Circular repetition thresholds on some small alphabets: Last cases of Gorbunova's conjecture

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Abstract

A word is called β-free if it has no factors of exponent greater than or equal to β. The *repetition threshold* $\mathrm{RT}(k)$ is the infimum of the set of all β such that there are arbitrarily long k-ary β-free words (or equivalently, there are k-ary β-free words of every sufficiently large length, or even every length). These three equivalent definitions of the repetition threshold give rise to three natural definitions of a repetition threshold for *circular words.* The infimum of the set of all β such that

(a) there are arbitrarily long k-ary β-free circular words is called the *weak circular repetition threshold*, denoted $\mathrm{CRT}_\mathrm{W}(k)$;

(b) there are k-ary β-free circular words of every sufficiently large length is called the *intermediate circular repetition threshold*, denoted $\mathrm{CRT}_\mathrm{I}(k)$;

(c) there are k-ary β-free circular words of every length is called the *strong circular repetition threshold*, denoted $\mathrm{CRT}_\mathrm{S}(k)$.

We prove that $\mathrm{CRT}_\mathrm{S}(4) = \frac{3}{2}$ and $\mathrm{CRT}_\mathrm{S}(5) = \frac{4}{3}$, confirming a conjecture of Gorbunova and providing the last unknown values of the strong circular repetition threshold. We also prove that $\mathrm{CRT}_\mathrm{I}(3) = \mathrm{CRT}_\mathrm{W}(3) = \mathrm{RT}(3) = \frac{7}{4}$.

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1 Introduction

A (linear) word is simply a sequence of letters over some finite alphabet. A word $w = w_1 \dots w_n$, where the w_i are letters, is *periodic* if for some positive integer p, $w_{i+p} = w_i$ for all $1 \leqslant i \leqslant n - p$. In this case, p is called a *period* of w. Note that every word of length n trivially has period n. The *exponent* of a word w, denoted $\exp(w)$, is the ratio between its length and its minimal period. If $r = \exp(w)$, then w is called an *r-power*. For example, the English word `alfalfa` has minimal period 3 and exponent $\frac{7}{3}$, so it is a $\frac{7}{3}$-power.

A word is called β-free if it has no factors of exponent greater than or equal to β, i.e. if it has no r-powers for $r \geqslant \beta$. It is called β^+-free if it has no factors of exponent strictly greater than β. The *repetition threshold* function is given by

$$\mathrm{RT}(k) = \inf\{\beta\colon \text{there are arbitrarily long } \beta\text{-free words on } k \text{ letters}\}.$$

Equivalently, $\mathrm{RT}(k)$ is the smallest β such that there is an infinite β^+-free word on k letters. It is well-known that the Thue-Morse sequence avoids overlaps [3], from which it follows that $\mathrm{RT}(2) = 2$. The repetition threshold function was introduced by Dejean [10], who conjectured that

$$\mathrm{RT}(k) = \begin{cases} 7/4 \text{ if } k = 3; \\ 7/5 \text{ if } k = 4; \text{ and} \\ k/(k-1) \text{ if } k \geqslant 5. \end{cases}$$

Certain cases of the conjecture were proven by various authors [5, 7, 8, 10, 13, 14, 16], and the last remaining cases were finally proven independently by Currie and Rampersad [9] and Rao [17].

Interest in Dejean's conjecture has given rise to a number of similar threshold problems in combinatorics on words. These include the *generalized repetition threshold* introduced by Ilie, Ochem, and Shallit [12], and the *Abelian repetition threshold* introduced by Samsonov and Shur [18]. Further, the notion of repetition threshold has been extended to coloured graphs [15], where the "factors" of a coloured graph are the sequences of colours that arise from walking paths in the graph. In this article, we are concerned with repetition thresholds for *circular words*. From a graph theoretical perspective, circular words can be thought of as coloured cycles, so our work fits into the notion of repetition threshold for coloured graphs. However, we view the problem purely from a combinatorics on words perspective.

Two words x and y are said to be *conjugates* if there are words u and v such that $x = uv$ and $y = vu$. The conjugates of a word w can be obtained by rotating the letters of w cyclically. For a word w, the *circular word* (w) is the set of all conjugates of w. Intuitively, one can think of a circular word as being obtained from a linear word by linking the ends, giving a cyclic sequence of letters.

A word is a *factor* of a circular word (w) if it is a factor of some conjugate of w. As for linear words, a circular word is β-free if it has no factors of exponent greater than or equal to β, and β^+-free if it has no factors of exponent strictly greater than β.

Note that if the linear word w is β-free, then so are all of its factors. This means that there are three equivalent definitions of the repetition threshold $\mathrm{RT}(k)$. It is the infimum of the set of all β such that

(a) there are arbitrarily long k-ary β-free words;

(b) there are k-ary β-free words of every sufficiently large length; or

(c) there are k-ary β-free words of every length.

On the other hand, if x is a factor of some β-free circular word (w), it is not necessarily true that the circular word (x) is β-free (even though the linear word x must be β-free). Thus, there are three natural definitions of a repetition threshold for circular words:

(a) the *weak circular repetition threshold*,

$$\mathrm{CRT_W}(k) = \inf\{\beta\colon \text{ there are arbitrarily long } k\text{-ary } \beta\text{-free circular words}\};$$

(b) the *intermediate circular repetition threshold*,

$$\begin{aligned}\mathrm{CRT_I}(k) = \inf\{\beta\colon \text{ there are } k\text{-ary } \beta\text{-free circular words}\\ \text{of every sufficiently large length}\};\end{aligned}$$

(c) and the *strong circular repetition threshold*,

$$\mathrm{CRT_S}(k) = \inf\{\beta\colon \text{ there are } k\text{-ary } \beta\text{-free circular words of every length}\}.$$

Clearly we have

$$\mathrm{RT}(k) \leqslant \mathrm{CRT_W}(k) \leqslant \mathrm{CRT_I}(k) \leqslant \mathrm{CRT_S}(k) \tag{1}$$

for all $k \geqslant 2$.

Almost all values of the strong circular repetition threshold are known. Aberkane and Currie [1] demonstrated that $\mathrm{CRT_S}(2) = 5/2$, while the fact that $\mathrm{CRT_S}(3) = 2$ follows from the work of Currie [6] along with a finite search, or alternatively from the work of Shur [20]. Gorbunova [11] demonstrated that $\mathrm{CRT_S}(k) = \frac{\lceil k/2\rceil+1}{\lceil k/2\rceil}$ for all $k \geqslant 6$, and conjectured that this formula holds for $k = 4$ and $k = 5$ as well. In this article, we demonstrate that $\mathrm{CRT_S}(4) = \frac{3}{2}$ and $\mathrm{CRT_S}(5) = \frac{4}{3}$, confirming Gorbunova's conjecture. Since $\mathrm{CRT_S}(4) \geqslant \frac{3}{2}$ and $\mathrm{CRT_S}(5) \geqslant \frac{4}{3}$ are already known [11], we only need to show that there are $\frac{3}{2}^+$-free circular 4-ary words of every length, and $\frac{4}{3}^+$-free circular 5-ary words of every length.

Relatively less is known about the intermediate and weak circular repetition thresholds, even though the fact that $\mathrm{CRT_W}(2) = 2$ was noted in the work of Thue (see [3]). Aberkane and Currie [2] demonstrated that $\mathrm{CRT_I}(2) = 7/3$. It follows from [15, Theorem 3.1] that $\mathrm{CRT_W}(3) = \mathrm{RT}(3) = \frac{7}{4}$ and $\mathrm{CRT_W}(4) \leqslant \frac{3}{2}$. In [20, Section 4], it is stated that $\mathrm{CRT_I}(3) = \frac{7}{4}$, and that the result is obtained by adapting the technique of [19], though the proof is omitted due to space constraints. We give an alternate proof that $\mathrm{CRT_I}(3) = \mathrm{CRT_W}(3) = \mathrm{RT}(3) = \frac{7}{4}$ by adapting the method we use to prove $\mathrm{CRT_S}(4) = \frac{3}{2}$.

All that is known about $\mathrm{CRT_I}(k)$ and $\mathrm{CRT_W}(k)$ for $k \geqslant 5$ are the bounds given by (1); note that $\mathrm{RT}(k) = \frac{k}{k-1}$ for all $k \geqslant 5$ and $\mathrm{CRT_S}(k) = \frac{\lceil k/2\rceil+1}{\lceil k/2\rceil}$ for all $k \geqslant 6$, so

$$\frac{k}{k-1} \leqslant \mathrm{CRT_W}(k) \leqslant \mathrm{CRT_I}(k) \leqslant \frac{\lceil k/2\rceil + 1}{\lceil k/2\rceil} \text{ for all } k \geqslant 6.$$

With the knowledge that $\mathrm{CRT_I}(3) = \mathrm{CRT_W}(3) = \mathrm{RT}(3)$, it seems reasonable to conjecture that $\mathrm{CRT_I}(k) = \mathrm{CRT_W}(k) = \mathrm{RT}(k)$ for all $k \geqslant 4$. We note that this strengthens statement (2) of Conjecture 1 in [20]. However, it is likely that different techniques than those used here will be needed to prove this conjecture.

2 $\mathrm{CRT_S}(4) = \frac{3}{2}$

We first give a short description of the technique used to achieve the main result of this section. We use a strong inductive argument to demonstrate that there are circular $\frac{3}{2}^+$-free 4-ary words of every length. The inductive step involves constructing longer circular $\frac{3}{2}^+$-free words from shorter ones. We use uniform morphisms to do so. However, we need two uniform morphisms of distinct (and relatively prime) sizes in order to show that there is a circular $\frac{3}{2}^+$-free 4-ary word of every length. First, we find a 9-uniform morphism f_9 and an 11-uniform morphism f_{11} that preserve $\frac{3}{2}^+$-freeness. To construct a $\frac{3}{2}^+$-free word of length n, we write $n = 9k + 11\ell$ for integers k and ℓ. This is possible by the following well-known lemma; we use the version stated in [21].

Lemma 1. *If $a, b \in \mathbb{N}$ and $\gcd(a, b) = 1$, then for each $n \geqslant (a-1)(b-1)$, there is exactly one pair of nonnegative integers r and s such that $s < a$ and $n = ra + bs$.*

Finally, we take a $\frac{3}{2}^+$-free circular word (w) of length $k + \ell$, write $w = uv$ where $|u| = k$ and $|v| = \ell$, and show that $(f_9(u)f_{11}(v))$ is also $\frac{3}{2}^+$-free. We introduce some terminology for dealing with these "mixed" images of f_9 and f_{11}.

Let A and B be alphabets, and let $h\colon A^* \to B^*$ be a morphism. Using the standard notation for images of sets, we have $h(A) = \{h(a)\colon\ a \in A\}$, which we refer to as the set of *building blocks* of h. If H is a *set* of morphisms from A^* to B^*, then we let $H(A)$ denote the set of all images of letters of A under all morphisms in H; that is,

$$H(A) = \bigcup_{h\in H} h(A) = \{h(a)\colon\ h \in H, a \in A\}.$$

We call the words in $H(A)$ the *building blocks* of H.

Definition 2. Let A and B be alphabets and let H be a set of nonerasing morphisms from A^* to B^*. An *H-image* of a word $w = a_1 \dots a_n$ with $a_i \in A$ is a word of the form

$$h_1(a_1)\dots h_n(a_n),$$

where $h_i \in H$. (Note that the h_i are not necessarily distinct.) Let u be a nonempty factor of some word $v \in H(A)^+$. We say that $w = a_1 \dots a_n$ is an *H-preimage* of u if there is an H-image $h_1(a_1) \dots h_n(a_n)$ of w that has u as a factor, and w is minimal in the sense that $h_1(a_1) \dots h_{n-1}(a_{n-1})$ and $h_2(a_2) \dots h_n(a_n)$ do not have u as a factor. The H-preimage of the empty word ϵ is simply ϵ.

A set of words $C \subseteq A^*$ is called a *prefix code* (respectively, *suffix code*) if no element of C is the prefix (respectively, *suffix*) of another element of C. We call C a *bifix code* if it is both a prefix code and a suffix code.

We are now ready to define f_9 and f_{11} and start working towards the proof that $\mathrm{CRT_S}(4) = \frac{3}{2}$. Let $\Sigma_4 = \{\mathtt{0}, \mathtt{1}, \mathtt{2}, \mathtt{3}\}$. We define the morphism $f_9 : \Sigma_4^* \to \Sigma_4^*$ by

$$\begin{aligned} \mathtt{0} &\mapsto \mathtt{0121323\ 10} \\ \mathtt{1} &\mapsto \mathtt{1232030\ 21} \\ \mathtt{2} &\mapsto \mathtt{2303101\ 32} \\ \mathtt{3} &\mapsto \mathtt{3010212\ 03} \end{aligned}$$

and the morphism $f_{11} : \Sigma_4^* \to \Sigma_4^*$ by

$$\begin{aligned} \mathtt{0} &\mapsto \mathtt{0121323\ 12\ 10} \\ \mathtt{1} &\mapsto \mathtt{1232030\ 23\ 21} \\ \mathtt{2} &\mapsto \mathtt{2303101\ 30\ 32} \\ \mathtt{3} &\mapsto \mathtt{3010212\ 01\ 03} \end{aligned}$$

Throughout this section, we let $F = \{f_9, f_{11}\}$. We make the following observations:

- The morphisms f_9 and f_{11} both have cyclic structure: for $a \in \{\mathtt{1}, \mathtt{2}, \mathtt{3}\}$, $f_9(a)$ is obtained from $f_9(\mathtt{0})$ by adding a to each letter of $f_9(\mathtt{0})$ modulo 4, and likewise for f_{11}.
- The word $f_{11}(\mathtt{0})$ is obtained from $f_9(\mathtt{0})$ by inserting the factor 12 after the length 7 prefix (indicated above with spacing).
- For all $a \in \Sigma_4$, $f_9(a)$ and $f_{11}(a)$ have the same prefix of length 8, and the same suffix of length 2.
- $F(\Sigma_4)$ is a prefix code; the length 9 prefixes of all building blocks of F are distinct.
- $F(\Sigma_4)$ is a suffix code; the length 3 suffixes of all building blocks of F are distinct.

