Homotopy type of the neighborhood complexes of graphs of maximal degree at most 3 and 4-regular circulant graphs

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Abstract

To estimate the lower bound for the chromatic number of a graph G, Lovász associated a simplicial complex $\mathcal{N}(G)$ called the neighborhood complex and related the topological connectivity of $\mathcal{N}(G)$ to the chromatic number of G. More generally he proved that the chromatic number of G is bounded below by the topological connectivity of $\mathcal{N}(G)$ plus 3.

In this article, we consider the graphs of maximal degree at most 3 and 4-regular circulant graphs. We show that each connected component of the neighborhood complexes of these graphs is homotopy equivalent either to a point, to a wedge sum of circles, to a wedge sum of 2-spheres S^2, to S^3, to a garland of 2-spheres S^2 or to a connected sum of tori.

Mathematics Subject Classifications: 05C15, 57M15

1 Introduction

The *neighborhood complex*, $\mathcal{N}(G)$ of a graph G is the simplicial complex whose vertices are all non isolated vertices of G and simplices are those subsets of $V(G)$, which have a common neighbor. The concept of neighborhood complex was introduced by Lovász [8] in his proof of the Kneser conjecture, that if we split the n-subsets of a $(2n+k)$-elements set into $k+1$ classes, then one of the classes will contain two disjoint subsets. To prove this conjecture, Lovász first converted this set theoretic problem into an equivalent problem of

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the computation of chromatic number of a class of graphs, called Kneser graphs and then related the connectivity of neighborhood complex to the chromatic number of graph.

A topological space X is said to be *k-connected* if every map from a m-dimensional sphere $S^m \to X$ can be extended to a map from the $(m+1)$-dimensional ball $\mathbb{B}^{m+1} \to X$ form $m = 0, 1, \ldots, k$.

Theorem 1. *(Lovász) If $\mathcal{N}(G)$ is $k-$connected, then $\chi(G) \geqslant k+3$.*

Lovász generalized the notion of neighborhood complex to a polyhedral complex $\mathrm{Hom}(G, H)$, called the hom complex, for graphs G and H. In particular $\mathrm{Hom}(K_2, G)$ and $\mathcal{N}(G)$ are homotopy equivalent. The 0-dimensional cells of $\mathrm{Hom}(G, H)$ are the graph homomorphisms from G to H. For more details about hom complexes we refer the reader to [1]. In [3], Björner and Longueville showed that the neighborhood complexes of a family of vertex critical subgraphs of Kneser graphs, the stable Kneser graphs, are spheres up to homotopy. In [10], Nilakantan and author studied the neighborhood complexes of the exponential graphs $K_{n+1}^{K_n}$. In this article we compute the homotopy type of the neighborhood complexes of 4-regular circulant graphs.

Let $n \geqslant 2$ be a positive integer and $S \subset \{1, 2, \ldots, n-1\}$. The *circulant graph* $C_n(S)$ is the graph, whose set of vertices $V(C_n(S)) = \{1, 2, \ldots, n\}$ and any two vertices x and y adjacent if and only if $x - y \pmod{n} \in S \cup -S$, where $-S = \{n - a \mid a \in S\}$. Circulant graphs are also Cayley graphs of $\mathbb{Z}_n$, the cyclic group on n elements. Since $n \notin S$, $C_n(S)$ is a simple graph, *i.e.*, does not contains any loop. Further, $C_n(S)$ are $|S \cup -S|$-regular graphs, here $|\cdot|$ denoting the cardinality. It can be easily verify that $C_n(S)$ is connected if and only if $S \cup -S$ generates $\mathbb{Z}_n$. In this article we restrict ourselves to $|S| = 2$ and for the convenience of notation, we write $C_n(s,t)$ in place of $C_n(\{s,t\})$. Since $C_n(s,t) = C_n(n-s,t) = C_n(s,n-t)$, we assume that $s, t \leqslant \frac{n}{2}$.

2 Statement of results

We use the following definition of garland of topological spaces in Theorem 5 and Theorem 6. Let $X_1, X_2, \ldots, X_n$ be topological spaces. A topological space $X = \bigcup_{i=1}^{n} X_i$ is said to be a *garland* of $X_1, \ldots, X_n$, if $|X_i \cap X_j| = 1$ when $|i-j| = 1$ $(mod\ n)$ and $X_i \cap X_j = \emptyset$ when $|i-j| \neq 1$ $(mod\ n)$ for all $1 \leqslant i \neq j \leqslant n$.

