Many triangles with few edges

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Abstract

Extremal problems concerning the number of independent sets or complete subgraphs in a graph have been well studied in recent years. Cutler and Radcliffe proved that among graphs with n vertices and maximum degree at most r, where $n = a(r+1) + b$ and $0 \leqslant b \leqslant r$, $aK_{r+1} \cup K_b$ has the maximum number of complete subgraphs, answering a question of Galvin. Gan, Loh and Sudakov conjectured that $aK_{r+1} \cup K_b$ also maximizes the number of complete subgraphs K_t for each fixed size $t \geqslant 3$, and proved this for $a = 1$. Cutler and Radcliffe proved this conjecture for $r \leqslant 6$.

We investigate a variant of this problem where we fix the number of edges instead of the number of vertices. We prove that $aK_{r+1} \cup \mathcal{C}(b)$, where $\mathcal{C}(b)$ is the colex graph on b edges, maximizes the number of triangles among graphs with m edges and any fixed maximum degree $r \leqslant 8$, where $m = a\binom{r+1}{2} + b$ and $0 \leqslant b < \binom{r+1}{2}$.

Mathematics Subject Classifications: 05

1 Introduction

The problem of determining which graphs contain the largest number of complete subgraphs of size t has a long history. One can think of it as starting with the Kruskal-Katona theorem concerning the shadows of uniform hypergraphs. Here we should introduce the colex order on finite subsets of $\mathbb{N}$, defined by $A <_{\text{colex}} B$ if and only if $\max(A \triangle B) \in B$. Thus the first few 2-sets in colex order are

$$\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,5\}, \ldots.$$

Theorem 1 (Kruskal [16], Katona [15])**.** *Suppose that $1 \leqslant \ell \leqslant t$. If $\mathcal{H} \subseteq \binom{[n]}{t}$ has size N, and we define*

$$\partial^{(\ell)}\mathcal{H} = \left\{ A \in \binom{[n]}{\ell} : \exists\, B \in \mathcal{H} \text{ such that } A \subseteq B \right\},$$

then $\partial^{(\ell)}\mathcal{H}$ is at least as large as $\partial^{(\ell)}\mathcal{C}$, where $\mathcal{C} \subseteq \binom{[n]}{t}$ consists of the first N t-sets in colex order, and moreover $\partial^{(\ell)}\mathcal{C}$ is an initial segment of ℓ-sets in the colex order.

This immediately implies, by taking $\ell = 2$ and $\mathcal{H}$ to be the set of complete subgraphs of size t in G, that any graph containing N complete subgraphs of size t must have at least $\left|\partial^{(2)}\mathcal{C}\right|$ edges, where $\mathcal{C}$ consists of the first N t-sets in colex order. This in turn implies an upper bound on the number of copies of K_t in a graph on m edges. Similarly, since the optimal graphs are the same for all t, the colex graph $\mathcal{C}(m)$, whose edges are the first m pairs in colex order, has the largest number of complete subgraphs among all graphs with m edges. Note that if $\binom{k}{2} \leqslant m < \binom{k+1}{2}$ then $\mathcal{C}(m)$ is a clique on $\{1, 2, \ldots, k\}$ together with edges joining vertex $k+1$ to vertices $1, 2, \ldots, m - \binom{k}{2}$.

These results have been extended in a number of directions, usually by restricting the class of graphs considered. If we write $k(G)$ for the number of complete subgraphs in a graph G, $k_t(G)$ for the number of complete subgraphs of size t, and $\mathcal{K}_t(G)$ for the set of complete subgraphs of size t, we wish to find upper bounds on $k(G)$ and $k_t(G)$ over the class of graphs satisfying some constraints. One example is the following result due to Zykov [20] (see also [7, 13, 17, 18]), which bounds the number of complete subgraphs in graphs with bounded clique number, $\omega(G)$.

Theorem 2 (Zykov [20])**.** *If $t \geqslant 2$ and G is a graph with n vertices and $\omega(G) \leqslant \omega$, then*

$$k_t(G) \leqslant k_t(T_{n,\omega}),$$

where $T_{n,\omega}$ is the Turán graph with ω parts. The extremal graph is unique except when $n < t$ or $\omega < t$.

Galvin [11] made the following conjecture[1], and proved it in a wide range of cases.

Conjecture 3 (Galvin [11])**.** If G is a graph on n vertices with maximum degree at most r, where $r \geqslant n/2 - 1$, then $k(G) \leqslant k(K_{r+1} \cup K_{n-r-1})$, the union of two complete graphs.

Cutler and the second author [2] proved this conjecture (indeed, without the lower bound on r) showing that the extremal graph is the union of complete graphs $aK_{r+1} \cup K_b$ where $n = a(r+1) + b$ and $0 \leqslant b < r+1$. Soon thereafter Gan, Loh, and Sudakov [12] considered the question of maximizing $k_t(G)$ over this same class of graphs. They made substantial progress on the following conjecture.

Conjecture 4. For all $t \geqslant 3$ and $n, r \geqslant 1$, if G is a graph on n vertices with maximum degree at most r, then

$$k_t(G) \leqslant k_t(aK_{r+1} \cup K_b),$$

where $n = a(r+1) + b$ and $0 \leqslant b < r+1$.

Note that the conjecture is false for $t = 2$, since whenever n is not divisible by $r+1$ the conjectured extremal graph has fewer edges than an r-regular graph on n vertices. Gan, Loh, and Sudakov proved Conjecture 4 for $a = 1$, the case conjectured by Engbers and Galvin [6], and also demonstrated that if the conjecture holds for $t = 3$, then it holds for all $t \geqslant 3$. Cutler and Radcliffe [3] proved the conjecture for $r \leqslant 6$.

[1] To be precise, Galvin's conjecture was about maximizing the number of independent sets in a graph of bounded minimum degree, but his conjecture is easily seen to be equivalent to the one here.

1.1 The edge analogue of Conjecture 4

In this paper we work on the edge, or Kruskal-Katona, version of Conjecture 4. We fix the number of edges of G (and allow the number of vertices to be arbitrary) and ask which graphs maximize the number of complete subgraphs of size t.

Eckhoff [4, 5] (see also related work by Fisher and Ryan [8]) gave partial results on the edge analogue of Zykov's theorem (Theorem 2). The result is now known—it is an immediate consequence of the 'rainbow' Kruskal-Katona theorem of Frankl, Füredi, and Kalai [9] and a more recent theorem of Frohmader [10]. For convenience in stating the theorems it is helpful to make a temporary definition.

Definition 5. A subset $A \subseteq \mathbb{N}$ is *ω-rainbow* if no two elements of A are congruent modulo ω. We write $\mathcal{R}_\omega$ for the collection of all ω-rainbow subsets of $\mathbb{N}$.

We first state the rainbow Kruskal-Katona theorem.

Theorem 6 (Frankl, Füredi, and Kalai [9]). *Suppose that $1 \leqslant \ell \leqslant t$. If $\mathcal{H} \subseteq \binom{\mathbb{N}}{t}$ has size N, and moreover $\mathcal{H}$ is ω-*partite*—i.e., we can partition $\mathbb{N}$ into ω subsets such that no set in $\mathcal{H}$ contains more than one element from each part—then $\partial^{(\ell)}\mathcal{H}$ is at least as large as $\partial^{(\ell)}\mathcal{C}$, where $\mathcal{C}$ consists of the first N t-sets in $\mathcal{R}_\omega$ in colex order, and moreover $\partial^{(\ell)}\mathcal{C}$ is an initial segment of $\mathcal{R}_\omega \cap \binom{\mathbb{N}}{\ell}$ in colex order.*

Frohmader's result exploits this theorem to extend its conclusion to flag complexes—set systems defined by the set of complete subgraphs in a graph.

Theorem 7 (Frohmader [10]). *Let G be a graph having $\omega(G) \leqslant \omega$. If we let $\mathcal{H} = \mathcal{K}_t(G)$ then $\partial^{(\ell)}\mathcal{H} \subseteq \mathcal{K}_\ell(G)$ satisfies the inequality of the previous theorem (though $\mathcal{H}$ need not be ω-partite).*

Corollary 8. *If G is a graph with m edges having $\omega(G) \leqslant \omega$ then for all $t \geqslant 2$ we have $k_t(G) \leqslant k_t(\mathcal{R}_\omega(m))$, where $\mathcal{R}_\omega(m)$ is the graph whose edges are the first m 2-sets in $\mathcal{R}_\omega$ in colex order.*

Proof. Define $\mathcal{C}$ to be the first $k_t(G)$ t-sets from $\mathcal{R}_\omega$ in colex order, and set

$$m' = |\partial^{(2)}\mathcal{C}|.$$

We must have $m \geqslant m'$ since, from the fact that $E(G) \supseteq \partial^{(2)}\mathcal{K}_t(G)$, we get

$$m = e(G) \geqslant |\partial^{(2)}\mathcal{K}_t(G)| \geqslant |\partial^{(2)}\mathcal{C}| = m'.$$

Thus

$$k_t(R_\omega(m)) \geqslant k_t(R_\omega(m')) = k_t(G),$$

where the last identity is the simple fact that $\mathcal{K}_t(\partial^{(2)}\mathcal{C}) = \mathcal{C}$; every edge of $\partial^{(2)}\mathcal{C}$ is in a copy of K_t. □

1.2 Results and Notation

We conjecture the following.