If P is a prefix code and w is a nonempty factor of some element of P^+, a *cut* of w is a pair (x, y) such that (i) $w = xy$; and (ii) for any words p, s with $pws \in P^+$, $px \in P^*$. We use vertical bars to denote cuts. For example, over the code $\{\mathtt{01}, \mathtt{10}\}$, the word 11 has cut $\mathtt{1}|\mathtt{1}$.

We first prove that any sufficiently long factor of a word in $(F(\Sigma_4))^+$ has a cut, and that any factor of a word in $(F(\Sigma_4))^+$ with a cut has unique F-preimage. These results are used frequently in the material that follows, sometimes without reference.

Lemma 3. *Let u be a factor of some F-image. If $|u| \geqslant 10$, then u has a cut.*

Proof. Consider the set $P = \{\mathtt{0121}, \mathtt{1232}, \mathtt{2303}, \mathtt{3010}\}$ of all length 4 prefixes of the building blocks of F. Observe that there is a cut to the left of every appearance of a member of P in u, since each member of P appears as a factor in an F-image only as the prefix of a building block. (It suffices to check all F-images of words of length 2.) On the other hand, consider the set

$$S = \{\mathtt{2310}, \mathtt{3021}, \mathtt{0132}, \mathtt{1203}, \mathtt{1210}, \mathtt{2321}, \mathtt{3032}, \mathtt{0103}\}$$

of all length 4 suffixes of the building blocks of F. There is a cut to the right of every appearance of a member of S in u, since each member of S appears as a factor in an F-image only as the suffix of a building block. (Again, we need only check all F-images of words of length 2.)

Let $|u| \geqslant 10$. Since the building blocks of F have length at most 11, u must contain either the prefix of length 4 of some building block of F (in which case there is a cut to the left of this factor), or the suffix of length 4 of some building block of F (in which case there is a cut to the right of this factor). ☐

Lemma 4. *Let u be a factor of some F-image. If u has a cut, then u has unique F-preimage.*

Proof. Suppose that u has a cut. Since $F(\Sigma_4)$ is a bifix code, we can write $u = s|v|p$, where s is a proper suffix of some word in $F(\Sigma_4)$, $v \in F(\Sigma_4)^*$, and p is a proper prefix of some word in $F(\Sigma_4)$. Since $v \in F(\Sigma_4)^*$ and $F(\Sigma_4)$ is a code, v has unique F-preimage. Note that for any letter $a \in \Sigma_4$, both $f_9(a)$ and $f_{11}(a)$ begin and end in a. Thus if s is nonempty, then the F-preimage of s is completely determined by the last letter of s, while if p is nonempty, then the F-preimage of p is completely determined by the first letter of p. ☐

Next, we show that the individual morphisms f_9 and f_{11} are $\frac{3}{2}^+$-free. That is, they preserve $\frac{3}{2}^+$-freeness for linear words.

Lemma 5. *Let $w \in \Sigma_4^+$. If w is $\frac{3}{2}^+$-free, then $f_9(w)$ and $f_{11}(w)$ are $\frac{3}{2}^+$-free.*

Proof. Let w be $\frac{3}{2}^+$-free, and let $h \in \{f_9, f_{11}\}$. Suppose towards a contradiction that $h(w)$ has a factor of exponent greater than $\frac{3}{2}$. Then in particular, $h(w)$ has a factor of the form xyx for words $x, y \in \Sigma_4^*$ with $|xyx| > \frac{3}{2}|xy|$, or equivalently, $|x| > |y|$.

First suppose $|x| \leqslant 9$. Then $|y| \leqslant 8$ and $|xyx| \leqslant 26$, so xyx appears in some word $h(u)$ with $u \in \Sigma_4^+$ a $\frac{3}{2}^+$-free word of length 4. Eliminating this possibility by an exhaustive search, we may assume that $|x| \geqslant 10$. Then by Lemma 3, x has a cut. So x has the form $s_x|m_x|p_x$, where s_x is a proper suffix of some building block of h and p_x is a proper prefix of some building block of h. This means that $y = s_y|m_y|p_y$, where $p_x s_y$ and $p_y s_x$ are either empty, or building blocks of h. Let m_x and m_y have h-preimages x' and y', respectively, and let $p_x s_y$ and $p_y s_x$ have h-preimages a and b, respectively. Note that $a, b \in \Sigma_4 \cup \{\epsilon\}$. Then xyx has h-preimage

$$bx'ay'bx'a.$$

Since h is uniform and $|x| > |y|$, it follows that $|bx'a| > |y'|$, and this contradicts the assumption that w is $\frac{3}{2}^+$-free. □

Now that we know that f_9 and f_{11} preserve $\frac{3}{2}^+$-freeness for linear words, we are ready to show that we can construct longer $\frac{3}{2}^+$-free circular words from shorter ones using f_9 and f_{11}. For a word w of length n and an integer $m \in \{1, \dots, n\}$, we let $p_m(w)$ denote the prefix of w of length m, and we let $s_m(w)$ denote the suffix of w of length m.

Theorem 6. *Let $k \geqslant 8$ and $2 \leqslant \ell \leqslant 10$. Let $u, v \in \Sigma_4^+$ be words of length k and ℓ, respectively. If the circular word (uv) is $\frac{3}{2}^+$-free, then so is the circular word $(f_9(u)f_{11}(v))$.*

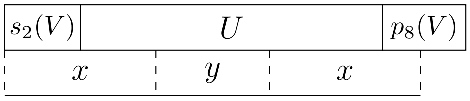
Proof. Let $u = u_1u_2 \dots u_k$ and $v = v_1v_2 \dots v_\ell$ and suppose that (uv) is $\frac{3}{2}^+$-free. For ease of notation, let $U = f_9(u)$ and $V = f_{11}(v)$. Suppose towards a contradiction that (UV) has a factor of exponent greater than $\frac{3}{2}$. Then some conjugate of UV has a factor xyx with $|x| > |y|$. Suppose first that $|x| \leqslant 10$. Then $|y| \leqslant 9$ and thus $|xyx| \leqslant 29$. It follows that xyx is a factor of some F-image $f_9(w_1)f_{11}(w_2)f_9(w_3)$, where $w_1, w_2, w_3 \in \Sigma_4^*$ satisfy $|w_1| + |w_2| + |w_3| = 5$ and $w_1w_2w_3$ is $\frac{3}{2}^+$-free. By exhaustive search, we eliminate this possibility and we may now assume that $|x| \geqslant 11$.

We first claim that x cannot contain the factor $s_3(U)p_9(V)$. By inspection, $s_3(U)p_9(V)$ has the cut $s_3(U)|p_9(V)$. No matter the exact identity of u_k, the suffix $s_3(U)$ is not a suffix of any building block of f_{11}. Similarly, the prefix $p_9(V)$ is not a prefix of any building block of f_9, no matter the identity of v_1. So the only place that the factor $s_3(U)p_9(V)$ appears as a factor of the circular word (UV) is at the boundary between U and V. Since x appears at least twice in some conjugate of UV (as xyx is a factor of some conjugate of UV), we conclude that $s_3(U)p_9(V)$ is not a factor of x. By a similar argument, x cannot contain the factor $s_3(V)p_9(U)$.

U				V			
$f_9(u_1)$	$f_9(u_2)$	$\cdots$	$f_9(u_k)$	$f_{11}(v_1)$	$f_{11}(v_2)$	$\cdots$	$f_{11}(v_\ell)$
$p_9(U)$			$s_3(U)$	$p_9(V)$			$s_3(V)$

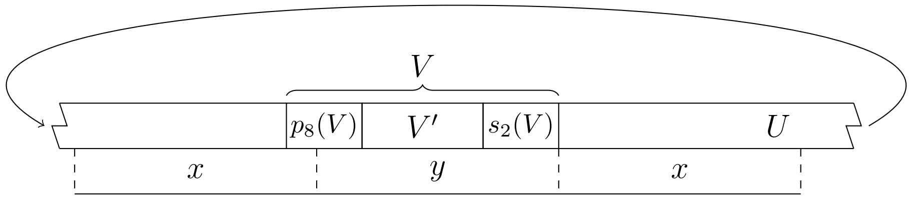
Since $|x| \geqslant 11$, x must contain either the prefix of length 9 of some building block of F, or the suffix of length 3 of some building block of F. If x contains the length 9 prefix or the length 3 suffix of some building block of f_9, then x appears only inside $s_2(V)Up_8(V)$ by the argument of the previous paragraph. On the other hand, if x contains the length 9 prefix or the length 3 suffix of some building block of f_{11}, then x appears only inside $s_2(U)Vp_8(U)$. This analysis leads to four separate cases, depending on the positioning of the entire factor xyx.

Case 1: xyx is a factor of $s_2(V)Up_8(V)$.



Observe that $s_2(V)$ is the suffix of length 2 of $f_{11}(v_\ell)$, and that this is the same as the length 2 suffix of $f_9(v_\ell)$. Similarly, $p_8(V)$ is the prefix of length 8 of both $f_{11}(v_1)$ and $f_9(v_1)$. Thus xyx is a factor of $f_9(v_\ell)Uf_9(v_1) = f_9(v_\ell)f_9(u)f_9(v_1) = f_9(v_\ell uv_1)$. By Lemma 5, $v_\ell uv_1$ must have a factor with exponent greater than $\frac{3}{2}$. But since $\ell \geqslant 2$, $v_\ell uv_1$ is a factor of (uv), and this contradicts the assumption that (uv) is $\frac{3}{2}^+$-free.

Case 2: x is a factor of $s_2(V)Up_8(V)$ and xyx has V as a factor.



Let $x = s_x|m_x|p_x$, and $y = s_y|m_y|p_y$, where s_x and s_y are proper suffixes of building blocks of F and p_x and p_y are proper prefixes of building blocks of F. Then

$$xyx = s_x|m_x|p_xs_y|m_y|p_ys_x|m_x|p_x$$

Let m_x have F-preimage x' and m_y have F-preimage y'. Then $|m_x| = 9|x'|$ and $|m_y| \geqslant 9|y'|$. Now p_xs_y and p_ys_x are each either a single building block or the empty word. Let p_xs_y have F-preimage $a \in \Sigma_4 \cup \{\epsilon\}$ and p_ys_x have F-preimage $b \in \Sigma_4 \cup \{\epsilon\}$. Then the F-preimage of xyx is

$$bx'ay'bx'a.$$

Note that this preimage may not be a factor of the circular word (uv) if xyx is so long that the factor p_x at the end of xyx and the factor s_x at the beginning of xyx are actually part of the same building block $f_9(u_i)$. We break into two subcases:

Subcase 2a: $bx'ay'bx'a$ is a factor of (uv).

We argue that $|bx'a| > |y'|$, which contradicts the assumption that (uv) is $\frac{3}{2}^+$-free. We have

$$9|bx'a| \geqslant |s_x| + |m_x| + |p_x| = |x| > |y| \geqslant |m_y| \geqslant 9|y'|,$$

from which the desired inequality follows.

Subcase 2b: $bx'ay'bx'a$ is not a factor of (uv).

Then neither a nor b is equal to ϵ, since the ends of xyx are in the same building block $f_9(u_i)$. Note that both $x'ay'bx'a$ and $bx'ay'bx'$ appear in (uv). We claim that $|x'a| > |y'b|$, which contradicts the assumption that (uv) is $\frac{3}{2}^+$-free. Since $|x| > |y|$, we have $|s_xm_xp_x| > |s_ym_yp_y|$. Thus, $|m_x| > |m_y| + |s_y| + |p_y| - |s_x| - |p_x|$. Further, since p_x and s_x appear in the same building block $f_9(u_i)$ and must not overlap, we have $|p_x| + |s_x| \leqslant 9$. Since $|p_xs_y| \geqslant 9$ and $|p_ys_x| \geqslant 9$ we have $|s_y| \geqslant 9 - |p_x|$ and $|p_y| \geqslant 9 - |s_x|$, so

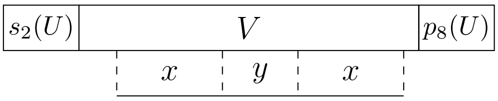
$$|m_x| > |m_y| + |s_y| + |p_y| - |s_x| - |p_x| \geqslant |m_y| + 18 - 2(|p_x| + |s_x|) \geqslant |m_y|.$$

Now

$$9|x'| = |m_x| > |m_y| \geqslant 9|y'|,$$

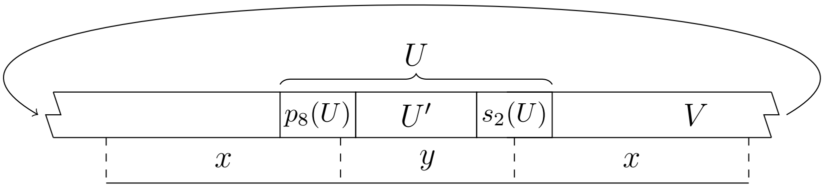
and $|x'a| = |x'| + 1 > |y'| + 1 = |y'b|$ follows.