Theorem 2. *Let G be a connected graph with maximal degree at most 3 and $G \neq K_4, T$ (see Figure 1a and 1b). Each connected component on $\mathcal{N}(G)$ is either contractible or homotopy equivalent to wedge sum of circles.*

Let $n \geqslant 5$ and $s, t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}, s \neq t$. To compute the homotopy type of the neighborhood complexes of 4-regular circulant graphs we divide the set $I = \{\{s,t\} \mid s, t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}, s \neq t\}$ in to the following disjoint classes: $I_1 = \{\{s,t\} \in I \mid n \in \{2s, 2t, 2(s+t)\}\}$, $I_2 = \{\{s,t\} \in I \setminus I_1 \mid 3s = t \text{ or } 3t = s\}$, $I_3 = \{\{s,t\} \in I \setminus (I_1 \cup I_2) \mid 3s = 5t \text{ or } 3t = 5s\}$ and $I_4 = I \setminus (I_1 \cup I_2 \cup I_3)$.

Theorem 3. *Let* $\{s,t\} \in I_1$*. Each connected component of* $\mathcal{N}(C_n(s,t))$ *is homotopy equivalent to*

(A) a point or wedge sum of circles, if $2s = n$ *or* $2t = n$*.*

(B) a point or S^1*, if* $2s, 2t \neq n$ *and* $2(s+t) = n$*.*

Theorem 4. *Let* $\{s,t\} \in I_2$*. Each connected component of* $\mathcal{N}(C_n(s,t))$ *is homotopy equivalent to*

(A) S^3*, if* $3s = t$ *and* $n = 10s$ *(or* $3t = s$ *and* $n = 10t$*).*

(B) $S^2 \vee S^2$*, if* $3s = t$ *and* $n = 12s$ *(or* $3t = s$ *and* $n = 12t$*).*

(C) wedge sum of circles, if $3s = t$ *and* $n \neq 8s, 10s, 12s$ *(or* $3t = s$ *and* $n \neq 8t, 10t, 12t$*).*

Theorem 5. *Let* $\{s,t\} \in I_3$*. Each connected component of* $\mathcal{N}(C_n(s,t))$ *is homotopy equivalent to*

(A) a garland of the 2-dimensional spheres S^2*, if* $5s = 3t$ *and* $n = 4t$ *(or* $5t = 3s$ *and* $n = 4s$*).*

(B) $S^2 \vee S^2$*, if* $5s = 3t$ *and* $n = 4s$ *(or* $5t = 3s$ *and* $n = 4t$*).*

(C) wedge sum of circles, if $5s = 3t$ *and one of* $6s$ *or* $\frac{14s}{3}$ *is equal to* n *(or* $5t = 3s$ *and one of* $6t$ *or* $\frac{14t}{3}$ *is equal to* n*).*

(D) connected sum of tori, if $5s = 3t$ *and* $4s, 6s, \frac{14s}{3}, 4t \neq n$ *(or* $5t = 3s$ *and* $4t, 6t, \frac{14t}{3}, 4s \neq n$*).*

Theorem 6. *Let* $\{s,t\} \in I_4$*. Each connected component of* $\mathcal{N}(C_n(s,t))$ *is homotopy equivalent to*

(A) either S^1 *or* S^3*, if one of the* $3s-t, 3t-s, 3s+t$ *or* $3t+s$ *is equal to* n*.*

(B) a garland of the 2-dimensional spheres S^2*, if* $3s-t, 3t-s, 3s+t, 3t+s \neq n$ *and,* $4t = n$ *or* $4s = n$*.*

(C) connected sum of tori, if $3s-t, 3t-s, 3s+t, 3t+s, 4s, 4t \neq n$*.*

The following theorem can be considered as a special case of Theorem 6 (C).

