x^2H_{k-1}(x) + xQ'_k(x),\end{aligned}$$

for all $a = 1, 2, \ldots, k-1$. Therefore, by summing over $a = 1, 2, \ldots, k-1$ we obtain

$$\begin{aligned}Q'_k(x) &= \frac{1}{12}x(k-1)(k(k-2)(k-3)x^2+2k(k+1)x+12)H_k(x)\\ &\quad + Q'_k(x|k) + (k-1)xQ'_k(x).\end{aligned} \tag{12}$$

Now, we focus on $Q'_k(x|k)$. By differentiating (8)-(9), we have

$$Q'_k(x|k) = \sum_{b=1}^{k-1}Q'_k(x|bk) + 2xQ_k(x,1|k) + xQ'_k(x|k),$$

$$\begin{aligned}Q'_k(x|bk) &= x(2k-2b+1)\sum_{c=1}^{b}(Q_k(x,1|cb)+Q_{k-1}(x,1|cb))\\ &\quad + x\sum_{c=b+1}^{k-1}(2k-b-c+2)(Q_k(x,1|cb)+Q_{k-1}(x,1|cb))\\ &\quad + x(k-b+2)Q_k(x,1|kb) + xQ'_k(x|b) + xQ'_{k-1}(x|b),\end{aligned}$$

where $1 \leqslant b \leqslant k-1$. Thus, by substituting expression of $Q'_k(x|bk)$ into expression of $Q'_k(x|k)$ with using (10)-(11), we obtain

$$\begin{aligned}Q'_k(x|k) &= \frac{1}{6}(2k(k-2)(k-4)x^2+3(k^2-k-4)x+18)(1-(k-1)x)H_k(x)\\ &\quad + xQ'_k(x) + xQ'_{k-1}(x).\end{aligned} \tag{13}$$

By substituting (13) into (12), we obtain that the generating function $Q'_k(x)$ satisfies

$$\begin{aligned}Q'_k(x) &= \frac{36+6k(k-5)x-12(k^2-4k+2)x^2-k(k-1)(k-2)(3k-13)x^3}{12(1-kx)}H_k(x)\\ &\quad + \frac{x}{1-kx}Q'_{k-1}(x),\end{aligned} \tag{14}$$

for all $k \geqslant 3$. Note that $Q'_0(x) = 0$, $Q'_1(x) = \frac{2(2-x)}{(1-x)^2}$ and $Q'_2(x) = \frac{x^2(7-16x+9x^2-2x^3)}{(1-2x)^2(1-x)^2}$. By (14), we can introduce first values of the total of the site-perimeter over all set partitions of $[n]$ with k blocks, see Table 1.

$k\backslash n$	1	2	3	4	5	6	7
1	4	6	8	10	12	14	16
2	0	7	26	74	188	450	1040
3	0	0	10	71	345	1426	5398
4	0	0	0	13	151	1122	6816
5	0	0	0	0	16	276	2915

Table 1: The total of the site-perimeter over all set partitions of $[n]$ with k blocks, where $n = 1, 2, \dots, 7$ and $k = 1, 2, \dots, 5$.

To study further the total of the site-perimeter over all set partitions of $[n]$, we consider the exponential generating function $R_k(x) = \sum_{n\geqslant 0}[x^n](Q'_k(x))\frac{x^n}{n!}$. By (14), we have

$$\begin{aligned}&(1-kx)Q'_k(x)\\&=\left(3+\frac{1}{2}k(k-5)x-(k^2-4k+2)x^2-\frac{1}{12}k(k-1)(k-2)(3k-13)x^3\right)\sum_{n\geqslant k}S_{n,k}x^n\\&+xQ'_{k-1}(x)\end{aligned}$$

with $Q'_0(x) = 0$, $Q'_1(x) = \frac{2(2-x)}{(1-x)^2}$ and $Q'_2(x) = \frac{x^2(7-16x+9x^2-2x^3)}{(1-2x)^2(1-x)^2}$. Note that $H_k(x) = \frac{x^k}{\prod_{j=1}^{k}(1-jx)} = \sum_{n\geqslant 0} S_{n,k}x^n$ and $\frac{1}{k!}(e^x-1)^k = \sum_{n\geqslant 0} S_{n,k}\frac{x^n}{n!}$, where $S_{n,k}$ denotes the Stirling number of the second kind (for example, see [12]). Thus, the exponential generating function $R_k(x)$ satisfies,

$$\begin{aligned}R_k(x) &= k\int_0^x R_k(t)dt + \int_0^x R_{k-1}(t)dt + 3\frac{(e^x-1)^k}{k!}\\&+\frac{1}{2}k(k-5)\int_0^x\frac{(e^t-1)^k}{k!}dt-(k^2-4k+2)\int_0^x\int_0^t\frac{(e^r-1)^k}{k!}drdt\\&-\frac{1}{12}k(k-1)(k-2)(3k-13)\int_0^x\int_0^t\int_0^r\frac{(e^s-1)^k}{k!}dsdrdt,\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{d^3}{dx^3}R_k(x) &= k\frac{d^2}{dx^2}R_k(x)+\frac{d^2}{dx^2}R_{k-1}(x)+3\frac{d^3}{dx^3}\frac{(e^x-1)^k}{k!}\\&+\frac{1}{2}k(k-5)\frac{d^2}{dx^2}\frac{(e^x-1)^k}{k!}-(k^2-4k+2)\frac{d}{dx}\frac{(e^x-1)^k}{k!}\\&-\frac{1}{12}k(k-1)(k-2)(3k-13)\frac{(e^x-1)^k}{k!}\end{aligned}$$