Case 3: xyx is a factor of $s_2(U)Vp_8(U)$.



Observe that $s_2(U)$ is the suffix of length 2 of $f_9(u_k)$, and that this is the same as the length 2 suffix of $f_{11}(u_k)$. Similarly, $p_8(U)$ is the prefix of length 8 of both $f_9(u_1)$ and $f_{11}(u_1)$. Thus, xyx is a factor of $f_{11}(u_k)Vf_{11}(u_1) = f_{11}(u_k)f_{11}(v)f_{11}(u_1) = f_{11}(u_kvu_1)$. By Lemma 5, u_kvu_1 must have a factor with exponent greater than $\frac{3}{2}$. But since $k \geqslant 8$, u_kvu_1 is a factor of (uv), and this contradicts the assumption that (uv) is $\frac{3}{2}^+$-free.

Case 4: x is a factor of $s_2(U)Vp_8(U)$ and xyx has U as a factor.



Since y contains U' as a factor,

$$|y| \geqslant |U'| = |U| - 10 = |f_9(u)| - 10 = 9|u| - 10 = 9k - 10 \geqslant 62,$$

from the assumption that $|u| = k \geqslant 8$. On the other hand, since x appears twice (without overlapping itself) in $s_2(U)Vp_8(U)$, we conclude that

$$|x| \leqslant \frac{|V| + 10}{2} = \frac{|f_{11}(v)| + 10}{2} = \frac{11|v| + 10}{2} = \frac{11\ell + 10}{2} \leqslant 60,$$

from the assumption that $|v| = \ell \leqslant 10$. But this contradicts the assumption that $|x| > |y|$. □

We are now ready to prove the main result of this section.

Theorem 7. *For every $n \in \mathbb{N}$, there is a $\frac{3}{2}^+$-free circular 4-ary word of length n.*

Proof. The proof is by strong induction on n. For $n \leqslant 173$ we found a $\frac{3}{2}^+$-free circular 4-ary word of length n by computer search.

Assume that for some $n \geqslant 174$, there is a $\frac{3}{2}^+$-free circular 4-ary word of every length $m < n$. Then $n - 9(8) - 11(2) \geqslant 174 - 72 - 22 = 80$, so by Lemma 1, we can write $n - 9(8) - 11(2) = 9r + 11s$, or equivalently $n = 9(r + 8) + 11(s + 2)$, for integers $r \geqslant 0$ and $0 \leqslant s \leqslant 8$. Let $k = r + 8$ and $\ell = s + 2$, and note that $k \geqslant 8$ and $2 \leqslant \ell \leqslant 10$. Clearly, $k + \ell < n$, so by the inductive hypothesis, there is a $\frac{3}{2}^+$-free circular 4-ary word (w) of length $k + \ell$. Let $w = uv$, with $|u| = k$ and $|v| = \ell$. By Theorem 6, the circular 4-ary word $(f_9(u)f_{11}(v))$ is also $\frac{3}{2}^+$-free, and has length $9k + 11\ell = n$. □

3 $\mathrm{CRT_I}(3) = \frac{7}{4}$

It follows immediately from the fact that $\mathrm{RT}(3) = \frac{7}{4}$ that $\mathrm{CRT_I}(3) \geqslant \frac{7}{4}$. Here, we demonstrate that there are $\frac{7}{4}^+$-free ternary words of every length $n \geqslant 23$, from which we conclude that $\mathrm{CRT_I}(3) = \frac{7}{4}$. The only lengths which do not admit a $\frac{7}{4}^+$-free circular ternary word are 5, 7, 9, 10, 14, 16, 17, and 22.

We use a construction very similar to the one used in the previous section. Let $\Sigma_3 = \{\mathtt{0}, \mathtt{1}, \mathtt{2}\}$. Define the morphism $d_{19} : \Sigma_3^* \to \Sigma_3^*$ by

$$\begin{aligned} \mathtt{0} &\mapsto \mathtt{012021201\ 2102120210} \\ \mathtt{1} &\mapsto \mathtt{120102012\ 0210201021} \\ \mathtt{2} &\mapsto \mathtt{201210120\ 1021012102} \end{aligned}$$

and the morphism $d_{23} : \Sigma_3^* \to \Sigma_3^*$ by

$$\begin{aligned} \mathtt{0} &\mapsto \mathtt{012021201\ 0201\ 2102120210} \\ \mathtt{1} &\mapsto \mathtt{120102012\ 1012\ 0210201021} \\ \mathtt{2} &\mapsto \mathtt{201210120\ 2120\ 1021012102} \end{aligned}$$

Throughout this section, let $D = \{d_{19}, d_{23}\}$. Note that d_{19} is the morphism used by Dejean [10] to prove that $\mathrm{RT}(3) = \frac{7}{4}$. In particular, Dejean proved that d_{19} is $\frac{7}{4}^+$-free. The image $d_{23}(\mathtt{0})$ is obtained by inserting the factor `0201` into the middle of $d_{19}(\mathtt{0})$ (indicated above by spacing), and for $a \in \{\mathtt{1}, \mathtt{2}\}$, $d_{23}(a)$ is obtained from $d_{23}(\mathtt{0})$ by adding a to each letter of $d_{23}(\mathtt{0})$ modulo 3. Note that d_{19} and d_{23} have similar properties to f_9 and f_{11} used in Section 2:

- Both d_{19} and d_{23} have the cyclic structure described above for d_{23}.
- For all $a \in \Sigma_3$, $d_{19}(a)$ and $d_{23}(a)$ have the same prefix of length 9, and the same suffix of length 13.
- $D(\Sigma_3)$ is a prefix code; the length 10 prefixes of all building blocks of D are distinct.
- $D(\Sigma_3)$ is a suffix code; the length 14 suffixes of all building blocks of D are distinct.

Lemma 8. *Let u be a factor of some D-image.*

(a) *If $|u| \geqslant 22$, then u has a cut.*

(b) *If u has a cut, then u has unique D-preimage.*

Proof. The proof of (a) is similar to that of Lemma 3. If $|u| \geqslant 22$ then u contains the length 9 prefix of some building block (and there is a cut to the left of this prefix) or the length 9 suffix of some building block (and there is a cut to the right of this prefix). The proof of (b) is similar to that of Lemma 4. □

Theorem 9. *Let $k \geqslant 6$ and $2 \leqslant \ell \leqslant 20$. Let $u, v \in \Sigma_3^+$ be words of length k and ℓ, respectively. If the circular word (uv) is $\frac{7}{4}^+$-free, then so is the circular word $(d_{19}(u)d_{23}(v))$.*

Proof. Let $u = u_1 \dots u_k$ and $v = v_1 \dots v_\ell$ and suppose that (uv) is $\frac{7}{4}^+$-free. Let $U = d_{19}(u)$ and $V = d_{23}(v)$. Suppose towards a contradiction that (UV) has a factor of exponent greater than $\frac{7}{4}$. Then some conjugate of UV has a factor xyx with $|x| > 3|y|$. Suppose first that $|x| \leqslant 22$. Then $|y| \leqslant 7$ and $|xyx| \leqslant 51$. It follows that xyx is a factor of some D-image of the form $d_{19}(w_1)d_{23}(w_2)d_{19}(w_3)$, where $w_1, w_2, w_3 \in \Sigma_3^*$ satisfy $|w_1| + |w_2| + |w_3| = 4$ and $w_1w_2w_3$ is $\frac{7}{4}^+$-free. We eliminate this possibility by exhaustive search and may now assume that $|x| \geqslant 23$.

Notice that x contains neither $s_{14}(U)p_{10}(V)$ nor $s_{14}(V)p_{10}(U)$ as a factor, since each of these factors appears at most once in any conjugate of UV. It follows that x appears only inside $s_{13}(V)Up_9(V)$, or only inside $s_{13}(U)Vp_9(U)$. We have four cases, as in Theorem 6.

Case 1: xyx is a factor of $s_{13}(V)Up_9(V)$.
Then xyx is a factor of $d_{19}(v_\ell uv_1)$, and since d_{19} is $\frac{7}{4}^+$-free [10], we reach a contradiction as in Case 1 in the proof of Theorem 6.

Case 2: x is a factor of $s_{13}(V)Up_9(V)$ and xyx has V as a factor.
The argument is analogous to that of Case 2 in the proof of Theorem 6.

Case 3: xyx is a factor of $s_{13}(U)Vp_9(U)$.
Since $|v| = \ell \leqslant 20$, it suffices to check that $d_{23}(w)$ is $\frac{7}{4}^+$-free for all $\frac{7}{4}^+$-free words $w \in \Sigma_3^*$ with $|w| \leqslant 22$.

Case 4: x is a factor of $s_{13}(U)Vp_9(U)$ and xyx has U as a factor.
Write $U = p_9(U)U's_{13}(U)$. Then y contains U' as a factor, so

$$|y| \geqslant |U'| = |U| - 22 = 19k - 22 \geqslant 92$$

from the assumption that $k \geqslant 6$. On the other hand, since x appears twice (without overlapping itself) in $s_{13}(U)Vp_9(U)$, we conclude that

$$|x| \leqslant \frac{|V| + 22}{2} = \frac{23\ell + 22}{2} \leqslant 241,$$

from the assumption that $\ell \leqslant 20$. But then $3|y| > |x|$, a contradiction. □

Theorem 10. *For every $n \geqslant 23$, there is a $\frac{7}{4}^+$-free circular ternary word of length n.*

Proof. The proof is by strong induction on n. For $23 \leqslant n < 555$ we found a $\frac{7}{4}^+$-free circular ternary word of length n by computer search.

Assume that for some $n \geqslant 555$, there is a $\frac{7}{4}^+$-free circular ternary word of every length m such that $23 \leqslant m < n$. Then $n - 19(6) - 23(2) \geqslant 555 - 114 - 46 = 395$, so by Lemma 1, we can write $n - 19(6) - 23(2) = 19r + 23s$, or equivalently $n = 19(r + 6) + 23(s + 2)$, for integers $r \geqslant 0$ and $0 \leqslant s \leqslant 18$. Let $k = r + 6$ and $\ell = s + 2$, and note that $k \geqslant 6$ and $2 \leqslant \ell \leqslant 20$. Now since $555 > 23^2$, we have $23 \leqslant k + \ell < n$, so by the inductive hypothesis, there is a $\frac{7}{4}^+$-free circular ternary word (w) of length $k + \ell$. Let $w = uv$, with $|u| = k$ and $|v| = \ell$. By Theorem 9, the circular ternary word $(d_{19}(u)d_{23}(v))$ is also $\frac{7}{4}^+$-free, and has length $19k + 23\ell = n$. □

4 $\mathbf{CRT_S(5) = \frac{4}{3}}$

In this section, we prove that the strong circular repetition threshold for 5 letters is $\frac{4}{3}$. The construction is more complicated than the one used in Sections 2 and 3, so we first explain why a different type of construction is needed. In order to apply the technique of Sections 2 and 3 here, we would need an r-uniform $\frac{4}{3}^+$-free morphism from Σ_5^* to Σ_5^* for some $r \geqslant 2$. However, no such morphism exists. Brandenburg [4] demonstrated that for every $k \geqslant 4$, there is no growing $\mathrm{RT}(k)^+$-free morphism from Σ_k^* to Σ_k^* (a morphism $h : \Sigma \to \Delta$ is *growing* if $h(a) > 1$ for all $a \in \Sigma$). By a minor modification of his proof, one can show that there is no growing $\frac{4}{3}^+$-free morphism from Σ_5^* to Σ_5^*.

So in order to prove this last case of Gorbunova's conjecture, we adapt the technique used by Gorbunova [11] for larger alphabets. Throughout this section, for a (finite or infinite) word $u = u_1u_2u_3\ldots$, where the u_k are letters, we define $u[i] = u_i$ and $u[i : j] = u_i \ldots u_j$ for all positive integers i and j with $i \leqslant j$. For a finite word $v = v_1v_2 \ldots v_n$, we define $v[-j] = v[n-j+1] = v_{n-j-1}$. In other words, $v[-j]$ is the jth letter of w counting from the right (and starting at 1). For positive integers i and j with $i \geqslant j$, we define $v[-i : -j] = v[n-i+1 : n-j+1] = v_{n-i+1} \ldots v_{n-j+1}$. Finally, for positive integers i and j with $i + j \leqslant n + 1$, we define $v[i : -j] = v_i \ldots v_{n-j+1}$. Throughout this section, words are always assumed to start at index 1.

Define $\varphi : \{\mathtt{0}, \mathtt{1}\}^* \to \{\mathtt{0}, \mathtt{1}\}^*$ by $\varphi(\mathtt{0}) = \mathtt{101101}$ and $\varphi(\mathtt{1}) = \mathtt{10}$. Define

$$\mathbf{p} = \varphi^\infty(\mathtt{1}) = \mathtt{10\ 101101\ 10\ 101101\ 10\ 10\ 101101\ 10\ 10\ 101101}\ldots$$

Throughout this section, let $A = \{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$ and let $s = \mathtt{abc}$. For a (finite or infinite) binary word w, define

$$M(w)[i] = \begin{cases} s[i] \text{ if } i \leqslant 3; \\ M(w)[i-3] \text{ if } i > 3 \text{ and } w[i] = \mathtt{0}; \\ \text{the unique element of} \\ A \backslash \{M(w)[i-1], M(w)[i-2], M(w)[i-3]\} \text{ otherwise.} \end{cases}$$

This definition originates with Pansiot [16], though we use different terminology. The word w is called the *encoding* of $M(w)$. Note that the encoding w can be recovered from $M(w)$, i.e. M is invertible. Throughout this section, let $\mathbf{P} = M(\mathbf{p})$. The following results are due to Pansiot [16].