Theorem 7. *Let* $n = pq$*, where* $gcd(p,q) = 1$*. Let* $s, t \in \{1, \ldots, n-1\}$ *such that* $2s, 2t, 2(s+t), 3s-t, 3t-s, 3s+t, 3t+s, 4s, 4t \not\equiv 0 \pmod{n}$*. If* $s = \frac{p-q}{2}$ *and* $t = \frac{p+q}{2}$ *or,* $s = \frac{p^2-q}{2}$ *and* $t = \frac{p^2+q}{2}$*, then* $\mathcal{N}(C_n(s,t))$ *is homotopy equivalent to torus.*

3 Preliminaries

3.1 Graph

A graph G is a pair $(V(G), E(G))$, where $V(G)$ is the set of vertices of G and $E(G) \subset V(G) \times V(G)$ denotes the set of edges. If $(x, y) \in E(G)$, it is also denoted by $x \sim y$. A *subgraph* H of G is a graph with $V(H) \subset V(G)$ and $E(H) \subset E(G)$. For a subset $U \subset V(G)$, the induced subgraph $G[U]$ is the subgraph whose set of vertices $V(G[U]) = U$ and the set of edges $E(G[U]) = \{(a, b) \in E(G) \mid a, b \in U\}$.

A *graph homomorphism* from G to H is a function $\phi : V(G) \to V(H)$ such that, $(v, w) \in E(G) \implies (\phi(v), \phi(w)) \in E(H)$. A graph homomorphism f is called an *isomorphism* if f is bijective and f^{-1} is also a graph homomorphism. Two graphs are called *isomorphic*, if there exists an isomorphism between them. If G and H are isomorphic, we write $G \cong H$. The *chromatic number* $\chi(G)$ of a graph G is defined as $\chi(G) := \min\{n \mid \exists \text{ a graph homomorphism from } G \text{ to } K_n\}$. Here, K_n denotes a complete graph on n vertices.

Let G be a graph and v be a vertex of G. The *neighbourhood of* v is defined as $N(v) = \{w \in V(G) \mid (v, w) \in E(G)\}$. If $A \subset V(G)$, the set of neighbours of A is defined as $N(A) = \{x \in V(G) \mid (x, a) \in E(G) \ \forall \ a \in A\}$. The *degree* of a vertex v is $|N(v)|$. A graph is said to be *d-regular*, if each vertex has degree d. The *maximal degree* of G is the maximum of the degree of vertices of G.

3.2 Simplicial complex

A *finite abstract simplicial complex* X is a collection of finite sets such that if $\tau \in X$ and $\sigma \subset \tau$, then $\sigma \in X$. The elements of X are called *simplices* of X. The dimension of a simplex σ is equal to $|\sigma| - 1$. The dimension of an abstract simplicial complex is the maximum of the dimensions of its simplices. The 0-dimensional simplices are called vertices of X. If $\sigma \subset \tau$, we say that σ is a face of τ. If a simplex has dimension k, it is said to be *k-dimensional* or *k-simplex*. The *boundary* of a k-simplex σ is the simplicial complex, consisting of all faces of σ of dimension $\leqslant k - 1$ and it is denoted by $Bd(\sigma)$. A simplex which is not a face of any other simplex is called a *maximal simplex*. The set of maximal simplices of X is denoted by $M(X)$. A simplicial complex is called *pure d-dimensional*, if all of its maximal simplices are of dimension d.

Let X be a simplicial complex and $\tau, \sigma \in X$ such that $\sigma \subsetneq \tau$ and τ is the only maximal simplex in X that contains σ. A *simplicial collapse* of X is the simplicial complex Y obtained from X by removing all those simplices γ of X such that $\sigma \subseteq \gamma \subseteq \tau$. Here, σ is called a *free face* of τ and (σ, τ) is called a *collapsible pair*. We denote this collapse by $X \searrow Y$. In particular, if $X \searrow Y$, then $X \simeq Y$.

3.3 Shellability

A pure d-dimensional simplicial complex X is said to be *shellable*, if its maximal simplices can be ordered $\Gamma_1, \Gamma_2 \dots, \Gamma_t$ in such a way that the subcomplex $(\bigcup_{i=1}^{k-1} \Gamma_i) \cap \Gamma_k$ is pure and $(d-1)$-dimensional for all $k = 2, \ldots, t$. Here, this ordering $\Gamma_1, \ldots, \Gamma_t$ of maximal simplices is called a *shelling order*. A maximal simplex Γ_k is called spanning with respect to the given shelling order if $Bd(\Gamma_k) \subseteq \bigcup_{i=1}^{k-1} \Gamma_i$.