Conjecture 9. For any $t \geqslant 3$, if G is a graph with m edges and maximum degree at most r, then

$$k_t(G) \leqslant k_t(aK_{r+1} \cup \mathcal{C}(b)),$$

where $m = a\binom{r+1}{2} + b$ and $0 \leqslant b < \binom{r+1}{2}$.

Conjecture 9 is the exact analogue of Conjecture 4: we build as many K_{r+1}'s as we can, and then use our remaining resources optimally. Let's define

$$f_t(m, r) = \max\{k_t(G) : G \text{ has } m \text{ edges and } \Delta(G) \leqslant r\}.$$

First note that the conjecture is easily seen to be true asymptotically as $m \to \infty$. Wood [19] used a similar proof to give an upper bound on the number of cliques in n-vertex, m-edge graphs with given maximum degree.

Theorem 10. *For all* $3 \leqslant t \leqslant r + 1$,

$$f_t(m, r) \leqslant m \frac{\binom{r+1}{t}}{\binom{r+1}{2}},$$

and moreover for fixed t *and* r*,*

$$f_t(m, r) = (1 - o_m(1))\, m \frac{\binom{r+1}{t}}{\binom{r+1}{2}}.$$

Proof. For the first bound, note that if G is a graph on m edges with $\Delta(G) \leqslant r$ then the endpoints of an edge e of G have at most $r-1$ common neighbors, and complete subgraphs of size t in G containing e correspond to K_{t-2}'s in this set of common neighbors. There are at most $\binom{r-1}{t-2}$ such K_{t-2}'s, so, counting pairs (e, K) with e an edge of G and $K \in \mathcal{K}_t(G)$ containing e, we have

$$\binom{t}{2} k_t(G) \leqslant m \binom{r-1}{t-2}.$$

Thus

$$k_t(G) \leqslant m \frac{\binom{r-1}{t-2}}{\binom{t}{2}} = m \frac{\binom{r+1}{t}}{\binom{r+1}{2}}.$$

Now we have

$$\begin{aligned} m \frac{\binom{r+1}{t}}{\binom{r+1}{2}} &\leqslant (1 + o(1)) \left(\frac{m}{\binom{r+1}{2}} - 1 \right) \binom{r+1}{t} \\ &\leqslant (1 + o(1)) \left\lfloor \frac{m}{\binom{r+1}{2}} \right\rfloor \binom{r+1}{t} \\ &\leqslant (1 + o(1)) f_t(m, r), \end{aligned}$$

where the final inequality comes from considering the graph that is the disjoint union of $\left\lfloor \frac{m}{\binom{r+1}{2}} \right\rfloor$ copies of K_{r+1} and a matching to make the edge count up to m. $\square$

We were not able to show that it is sufficient to prove Conjecture 9 only for $t = 3$. Our main result is that the conjecture is true for triangles ($t = 3$) for $r \leqslant 8$.

Main Theorem. If G is a graph with m edges and maximum degree at most r for any fixed $r \leqslant 8$, then

$$k_3(G) \leqslant k_3(aK_{r+1} \cup \mathcal{C}(b)),$$

where $m = a\binom{r+1}{2} + b$ and $0 \leqslant b < \binom{r+1}{2}$. That is, the graphs with the maximum number of triangles consist of as many disjoint copies of K_{r+1} as possible, with the remaining edges formed into a colex graph.

Our proof does not show that the extremal example is unique, although we conjecture that it is (up to the trivial changes of adding isolated vertices or disconnecting a pendant edge of a colex component).

Most of our graph theory notation is standard; see for instance Bollobás [1] for a reference. In particular we will write (as we have done above) $G \cup H$ for the disjoint union of G and H, and also nG for the disjoint union of n copies of G.

We write $\mathcal{G}(m, r)$ for the set of graphs G with m edges and having $\Delta(G) \leqslant r$. In this class it will be handy to single out the connected ones; we write $\mathcal{G}_C(m, r)$ for these.

In Section 2 we prove some general results saying that in proving the conjecture we may restrict our attention to connected graphs that achieve maximum degree r. In Section 3 we introduce an approach parallel to the folding technique in [2], and in Section 4 we discuss how this restricts the class of potentially extremal graphs. In Section 5 we give constraints on the extremal graphs in terms of their degree multisets. In Section 6, we combine these two approaches to prove the main theorem. The approaches from Sections 3 and 5 combine to cover all cases when $r \leqslant 8$. A gap opens up for $r \geqslant 9$, and it seems that a new approach is necessary for r substantially greater than 8. Finally, in Section 7 we briefly discuss some related open problems.

2 Disconnected graphs and graphs with small maximum degree

In this section, we prove two results that hold for all r and allow us to restrict our attention to connected graphs with maximum degree equal to r. Both are corollaries of the following lemma concerning colex graphs. The colex graph $\mathcal{C}(b)$ consists of a complete graph of size c, where $\binom{c}{2} \leqslant b < \binom{c+1}{2}$, and then potentially one more vertex, joined to d vertices of the K_c, where $d = b - \binom{c}{2}$. Given this structure it is often useful to think of b as written in the form $b = \binom{c}{2} + d$ where $0 \leqslant d < c$. We abbreviate this fact as $b = [c, d]$, and write $\mathcal{C}(c, d)$ for $\mathcal{C}([c, d])$. It is easy to check that

$$k_t(\mathcal{C}(c, d)) = \binom{c}{t} + \binom{d}{t-1}.$$

We in fact also allow $d = c$: we have $[c, c] = [c + 1, 0]$, and the above formula for k_t still applies.

Thus, the number of K_t's in the conjectured extremal graph, which we denote by $g_t(m,r)$, can be written as

$$g_t(m,r) = k_t(aK_{r+1} \cup \mathcal{C}(b)) = a\binom{r+1}{t} + \binom{c}{t} + \binom{d}{t-1},$$

where $m = a\binom{r+1}{2} + b$ and $b = [c,d]$.

Lemma 11. *Suppose* $1 \leqslant b_i \leqslant \binom{r+1}{2} - 1$ *for* $i = 1,2$ *and* $t \geqslant 3$. *Letting* G *be the graph* $\mathcal{C}(b_1) \cup \mathcal{C}(b_2)$ *then* $k_t(G) < g_t(b_1 + b_2, r)$, *unless* $b_1 = \binom{c_1}{2}$ *for some* $c_1 \in \mathbb{N}$ *and* $b_2 = 1$ *(or vice versa), in which case* $k_t(G) = k_t(\mathcal{C}(b_1 + b_2)) = g_t(b_1 + b_2, r) = f_t(b_1 + b_2, r)$.

Proof. Let's write $b_i = [c_i, d_i]$ for $i = 1,2$ with $0 \leqslant d_i < c_i$ and $b_1 \geqslant b_2$ (so $c_1 \geqslant c_2$). We split into cases depending on the values of the d_i.

Case 11.1: $d_1, d_2 \geqslant 1$.

If $d_1 < d_2$, then since $b_1 \geqslant b_2$ we have $c_1 > c_2$, and $d_1 < d_2 < c_2 < c_1$. Then

$$\binom{c_1}{2} + d_2 < \binom{c_1}{2} + c_1 = \binom{c_1+1}{2}$$
$$\binom{c_2}{2} + d_1 < \binom{c_2}{2} + c_2 = \binom{c_2+1}{2},$$

and we may swap d_1 and d_2 as $k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2)) = k_t(\mathcal{C}(c_1, d_2) \cup \mathcal{C}(c_2, d_1))$, with $[c_1, d_2] > [c_2, d_1]$. Therefore we may assume $d_1 \geqslant d_2$.

First note that $\mathcal{C}(b_1 + 1) \cup \mathcal{C}(b_2 - 1) \in \mathcal{G}(m, r)$ since $b_1 \leqslant \binom{r+1}{2} - 1$. We compare the number of K_t's in $\mathcal{C}(b_1) \cup \mathcal{C}(b_2)$ to those in $\mathcal{C}(b_1 + 1) \cup \mathcal{C}(b_2 - 1)$. Observe that $k_t(\mathcal{C}(b_1 + 1)) = k_t(\mathcal{C}(b_1)) + \binom{d_1}{t-2}$ and $k_t(\mathcal{C}(b_2 - 1)) = k_t(\mathcal{C}(b_2)) - \binom{d_2-1}{t-2}$, so

$$k_t(\mathcal{C}(b_1 + 1) \cup \mathcal{C}(b_2 - 1)) = k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2)) + \binom{d_1}{t-2} - \binom{d_2 - 1}{t-2} > k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2))$$

since $d_1 > d_2 - 1$.

Case 11.2: Exactly one of d_1, d_2 is zero.