Theorem 11. (a) *The word* 00 *is not a factor of* $\mathbf{p}$.

(b) *The word* $\mathbf{P}$ *is* $\frac{7}{5}^+$*-free.*

(c) *The only factors of* $\mathbf{P}$ *with exponent greater than* $\frac{4}{3}$ *are encoded by*

$$t_1 = \mathtt{10110101101}.$$

This factor of $\mathbf{p}$ *gives rise to* $\frac{7}{5}$*-powers in* $\mathbf{P}$ *equal to*

$$T_1 = M(t_1) = \mathtt{abcd\ bacbdc\ abcd}$$

up to permutation of A.

(d) *If a finite word u appears as a factor of $\mathbf{P}$, then so does $\sigma(u)$, where σ is any permutation of A.*

We also require one further result which can be derived from Pansiot's work fairly easily. To state the result, we require some additional terminology related to the generalized repetition threshold [12]. For a rational number β such that $1 < \beta \leqslant 2$, a β-power u has *period p* if we can write $u = xx'$, where x' is a prefix of x, $|x| = p$, and $|u| = \beta|x|$; in this case, we call u a *(β, p)-power*, and we call x' the *excess* of the β-power. A word is called *(β^+, p)-free* if it contains no factor that is a (β', p')-power for $\beta' > \beta$ and $p' \geqslant p$.

Lemma 12. *The word $\mathbf{P}$ is $(\frac{14}{11}^+, 11)$-free.*

Proof. First of all, if $\mathbf{P}$ has a (β, p)-power with excess of length at most 2, and if $p \geqslant 11$, then $\beta \leqslant \frac{13}{11}$. So we need only consider powers with excess at least 3. The proof is analogous to that of [16, Propriété 4.14], so we omit some details. Define $\mu : \{0,1\}^* \to \{0,1\}^*$ by $\mu(w) = \varphi(w)\texttt{101}$, as in [16]. By [16, Propriété 4.9, 4.10, and 4.12], it suffices to check that $\mu(uvu)$ and $\mu^2(uvu)$ have exponent less than $\frac{14}{11}$ for $u = \varepsilon$ and $v = \texttt{101101101}$, and $u = \texttt{1}$ and $v = \texttt{011010110}$. This allows us to conclude that the only *inextensible* repetitions of $\mathbf{P}$ (see [16] for the precise definition) with exponent greater than $\frac{14}{11}$ and excess at least 3 are

$$\begin{aligned} M(\texttt{101101101}) &= \texttt{abc dbacbd abc}, \text{ and} \\ M(\texttt{10110101101}) &= \texttt{abcd bacbdc abcd}, \end{aligned}$$

up to permutation of A. These are a $\frac{4}{3}$-power of period 9 and a $\frac{7}{5}$-power of period 10, respectively. It follows that any factor with exponent greater than $\frac{14}{11}$ has period less than 11. □

We note that the factor

$$t_2 = \texttt{1}\rlap{$\overbrace{\phantom{\texttt{0110101101}}}^{t_1}$}\texttt{01101011}\underbrace{\texttt{0110101101}}_{t_1}$$

also appears in $\mathbf{p}$, which encodes the factor

$$T_2 = M(t_2) = \overline{\texttt{abcd}}\texttt{bacb}\underline{\texttt{dc}\overline{\texttt{ab}}}\overline{\texttt{cd}}\texttt{acba}\underline{\texttt{dcab}} \tag{2}$$

up to permutation of the letters. The overlining and underlining is to emphasize the two $\frac{7}{5}$-powers. However, by exhaustive search, this is the only way that two appearances of t_1 in $\mathbf{p}$ can overlap, and no appearance of t_2 overlaps with a third appearance of t_1. Further, if t_1vt_1 is a factor of $\mathbf{p}$, then $|v| \geqslant 17$. It follows that any two nonoverlapping $\frac{7}{5}$-powers of $\mathbf{P}$ have a factor of length at least 14 between them.

```
input: $w$, a finite factor of $\mathbf{P}$ of length $n$
output: $\eta(w)$, a $\frac{4}{3}^{+}$-free word of length $n$ on $A \cup \{\mathtt{e}\}$

for $i$ from 1 to $n-13$:
    if $w[i:i+3] = w[i+10:i+13]$, then
        if $i \leqslant n-21$ and $w[i+8:i+11] = w[i+18:i+21]$, then
          if $w[i] = \mathtt{d}$ or $w[i+1] = \mathtt{d}$, then
            change the appearance of d in $w[i+10:i+11]$ to e
          else
            change the appearance of d in $w[i:i+3]$ to e, and
            change the appearance of d in $w[i+18:i+21]$ to e
        else
          if $i < 19$, then
            change the appearance of d in $w[i+10:i+13]$ to e
          if $i \geqslant 19$, then
            change the appearance of d in $w[i:i+3]$ to e

return $w$
```

Figure 1: The defining algorithm for $\eta(w)$.

We first describe a systematic way of eliminating the $\frac{7}{5}$-powers in a finite factor of $\mathbf{P}$ by introducing a fifth letter e. The key is to note that the excess of every $\frac{7}{5}$-power in $\mathbf{P}$ contains every letter from A exactly once. So we can eliminate the $\frac{7}{5}$-powers, and leave repetitions of exponent at most $\frac{4}{3}$, by changing the d in either the length 4 prefix or the length 4 suffix (but not both) of every $\frac{7}{5}$-power to an e. The algorithm given in Figure 1 accomplishes this, and the output word also satisfies an additional structural property.

Essentially, we search through the factors of w of length 14 for the $\frac{7}{5}$-powers. When we find a $\frac{7}{5}$-power u, we first check whether it overlaps with another $\frac{7}{5}$-power v, in which case the factor of w at hand is equal to T_2 (see (2)) up to permutation of A. If d is the first or second letter of u, then we change the appearance of d that lies in the length 4 suffix of u (and also the length 4 prefix of v) to an e. This one change eliminates both of the overlapping $\frac{7}{5}$-powers. Otherwise, if d is the third or fourth letter of u, we change the appearance of d in the length 4 prefix of u to an e and the appearance of d in the length 4 suffix of v to an e. On the other hand, if u does not overlap with another $\frac{7}{5}$-power, we consider where u occurs in w. If the $\frac{7}{5}$-power at hand starts in the first eighteen letters of w, then we replace the appearance of d in the length 4 suffix with an e. Otherwise, we replace the appearance of d in the length 4 prefix with an e.

For a given factor w of $\mathbf{P}$, the word $\eta(w)$ is not only $\frac{4}{3}^{+}$-free, but any two appearances of e in $\eta(w)$ are relatively far apart. The algorithm is also structured so that the letter e does not appear too close to the beginning nor the end of $\eta(w)$ very often; this simply reduces the number of exceptional cases that we need to deal with later.

Lemma 13. *Let w be a factor of $\mathbf{P}$. Then the following hold:*

(a) *$\eta(w)$ is $\frac{4}{3}^+$-free.*

(b) *Every factor of $\eta(w)$ of length 15 contains at least two* `d`*'s and at most one* `e`*.*

Proof. First we prove (a). We can recover w from $\eta(w)$ by changing all `e`'s to `d`'s, so every β-power in $\eta(w)$ gives rise to a β-power in w. Thus, by Result 11(b), $\eta(w)$ is $\frac{7}{5}^+$-free. Further, Result 11(c) describes the factors with exponent greater than $\frac{4}{3}$ in w. We change either the prefix of length 4 or the suffix of length 4 (but not both) of every $\frac{7}{5}$-power in w when we construct $\eta(w)$, so we conclude that $\eta(w)$ is $\frac{4}{3}^+$-free.

For (b), by exhaustive check, every factor of length 15 in $\mathbf{P}$ contains at least three `d`'s. It suffices to show that there is a factor of length at least 14 between any two appearances of `e` in $\eta(w)$. When constructing $\eta(w)$ from w, we only change an appearance of `d` to `e` if it lies inside of a $\frac{7}{5}$-power in w. Moreover, we only change a single `d` to an `e` in each $\frac{7}{5}$-power. Take any two distinct appearances of `e` in $\eta(w)$. If they arise due to overlapping appearances of $\frac{7}{5}$-powers in w, then the factor of w we are dealing with is equal to $\sigma(T_2)$ for some permutation σ of A. Since we are concerned with two distinct appearances of `e`, we must have changed some letter in the length 4 prefix of $\sigma(T_2)$ and the length 4 suffix of $\sigma(T_2)$. This leaves a factor of length at least 14 between the two `e`'s. Otherwise, the two appearances of `e` arise due to nonoverlapping appearances of $\frac{7}{5}$-powers in w. But by exhaustive check, there is a factor of length at least 14 between any pair of nonoverlapping appearances of $\frac{7}{5}$-powers in w. ☐

Our constructions of $\frac{4}{3}^+$-free circular words on five letters will require factors of $\mathbf{P}$ that begin and end in the same letter. The next lemma concerns the existence of such factors.

Lemma 14. *For every $n \geqslant 3$, $\mathbf{P}$ either has a factor of length n that begins and ends in* `d`*, or a factor of length $n+1$ that begins and ends in* `d`*.*

Proof. Let $n \geqslant 3$. First we demonstrate that $\mathbf{p}$ has a factor of length n which begins and ends in `1`. Recall that $\mathbf{p} = \texttt{10101101}\ldots$, so in particular $\mathbf{p}[5] = \mathbf{p}[6] = \texttt{1}$. By Result 11(a), either $\mathbf{p}[n+4] = \texttt{1}$ or $\mathbf{p}[n+5] = \texttt{1}$, and hence at least one of the length n factors $\mathbf{p}[5 : n+4]$ or $\mathbf{p}[6 : n+5]$ begins and ends in `1`.

By Result 11(d), it suffices to show that $\mathbf{P}$ either has a factor of length n that begins and ends in the same letter, or a factor of length $n+1$ that begins and ends in the same letter. Let $\mathbf{q}$ be a suffix of $\mathbf{p}$ that satisfies $\mathbf{q}[1] = \mathbf{q}[n] = \texttt{1}$, and consider the word $\mathbf{Q} = M(\mathbf{q})$. (Note that $\mathbf{Q}$ is a suffix of $\mathbf{P}$ under a permutation of A, but by Result 11(d), every finite factor of $\mathbf{Q}$ is a factor of $\mathbf{P}$.) Since $\mathbf{q}[1] = \mathbf{q}[n] = \texttt{1}$, we see that $\mathbf{Q}[1 : 4] = \texttt{abcd}$, and $\mathbf{Q}[n : n+3]$ contains all 4 letters of A. If $\mathbf{Q}[n] = \texttt{a}$, then $\mathbf{Q}[1 : n]$ begins and ends with `a`, and we are done. So we may assume that $\mathbf{Q}[n] \neq \texttt{a}$. If $\mathbf{Q}[n+1] = \texttt{a}$, then $\mathbf{Q}[1 : n+1]$ begins and ends in `a`, and if $\mathbf{Q}[n+1] = \texttt{b}$, then $\mathbf{Q}[2 : n+1]$ begins and ends in `b`. So we may assume that $\mathbf{Q}[n+1] \in \{\texttt{c}, \texttt{d}\}$. By similar arguments, we may assume that $\mathbf{Q}[n+2] \in \{\texttt{a}, \texttt{d}\}$ and $\mathbf{Q}[n+3] \in \{\texttt{a}, \texttt{b}\}$. By inspection, there are only three possibilities for $\mathbf{Q}[n : n+3]$:

$$\texttt{bcda}, \ \texttt{cdab}, \ \text{and } \texttt{dcab}.$$

	$\mathbf{Q}[k-3]$ α	$\mathbf{Q}[k-2]$ β	$\mathbf{Q}[k-1]$ γ	$\mathbf{Q}[k]$	
$\mathbf{Q}[n+k-5]$ α	$\mathbf{Q}[n+k-4]$ β	$\mathbf{Q}[n+k-3]$ γ	$\mathbf{Q}[n+k-2]$	$\mathbf{Q}[n+k-1]$	$\mathbf{Q}[n+k]$

Table 1: Visual aid for Case I of the proof of Lemma 14. Columns show first and last letter of factors of length n in $\mathbf{Q}$.

$\mathbf{Q}[k-3]$ α	$\mathbf{Q}[k-2]$ β	$\mathbf{Q}[k-1]$ γ	$\mathbf{Q}[k]$	$\mathbf{Q}[k+1]$	$\mathbf{Q}[k+2]$
		$\mathbf{Q}[n+k-2]$ α	$\mathbf{Q}[n+k-1]$ β	$\mathbf{Q}[n+k]$ γ	$\mathbf{Q}[n+k+1]$

Table 2: Visual aid for Case II of the proof of Lemma 14. Columns show first and last letter of factors of length n in $\mathbf{Q}$.