From [1, Theorem 12.3] we have the following result which tells us about the homotopy type of shellable complexes.

Proposition 8. *Assume that X be a shellable simplicial complex, with $\Gamma_1, \Gamma_2, \ldots, \Gamma_t$ being the corresponding shelling order of the maximal simplices, and $\sum$ being the set of spanning simplices. Then*

$$X \simeq \bigvee_{\sigma \in \sum} S^{dim\ \sigma}$$

Remark 9. Any connected 1-dimensional simplicial complex is always shellable and therefore it is either contractible or homotopy equivalent to wedge sum of circles.

3.4 Folding

Let G be a graph and $N(u) \subset N(v)$ for $u, v \in V(G), u \neq v$. In this case, the graph homomorphism $V(G) \to V(G) \setminus \{u\}$, which sends u to v and fixes all other vertices, called *folding* and the graph $G \setminus \{u\}$ is called a *fold* of G. Here, $V(G \setminus \{u\}) = V(G) \setminus \{u\}$ and the edges in the subgraph $G \setminus \{u\}$ are all those edges of G which do not contain u.

Proposition 10. *([1], Proposition 4.2 and Proposition 5.1)*
Let G be a graph and $u \in V(G)$. If G is folded on to $G \setminus \{u\}$, then $\mathcal{N}(G)$ is of same homotopy type as $\mathcal{N}(G \setminus \{u\})$.

4 Proofs

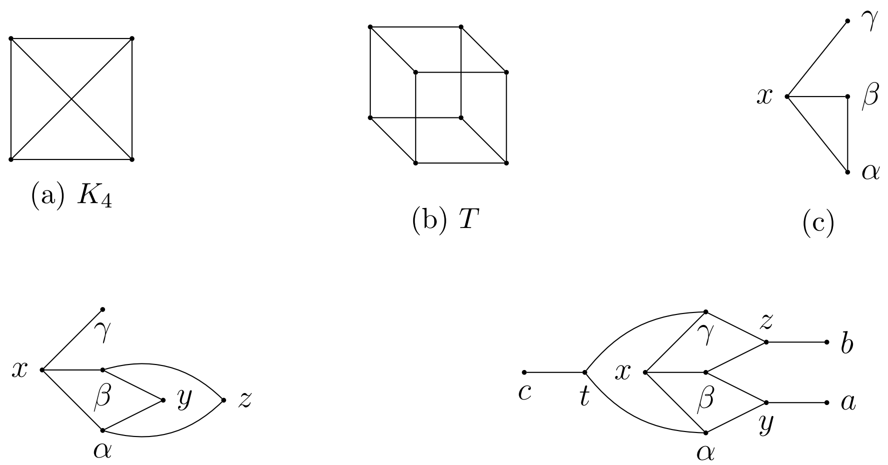
In this article $[n]$ denotes the set $\{1, 2, \ldots, n\}$. Throughout this paper, all the graphs are finite and simple, *i.e.*, not contain loops.

Proof of Theorem 2. Since, by Proposition 10, folding preserve the homotopy type in neighborhood complex, without loss of generality we assume that G cannot be folded onto any of its subgraphs. If the maximal degree of G is 1, then $G \cong K_2$. If the maximal degree of G is 2, then $\mathcal{N}(G)$ is a 1-dimensional complex and the result follows from Remark 9. So, assume that maximal degree of G is 3. In this case, $\mathcal{N}(G)$ is 2-dimensional. We show that $\mathcal{N}(G)$ collapses to a 1-dimensional subcomplex. The 2-dimensional simplices of $\mathcal{N}(G)$ are the neighborhoods of vertices of degree 3. Let $\sigma = \{\alpha, \beta, \gamma\}$ be a 2-simplex of $\mathcal{N}(G)$. Then, there exists $x \in V(G)$ such that $N(x) = \{\alpha, \beta, \gamma\}$. We consider the following two cases.