Subcase 11.2.1: $d_2 \neq 0$ or $d_1 < c_2$

If $d_2 = 0$ and $d_1 < c_2$,

$$k_t(\mathcal{C}(c_1, d_1) \cup \mathcal{C}(c_2, 0)) = k_t(\mathcal{C}(c_1, 0) \cup \mathcal{C}(c_2, d_1)),$$

so we may assume that it is d_1 that is zero, and $d_2 \geqslant 1$.

We compare G to the graph

$$G' = \mathcal{C}(b_1 + c_2) \cup \mathcal{C}(b_2 - c_2) = \mathcal{C}(c_1, c_2) \cup \mathcal{C}(c_2 - 1, d_2 - 1) \in \mathcal{G}(b_1 + b_2, r).$$

Note first that there are enough edges in $\mathcal{C}(b_2)$ to remove c_2 of them because $c_2 \geqslant d_2 + 1 \geqslant 2$, and $d_2 \geqslant 1$, so if $c_2 = 2$ then $b_2 = 2$, and if $c_2 \geqslant 3$ then $b_2 > \binom{c_2}{2} \geqslant c_2$. Note also that

$\Delta(\mathcal{C}(b_1+c_2)) \leqslant r$ because $d_1 = 0$ implies $b_1 + c_2 \leqslant \binom{r}{2} + r = \binom{r+1}{2}$. To prove the second equality above we have

$$[c_2, d_2] - c_2 = \binom{c_2}{2} + d_2 - c_2 = \binom{c_2}{2} - \binom{c_2-1}{1} + d_2 - 1 = \binom{c_2-1}{2} + d_2 - 1 = [c_2-1, d_2-1].$$

Note that in the representation $\mathcal{C}(c_1, c_2) \cup \mathcal{C}(c_2-1, d_2-1)$ we might have $c_2 = c_1$. The net change in k_t is

$$\begin{aligned}
k_t(G') - k_t(G) &= \binom{c_1}{t} + \binom{c_2}{t-1} + \binom{c_2-1}{t} + \binom{d_2-1}{t-1} - \left(\binom{c_1}{t} + \binom{c_2}{t} + \binom{d_2}{t-1}\right) \\
&= \binom{c_2}{t-1} + \binom{c_2-1}{t} - \binom{c_2}{t} + \binom{d_2-1}{t-1} - \binom{d_2}{t-1} \\
&= \binom{c_2}{t-1} - \binom{c_2-1}{t-1} - \left(\binom{d_2}{t-1} - \binom{d_2-1}{t-1}\right) \\
&= \binom{c_2-1}{t-2} - \binom{d_2-1}{t-2} \\
&> 0
\end{aligned}$$

because $c_2 > d_2$.

Subcase 11.2.2: $d_2 = 0$ and $d_1 \geqslant c_2$

We compare G to the graph

$$G' = \mathcal{C}(b_1+1) \cup \mathcal{C}(b_2-1) = \mathcal{C}(c_1, d_1+1) \cup \mathcal{C}(c_2-1, c_2-2) \in \mathcal{G}(b_1+b_2, r).$$

To prove the second equality, we have

$$b_2 - 1 = \binom{c_2}{2} - 1 = \binom{c_2-1}{2} + \binom{c_2-1}{1} - 1 = \binom{c_2-1}{2} + c_2 - 2.$$

Note that we might have $[c_1, d_1+1] = [c_1+1, 0]$. The net change in k_t is

$$\begin{aligned}
k_t(G') - k_t(G) &= \binom{c_1}{t} + \binom{d_1+1}{t-1} - \binom{c_1}{t} - \binom{d_1}{t-1} + \binom{c_2-1}{t} + \binom{c_2-2}{t-1} - \binom{c_2}{t} \\
&= \binom{d_1+1}{t-1} - \binom{d_1}{t-1} - \left(\binom{c_2}{t} - \binom{c_2-1}{t}\right) + \binom{c_2-2}{t-1} \\
&= \binom{d_1}{t-2} - \left(\binom{c_2-1}{t-1} - \binom{c_2-2}{t-1}\right) \\
&= \binom{d_1}{t-2} - \binom{c_2-2}{t-2} \\
&> 0
\end{aligned}$$

because $d_1 > c_2 - 2$.

Case 11.3: $d_1 = d_2 = 0$

If $c_2 = 2$, then $b_2 = 1$. In this case, $\mathcal{C}(b_2)$ is a single edge, and $\mathcal{C}(b_1)$ is a complete graph, so

$$k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2)) = k_t(\mathcal{C}(b_1 + b_2)) = f_t(b_1 + b_2, r)$$

by the Kruskal-Katona theorem.

Otherwise, $c_2 \geqslant 3$, so $b_2 = \binom{c_2}{2} \geqslant c_2$, and we compare G to

$$G' = \mathcal{C}(b_1 + c_2) \cup \mathcal{C}(b_2 - c_2) = \mathcal{C}(c_1, c_2) \cup \mathcal{C}(c_2 - 2, c_2 - 3) \in \mathcal{G}(b_1 + b_2, r).$$

As in Subcase 11.2.1, $\Delta(\mathcal{C}(b_1 + c_2)) \leqslant r$. To prove the second equality, we have

$$b_2 - c_2 = \binom{c_2}{2} - c_2 = \left(\binom{c_2 - 2}{2} + \binom{c_2 - 2}{1}\right) + \binom{c_2 - 1}{1} - c_2 = \binom{c_2 - 2}{2} + c_2 - 3.$$

This move yields a net gain of

$$\begin{aligned}
k_t(G') - k_t(G) &= \binom{c_1}{t} + \binom{c_2}{t-1} + \binom{c_2 - 2}{t} + \binom{c_2 - 3}{t-1} - \left(\binom{c_1}{t} + \binom{c_2}{t}\right) \\
&= \binom{c_2}{t-1} + \left(\binom{c_2 - 2}{t} - \binom{c_2}{t}\right) + \binom{c_2 - 3}{t-1} \\
&= \binom{c_2}{t-1} - \binom{c_2 - 1}{t-1} - \left(\binom{c_2 - 2}{t-1} - \binom{c_2 - 3}{t-1}\right) \\
&= \binom{c_2 - 1}{t-2} - \binom{c_2 - 3}{t-2} \\
&> 0.
\end{aligned}$$

In all cases except $b_1 = \binom{c_1}{2}$ and $b_2 = 1$ we have shown that there exists $\beta \geqslant 1$ such that $k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2)) < k_t(\mathcal{C}(b_1 + \beta) \cup \mathcal{C}(b_2 - \beta))$ and also that $\mathcal{C}(b_1 + \beta) \cup \mathcal{C}(b_2 - \beta) \in \mathcal{G}(b_1 + b_2, r)$. We have that, except in the special case,

$$\begin{aligned}
k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2)) &< \begin{cases} k_t(\mathcal{C}(b_1 + b_2)) & \text{if } b_1 + b_2 \leqslant \binom{r+1}{2} \\ k_t(\mathcal{C}(\binom{r+1}{2}) \cup \mathcal{C}(b_1 + b_2 - \binom{r+1}{2})) & \text{otherwise} \end{cases} \\
&= g_t(b_1 + b_2, r).
\end{aligned}$$

□

Corollary 12. *For $t \geqslant 3$, if Conjecture 9 holds for numbers of edges up through $m - 1$, and $G \in \mathcal{G}(m, r)$ is not connected, then $k_t(G) \leqslant g_t(m, r)$.*

Proof. Suppose $G = N \cup M$, where $0 \neq e(N) = a_1\binom{r+1}{2} + b_1$, $0 \neq e(M) = a_2\binom{r+1}{2} + b_2$, and $m = e(G) = e(M) + e(N) = a\binom{r+1}{2} + b$ with $0 \leqslant b_1, b_2, b \leqslant \binom{r+1}{2} - 1$. By Conjecture 9 for smaller values of m,

$$\begin{aligned}
k_t(G) &= k_t(N) + k_t(M) \\
&\leqslant k_t(a_1 K_{r+1} \cup \mathcal{C}(b_1)) + k_t(a_2 K_{r+1} \cup \mathcal{C}(b_2)) \\
&= k_t((a_1 + a_2) K_{r+1}) + k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2)).
\end{aligned}$$

If $b_1, b_2 \leqslant 1$, then $a = a_1 + a_2$, $b = b_1 + b_2 \leqslant 2$, and $k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2)) = 0 = k_t(\mathcal{C}(b_1 + b_2))$, so we have shown $k_t(G) \leqslant k_t(aK_{r+1} \cup \mathcal{C}(b)) = g_t(m, r)$.

If $b_1 = \binom{c_1}{2}$ and $b_2 = 1$, then $a = a_1 + a_2$, $b = b_1 + b_2$, and $k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2)) = k_t(\mathcal{C}(b_1 + b_2))$, so we have shown $k_t(G) \leqslant k_t(aK_{r+1} \cup \mathcal{C}(b)) = g_t(m, r)$.