Case I: $\mathbf{Q}[n : n+3] = \texttt{bcda}$ (see Table 1)

Note that $\mathbf{Q}[n : n+2] = \texttt{bcd} = \mathbf{Q}[2 : 4]$. However, $\mathbf{Q}[2 : n-1]$ and $\mathbf{Q}[n : 2n-3]$ must differ at some point, as otherwise $\mathbf{Q}[2 : 2n-3]$ is a square, contradicting Result 11(b). Let k be the smallest number greater than 2 such that $\mathbf{Q}[k] \neq \mathbf{Q}[n+k-2]$. Let $\mathbf{Q}[k-3 : k-1] = \mathbf{Q}[n+k-5 : n+k-3] = \alpha\beta\gamma$, where α, β, γ, and δ are the images of $\texttt{a}$, $\texttt{b}$, $\texttt{c}$, and $\texttt{d}$, respectively, under some permutation of A. If $\mathbf{Q}[k] = \delta$ and $\mathbf{Q}[n+k-2] = \alpha$, then it follows that $\mathbf{Q}[n+k-1] = \delta$. But then $\mathbf{Q}[k : n+k-1]$ has length n and begins and ends with δ. On the other hand, if $\mathbf{Q}[k] = \alpha$ and $\mathbf{Q}[n+k-2] = \delta$, then either $\mathbf{Q}[n+k-1] = \alpha$, or $\mathbf{Q}[n+k] = \alpha$. But then $\mathbf{Q}[k : n+k-1]$, or $\mathbf{Q}[k : n+k]$, respectively, begins and ends with α.

Case II: $\mathbf{Q}[n : n+3] \in \{\texttt{cdab}, \texttt{dcab}\}$ (see Table 2)

If $\mathbf{Q}[n+4] = \texttt{d}$, then $\mathbf{Q}[4 : n+4]$ begins and ends in $\texttt{d}$, and we are done, so we may assume that $\mathbf{Q}[n+4] = \texttt{c}$. Then $\mathbf{Q}[1 : 3] = \texttt{abc} = \mathbf{Q}[n+2 : n+4]$. However, $\mathbf{Q}[1 : n+1]$ and $\mathbf{Q}[n+2 : 2n+2]$ must differ at some point, as otherwise $\mathbf{Q}[1 : 2n+2]$ is a square, contradicting Result 11(b). Let k be the smallest number such that $\mathbf{Q}[k] \neq \mathbf{Q}[n+k+1]$. Let $\mathbf{Q}[k-3 : k-1] = \mathbf{Q}[n+k-2 : n+k] = \alpha\beta\gamma$, where α, β, γ, andδ are the images of $\texttt{a}$, $\texttt{b}$, $\texttt{c}$, and $\texttt{d}$, respectively, under some permutation of A. If $\mathbf{Q}[k] = \alpha$ and $\mathbf{Q}[n+k+1] = \delta$, then it follows that $\mathbf{Q}[k+1] = \delta$, and $\mathbf{Q}[k+1 : n+k+1]$ begins and ends with δ. Otherwise, $\mathbf{Q}[k] = \delta$ and $\mathbf{Q}[n+k+1] = \alpha$. Then either $\mathbf{Q}[k+1] = \alpha$, or $\mathbf{Q}[k+2] = \alpha$. But then either $\mathbf{Q}[k+1 : n+k+1]$, or $\mathbf{Q}[k+2 : n+k+1]$, respectively, begins and ends in α. $\square$

We believe that a stronger version of Lemma 14 holds. We conjecture that for every $n \geqslant 4$, there is a factor of length n in $\mathbf{P}$ that begins and ends in the same letter. We have verified this statement for $4 \leqslant n \leqslant 2000$. A proof of this fact would reduce the amount of case work required in the proof of the main result of this section, which we are now ready to prove.

We first show that there is a $\frac{4}{3}^+$-free circular word on 5 letters of every even length. We then construct words of every odd length from the words of even length by inserting a single letter `e` in a carefully chosen location.

Theorem 15. *For every $n \in \mathbb{N}$, there is a $\frac{4}{3}^+$-free circular word on 5 letters of length $2n$.*

Proof. For every $n < 73$, we find such a word by computer search, so we may assume that $n \geqslant 73$. By Lemma 14, $\mathbf{P}$ either has a factor of length $n - 4$ that begins and ends in `d`, or a factor of length $n - 5$ that begins and ends in `d`. This gives us two cases:

Case I: $\mathbf{P}$ has a factor w' of length $n - 4$ that begins and ends in `d`

Let $w = \alpha w' \beta$ be a factor of $\mathbf{P}$ containing w', where $\alpha, \beta \in A$. By Result 11(d), we may assume that $\alpha = \mathtt{a}$, and $\beta \in \{\mathtt{a}, \mathtt{b}\}$. Let σ be the permutation of $A \cup \{\mathtt{e}\}$ defined by $(\mathtt{a}, \mathtt{b}, \mathtt{c})(\mathtt{d}, \mathtt{e})$ in cycle notation. Consider the word

$$W_{2n} = \mathtt{de}\ \eta(w)\ \mathtt{ed}\ [\sigma(\eta(w))]^R.$$

We claim that the circular word (W_{2n}) is $\frac{4}{3}^+$-free, except in a small number of exceptional cases which are handled later by making minor adjustments to $\eta(w)$.

Suppose towards a contradiction that some conjugate of W_{2n} has a factor xyx with $2|x| > |y|$. First of all, note that x does not contain the factor `de` or the factor `ed`, since each of these factors appears at most once in any conjugate of W_{2n}. So we may assume that x is a factor of either $\mathtt{e}\eta(w)\mathtt{e}$ or $\mathtt{d}[\sigma(\eta(w))]^R\mathtt{d}$. Further, note that the length 3 prefix and the length 3 suffix of $\mathtt{e}\eta(w)\mathtt{e}$, namely `ead` and $\mathtt{d}\beta\mathtt{e}$, respectively, and the length 3 prefix and the length 3 suffix of $\mathtt{d}[\sigma(\eta(w))]^R\mathtt{d}$, namely $\mathtt{d}\sigma(\beta)\mathtt{e}$ and $\mathtt{e}\sigma(\mathtt{a})\mathtt{d}$, respectively, appear at most once in any conjugate of W_{2n}. This is independent of the identity of β, and simply relies on the fact that $\sigma(\mathtt{a}) \neq \mathtt{a}$ and $\sigma(\mathtt{b}) \neq \mathtt{b}$. Therefore, x does not contain any of these factors. The possibility that x appears as a prefix or suffix of $\mathtt{e}\eta(w)\mathtt{e}$ or $\mathtt{d}\sigma(\eta(w))^R\mathtt{d}$ with $|x| = 2$ is eliminated by a later exhaustive search, so we may assume that x is a factor of either $\eta(w)$ or $\sigma(\eta(w))^R$.

Suppose first that the factor xyx appears in (W_{2n}) so that both appearances of x lie in $\eta(w)$. By Lemma 13(a), xyx is not a factor of $\eta(w)$. On the other hand, if y contains all of $\mathtt{ed}[\sigma(\eta(w))]^R\mathtt{de}$, then $|y| \geqslant n + 2$, which is over half the length of W_{2n}, and contradicts the assumption that $2|x| > |y|$. So the factor xyx does not appear in such a way that both appearances of x lie in $\eta(w)$. By a similar argument, the factor xyx does not appear in such a way that both appearances of x lie in $\sigma(\eta(w))^R$. (Note that $\sigma(\eta(w))^R$ is also $\frac{4}{3}^+$-free, because this property is preserved under permutation of the letters and reversal.)

So we may assume that xyx appears in (W_{2n}) in such a way that one appearance of x is in $\eta(w)$ and the other is in $\sigma(\eta(w))^R$. By Lemma 13(b), every factor of length 15 in $\eta(w)$ contains at least two `d`'s and at most one `e`. Since σ swaps `d` and `e`, every factor of length 15 of $\sigma(\eta(w))^R$ contains at least two `e`'s and at most one `d`. Since x must appear in both $\eta(w)$ and $\sigma(\eta(w))^R$, we must have $|x| \leqslant 14$.

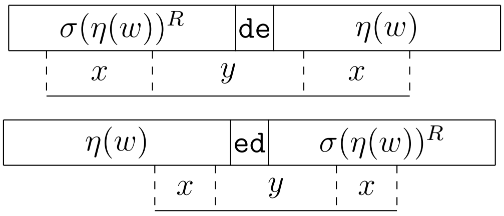


Figure 2: Possible appearances of xyx in (W_{2n}).

The remainder of the proof is completed by a finite search. Since $|x| \leqslant 14$ and $|y| < 2|x|$, we must have $|xyx| < 56$. Further, since xyx must appear as in Figure 2, we only need to search a short factor of $\sigma(\eta(w))^R\mathtt{de}\eta(w)$ (and $\eta(w)\mathtt{ed}\sigma(\eta(w))^R$) around $\mathtt{de}$ ($\mathtt{ed}$, respectively). In fact, since the two appearances of x must lie on opposite sides of $\mathtt{de}$ ($\mathtt{ed}$, respectively), xyx can extend at most 39 letters away from $\mathtt{de}$ ($\mathtt{ed}$, respectively).

So it suffices to check that $\sigma(p_{39}(\eta(w)))^R\mathtt{de}p_{39}(\eta(w))$ and $s_{39}(\eta(w))\mathtt{ed}\sigma(s_{39}(\eta(w)))^R$ are $\frac{4}{3}^+$-free. We know that $\eta(w)$ begins in $\mathtt{ad}$ and ends in either $\mathtt{da}$ or $\mathtt{db}$, but we can't assume anything else about the prefix or the suffix of $\eta(w)$; we just know that w is a factor of $\mathbf{P}$. The length 39 prefix of $\eta(w)$ is completely determined by the length 60 prefix of w. The length 60 prefix of w is required because factors of length 14 (nonoverlapping $\frac{7}{5}$-powers) and 22 (overlapping $\frac{7}{5}$-powers) determine whether or not we change a given $\mathtt{d}$ in w to an $\mathtt{e}$ in $\eta(w)$. Similarly, the length 39 suffix of $\eta(w)$ is completely determined by the length 60 suffix of w. This is where the assumption that $n \geqslant 73$ is used, so that we may assume that all nonoverlapping $\frac{7}{5}$-powers in $s_{52}(w)$ (i.e. all those that could impact $s_{39}(\eta(w))$) are eliminated in $\eta(w)$ by changing a $\mathtt{d}$ to an $\mathtt{e}$ in the prefix (and not the suffix) of the $\frac{7}{5}$-power.

We run through all possible prefixes $p_{39}(\eta(w))$, and find that

$$\sigma(p_{39}(\eta(w)))^R\mathtt{de}p_{39}(\eta(w))$$

is $\frac{4}{3}^+$-free in all but a small number of cases. In each of these exceptional cases, we make a minor adjustment in a short prefix of $\eta(w)$ to fix the issue; see Appendix A.1 for details. Similarly, we run through all possible suffixes $s_{39}(\eta(w))$, and find that

$$s_{39}(\eta(w))\mathtt{ed}\sigma(s_{39}(\eta(w)))^R$$

is $\frac{4}{3}^+$-free in all but a small number of cases. In each of these exceptional cases, we make a minor adjustment in a short suffix of $\eta(w)$ to fix the issue; see Appendix A.1 for details.

Case II: $\mathbf{P}$ has a factor w' of length $n-5$ that begins and ends in $\mathtt{d}$

Let $w = \alpha\beta w'\gamma$ be a factor of $\mathbf{P}$ containing w', where $\alpha, \beta, \gamma \in A$. By Result 11(d), we may assume that $\alpha = \mathtt{a}$, $\beta = \mathtt{b}$, and $\gamma \in \{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$. We have two subcases:

Case II(a): $\gamma \in \{\mathtt{a}, \mathtt{c}\}$.

Let τ be the permutation of $A \cup \{\mathtt{e}\}$ defined by $(\mathtt{a}, \mathtt{c})(\mathtt{d}, \mathtt{e})$ in cycle notation. (Note that b is fixed by τ). Consider the word

$$W_{2n} = \mathtt{de}\ \eta(w)\ \mathtt{ed}\ [\tau(\eta(w))]^R.$$

We claim that the circular word (W_{2n}) is $\frac{4}{3}^+$-free, except in a small number of exceptional cases which are handled later by making minor adjustments to $\eta(w)$.

Suppose towards a contradiction that some conjugate of W has a factor xyx with $2|x| > |y|$. As in Case I, x does not contain the factor $\mathtt{de}$ or the factor $\mathtt{ed}$. So we may assume that x is a factor of either $\mathtt{e}\eta(w)\mathtt{e}$ or $\mathtt{d}[\tau(\eta(w))]^R\mathtt{d}$. Further, note that the length 3 suffix of $\mathtt{e}\eta(w)\mathtt{e}$, namely $\mathtt{d}\gamma\mathtt{e}$, and the length 3 prefix of $\mathtt{d}[\tau(\eta(w))]^R\mathtt{d}$, namely $\mathtt{d}\tau(\gamma)\mathtt{e}$ appear only once in (W_{2n}). Therefore, x does not contain either of these factors, and we may assume that x is a factor of either $\mathtt{e}\eta(w)$ or $\tau(\eta(w))^R\mathtt{d}$. Note that the length 4 prefix of $\mathtt{e}\eta(w)$, namely $\mathtt{eabd}$, and the length 4 suffix of $\tau(\eta(w))^R\mathtt{d}$, namely $\mathtt{ebcd}$, may both appear elsewhere in (W_{2n}), so we cannot immediately eliminate the possibility that x contains one of these factors (as we did in Case I).