Case 1. At least one of the $(\alpha, \beta), (\beta, \gamma)$ or $(\alpha, \gamma) \in E(G)$.

Without loss of generality assume that $(\alpha, \beta) \in E(G)$ (see Figure 1c). If α and β do not have any common neighbor other than x, then $(\{\alpha, \beta\}, \sigma)$ is a collapsible pair and therefore $\mathcal{N}(G) \searrow \mathcal{N}(G) \setminus \{\sigma, \{\alpha, \beta\}\}$.

Assume α, β have a common neighbor $y \neq x$. Since, $G \neq K_4, y \neq \gamma$. If $y \sim \gamma$, then $N(y) = \{\alpha, \beta, \gamma\}$ and $(\{\alpha, \beta\}, \sigma)$ is a collapsible pair. If $y \nsim \gamma$, then since $N(\alpha) = \{x, \beta, y\}, N(\beta) = \{x, \alpha, y\}$ and maximal degree of G is 3, we see that $(\{\beta, \gamma\}, \sigma)$ is a collapsible pair.



(a) K_4 (b) T (c) (d) (e)

Figure 1

Case 2. None of the $(\alpha, \beta), (\beta, \gamma)$ or $(\alpha, \gamma) \in E(G)$.

Since maximal degree of G is 3, any 1-dimensional simplex of $\mathcal{N}(G)$ can be a face of at most three 2-simplices. If there exists a face, say $\{\alpha, \beta\}$ of $\{\alpha, \beta, \gamma\}$ which is a face of three 2-simplices of $\mathcal{N}(G)$ (see Figure 1d), then since x is not contained in any 3-cycle (a graph on 3 vertices, where each vertex has degree 2), $\gamma \neq y, z$. In this case, $(\{\alpha, \gamma\}, \sigma)$ is a collapsible pair.

Assume none of the 1-dimensional face of $\{\alpha, \beta, \gamma\}$ is contained in three maximal simplices, *i.e.*, each face is contained in at most two 2-simplices of $\mathcal{N}(G)$. If some face of σ is contained in only one 2-simplex, then clearly that face will be a free face of σ. So, assume that each face of σ contained in exactly two 2-simplices of $N(G)$ (see Figure 1e). If $y = z = t$, then since maximal degree of G is 3, at least one of the 1-dimensional face of $\{\alpha, \beta, \gamma\}$ will be a free face.

Suppose exactly two elements of $\{y, z, t\}$ are same, say $y = z$. In this case $\{\beta, \gamma\}$ is a free face of $\{\alpha, \beta, \gamma\}$. Assume $|\{y, z, t\}| = 3$. Since, G cannot be folded onto any of its subgraph, degree of y, z and t must be 3. Let a, b and c are as depicted in Figure 1e and $\{a, b, c\} \cap \{\alpha, \beta, \gamma\} = \emptyset$.

If $|\{a, b, c\}| = 3$, then since the common neighbor of γ and c is only t, we see that $(\{\gamma, c\}, N(t))$ is a collapsible pair. Hence, $\mathcal{N}(G) \searrow \mathcal{N}(G) \setminus \{N(t), \{\gamma, c\}\}$. Now $(\{\alpha, \gamma\}, N(x))$ is a collapsible pair in $\mathcal{N}(G) \setminus \{N(t), \{\gamma, c\}\}$ and therefore $\mathcal{N}(G) \searrow \mathcal{N}(G) \setminus \{N(x), N(t), \{\alpha, \gamma\}, \{\gamma, c\}\}$.

Now, let $|\{a, b, c\}| = 2$. Without loss of generality we can assume that $b = c$. Then $(\{\beta, a\}, N(y))$ is a collapsible pair and therefore $\mathcal{N}(G) \searrow \mathcal{N}(G) \setminus \{N(y), \{\beta, a\}\}$. Now

$(\{\alpha,\beta\}, N(x))$ is a collapsible pair in $\mathcal{N}(G)\setminus\{N(y),\{\beta,a\}\}$ and therefore $\mathcal{N}(G)\searrow\mathcal{N}(G)\setminus\{N(x), N(y), \{\alpha,\beta\},\{\beta,a\}\}$. If $a=b=c$, then $G\cong T$, which is not possible.