In all other cases, by Lemma 11 we have $k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2)) < g_t(b_1 + b_2, r)$, so

$$\begin{aligned} k_t(G) &\leqslant k_t((a_1 + a_2)K_{r+1}) + k_t(\mathcal{C}(b_1) \cup \mathcal{C}(b_2)) \text{ [shown above]} \\ &< k_t((a_1 + a_2)K_{r+1}) + g_t(b_1 + b_2, r) \\ &= g_t\big((a_1 + a_2)\binom{r+1}{2} + b_1 + b_2, r\big) \\ &= g_t(m, r). \end{aligned}$$

□

Corollary 13. *If $t \geqslant 3$, $r \geqslant 2$, $m \geqslant \binom{r+1}{2} + 1$, $G \in \mathcal{G}(m, r)$ has $k_t(G) = f_t(m, r)$, and Conjecture 9 holds for maximum degree at most $r - 1$ and for numbers of edges up through $m - 1$, then $\Delta(G) = r$.*

Proof. The statement is trivial for $r = 2$, so assume $r \geqslant 3$. Suppose $\Delta(G) \leqslant r - 1$, so $G \in \mathcal{G}(m, r - 1)$. By Conjecture 9 for $r - 1$, we have $k_t(G) \leqslant k_t(aK_r \cup \mathcal{C}(b))$ for $m = a\binom{r}{2} + b$ and $0 \leqslant b < \binom{r}{2}$.

If $b \geqslant 2$, then $k_t(K_r \cup \mathcal{C}(b)) < f_t(\binom{r}{2} + b, r)$ by Lemma 11. Therefore

$$\begin{aligned} k_t(aK_r \cup \mathcal{C}(b)) &= k_t((a-1)K_r) + k_t(K_r \cup \mathcal{C}(b)) \\ &< k_t((a-1)K_r) + f_t\big(\binom{r}{2} + b, r\big) \\ &\leqslant f_t\big((a-1)\binom{r}{2}, r\big) + f_t\big(\binom{r}{2} + b, r\big) \\ &\leqslant f_t\big((a-1)\binom{r}{2} + \binom{r}{2} + b, r\big) = f_t(m, r) \end{aligned}$$

since f_t is a superadditive function of m.

Otherwise, $b \leqslant 1$, and $m \geqslant \binom{r+1}{2} + 1$, so $a \geqslant 2$. Notice that $k_t(K_r \cup \mathcal{C}(b)) = k_t(\mathcal{C}(r, b))$.

$$\begin{aligned} k_t(aK_r \cup \mathcal{C}(b)) &= k_t((a-2)K_r) + k_t(K_r) + k_t(K_r \cup \mathcal{C}(b)) \\ &= k_t((a-2)K_r) + k_t\big(\mathcal{C}(r, 0)\big) + k_t(\mathcal{C}(r, b)) \\ &= k_t((a-2)K_r) + k_t(\mathcal{C}(r, 0) \cup \mathcal{C}(r, b)) \\ &< k_t((a-2)K_r) + f_t\Big(2\binom{r}{2} + b, r\Big) \text{ [by Lemma 11 since } r \geqslant 3] \\ &\leqslant f_t\Big((a-2)\binom{r}{2}, r\Big) + f_t\Big(2\binom{r}{2} + b, r\Big) \\ &\leqslant f_t\Big(a\binom{r}{2} + b, r\Big) = f_t(m, r), \end{aligned}$$

since f_t is a superadditive function of m. Therefore any $G \in \mathcal{G}(m, r - 1)$ is suboptimal, and $k_t(G) = f_t(m, r)$ implies $\Delta(G) = r$. □

3 Edge Weights, Clusters, and Folding

In this section we introduce a 'folding' operation that acts on a graph containing a large subset of vertices with as many common neighbors as is possible. To this end we define the *weight* of a pair of vertices to be the number of common neighbors they have: $w(xy) = |N(x) \cap N(y)|$ for any $x, y \in V(G)$. In particular if xy is an edge this is the number of triangles containing that edge. If xy is a non-edge this is the number of triangles we would gain by adding that pair as an edge. The maximum possible weight of an edge xy is $w(xy) = r - 1$, which occurs exactly when $d(x) = d(y) = r$ and $N[x] = N[y]$. (Note that a non-edge can have weight r. If $x \not\sim y$, $d(x) = d(y) = r$, and $N(x) = N(y)$, then x and y have r common neighbors, but we cannot add the edge xy to complete r triangles as x and y already have the maximum degree.)

Definition 14. An edge $xy \in E(G)$ is called *tight* if $w(xy) = r - 1$. A complete subgraph in G, all of whose edges are tight, is called a *tight clique*, and a maximal tight clique is called a *cluster*.

For any cluster T, let $S_T = \bigcap_{v \in T} N(v)$, the set of common neighbors of T. For $v \in T$, every other vertex $x \in T$ has $r - 1$ neighbors in common with v. They must be the same common neighbors for each x since $d(v) \leqslant r$, so $|T \cup S_T| = r + 1$. For any $v \in T$, the closed neighborhood of v is $N[v] = T \cup S_T$.

If G contains a cluster of size $r + 1$, then it contains a copy of K_{r+1}, and it is disconnected. What we'll investigate is the situation in which there is a cluster T with $|T| < r+1$, and try to understand the edges missing from S_T. Let $R_T = \overline{G[S_T]}$. Since T is maximal, $\delta(R_T) \geqslant 1$. The vertices in S_T may have neighbors in T, S_T, and $V(G) \setminus (T \cup S_T)$. Let B_T be the graph of edges uv such that $u \in S_T$ and $v \in V(G) \setminus (T \cup S_T)$.

We will consistently write t for $|T|$, S for S_T, s for $|S_T|$, R for R_T, and B for B_T. Thus we will always have

$$t + s = r + 1. \tag{1}$$

We refer to R as the *red graph* and the edges of B as *blue edges*. We will also define a, b, c, and d from m and r by $m = a\binom{r+1}{2} + b$, $0 \leqslant b < \binom{r+1}{2}$, and $b = [c, d]$.

The following simple bound on the weight of a blue edge will aid in determining effects of local moves.

Lemma 15. *Each blue edge has weight at most $s - 2$.*

Proof. Let xy be a blue edge with $x \in S$. We will separately count the possible common neighbors z in and out of S. There are at most $s - 1 - d_R(x)$ neighbors z of x that are in S. For $z \notin S$, xz is also a blue edge, and $d_B(x) \leqslant d_R(x)$ to maintain $d_G(x) \leqslant r$. The maximum number of such z's then is $d_R(x) - 1$. In total, xy is in at most $s - 1 - d_R(x) + d_R(x) - 1 = s - 2$ triangles. □

We will often find it useful to delete all the blue edges from a cluster and add all the red edges. We call this operation *folding*:

Definition 16. For $G \in \mathcal{G}(m,r)$ with a cluster T and $e(B) \geqslant e(R)$, we define a new graph $G_T \in \mathcal{G}(m,r)$ by converting $T \cup S_T$ into a complete subgraph (of size $r+1$) and deleting all the edges in B_T. In other words, we define the *folding of G at T* by

$$G_T = G + \binom{S_T}{2} - E(B_T).$$

The graph G_T contains a K_{r+1}, has maximum degree at most r, and has at most m edges since $e(B) \geqslant e(R)$. If we can show that $e(B) \geqslant e(R)$ and $k_3(G_T) \geqslant k_3(G)$, then by induction on m, we have, writing $G_T = K_{r+1} \cup G'$,

$$k_3(G_T) = \binom{r+1}{3} + k_3(G') \leqslant \binom{r+1}{3} + g_3\Big(e(G_T) - \binom{r+1}{2}, r\Big) = g_3(e(G_T), r).$$

Thus

$$k_3(G) \leqslant k_3(G_T) \leqslant g_3(e(G_T), r) \leqslant g_3(m, r),$$

noting, for the final inequality, that g_3 is weakly increasing in m.

4 Excluded Red Graphs R

In this section we identify several graphs that cannot occur as R in an extremal graph G because folding (when $e(B) \geqslant e(R)$) or another local move (when $e(B) < e(R)$) would increase the number of triangles. Our first step toward identifying when folding increases the number of triangles in G will be to give an upper bound on the number of *blue triangles*, or triangles containing two blue edges. We will use a compression argument, in which we determine which configuration of blue edges is least helpful to us.

Definition 17. For vertices $x \not\sim_G y$, the *compression of G from x to y*, denoted $G_{x \to y}$, is the graph obtained from G by deleting all edges between x and $N(x) \setminus N(y)$ and adding all edges from y to $N(x) \setminus N(y)$.

We define an auxiliary function to use in the compression argument.

Definition 18. For a graph G, let $d_2(G) := \sum_{v \in G} (d(v))^2$.

We will use compressions to maximize the following function and bound the number of blue triangles.

Definition 19. For a graph H and a bipartite graph B with bipartition $(V(H), Y)$, we define

$$\psi_H(B) = \sum_{v \in V(H)} \binom{d_B(v)}{2} + \sum_{v \in Y} |\{i, j \in V(H) : i \neq j, iv, jv \in E(B), ij \notin E(H)\}|.$$

This function counts the number of blue triangles when applied to the situation where H is the red graph R_T and $\left(\bigcup_{e \in E(B_T)} e\right) \setminus S_T$ induces a complete subgraph. Thus it serves as an upper bound on the number of blue triangles for a given red graph.