Suppose first that the factor xyx appears in (W_{2n}) so that both appearances of x lie in $\mathtt{e}\eta(w)$. By Lemma 13(a), xyx is not a factor of $\eta(w)$. Further, it cannot be the case that y contains all of $\mathtt{ed}[\tau(\eta(w))]^R\mathtt{d}$, because then $|y| \geqslant n + 1$, which is over half the length of W_{2n}. The only possibility that remains is that xyx is a prefix of $\mathtt{e}\eta(w)$. (Note that $\mathtt{d}w$ is not necessarily a factor of $\mathbf{P}$ so we cannot apply Lemma 13(a) directly here.) By exhaustive search of the possible prefixes of $\eta(w)$ of length 19, we may assume that $|x| \geqslant 6$ and $|xy| \geqslant 11$. Write $x = \mathtt{e}x'$, so that $xyx = \mathtt{e}x'y\mathtt{e}x'$. We see that $x'y\mathtt{e}x'$ is a repetition of exponent $\frac{|xyx|-1}{|xy|}$. Since $|xy| \geqslant 11$, by Lemma 12, we must have

$$\frac{|xyx| - 1}{|xy|} \leqslant \tfrac{14}{11} \quad \Rightarrow \quad 8|x| - 11 \leqslant 3|y|.$$

Since $|x| \geqslant 6$, we have $8|x|-11 \geqslant 6|x|+12-11 > 6|x|$, and thus $2|x| < |y|$, a contradiction. By a similar argument, the factor xyx cannot appear in such a way that both appearances of x lie in $\tau(\eta(w))^R\mathtt{d}$.

So we may assume that xyx appears in (W_{2n}) in such a way that one appearance of x is in $\mathtt{e}\eta(w)$ and the other is in $\tau(\eta(w))^R\mathtt{d}$. By Lemma 13(b), every factor of $\eta(w)$ of length 15 contains at least two $\mathtt{d}$'s and at most one $\mathtt{e}$. Since τ swaps $\mathtt{d}$ and $\mathtt{e}$, every factor of length 15 of $\tau(\eta(w))^R$ contains at least two $\mathtt{e}$'s and at most one $\mathtt{d}$.

Suppose that xyx appears so that one appearance of x is a prefix of $\mathtt{e}\eta(w)$ and $|x| \geqslant 17$. Then $x[2:16]$ is a factor of $\eta(w)$ of length 15, and hence is not a factor of $\tau(\eta(w))^R$. But then x is not a factor of $\tau(\eta(w))^R\mathtt{d}$. If x is a prefix of $\mathtt{e}\eta(w)$ and $|x| \leqslant 16$, then certainly xyx appears in $\tau(\eta(w))^R\mathtt{de}\eta(w)$ and not $\mathtt{e}\eta(w)\mathtt{ed}\tau(\eta(w))^R\mathtt{d}$, since $n \geqslant 73$. Since $|x| \leqslant 16$, xyx must in fact appear in $\tau(\eta(w)[1:46])^R\mathtt{de}\eta(w)[1:15]$. We eliminate this possibility by exhaustively checking the possible prefixes of $\eta(w)$. By a similar argument, we may assume that x is not a suffix of $\tau(\eta(w))^R\mathtt{d}$.

But then xyx appears in (W_{2n}) in such a way that one appearance of x is in $\eta(w)$ and the other is in $\tau(\eta(w))^R$. We then have $|x| \leqslant 14$, and the proof is completed by an exhaustive search as in Case I. Here we check that

$$\tau(p_{46}(\eta(w)))^R\,\mathtt{de}\,p_{46}(\eta(w))$$

is $\frac{4}{3}^+$-free (We check a longer prefix than in Case I as this is required in the previous paragraph.), and that

$$s_{39}(\eta(w))\,\mathtt{ed}\,\tau(s_{39}(\eta(w)))^R$$

is $\frac{4}{3}^+$-free. Once again, there are several exceptional cases in which we need to make a minor adjustment in a short prefix and/or suffix of $\eta(w)$; see Appendix A.2 for details.

Case II(b): $\gamma = \mathtt{b}$.

Here, we have $w = \mathtt{ab}w'\mathtt{b}$, and recall that $|w| = n-2$. While we would have liked to find a way to handle this case as we did for Case I and Case II(a), we must do something slightly different here to avoid repetitions of short length close to the 'buffers' de and ed. Let $\delta \in A$ be a letter such that $w\delta$ is a factor of $\mathbf{P}$. Since w ends in db, we must have $\delta \in \{a, c\}$, meaning two cases, though the second reduces to an already completed case.

Case II(b1): $\delta = \mathtt{a}$

Let $\dot{w} = \mathtt{b}w'\mathtt{b}$ and $\ddot{w} = \mathtt{ab}w'\mathtt{ba}$. Note that $|\dot{w}| = n-3$ and $|\ddot{w}| = n-1$. Let π be the permutation of A defined by $(\mathtt{b}, \mathtt{d}, \mathtt{c})$ and let ρ be the permutation of $A \cup \{\mathtt{e}\}$ defined by $(\mathtt{d}, \mathtt{e})$ (i.e. ρ swaps d and e). Define

$$W_{2n} = \mathtt{de}\ \eta(\dot{w})\ \mathtt{ed}\ \rho\left(\eta(\pi(\ddot{w}))\right)^R.$$

We claim that the circular word (W_{2n}) has no $\frac{4}{3}^+$-powers, except in a small number of cases which are handled later by making small adjustments to $\eta(\dot{w})$ and/or $\rho(\eta(\pi(\ddot{w})))^R$. Most of the proof is similar to that of Case I, so we omit some details.

Suppose towards a contradiction that some conjugate of W_{2n} has a factor xyx with $2|x| > |y|$. First of all, note that x does not contain the factor de or the factor ed, so we may assume that x is a factor of either $\mathtt{e}\eta(\dot{w})\mathtt{e}$ or $\mathtt{d}\rho\left(\eta(\pi(\ddot{w}))\right)^R\mathtt{d}$. Further, note that the length 3 prefix and the length 3 suffix of $\mathtt{e}\eta(\dot{w})\mathtt{e}$, namely ebd and dbe, respectively, and the length 3 prefix and the length 3 suffix of $\mathtt{d}\rho(\eta(\pi(\ddot{w})))^R\mathtt{d}$, namely dae and ead, respectively, appear at most once in any conjugate of W_{2n}. Therefore, x does not contain any of these factors, and we may assume that x is a factor of either $\eta(\dot{w})$ or $\rho(\eta(\pi(\ddot{w})))^R$.

By an argument similar to the one used in Case I, the factor xyx cannot appear in such a way that both appearances of x lie in $\eta(\dot{w})$ (or $\rho(\eta(\pi(\ddot{w})))^R$). So we may assume that xyx appears in (W_{2n}) in such a way that one appearance of x is in $\eta(\dot{w})$ and the other is in $\rho(\eta(\pi(\ddot{w})))^R$. By Lemma 13(b), every factor of length 15 in $\eta(\dot{w})$ or $\eta(\pi(\ddot{w}))$ contains at least two d's and at most one e. Since ρ swaps d and e, every factor of length 15 of $\rho(\eta(\pi(\ddot{w})))^R$ contains at least two e's and at most one d. Since x must appear in both $\eta(\dot{w})$ and $\rho(\eta(\pi(\ddot{w})))^R$, we must have $|x| \leqslant 14$.

The remainder of the proof is completed by a finite search, as in Case I. It suffices to check that

$$\rho(p_{39}(\eta(\pi(\ddot{w}))))^R\mathtt{de}p_{39}(\eta(\dot{w}))$$

	$\mathbf{P}[k]$	$\mathbf{P}[k+1]$	$\mathbf{P}[k+2]$	$\mathbf{P}[k+3]$	$\mathbf{P}[k+4]$	$\mathbf{P}[k+5]$	$\mathbf{P}[k+6]$	$\mathbf{P}[k+7]$	$\mathbf{P}[k+8]$	
	$\mathtt{a}$	$\mathtt{b}$	$\mathtt{d}$							
encoding										
	$\mathbf{P}[n+k-5]$	$\mathbf{P}[n+k-4]$	$\mathbf{P}[n+k-3]$	$\mathbf{P}[n+k-2]$	$\mathbf{P}[n+k-1]$	$\mathbf{P}[n+k]$	$\mathbf{P}[n+k+1]$	$\mathbf{P}[n+k+2]$	$\mathbf{P}[n+k+3]$	$\mathbf{P}[n+k+4]$
		$\mathtt{d}$	$\mathtt{b}$	$\mathtt{c}$						
encoding										

Table 3: Visual aid for Case II(b2) of the proof of Theorem 15. Columns contain the first and last letter of factors of length $n-4$ in $\mathbf{P}$. The reader can complete the table as they read the proof.

and

$$s_{39}(\eta(\dot{w}))\mathtt{ed}\rho(s_{39}(\eta(\pi(\ddot{w}))))^R$$

are $\frac{4}{3}^+$-free. As in Case I and Case II(a), there are several exceptional cases in which we need to make a minor adjustment in a short prefix and/or suffix of $\eta(\dot{w})$ or $\eta(\pi(\ddot{w}))$; see Appendix A.3 for details.

Case II(b2): $\delta = \mathtt{c}$

We show that $\mathbf{P}$ either has a factor of length $n-4$ that begins and ends in $\mathtt{d}$ (and we are back in Case I), or has a factor z of length $n-5$ that begins and ends in $\mathtt{d}$ and appears internally as $\mathtt{ab}z\mathtt{ba}$, i.e. a factor of length $n-1$ that begins in $\mathtt{abd}$ and ends in $\mathtt{dba}$ (and we are back in Case II(b1)).

Suppose otherwise that $\mathbf{P}$ has no factors of either of these forms. By Lemma 11(d), $\mathbf{P}$ also does not contain a factor of either of these forms under any permutation of A. Let $k \in \mathbb{N}$ satisfy $\mathbf{P}[k : n+k-2] = \mathtt{ab}w'\mathtt{bc}$. For ease of reading, we provide Table 3, which can be filled in as follows. First of all, note that $\mathbf{P}[n+k-5] = \mathtt{c}$, since otherwise $\mathbf{P}[k : n+k-5]$ is a factor of length $n-4$ that begins and ends in $\mathtt{a}$. Hence, $\mathbf{P}[n+k-2]$ is encoded by $\mathtt{0}$, and by Lemma 11(a), $\mathbf{P}[n+k-1]$ must be encoded by $\mathtt{1}$. Hence, $\mathbf{P}[n+k-1] = \mathtt{a}$. Now we see that $\mathbf{P}[k+3] = \mathtt{a}$, since otherwise $\mathbf{P}[k+3 : n+k-2]$ begins and ends in $\mathtt{c}$, and the encoding for $\mathbf{P}[k+3]$ is $\mathtt{0}$. Hence, the encoding for $\mathbf{P}[k+4]$ is $\mathtt{1}$, and $\mathbf{P}[k+4] = \mathtt{c}$.

Now if $\mathbf{P}[n+k] = \mathtt{d}$, then the factor $\mathbf{P}[k+2 : n+k]$ under the permutation $(\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d})$ puts us back in Case II(b1), so we may assume that $\mathbf{P}[n+k] = \mathtt{b}$, and since this is encoded by $\mathtt{0}$, the next letter in the encoding is $\mathtt{1}$, and hence $\mathbf{P}[n+k+1] = \mathtt{d}$. Now $\mathbf{P}[k+5]$ cannot be $\mathtt{b}$, so it must be $\mathtt{d}$. Hence, $\mathbf{P}[k+5]$ is encoded by $\mathtt{0}$ and it follows that $\mathbf{P}[k+6]$ is encoded by $\mathtt{1}$, so that $\mathbf{P}[k+6] = \mathtt{b}$.

Now if $\mathbf{P}[n+k+2] = \mathtt{c}$, then the factor $\mathbf{P}[k+4 : n+k+2]$ under the permutation $(\mathtt{a}, \mathtt{c})(\mathtt{b}, \mathtt{d})$ puts us back in Case II(b1), so we may assume that $\mathbf{P}[n+k+2] = \mathtt{a}$. Then $\mathbf{P}[n+k+2]$ is encoded by $\mathtt{0}$, and $\mathbf{P}[n+k+3]$ is encoded by $\mathtt{1}$, so $\mathbf{P}[n+k+3] = \mathtt{c}$. Now $\mathbf{P}[k+7]$ cannot be $\mathtt{a}$, so it must be $\mathtt{c}$. Hence $\mathbf{P}[k+7]$ is encoded by $\mathtt{0}$ and $\mathbf{P}[k+8] = \mathtt{a}$ is encoded by $\mathtt{1}$. Finally, if $\mathbf{P}[n+k+4] = \mathtt{b}$, then the factor $\mathbf{P}[k+6 : n+k+4]$ under the permutation $(\mathtt{a}, \mathtt{d}, \mathtt{c}, \mathtt{b})$ puts us back in Case II(b1), so we may assume that $\mathbf{P}[n+k+4] = \mathtt{d}$, and hence is encoded by $\mathtt{0}$. However, this is impossible, because $\mathbf{p}$ does not have $\mathtt{0101010}$ as a factor. ☐

Theorem 16. *For every $n \geqslant 0$, there is a $\frac{4}{3}^+$-free circular word on 5 letters of length $2n+1$.*

Proof. We first verify the statement directly for $n \leqslant 644$ by computer. Now suppose $n \geqslant 645$. Take the $\frac{4}{3}^+$-free circular word (W_{2n}) on $A \cup \{e\}$ of length $2n$ constructed in Theorem 15. Note that W_{2n} may have been constructed with one of the adjustments outlined in Appendix A; our arguments still apply in each of these exceptional cases. In particular, W_{2n} contains the factor $\eta(w)[23:-23]$ (Case I or Case II(a)) or $\eta(\dot{w})[23:-23]$ (Case II(b1)), where w is a factor of length $n-2 \geqslant 643$ of $\mathbf{P}$ and $\dot{w}$ is a factor of length $n-3 \geqslant 642$ of $\mathbf{P}$.