with $R_0(x) = 0$, $R_1(x) = 2xe^x + 2e^x - 2$ and $R_2(x) = \frac{3}{4} - \frac{x}{2} - 2(1+x)e^x + (2x + \frac{5}{4})e^{2x}$.

Define $R(x,y)=\sum_{k\geqslant 0}R_k(x)y^k$. Multiplying by y^k and summing over $k\geqslant 3$ gives

$$\begin{aligned}&\frac{\partial^3}{\partial x^3}R(x,y)-y\frac{\partial^2}{\partial x^2}R(x,y)-y\frac{\partial^3}{\partial x^2 y}R(x,y)\\&=\frac{y(-y^2(3y+4)+(12y^3+6y^2+24)e^x-12y(y^2-6)e^{2x}+34y^2e^{3x}+3y^3e^{4x})e^{y(e^x-1)}}{12}\end{aligned}$$

By solving for $\frac{\partial^2}{\partial x^2}R(x,y)$, we obtain the following result.

Theorem 6. *The exponential generating function $R(x,y)$ for the total of the site-perimeter over all set partitions of $[n]$ with exactly k blocks satisfies*

$$\begin{aligned}\frac{\partial^2}{\partial x^2}R(x,y)=\frac{y}{144}e^{y(e^x-1)}\Big(&y^2(9y+16)-12(4y^3+3y^2-24x-72)e^x\\&+72y(y^2+12x+14)e^{2x}+4y^2(102x+5)e^{3x}+3y^3(12x-11)e^{4x}\Big).\end{aligned}$$

By Theorem 6, we have

$$\begin{aligned}\frac{\partial^2}{\partial x^2}R(x,y)=&\left(\frac{x}{4}-\frac{11}{48}\right)\frac{\partial^4}{\partial x^4}e^{y(e^x-1)}+\left(\frac{4x}{3}+\frac{109}{72}\right)\frac{\partial^3}{\partial x^3}e^{y(e^x-1)}\\&+\left(\frac{y^2}{2}+\frac{x}{4}+\frac{65}{16}\right)\frac{\partial^2}{\partial x^2}e^{y(e^x-1)}+\left(-\frac{y^3}{3}-\frac{3y^2}{4}+\frac{x}{6}+\frac{47}{72}\right)\frac{\partial}{\partial x}e^{y(e^x-1)}\\&+\left(\frac{y^4}{16}+\frac{y^3}{9}\right)e^{y(e^x-1)}.\end{aligned}$$

By comparing the coefficient of $x^ny^k/n!$ in both sides, we get the following formula.

Corollary 7. *The total of the site-perimeter over all set partitions of $[n+2]$ with exactly k blocks is given by*

$$\begin{aligned}&\left(\frac{n}{4}+\frac{109}{72}\right)S_{n+3,k}-\frac{11}{48}S_{n+4,k}+\left(\frac{4n}{3}+\frac{65}{16}\right)S_{n+2,k}+\frac{1}{2}S_{n+2,k-2}+\left(\frac{n}{4}+\frac{47}{72}\right)S_{n+1,k}\\&\qquad-\frac{1}{3}S_{n+1,k-3}-\frac{3}{4}S_{n+1,k-2}+\frac{n}{6}S_{n,k}+\frac{1}{16}S_{n,k-4}+\frac{1}{9}S_{n,k-3},\end{aligned}$$

and the total of the site-perimeter over all set partitions of $[n+2]$ is given by

$$\left(\frac{n}{4}+\frac{109}{72}\right)B_{n+3}-\frac{11}{48}B_{n+4}+\left(\frac{4n}{3}+\frac{73}{16}\right)B_{n+2}+\left(\frac{n}{4}-\frac{31}{72}\right)B_{n+1}+\left(\frac{n}{6}+\frac{25}{144}\right)B_n,$$

where $S_{n,k}$ denotes the Stirling number of the second kind and B_n denotes the nth Bell number.

Remark 8. Based on this work, we realized that there is a typo in the statement of Corollary 2.6 in [13]. More precisely, by (1) we have that, asymptotically, the total number of interior vertices (a vertex in B is called *an interior vertex* if it is adjacent to exactly four different cells of bargraph B) in set partitions of $[n+1]$is given by

$$\frac{n^2 B_{n+1}}{3(\log(n)-\log\log n)}\left(1-\frac{1}{3(\log n-\log\log n)}\right)\left(1+O\left(\frac{\log n}{n}\right)\right).$$

We end the paper by emphasizing that all the above results have been compared with exact enumerations.

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