Lemma 20. *For a graph H, a bipartite graph B with bipartition $(V(H), Y)$, and vertices $x, y \in Y$ with $N(x) \not\subseteq N(y)$ and $N(y) \not\subseteq N(x)$,*

$$\psi_H(B) \leqslant \psi_H(B_{x\to y}) \text{ and } d_2(B) < d_2(B_{x\to y}).$$

Proof. For all $v \in V(H)$, we have $d_B(v) = d_{B_{x\to y}}(v)$ and therefore $\sum_{v\in V(H)} \binom{d_{B_{x\to y}}(v)}{2} = \sum_{v\in V(H)} \binom{d_B(v)}{2}$. H is fixed, and x is the only vertex of Y that loses neighbors, so any decrease in $\sum_{v\in Y} |\{i, j \in V(H) : i \neq j, iv, jv \in E(B), xy \notin E(H)\}|$ would be from the x term: pairs $i, j \in H$ that are neighbors of x and not adjacent in H. If $i, j \in N(y)$, then the edges ix and jx remain after the compression, so the pair i, j is still counted in the x term of the summation. If one or both of $i, j \notin N(y)$, then the pair i, j is counted in the y term of the summation after the compression but not before, compensating for the loss in the x term.

Let $\ell := |N(x) \setminus N(y)| > 0$. Then

$$d_2(B_{x\to y}) - d_2(B) = (d(x) - \ell)^2 + (d(y) + \ell)^2 - d(x)^2 - d(y)^2 = 2\ell(\ell + d(y) - d(x)) > 0.$$

□

Bipartite threshold graphs can be defined in different ways, but the following is the one we will use.

Definition 21 (See [14]). A graph G is a *bipartite threshold graph* if and only if G is bipartite and the neighborhoods of vertices in one of the partite sets are linearly ordered by inclusion.

Lemma 22. *Suppose that $\mathcal{B}$ is a family of bipartite graphs on a fixed vertex set (X, Y) such that for any $B' \in \mathcal{B}$ and $x, y \in Y$ with $N_{B'}(x) \not\subseteq N_{B'}(y)$ and $N_{B'}(y) \not\subseteq N_{B'}(x)$, we also have $B'_{x\to y} \in \mathcal{B}$. If $B \in \mathcal{B}$ and $d_2(B) = \max\{d_2(B') : B' \in \mathcal{B}\}$, then B is a bipartite threshold graph.*

Proof. In B, suppose there are $x, y \in Y$, $N_B(x) \not\subseteq N_B(y)$, and $N_B(y) \not\subseteq N_B(x)$. Then $d_2(B) < d_2(B_{x\to y})$ by Lemma 20, but $d_2(B) \geqslant d_2(B_{x\to y})$ because $B_{x\to y} \in \mathcal{B}$ and $d_2(B) = \max\{d_2(B') : B' \in \mathcal{B}\}$. Therefore every pair $x, y \in Y$ has $N(x) \subseteq N(y)$ or $N(x) \supseteq N(y)$. By Definition 21, B is a bipartite threshold graph. □

Lemma 23. *Given a graph H, among bipartite graphs B with bipartition $(V(H), Y)$ and a fixed number of edges, some bipartite threshold graph B maximizes $\psi_H(B)$.*

Proof. Consider the family $\mathcal{B}$ of bipartite graphs on $(V(H), Y)$ with the specified number of edges that maximize $\psi_H(B)$. It is closed under compressions by Lemma 20. By Lemma 22, some $B \in \mathcal{B}$ is a bipartite threshold graph. □

Corollary 24. *For a given red graph R, the number of blue triangles intersecting $V(R) = S$ is at most*

$$\sum_{v\in R} \binom{d_R(v)}{2} + \frac{1}{2}\sum_{v\in R} d_R(v)(s - 1 - d_R(v)).$$

Proof. Let $Y = V(B) \setminus S$. There are two types of blue triangles.

1. One vertex in R and two blue edges at that vertex: There are at most $\sum_{v \in R} \binom{d_B(v)}{2}$ triangles of this type.

2. Two adjacent vertices in S (non-adjacent in R) and one blue edge at each of these vertices to the same third vertex outside S: There are

$$\sum_{\substack{i,j \in R \\ i \not\sim_R j \\ i \neq j}} |\{v \in Y : iv, jv \in E(B)\}| = \sum_{v \in Y} |\{i, j \in R : i \neq j, iv, jv \in E(B), ij \notin E(R)\}|$$

triangles of this type.

The total number of blue triangles intersecting $V(R)$ is exactly $\psi_R(B)$, which by Lemma 23 is maximized by some bipartite threshold blue graph B. The definition of R and $\Delta(G) \leqslant r$ imply $d_B(v) \leqslant d_R(v)$ for all $v \in R$. Adding edges can only increase the number of blue triangles, so we may assume that $d_B(v) = d_R(v)$ for all $v \in R$. These blue degrees in R completely determine the graph B because the neighborhoods of vertices in Y are nested. Any vertex $i \in R$ has $d_R(i)$ neighbors in Y, and a pair $i, j \in R$ has $\min(d_R(i), d_R(j))$ common neighbors in Y.

$$\begin{aligned}
\psi_R(B) &= \sum_{v \in R} \binom{d_B(v)}{2} + \sum_{\substack{i,j \in R \\ i \not\sim_R j \\ i \neq j}} |\{v \in Y : i \neq j, iv, jv \in E(B)\}| \\
&\leqslant \sum_{v \in R} \binom{d_R(v)}{2} + \sum_{\substack{i,j \in R \\ i \not\sim_R j \\ i \neq j}} \min(d_R(i), d_R(j)) \\
&\leqslant \sum_{v \in R} \binom{d_R(v)}{2} + \frac{1}{2} \sum_{\substack{i,j \in R \\ i \not\sim_R j \\ i \neq j}} (d_R(i) + d_R(j)) \\
&= \sum_{v \in R} \binom{d_R(v)}{2} + \frac{1}{2} \sum_{v \in R} d_R(v)(s - 1 - d_R(v)).
\end{aligned}$$

□

Having given an upper bound on the number of blue triangles, we turn our attention to a lower bound on the number of red triangles gained from folding G at T.

Definition 25. For a graph R with s vertices, we define

$$Q(R) = (r + 1 - s)e(R) + k_3(R) - \sum_{v \in R} \binom{d_R(v)}{2}.$$

Lemma 26. *If G has a cluster T with $e(B) \geqslant e(R)$, then $k_3(G_T) - k_3(G) \geqslant Q(R)$.*

Proof. We will bound $k_3(G_T) - k_3(G)$ by counting the triangles gained and lost by folding at T. Three types of triangles are gained by folding.

1. One red edge and one vertex in T: There are $te(R)$ triangles of this type.

2. One or two red edges, and all vertices in S: Each triangle of this type contains exactly two unordered pairs of an incident edge and non-edge, and each such pair occurs in a triangle of this type. By counting these pairs and dividing by two, we find there are $\frac{1}{2}\sum_{v\in R} d_R(v)(s-1-d_R(v))$ triangles of this type.

3. Three red edges: There are $k_3(R)$ triangles of this type.

Thus the total number of triangles gained from folding is

$$te(R) + \frac{1}{2}\sum_{v\in R} d_R(v)(s-1-d_R(v)) + k_3(R).$$

The total number of triangles lost from folding is at most

$$\sum_{v\in R}\binom{d_R(v)}{2} + \frac{1}{2}\sum_{v\in R} d_R(v)(s-1-d_R(v))$$

by Corollary 24.

The net gain from folding, $k_3(G_T) - k_3(G)$, is at least

$$\begin{aligned} te(R) + \frac{1}{2}\sum_{v\in R} d_R(v)(s-1-d_R(v)) + k_3(R) - \sum_{v\in R}\binom{d_R(v)}{2} - \frac{1}{2}\sum_{v\in R} d_R(v)(s-1-d_R(v)) \\ = te(R) + k_3(R) - \sum_{v\in R}\binom{d_R(v)}{2} = Q(R). \end{aligned}$$

□

Lemma 27. *If $s \leqslant \frac{r+2}{2}$, then $Q(R) \geqslant 0$, with equality if and only if $R = E_s$.*

Proof. Note $Q(E_s) = 0$. We will show that E_s is the unique minimizer of $Q(R)$. If R has an edge xy, then deleting it would strictly decrease $Q(R)$. We will consider the net change in $Q(R)$ term by term. The change in the $(r+1-s)e(R)$ term is $-(r+1-s)$ since we are losing one edge. The change in the $k_3(R)$ term is $-|N_R(x)\cap N_R(y)|$. The change in the final term is $d_R(x) + d_r(y) - 2$. Therefore

$$\begin{aligned} Q(R-xy) - Q(R) &= -(r+1-s) - |N_R(x)\cap N_R(y)| + d_R(x) + d_R(y) - 2 \\ &= -r-3+s-|N_R(x)\cap N_R(y)| + |N_R(x)| + |N_R(y)| \\ &= -r-3+s+|N_R(x)\cup N_R(y)| \\ &\leqslant -r-3+2s \\ &\leqslant -r-3+r+2 = -1 < 0. \end{aligned}$$

□

Theorem 28. *If $G \in \mathcal{G}_C(m,r)$ has a cluster T with $s \leqslant \frac{r+2}{2}$, then $k_3(G) < f_3(m,r)$.*

Proof. As G is connected and $m \geqslant \binom{r+1}{2}$, there is at least one blue edge.