By exhaustive search, if u is any factor of $\mathbf{P}$ of length 642, then $\eta(u)[23:-23]$ contains the factor

$$z = \gamma\beta\alpha\gamma\mathtt{d}\alpha\beta\mathtt{d}\gamma\alpha\mathtt{d}\beta\gamma\mathtt{d}\alpha\gamma\beta\alpha$$

where α, β, and γ are the images of $\mathtt{a}$, $\mathtt{b}$, and $\mathtt{c}$ under some permutation of $\{\mathtt{a},\mathtt{b},\mathtt{c}\}$. Note that we ignore the prefix of length 22 and the suffix of length 22 of $\eta(u)$ to ensure that this property extends to all factors of $\mathbf{P}$ of length greater than 642. Note that z contains the factors $\mathtt{d}\alpha\beta\mathtt{d}$, $\mathtt{d}\gamma\alpha\mathtt{d}$, and $\mathtt{d}\beta\gamma\mathtt{d}$.

Now for concreteness, assume that W_{2n} was created using the construction of Case I of Theorem 15. The other cases are handled by similar arguments. Recall that

$$W_{2n} = \mathtt{de}\,\eta(w)\,\mathtt{ed}\,\sigma(\eta(w))^R,$$

and that $\eta(w)$ begins in $\mathtt{ad}$ and ends in $\mathtt{d}\lambda$ for some letter $\lambda \in \{\mathtt{a},\mathtt{b}\}$. Hence, $\sigma(\eta(w))^R$ begins in $\sigma(\lambda\mathtt{d}) = \sigma(\lambda)\mathtt{e}$ and ends in $\sigma(\mathtt{da}) = \mathtt{eb}$. So the 'buffers' $\mathtt{de}$ and $\mathtt{ed}$ appear internally as

$$\mathtt{ebdead} \quad \text{and} \quad \mathtt{d}\lambda\mathtt{ed}\sigma(\lambda)\mathtt{e},$$

respectively. By construction, these factors contain the only appearances of factors of the form $\mathtt{d}\kappa\mathtt{e}$ or $\mathtt{e}\kappa\mathtt{d}$ in (W_{2n}), where κ is a single letter. Exactly one letter from $\{\alpha,\beta,\gamma\}$ is equal to λ, and likewise for $\mathtt{a}$, so at least one of the following must be true:

(a) $\alpha \neq \lambda$ and $\beta \neq \mathtt{a}$;

(b) $\gamma \neq \lambda$ and $\alpha \neq \mathtt{a}$; or

(c) $\beta \neq \lambda$ and $\gamma \neq \mathtt{a}$.

We obtain a word W_{2n+1} of length $2n+1$ by inserting a single $\mathtt{e}$ into W_{2n}, and claim that (W_{2n+1}) is $\frac{4}{3}^+$-free.

If (a) is true, insert an $\mathtt{e}$ into the first appearance of z in $\eta(w)[23:-23]$ as follows:

$$\gamma\beta\alpha\gamma\mathtt{d}\alpha\,\mathtt{e}\,\beta\mathtt{d}\gamma\alpha\mathtt{d}\beta\gamma\mathtt{d}\alpha\gamma\beta\alpha.$$

Suppose towards a contradiction that (W_{2n+1}) contains a factor xyx with $2|x| > |y|$. Since (W_{2n}) is $\frac{4}{3}^+$-free and we inserted a single letter $\mathtt{e}$ into W_{2n} to create W_{2n+1}, it must be the case that the new letter $\mathtt{e}$ appears inside one of the instances of x. We first claim that x cannot contain either of the factors $\mathtt{d}\alpha\mathtt{e}$ or $\mathtt{e}\beta\mathtt{d}$. Suppose first that x contains $\mathtt{d}\alpha\mathtt{e}$ as a factor. Then certainly $\mathtt{d}\alpha\mathtt{e}$ must appear twice in some conjugate of W_{2n+1}. Since $\alpha \neq \lambda$, we must have $\alpha = \sigma(\lambda)$. However, the factor $\mathtt{d}\alpha\mathtt{e} = \mathtt{d}\sigma(\lambda)\mathtt{e}$ appears only inside the factors $\mathtt{d}\alpha\mathtt{e}\beta\mathtt{d}$ and $\mathtt{d}\lambda\mathtt{ed}\sigma(\lambda)\mathtt{e}$, which each appear only once. To the left of $\mathtt{d}\alpha\mathtt{e}$ in $\mathtt{d}\alpha\mathtt{e}\beta\mathtt{d}$ is the letter γ, which is different from $\mathtt{d}$, the letter to the left of $\mathtt{d}\sigma(\lambda)\mathtt{e}$ in $\mathtt{d}\lambda\mathtt{ed}\sigma(\lambda)\mathtt{e}$. On the other hand, the length 2 factor to the right of $\mathtt{d}\alpha\mathtt{e}$ in $\mathtt{d}\alpha\mathtt{e}\beta\mathtt{d}$ is $\beta\mathtt{d}$, while the length 2 factor to the right of $\mathtt{d}\sigma(\lambda)\mathtt{e}$ in $\mathtt{d}\lambda\mathtt{ed}\sigma(\lambda)\mathtt{e}$ does not end in $\mathtt{d}$. We conclude that $|x| \leqslant 4$. However, since we inserted the new letter $\mathtt{e}$ somewhere in $\eta(w)[23 : -23]$, this certainly forces $|y| \geqslant 2|x|$, a contradiction. The proof that x cannot contain $\mathtt{e}\beta\mathtt{d}$ is similar.

The only remaining possibilities are $x = \mathtt{e}$, $x = \alpha\mathtt{e}$, $x = \mathtt{e}\beta$, and $x = \alpha\mathtt{e}\beta$, and these are all eliminated by inspection.

Otherwise, if (b) is true, insert an $\mathtt{e}$ into the first appearance of z in W_{2n} as follows:

$$\gamma\beta\alpha\gamma\mathtt{d}\alpha\beta\mathtt{d}\gamma\ \mathtt{e}\ \alpha\mathtt{d}\beta\gamma\mathtt{d}\alpha\gamma\beta\alpha.$$

Otherwise, if (c) is true, insert an $\mathtt{e}$ into the first appearance of z in W_{2n} as follows:

$$\gamma\beta\alpha\gamma\mathtt{d}\alpha\beta\mathtt{d}\gamma\alpha\mathtt{d}\beta\ \mathtt{e}\ \gamma\mathtt{d}\alpha\gamma\beta\alpha.$$

The proofs for (b) and (c) are similar to the proof for (a). □

5 Conclusion

In this article, we proved that $\mathrm{CRT_S}(4) = \frac{3}{2}$, and $\mathrm{CRT_S}(5) = \frac{4}{3}$, providing the last unknown values of the strong circular repetition threshold:

$$\mathrm{CRT_S}(k) = \begin{cases} \frac{5}{2} & \text{if } k = 2; \\ 3 & \text{if } k = 3; \text{ and} \\ \frac{\lceil k/2 \rceil + 1}{\lceil k/2 \rceil} & \text{if } k \geqslant 4. \end{cases}$$

We also gave a proof that $\mathrm{CRT_I}(3) = \mathrm{CRT_W}(3) = \frac{7}{4}$ by adapting the method used to prove $\mathrm{CRT_S}(4) = \frac{3}{2}$. While we conjecture that $\mathrm{CRT_I}(k) = \mathrm{CRT_W}(k) = \mathrm{RT}(k)$ for all $k \geqslant 4$, techniques different from those presented here will likely be needed.

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Appendix A. Exceptional Cases

A.1 Exceptions for Case I

There are 12 possible prefixes of length 39 of $\eta(w)$ for which we encounter a $\frac{4}{3}^{+}$-power in

$$\sigma(p_{39}(\eta(w)))^R\mathtt{de}p_{39}(\eta(w)),$$

and 16 possible suffixes of length 39 of $\eta(w)$ for which we encounter a $\frac{4}{3}^{+}$-power in

$$s_{39}(\eta(w))\mathtt{ed}\sigma(s_{39}(\eta(w)))^R.$$

We circumvent this problem by defining $\eta'(w)$ and $\eta''(w)$, which are both obtained from $\eta(w)$ by making a minor adjustment to a short prefix and/or suffix of $\eta(w)$. Begin by setting $\eta'(w) = \eta(w)$ and $\eta''(w) = \eta(w)$, and then make the changes described below. Redefine

$$W_{2n} = \mathtt{de}\ \eta'(w)\ \mathtt{ed}\ \sigma(\eta''(w))^R,$$

and we claim that (W_{2n}) is $\frac{4}{3}^{+}$-free.

(a) If

$$\begin{aligned}
p_{39}(\eta(w)) &= \mathtt{adbac\underline{d}abcadcbac\underline{e}bcabdacbadcabcebacbdabc},\\
p_{39}(\eta(w)) &= \mathtt{adbac\underline{d}abcadcbac\underline{e}bcabdacbadcabcdbacbdabc},\\
p_{39}(\eta(w)) &= \mathtt{adcab\underline{d}acbadbcab\underline{e}cbacdabcadbacbecabcdacb}, \text{ or}\\
p_{39}(\eta(w)) &= \mathtt{adcab\underline{d}acbadbcab\underline{e}cbacdabcadbacbdcabcdacb},
\end{aligned}$$

then swap $\eta'(w)[6] = \mathtt{d}$ and $\eta'(w)[16] = \mathtt{e}$.

(b) If

$$\begin{aligned}
p_{39}(\eta(w)) &= \mathtt{adcba\underline{c}dbcabecbadbcdacbdcabcdbaecbdacdbc},\\
p_{39}(\eta(w)) &= \mathtt{adcba\underline{c}dbcabecbadbcdacbdcabcdbadcbdacdbc}, \text{ or}\\
p_{39}(\eta(w)) &= \mathtt{adcba\underline{c}dbcabecbadbcdabdcadbacbdabcdbadca},
\end{aligned}$$

then change $\eta'(w)[6]$ from $\mathtt{c}$ to $\mathtt{e}$.

(c) If

$$\begin{aligned}
p_{39}(\eta(w)) &= \mathtt{adbca\underline{b}dcbacebcadcbdabcdbacbdcaebcdabdcb},\\
p_{39}(\eta(w)) &= \mathtt{adbca\underline{b}dcbacebcadcbdabcdbacbdcadbcdabdcb}, \text{ or}\\
p_{39}(\eta(w)) &= \mathtt{adbca\underline{b}dcbacebcadcbdacdbadcabcdacbdcadba},
\end{aligned}$$

then change $\eta''(w)[6]$ from $\mathtt{b}$ to $\mathtt{e}$.

(d) If

$$p_{39}(\eta(w)) = \texttt{adba\underline{ce}abcadcbacdbcabecbadbcdacbdcabcdba},$$

then change $\eta'(w)[5:6]$ from `ce` to `ed`.

(e) If

$$p_{39}(\eta(w)) = \texttt{adca\underline{be}acbadbcabdcbacebcadcbdabcdbacbdca},$$

then change $\eta''(w)[5:6]$ from `be` to `ed`.

(f) If

$$\begin{aligned} s_{39}(\eta(w)) &= \texttt{cbdcadbcdabdcbacdbcadcbdabcebacbd\underline{c}abcda}, \\ s_{39}(\eta(w)) &= \texttt{acdabdcbadbcabdacdbadcbdabcebacbd\underline{c}abcda}, \\ s_{39}(\eta(w)) &= \texttt{acdabdcadbcdacbadcabdacdbcaecbacd\underline{a}bcadb}, \text{ or} \\ s_{39}(\eta(w)) &= \texttt{badbcdacbdcabcdbadcbdacdbcaecbacd\underline{a}bcadb}, \end{aligned}$$

then change $\eta'(w)[-6]$ to `e`.

(g) If

$$\begin{aligned} s_{39}(\eta(w)) &= \texttt{bcdbadcbdacdbcabdcbadbcdacbecabcd\underline{b}acbda}, \\ s_{39}(\eta(w)) &= \texttt{abdacdbcadcbacdabdcadbcdacbecabcd\underline{b}acbda}, \\ s_{39}(\eta(w)) &= \texttt{cadcbdacdbadcabcdacbdcadbaceabcad\underline{c}bacdb}, \text{ or} \\ s_{39}(\eta(w)) &= \texttt{bcdbadcabdacbadbcdabdcadbaceabcad\underline{c}bacdb}, \end{aligned}$$

then change $\eta''(w)[-6]$ to `e`.

(h) If

$$\begin{aligned} s_{39}(\eta(w)) &= \texttt{bcadcbacdbcabdacbadcabc\underline{e}bacbdabca\underline{d}bacda}, \\ s_{39}(\eta(w)) &= \texttt{cbadbcabdcbacdabcadbacb\underline{e}cabcdacba\underline{d}cabda}, \\ s_{39}(\eta(w)) &= \texttt{acbdcabcdacbadbcabdcbac\underline{e}abcadbacb\underline{d}abcdb}, \text{ or} \\ s_{39}(\eta(w)) &= \texttt{cabdacbadcabcdbacbdabca\underline{e}cbacdbcab\underline{d}cbadb}, \end{aligned}$$

then swap $\eta'(w)[-6] = \texttt{d}$ and $\eta'(w)[-16] = \texttt{e}$.