Case 28.1: $0 < e(B) < e(R)$

We will delete all of the blue edges and add $e(B)$ of the red edges, which will maintain the number of edges and the bound on the maximum degree. The loss from deleting the blue edges is at most $(s-2)e(B)$ by Lemma 15. The gain from the red edges is at least $te(B)$, as each red edge has t common neighbors in T. Therefore the net gain is at least

$$\begin{aligned} te(B) - (s-2)e(B) &= (t-s+2)e(B) \\ &= (r+1-2s+2)e(B) = (r+3-2s)e(B) \\ &\geqslant (r+3-(r+2))e(B) \\ &= e(B) > 0, \end{aligned}$$

so the graph was not extremal.

Case 28.2: $e(B) \geqslant e(R)$

By Lemmas 26 and 27, since $\delta(R) \geqslant 1$, folding G at T strictly increases the number of triangles, so G is not extremal. □

Lemma 29. *If $G \in \mathcal{G}_C(m,r)$ has a cluster T with $\Delta(R) \leqslant 1$, then $k_3(G) < f_3(m,r)$.*

Proof. Any red graph has $\delta(R) \geqslant 1$, so $R = \frac{s}{2}K_2$.

Case 29.1: $0 < e(B) < e(R)$

There is a red edge that is not incident to any blue edges. Add that red edge, and delete a blue edge. The blue edge is in at most $s-2$ triangles by Lemma 15. The red edge has weight $r-1$. The net gain in number of triangles is at least $r-1-(s-2) = r+1-s \geqslant 1$, so $k_3(G) < f_3(m,r)$.

Case 29.2: $e(R) \leqslant e(B)$

By Lemma 26, $k_3(G_T) - k_3(G) \geqslant Q(\frac{s}{2}K_2) = (r+1-s)(\frac{s}{2}) + 0 - 0 = st/2 > 0$, so $k_3(G) < f_3(m,r)$. □

Lemma 30. *If $\Delta(R) \leqslant 2$, then $Q(R) \geqslant 0$, with equality only when $t = 1$ and R is a disjoint union of non-triangle cycles.*

Proof. Any red graph has $\delta(R) \geqslant 1$. Let k be the number of vertices with red degree 2, so $s-k$ is the number of vertices with red degree 1. By the degree sum formula, $2e(R) = 2k + (s-k) = s+k$.

$$\begin{aligned} Q(R) &= (r+1-s)e(R) + k_3(R) - \sum_{v \in R} \binom{d_R(v)}{2} \\ &\geqslant t\frac{s+k}{2} + k_3(R) - k \\ &= st/2 + k(t/2-1) + k_3(R). \end{aligned}$$

For $t \geqslant 2$, each term of this last expression is non-negative, and $st/2 > 0$, so $Q(R) > 0$. For $t = 1$,

$$st/2 + k(t/2 - 1) + k_3(R) = s/2 - k/2 + k_3(R) = (s - k)/2 + k_3(R) \geqslant 0,$$

with equality only when all vertices of R have red degree 2 and there are no triangles, i.e. R is a disjoint union of non-triangle cycles. $\square$

Lemma 31. *If $G \in \mathcal{G}_C(m, r)$ has a cluster T with $\Delta(R) \leqslant 2$ and $t \geqslant 2$, then $k_3(G) < f_3(m, r)$.*

Proof. There is at least one blue edge because G is connected and $m \geqslant \binom{r+1}{2}$. By Lemma 30, $Q(R) > 0$, so if $e(B) \geqslant e(R)$, then $k_3(G) < f_3(m, r)$. Now suppose $0 < e(B) < e(R)$.

We will show that there is a red edge that is incident to at most one blue edge. The average number of blue edges at a given red edge is

$$\begin{aligned} \frac{1}{e(R)} \sum_{f \in E(R)} |\{g \in E(B) : f \cap g \neq \emptyset\}| &= \frac{1}{e(R)} \sum_{g \in E(B)} |\{f \in E(R) : f \cap g \neq \emptyset\}| \\ &\leqslant \frac{1}{e(R)} \sum_{g \in E(B)} 2 \\ &= \frac{2e(B)}{e(R)} < 2. \end{aligned}$$

Thus there is a red edge xy incident to at most one blue edge. The vertices x and y may each have one other neighbor in R. The remaining $r + 1 - 2 - 2 = r - 3$ vertices in the cluster are neighbors of both x and y in G, so xy has weight at least $r - 3$. The blue edge has weight at most $s - 2$ by Lemma 15. Deleting the blue edge (or any blue edge, if there is none at xy) and adding the red edge xy yields a net gain of at least $r - 3 - (s - 2) = r - s - 1 = t - 2 \geqslant 0$ triangles since $t \geqslant 2$, a weak increase that reduces the number of red edges. By induction on the number of red edges, we are done. (The base case $e(R) = 1$ is done by Lemma 29.) $\square$

Theorem 32. *If $G \in \mathcal{G}_C(m, r)$ for $r \geqslant 3$ has a cluster with $e(R) \in \{1, 2\}$, then $k_3(G) < f_3(m, r)$.*

Proof. If $e(R) = 1$, then $R = K_2$, and by Lemma 29 we're done. If $e(R) = 2$, then $R = 2K_2$ or $R = P_3$. For $R = 2K_2$, we are again done by Lemma 29.

Consider $R = P_3$. G is connected and $m \geqslant \binom{r+1}{2}$, so $e(B) \geqslant 1$. Any blue edges have weight at most $s - 2 = 1$ by Lemma 15. Each red edge has weight $r - 2$. If $e(B) = 1$, delete the blue edge and add a red edge for a net gain of at least $r - 2 - 1 = r - 3$ triangles, a weak increase for $r \geqslant 3$. This reduces to the case $R = K_2$.

Otherwise, $e(B) \geqslant 2 = e(R)$, so we may fold. $Q(P_3) = (r + 1 - 3)2 + 0 - 1 = 2r - 5 \geqslant 1$ for $r \geqslant 3$, so $k_3(G) < f_3(m, r)$ by Lemma 26. $\square$

Theorem 33. *If $G \in \mathcal{G}_C(m, r)$ for $r \geqslant 7$ has a cluster with $e(R) = 3$, then $k_3(G) < f_3(m, r)$.*

Proof. Suppose G is extremal. By Theorem 28, we may assume $s \geqslant \frac{r+3}{2} \geqslant 5$. There are two graphs R with 3 edges, at least 5 vertices, and $\delta(R) \geqslant 1$: $R = 3K_2$ and $R = P_3 \cup K_2$. Both have $\Delta(R) \leqslant 2$ and $t \geqslant 2$, so by Lemma 31, $k_3(G) < f_3(m, r)$. □

Theorem 34. *If $G \in \mathcal{G}_C(m, r)$ for $r \geqslant 8$ has a cluster with $e(R) = 4$, then $k_3(G) < f_3(m, r)$.*

Proof. Suppose G is extremal. By Theorem 28, we may assume $s \geqslant \frac{r+3}{2} \geqslant 5.5$ so $s \geqslant 6$.

There are five graphs R with 4 edges, $\delta(R) \geqslant 1$, and $s \geqslant 6$ vertices. The ones on 6 vertices are $K_{1,3} \cup K_2$, $2P_3$, and $P_4 \cup K_2$. On 7 vertices, R can only be $2K_2 \cup P_3$, and on 8 vertices, R can only be $4K_2$. All of these except $K_{1,3} \cup K_2$ have either $\Delta(R) \leqslant 1$ or $\Delta(R) \leqslant 2$ and $t \geqslant 2$, so Lemmas 29 and 31 show G is not extremal.

Suppose $R = K_{1,3} \cup K_2$. Then $Q(R) = (r + 1 - 6)4 + 0 - 3 = 4r - 23 > 0$. If $e(B) \geqslant e(R)$, then folding will increase the number of triangles, and G was not extremal. Otherwise, $1 \leqslant e(B) \leqslant 3$. The isolated K_2 in R has weight $r - 1 \geqslant 7$. Any blue edge has weight at most $s - 2 = 4$. If there is only one blue edge incident to the red K_2 (or none), then delete it (or any blue edge) and add the edge corresponding to the red K_2, which increases the number of triangles by at least $7 - 4 = 3$. Otherwise, there are 2 blue edges incident to the red K_2, so 0 or 1 blue edges incident to the red $K_{1,3}$. Deleting a blue edge (from the $K_{1,3}$ if needed) and adding one of the red $K_{1,3}$ edges increases the number of triangles by at least $5 - 4 = 1$. Therefore G was not extremal. □

5 Degree Multiset Optimization

The number of triangles in a graph can be bounded easily (but crudely) as follows:

$$k_3(G) = \frac{1}{3} \sum_{v \in V(G)} k_3(v) = \frac{1}{3} \sum_{v \in V(G)} e(N(v)) \leqslant \frac{1}{3} \sum_{v \in V(G)} \binom{d(v)}{2}. \tag{2}$$

We have shown, in Section 4, that vertices of degree r lie in clusters with certain red graphs forbidden. In this section we consider the possible degree multisets to give an upper bound on the number of triangles in graphs G that do not have any of the excluded red graphs R considered in Section 4. We consider the constraints imposed by the trivial bound above.

For instance, when $r \geqslant 3$, Theorem 32 shows that in an extremal connected graph, every cluster is missing at least three edges. Similarly, for $r = 7$ and 8, Theorems 33 and 34 show that in an extremal connected graph, every cluster is missing at least four or five edges, respectively. We define upper bounds for $k_3(G)$ based on the degree multiset of G.

Definition 35. For given k, m, r and for $d \in \{0\} \cup [r]$, let

$$w(d) = \begin{cases} \binom{d}{2} & \text{if } d \neq r \\ \binom{r}{2} - k & \text{if } d = r. \end{cases}$$

Set

$$M_k(m,r) = \frac{1}{3}\max\Big\{\sum_{d\in D} w(d) : \sum_{d\in D} d = 2m\Big\} \text{ and}$$
$$M^*_k(m,r) = \frac{1}{3}\max\Big\{\sum_{d\in D} w(d) : \sum_{d\in D} d = 2m,\ r \in D\Big\},$$

where in both cases D is a multiset of arbitrary size from $\{0\} \cup [r]$.

Lemma 36. *For any $G \in \mathcal{G}(m,r)$ such that every cluster has $e(R) \geqslant k$ we have $k_3(G) \leqslant M_k(m,r)$. If in addition the Main Theorem holds for maximum degree at most $r-1$ and for numbers of edges up through $m-1$, and $m \geqslant \binom{r+1}{2}+1$, then we have $k_3(G) \leqslant \lfloor M^*_k(m,r)\rfloor$.*

Proof. Apply (2), and observe that (in each case) the degree multiset of G is one of the candidates in the maximization by Corollary 13. □

We will show in many cases the lower bounds we have on the number of edges in red graphs ensure that the upper bound from Lemma 36 is less than the number of triangles in $aK_{r+1} \cup \mathcal{C}(b)$, proving the Main Theorem in those cases. We say that a multiset D is *optimal* if it achieves the maximum $M_k(m,r)$ and has no 0 entries.

Lemma 37. *An optimal multiset D contains at most one entry from $[r-2]$.*

Proof. If an optimal multiset D had two elements in $[r-2]$, $x \leqslant y$, then changing these two elements to $x-1$ and $y+1$ maintains the sum of $2m$ while increasing the weighted sum $\sum_{d\in D} w(d)$, contradicting the optimality of D. If $x-1=0$, then the 0 is discarded, and $x=1$ and y are replaced by $y+1$. The weighted sum is increased because

$$\begin{aligned} w(x-1)+w(y+1)-(w(x)+w(y)) &= \binom{x-1}{2}+\binom{y+1}{2}-\binom{x}{2}-\binom{y}{2} \\ &= y-(x-1) = y-x+1 \geqslant 1. \end{aligned}$$
□

Lemma 38. *For any m and $r \leqslant 2k+1$, if an optimal multiset D contains an entry $d \in [r-2]$ and at least $r-1-d$ copies of r, then there is an optimal multiset containing no entries from $[r-2]$.*

Proof. We will change the $r-1-d$ copies of r and the $d \in [r-2]$ to $r-d$ copies of $r-1$. The resulting multiset contains no entries from $[r-2]$ by Lemma 37.

This maintains the sum of D since $d+(r-1-d)r = d+r^2-r-dr = (r-d)(r-1)$. The increase in the weighted sum is given by

$$\begin{aligned} &(r-d)\binom{r-1}{2} - \left(\binom{d}{2} + (r-1-d)\left(\binom{r}{2}-k\right)\right) \\ &\qquad = -\frac{1}{2}\left((r-d)(r-1)(2) + d^2 - d - r^2 + r - 2kr + 2dk + 2k\right) \\ &\qquad = -\frac{1}{2}\left(d^2 - 2dr + (1+2k)d + r^2 - (2k+1)r + 2k\right) \\ &\qquad = -\frac{1}{2}\left(d-(r-1)\right)\left(d-(r-2k)\right) \\ &\qquad \geqslant 0, \end{aligned}$$

for $r-2k \leqslant d \leqslant r-1$. Since $r \leqslant 2k+1$, we have $r-2k \leqslant 1 \leqslant d \leqslant r-2 < r-1$, and we weakly increased the weighted sum. □

Lemma 39. *If D is an optimal multiset containing only $r-1$'s and r's with $k \geqslant r/2$, then there is an optimal multiset containing at most $r-2$ copies of r, and the rest $r-1$'s.*

Proof. Changing $r-1$ copies of r to r copies of $r-1$ maintains the sum of D.

The weighted sum increases by

$$\begin{aligned} r\binom{r-1}{2} - (r-1)\left(\binom{r}{2} - k\right) &= (r-1)\left(\frac{1}{2}r(r-2) - \binom{r}{2} + k\right) \\ &= (r-1)\,(k - r/2) \geqslant 0. \end{aligned}$$

Change any $r-1$ copies of r to r copies of $r-1$ repeatedly to get an optimal multiset with at most $r-2$ copies of r. □

The next few results consider the case where $k = \lceil r/2 \rceil$.

Lemma 40. *If $m \geqslant \binom{r+1}{2} + 1$ and $k = \lceil r/2 \rceil$ then there is an optimal multiset containing no elements of $[r-2]$.*

Proof. Suppose D is an optimal multiset containing an entry $d \in [r-2]$.

Case 40.1: D contains at least $r-1-d$ copies of r.

Lemma 38 shows there is an optimal multiset containing no entries from $[r-2]$.

Case 40.2: D contains at least d copies of $r-1$.

We will change the d copies of $r-1$ and the $d \in [r-2]$ to d copies of r. The resulting multiset contains no entries from $[r-2]$ by Lemma 37. This maintains the sum: $d + d(r-1) = dr$. It also increases the weighted sum: $d\left(\binom{r}{2} - k\right) - \left(\binom{d}{2} + d\binom{r-1}{2}\right) = -\frac{d}{2}(d - 2r + (2k+1))$, which is positive between $d = 0$ and $d = 2r - (2k+1)$. Since $k < r$, we have $2r - (2k+1) > 0$.

Case 40.3: D contains at most $r-2-d$ copies of r and at most $d-1$ copies of $r-1$.

By Lemma 37, d is the only entry from $[r-2]$. Then $\sum_{d' \in D} d' \leqslant d + (r-2-d)r + (d-1)(r-1) = d + r^2 - 2r - dr + dr - r - d + 1 = r^2 - 3r + 1$. We also have $2m \geqslant 2\left(\binom{r+1}{2} + 1\right) = (r+1)r + 2 = r^2 + r + 2$. Thus $r^2 + r + 2 \leqslant 2m = \sum_{d \in D} d \leqslant r^2 - 3r + 1$, which implies $4r + 1 \leqslant 0$, i.e. $r \leqslant -\frac{1}{4}$, a contradiction. □

Theorem 41. *If $m \geqslant \binom{r+1}{2} + 1$ and $k = \lceil r/2 \rceil$ then $3M_k(m, r) \leqslant (r-2)m$.*

Proof. By Lemma 40 we may assume an optimal multiset D consists only of x copies of r and y copies of $r-1$. By Lemma 39, we may further assume $0 \leqslant x \leqslant r-2$. The degree sum formula implies $2m = xr + y(r-1)$, so $y = \frac{2m - xr}{r-1}$.