(i) If

$$\begin{aligned} s_{39}(\eta(w)) &= \texttt{acdbcabdcbadbcdacbecabcdbacbdabca\underline{eb}acda}, \\ s_{39}(\eta(w)) &= \texttt{abdcbacdbcadcbdabcebacbdcabcdacba\underline{ec}abda}, \\ s_{39}(\eta(w)) &= \texttt{bcdacbadcabdacdbcaecbacdabcadbacb\underline{ea}bcdb}, \text{ or} \\ s_{39}(\eta(w)) &= \texttt{badcabcdacbdcadbaceabcadcbacdbcab\underline{ec}badb}, \end{aligned}$$

then change both $\eta'(w)[-6:-5]$ and $\eta''(w)[-6:-5]$ to `de`.

We briefly explain why the proof of Theorem 15 Case I (with some slight modifications) still applies to the newly defined W_{2n}. The changes fall into two main types:

(A) We swap a letter $\mathtt{d}$ and a letter $\mathtt{e}$ in $\eta'(w)$, effectively eliminating the $\frac{7}{5}$-power that resulted in the given appearance of $\mathtt{e}$ in $\eta(w)$ by changing the appearance of $\mathtt{d}$ in the length 4 prefix to an $\mathtt{e}$ instead of the appearance of $\mathtt{d}$ in the length 4 suffix (or vice versa). This is what we do in cases (a) and (h).

(B) We change a letter beside a $\mathtt{d}$ to an $\mathtt{e}$. This is done in such a way that we get an appearance of $\mathtt{ed}$ in $\sigma(p_7(\eta(w)))^R\mathtt{de}p_7(\eta(w))$, or an appearance of $\mathtt{de}$ in $s_7(\eta(w))\mathtt{ed}\sigma(s_7(\eta(w)))^R$, respectively. This is what we do in cases (b), (c), (f), and (g). In cases (d), and (e), we need to change an $\mathtt{e}$ to a $\mathtt{d}$ first in order to make this idea work, which is why we change two consecutive letters in $\eta'(w)$ or $\eta''(w)$. Finally, in case (i), we need to make this change in both $\eta'(w)$ and $\eta''(w)$.

Here we need to modify the argument in the proof of Theorem 15 Case I that x cannot contain either of the factors $\mathtt{ed}$ or $\mathtt{de}$. We consider only $\mathtt{ed}$; the argument for $\mathtt{de}$ is analogous. Suppose that xyx is a factor of W_{2n} with $2|x| > |y|$, and that x contains the factor $\mathtt{ed}$. Observe that the original appearance of $\mathtt{ed}$ appears internally as $\mathtt{d}\beta\mathtt{ed}\sigma(\beta)\mathtt{e}$, where $\beta \in \{\mathtt{a}, \mathtt{b}\}$. On the other hand, the factors of length 2 to the left and right of the new appearance of $\mathtt{ed}$ contain neither $\mathtt{d}$ nor $\mathtt{e}$. Therefore, x must be contained in $\beta\mathtt{ed}\sigma(\beta)$. However, then we certainly have $|y| \geqslant 8$, contradicting the fact that $2|x| > |y|$.

In case (i), we introduce two new appearances of the factor $\mathtt{de}$, but we rule out the possibility that both appearances of x in xyx contain one of the new appearances of $\mathtt{de}$ by a finite search. Then the analogous argument to the one given above for $\mathtt{ed}$ applies once again.

We check that Lemma 13(a) still applies with $\eta'(w)$ and $\eta''(w)$ in place of $\eta(w)$. While Lemma 13(b) does not necessarily hold for $\eta'(w)$ and $\eta''(w)$, the desired consequence that no factor of $\eta'(w)$ of length 15 is a factor of $\sigma(\eta''(w))^R$ still holds. (We only need to check the factors that contain a letter that has been changed here.) This is all that matters in the remainder of the proof of Theorem 15 Case I. So it suffices to check that

$$\sigma(p_{39}(\eta''(w)))^R\mathtt{de}p_{39}(\eta'(w)) \quad \text{(in cases (a)-(e))},$$

and

$$s_{39}(\eta'(w))\mathtt{ed}\sigma(s_{39}(\eta''(w)))^R \quad \text{(in cases (f)-(i))}$$

are $\frac{4}{3}^+$-free for every exceptional prefix and suffix.

A.2 Exceptions for Case II(a)

There are 3 possible prefixes of length 46 of $\eta(w)$ for which we encounter a $\frac{4}{3}^+$-power in

$$\tau(p_{46}(\eta(w)))^R\mathtt{de}p_{46}(\eta(w)),$$

and 60 possible suffixes of length 39 of $\eta(w)$ for which we encounter a $\frac{4}{3}^+$-power in

$$s_{39}(\eta(w))\texttt{ed}\tau(s_{39}(\eta(w)))^R.$$

Of these 60 possibilities, there are 30 that end in `a`, and swapping `a` and `c` gives the other 30. We consider only the suffixes ending in `a`, as those ending in `c` are handled analogously.

Again, we define $\eta'(w)$ and $\eta''(w)$, which are both obtained from $\eta(w)$ by making a minor adjustment to a short prefix and/or suffix of $\eta(w)$. Begin by setting $\eta'(w) = \eta(w)$ and $\eta''(w) = \eta(w)$, and then make the changes described below. Redefine

$$W_{2n} = \texttt{de}\ \eta'(w)\ \texttt{ed}\ \tau(\eta''(w))^R,$$

and we claim that (W_{2n}) is $\frac{4}{3}^+$-free. The proof is similar to that of Case I and is omitted.

(a) If

$$p_{46}(\eta(w)) = \texttt{abdcb}\underline{\texttt{ae}}\texttt{bcabdacbadcabceacbdcadbacdabcadcbdacdba}, \text{ or}$$
$$p_{46}(\eta(w)) = \texttt{abdcb}\underline{\texttt{ae}}\texttt{bcabdacbadcabceacbdcadbacdabcadcbeacdba},$$

then change both $\eta'(w)[6:7]$ and $\eta''(w)[6:7]$ from `ae` to `ed`.

(b) If

$$p_{46}(\eta(w)) = \texttt{ab}\underline{\texttt{e}}\texttt{cbadbcda}\underline{\texttt{b}}\texttt{dcadbaceabcadcbdacdbadcabceacbdcad},$$

then change both $\eta'(w)[3]$ and $\eta''(w)[3]$ from `e` to `d`, and both $\eta'(w)[12]$ and $\eta''(w)[12]$ from `b` to `e`.

(c) There are 21 possibilities for $s_{39}(\eta(w))$ ending in one of the factors `dbacbda` or `dbcabda`. In each of these cases, change $\eta'(w)[-6]$ from `b` to `e`.

(d) If

$$s_{39}(\eta(w)) = \texttt{cdbadcabdacbadbcdabdcadbacb}\underline{\texttt{ea}}\texttt{bcdbadcbda},$$
$$s_{39}(\eta(w)) = \texttt{cdbadcabeacbadbcdabdcadbacb}\underline{\texttt{ea}}\texttt{bcdbadcbda}, \text{ or}$$
$$s_{39}(\eta(w)) = \texttt{bcdacbadbcabdcbacdabcadbacb}\underline{\texttt{ea}}\texttt{bcdbadcbda},$$

then change $\eta'(w)[-12:-11]$ from `ea` to `de`.

(e) If

$$s_{39}(\eta(w)) = \texttt{dcabdacdbcaecbacdabcadbacb}\underline{\texttt{ea}}\texttt{bcdbadcabda},$$

then change $\eta'(w)[-13:-12]$ from `ea` to `de`.

(f) If

$$s_{39}(\eta(w)) = \texttt{acbadcabdacdbcaecbacdabcadbacb}\underline{\texttt{ea}}\texttt{bcdbadc},$$

then change $\eta'(w)[-9 : -8]$ from $\texttt{ea}$ to $\texttt{de}$.

(g) If

$$s_{39}(\eta(w)) = \texttt{bcadcbacdbcabdacbadcabc}\underline{\texttt{e}}\texttt{bacbdabca}\underline{\texttt{d}}\texttt{bacda},$$

then swap $\eta'(w)[-6]$ and $\eta'(w)[-16]$, and swap $\eta''(w)[-6]$ and $\eta''(w)[-16]$.

(h) If

$$s_{39}(\eta(w)) = \texttt{dabcadbacbdcabcdacba}\underline{\texttt{e}}\texttt{bcabdcbac}\underline{\texttt{d}}\texttt{bcadcbda},$$

then swap $\eta'(w)[-9]$ and $\eta'(w)[-19]$.

(i) If

$$s_{39}(\eta(w)) = \texttt{cbadbcabdcbacdabcadbacb}\underline{\texttt{e}}\texttt{cabcdacba}\underline{\texttt{d}}\texttt{cabda},$$

then swap $\eta'(w)[-6]$ and $\eta'(w)[-16]$.

(j) If

$$s_{39}(\eta(w)) = \texttt{adcbdacdbadcabceacbdcadbcdabdcba}\underline{\texttt{eb}}\texttt{cabda},$$

then change both $\eta'(w)[-7 : -6]$ and $\eta''(w)[-7 : -6]$ from $\texttt{eb}$ to $\texttt{de}$.

A.3 Exceptions for Case II(b1)

There are 5 possible prefixes of length 39 of $\eta(\dot{w})$ for which we encounter a ${\frac{4}{3}}^{+}$-power in

$$\rho(p_{39}(\eta(\pi(\ddot{w}))))^R \texttt{de} p_{39}(\eta(\dot{w})),$$

and 4 possible suffixes of length 39 of $\eta(\dot{w})$ for which we encounter a ${\frac{4}{3}}^{+}$-power in

$$s_{39}(\eta(\dot{w}))\texttt{ed}\rho(s_{39}(\eta(\pi(\ddot{w}))))^R.$$

We define $\eta'(\dot{w})$ and $\eta''(\pi(\ddot{w}))$, which are obtained from $\eta(\dot{w})$ and $\eta(\pi(\ddot{w}))$, respectively, by making a minor adjustment to a short prefix and/or suffix of $\eta(\dot{w})$ or $\eta(\pi(\ddot{w}))$, respectively. Begin by setting $\eta'(\dot{w}) = \eta(\dot{w})$ and $\eta''(\pi(\ddot{w})) = \eta(\pi(\ddot{w}))$, and then make the changes described below. Redefine

$$W_{2n} = \texttt{de}\ \eta'(\dot{w})\ \texttt{ed}\ \rho(\eta''(\pi(\ddot{w})))^R,$$

and we claim that (W_{2n}) is ${\frac{4}{3}}^{+}$-free. The proof is similar to that of Case I and is omitted.

(a) If

$$\begin{aligned}
p_{39}(\eta(\dot{w})) &= \texttt{bdcadbcdac}\underline{\texttt{be}}\texttt{cabcdbacbdabcadcbacdbcabdac}, \\
p_{39}(\eta(\dot{w})) &= \texttt{bdcadbcdac}\underline{\texttt{be}}\texttt{cabcdbadcbdacdbcabecbadbcda}, \\
p_{39}(\eta(\dot{w})) &= \texttt{bdcadbcdac}\underline{\texttt{be}}\texttt{cabcdbacbdabcadcbacdbcabeac}, \text{ or} \\
p_{39}(\eta(\dot{w})) &= \texttt{bdcadbcdac}\underline{\texttt{be}}\texttt{cabcdbadcbdacdbcabecbadbcda},
\end{aligned}$$

then change $\eta'(\dot{w})[11 : 12]$ from $\texttt{be}$ to $\texttt{ed}$.

(b) If

$$p_{39}(\eta(\dot{w})) = \texttt{bdcba}\underline{\texttt{ce}}\texttt{bcadcbdacdbadcabeacbadbcdabdcadb},$$

then

$$p_{39}(\eta(\pi(\ddot{w}))) = \texttt{adcbdab}\underline{\texttt{ce}}\texttt{bacbdcabcdacbaecabdacdbcadcbac},$$

and we change $\eta'(\dot{w})[6 : 7]$ from $\texttt{ce}$ to $\texttt{ed}$ and $\eta''(\pi(\ddot{w}))[8 : 9]$ from $\texttt{ce}$ to $\texttt{ed}$.

(c) If

$$\begin{aligned}
s_{39}(\eta(\dot{w})) &= \texttt{cadbacbdcabcdacbadbcabdcbac}\underline{\texttt{eb}}\texttt{cadcbdacdb}, \\
s_{39}(\eta(\dot{w})) &= \texttt{adcbdabcdbacbdcadbcdabdcbac}\underline{\texttt{eb}}\texttt{cadcbdacdb}, \text{ or} \\
s_{39}(\eta(\dot{w})) &= \texttt{adcbdabcebacbdcadbcdabdcbac}\underline{\texttt{eb}}\texttt{cadcbdacdb},
\end{aligned}$$

then change $\eta'(w)[-12 : -11]$ from $\texttt{eb}$ to $\texttt{de}$.

(d) If

$$s_{39}(\eta(\dot{w})) = \texttt{bdacdbadcbdabcaebacdabdcadbcdacb}\underline{\texttt{ec}}\texttt{abcdb},$$

then

$$s_{39}(\eta(\pi(\ddot{w}))) = \texttt{cabcdacbdcadbaceabcadcbacdbcab}\underline{\texttt{ec}}\texttt{badbcda},$$

and we change $\eta'(\dot{w})[-7 : -6]$ from $\texttt{ec}$ to $\texttt{de}$ and $\eta''(\pi(\ddot{w}))[-9 : -8]$ from $\texttt{ec}$ to $\texttt{de}$.