The weighted sum of D gives the value of $3M_k(m, r)$, which is

$$\begin{aligned} y\binom{r-1}{2} + x\left(\binom{r}{2} - k\right) &= \frac{2m - xr}{r-1}(r-1)(r-2)/2 - x\left(k - \frac{r(r-1)}{2}\right) \\ &= (r-2)m - \frac{x}{2}(2k - r) \\ &\leqslant (r-2)m \end{aligned}$$

since $x \geqslant 0$ and $2k \geqslant r$. ☐

Lemma 42. *If $1 \leqslant r \leqslant 7$ and $a \geqslant 1$, then $(r-2)m < 3g_3(m,r)$.*

Proof.

$$\begin{aligned}
&3k_3(aK_{r+1} \cup C(b)) - (r-2)m \\
&\qquad = 3\left(a\binom{r+1}{3} + \binom{c}{3} + \binom{d}{2}\right) - (r-2)\left(a\binom{r+1}{2} + \binom{c}{2} + d\right) \\
&\qquad = a\binom{r+1}{2} + (c-r)\binom{c}{2} + d\left(\frac{3}{2}(d-1) - (r-2)\right) \\
&\qquad \geqslant \binom{r+1}{2} + (c-r)\binom{c}{2} + d\left(\frac{3}{2}(d-1) - (r-2)\right)
\end{aligned}$$

Define $h_r(c) = (c-r)\binom{c}{2}$ and $q_r(d) = d(\frac{3}{2}(d-1) - (r-2))$. By taking derivatives, we find the minimum values for these functions are attained at $c = \frac{1}{3}(r+1+\sqrt{r^2-r+1})$ and $d = \frac{2r-1}{6}$, with minimum values $q_r(d) \geqslant -\frac{1}{24}(2r-1)^2$ and $h_r(c) \geqslant -\frac{1}{54}\left[(2r-1)(r+1)(r-2) + 2(r^2-r+1)^{3/2}\right]$, resulting in

$$3g_3(m) - (r-2)m \geqslant \binom{r+1}{2} - \frac{1}{54}\left[(2r-1)(r+1)(r-2) + 2(r^2-r+1)^{3/2}\right] - \frac{1}{24}(2r-1)^2.$$

This function of r is positive for $1 \leqslant r \leqslant 7$ with positive roots at $r \approx 0.14$ and $r \approx 7.21$. ☐

Theorem 43. *If $1 \leqslant r \leqslant 7$ and $m \geqslant \binom{r+1}{2} + 1$, then $M_k(m,r) < f_3(m,r)$ for $k = \lceil r/2 \rceil$.*

Proof. Theorem 41 and Lemma 42 together show $3M_k(m,r) \leqslant (r-2)m < 3g_3(m,r) \leqslant 3f_3(m,r)$. ☐

Lemma 44. *For $r = 8$, $m \geqslant \binom{r+1}{2} + 1$, if the Main Theorem holds for $r \leqslant 7$ and numbers of edges up through $m-1$, and every cluster has $e(R) \geqslant 5$, then $k_3(G) \leqslant g_3(m,r)$.*

Proof. We will restrict the multiset optimization problem for $r = 8$, $k = 5$ to multisets that contain at least one copy of $r = 8$, since we know from Lemma 13 that G must have an 8 in its degree multiset. Let D be an optimal multiset that attains the maximum $M_5^*(m,8)$. It contains at most one element of $[6]$ by Lemma 37.

First, D contains at least one 8, so we may assume there is no 6: If there were both a 6 and an 8, then we could replace them with two 7's, increasing the weighted sum: $\binom{6}{2} + \binom{8}{2} - 5 = 38$, and $2\binom{7}{2} = 42$. Therefore D contains only 8's, 7's, and at most one element $d \in [5]$.

Second, if there are d copies of 7, then we change them and the $d \in [5]$ to d copies of 8. This maintains the sum of D. It also weakly increases the weighted sum:

$$d\left(\binom{8}{2} - 5\right) - \left(\binom{d}{2} + d\binom{7}{2}\right) = -\frac{d}{2}(d-5) \geqslant 0.$$

Suppose $d \in [5]$ is in an optimal multiset D. Since $m \geqslant \binom{9}{2}+1$, either there are at least $7-d$ copies of 8 in D, or there are at least d copies of 7 in D, but we have just handled the second case. In the first case, Lemma 38 gives the result. We conclude that there is an optimal multiset (among multisets containing an 8) that consists only of copies of 8 and copies of 7. By Lemma 39, changing seven 8's to eight 7's will increase $\sum_{d' \in D} w(d')$, so we have $x \in [7]$ copies of 7. By the degree sum formula, we have $y = \frac{2m-8x}{7}$ copies of 8.

The weighted sum of D gives the value of $3M_5^*(m,8)$, which is

$$\frac{2m-8x}{7}\binom{7}{2} + x\left(\binom{8}{2} - 5\right) = 6m - x < 6m.$$

By the proof of Lemma 42, $3f_3(m,8) - 6m \geqslant 36a + (c-8)\binom{c}{2} + d\left(\frac{3}{2}(d-1) - 6\right)$.

The minimum value of $(c-8)\binom{c}{2}$ for integer values of c between 0 and 8, inclusive, is -30, at $c = 5, 6$. For $c \neq 5, 6$, the minimum value is -24.

The minimum value of $d\left(\frac{3}{2}(d-1) - 6\right)$ for integer values of d between 0 and 7, inclusive, is -9, at $d = 2, 3$. For $d \neq 2, 3$, the minimum value is -6 at $d = 1, 4$. For $d \neq 1, 2, 3, 4$, the minimum value is less than -6.

For $a \geqslant 2$, $3f_3(m,8) - 6m \geqslant 72 - 30 - 9 > 0$. For $a = 1$ and $c \neq 5, 6$, $3f_3(m,8) - 6m \geqslant 36 - 24 - 9 > 0$. For $a = 1$, $c = 5, 6$, and $d \neq 1, 2, 3, 4$, $3f_3(m,8) - 6m > 36 - 30 - 6 = 0$.

For $a = 1$, $c = 5, 6$, and $d = 1, 2, 3, 4$, we calculate $3M_5^*(m,8) = 6m - x$ exactly and compare it to $3k_3(aK_{r+1} \cup \mathcal{C}(b))$. We can determine x for each m because $2m = 8x + 7y \equiv x \mod 7$ and $x \in [7]$.

m	x	$3M_5^*(m,8) = 6m - x$	$3g_3(m,r)$
47	3	279	282
48	5	283	285
49	7	287	291
50	2	298	300
52	6	306	312
53	1	317	315
54	3	321	321
55	5	325	330

In all cases, by Lemma 36, $k_3(G) \leqslant \lfloor M_5^*(m,8) \rfloor \leqslant k_3(aK_{r+1} \cup \mathcal{C}(b))$. □

6 Proof of Main Theorem

We now prove our main theorem.

Main Theorem. If G is a graph with m edges and maximum degree at most r for any fixed $r \leqslant 8$, then

$$k_3(G) \leqslant k_3(aK_{r+1} \cup \mathcal{C}(b)),$$

where $m = a\binom{r+1}{2} + b$ and $0 \leqslant b < \binom{r+1}{2}$. That is, the graphs with the maximum number of triangles consist of as many disjoint copies of K_{r+1} as possible, with the remaining edges formed into a colex graph.

Proof. We will induct on m and r. For $m \leqslant \binom{r+1}{2}$, the Kruskal-Katona Theorem implies the theorem, so we may assume $m \geqslant \binom{r+1}{2} + 1$. Let G be an extremal graph.

Note 45. If G contains a K_{r+1}, then it is a connected component of G, as $\Delta(G) \leqslant r$. Then

$$\begin{aligned} k_3(G) &= k_3(K_{r+1}) + k_3(G \setminus K_{r+1}) \\ &\leqslant k_3(K_{r+1}) + f_3(m - \binom{r+1}{2}, r) \\ &= k_3(K_{r+1}) + k_3((a-1)K_{r+1} \cup \mathcal{C}(b)) \text{ by induction} \\ &= k_3(aK_{r+1} \cup \mathcal{C}(b)). \end{aligned}$$

Thus we may assume G does not contain a K_{r+1}, and since $a \geqslant 1$, $G \neq aK_{r+1} \cup \mathcal{C}(b)$.

By Corollary 12 and induction on m, if G is not connected, then $k_3(G) \leqslant k_3(aK_{r+1} \cup \mathcal{C}(b))$, so we assume G is connected. In particular, G does not contain a K_{r+1}. By Corollary 13 and induction on m and r, G contains a vertex of degree r. Therefore G contains a cluster, and every cluster has at least one red edge and at least one blue edge.

For $r = 1$, every $G \in \mathcal{G}(m, 1)$ has $k_3(G) = 0 = k_3(aK_2 \cup \mathcal{C}(b))$. For $r = 2$, we have assumed G does not contain a $K_{r+1} = K_3$, so there are no triangles, and G is not extremal.

For $3 \leqslant r \leqslant 7$, Theorems 32 and 33 imply that every cluster of G has $e(R) \geqslant \lceil r/2 \rceil$. By Lemma 36 and Theorem 43, $k_3(G) < f_3(m, r)$.

For $r = 8$, Theorems 32, 33, and 34 show that every cluster has $e(R) \geqslant 5$. Lemma 44 implies $k_3(G) \leqslant k_3(aK_{r+1} \cup \mathcal{C}(b))$. □

7 Open Problems

The question "Which graphs with a fixed number of edges m and maximum degree at most r maximize the number of K_t's?" remains open for maximum degrees $r \geqslant 9$ and remains open for complete subgraph sizes $t \geqslant 4$. Similarly, the corresponding question when fixing the number of vertices n instead of the number of edges remains open for $r \geqslant 7$ except when $a = 1$. Both are extremely natural questions.

The (probably very hard) problem of determining which n-vertex, m-edge graphs with maximum degree at most r have the maximum number of triangles is also open for a range of values of n, even if we know the value of $f_3(m, r)$. If we write $N(m, r)$ for the number of vertices in our extremal graphs in $\mathcal{G}(m, r)$, then our theorem answers this question for $r \leqslant 8$ for $n \geqslant N(m, r)$. There are no n-vertex graphs in $\mathcal{G}(m, r)$ with $n < 2m/r$, so this question is open in the range $2m/r \leqslant n < N(m, r)$.

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