Thus $\mathcal{N}(G)$ collapses to a 1-dimensional subcomplex. From Remark 9, each connected component of $\mathcal{N}(G)$ is either contractible or homotopy equivalent to wedge sum of circles. □

Remark 11. It has been shown in [8] that the neighborhood complex of any non-bipartite graph is connected and is never homotopically trivial. Therefore, the nieghborhood complex of any non-bipartite graph of maximal degree at most 3 is homotopy equivalent to a wedge sum of circles.

We now fix some notations. Throughout this article, if we write an integer r as a vertex of $C_n(s,t)$, it is understood that we are taking r modulo n. Further, for any two vertices v_1 and v_2 of $C_n(s,t)$, $v_1=v_2$ means $v_1\equiv v_2 \pmod n$. Since n is fixed, for any two integers x and y such that $x\equiv y \pmod n$, if no confusion arises, we just write $x\equiv y$.

For any set $X\subset\mathbb{Z}$ and any integer r, let $X+r=\{x+r \mid x\in X\}$. Observe that, for any $k\in[n]=V(C_n(s,t))$, the neighborhood of k, $N(k)=\{s+k, t+k, n-s+k, n-t+k\}$. Since $n\geqslant 5$, if $s,t\neq\frac{n}{2}$, then $C_n(s,t)$ is a 4-regular graph, *i.e.*, $|\{s+k,t+k,n-s+k,n-t+k\}|=4$. Since $N(k+r)=N(k)+r$ for any $k,r\in[n]$, observe that any two connected components of $\mathcal{N}(C_n(s,t))$ are homeomorphic.

Proof of Theorem 3. If $2s=n$ or $2t=n$, then $C_n(s,t)$ will be a 3-regular graph and the result follows from Theorem 2. Assume $2s,2t\neq n$ and $2(s+t)=n$. We consider the following two cases.

Case 1. $|t-s|=\frac{n}{4}$.

Without loss of generality we can assume that $t>s$ and let $t-s=m$. Then $4m=n, s+t=2m, 2s=m$. Let $k\in[n]$. Now, $N(k+m)=\{s+k+m, t+k+m, n-s+k+m, n-t+k+m\}$ and $N(k)=\{s+k,t+k,n-s+k,n-t+k\}$. Here, $s+k+m=k+t$, $t+k+m=n-t+k$, $n-s+k+m\equiv m-s+k=s+k$ and $n-t+k+m=n-s+k$. Hence $N(k)=N(k+m)=N(k+2m)=N(k+3m)$ for all $k\in[n]$. For any $i,j\in[m], i\neq j$, it can be easily check that $N(i)\cap N(j)=\emptyset$.

Thus $\mathcal{N}(C_n(s,t))$ is consists of m disjoint simplices of dimension 3 and therefore it is homotopy equivalent to m distinct points.

Case 2. $|t-s|\neq\frac{n}{4}$.

Let $t+s=p$. Then $n=2p$. Since $n-t+k=n-t-s+s+k=p+s+k$, $n-s+k=p+t+k$, we see that $N(k)=N(k+p)$ for all $k\in[n]$. Since $N(s+k)=\{2s+k, s+t+k, k, 2s+p+k\}$ and $N(t+k)=\{s+t+k, 2t+k, 2t+p+k, k\}$, $N(s+k)\cap[p]=N(t+k)\cap[p]$ implies that $\{2s+k, 2s+p+k\}\cap[p]=\{2t+k,2t+p+k\}\cap[p]$. Since $s\neq t, 2s+k\not\equiv 2t+k$. However, $2s+k\equiv 2t+p+k$ implies that $3t\equiv s$. Since $t,s<\frac{n}{2}$, $3t=s$ or $3t=s+n$. If $3t=s$, then $p=4t$. But, then $s-t=2t=\frac{p}{2}=\frac{n}{4}$, which is a contradiction. If $3t=s+n=s+2t+2s$, then $3s=t$ and $p=4s$. Here, $t-s=2s=\frac{p}{2}=\frac{n}{4}$, which is not possible. By an argument similar as above, $2s+p+k\notin\{2t+k, 2t+p+k\}$. Hence $N(s+k)\cap[p]\neq N(t+k)\cap[p]\ \forall\ k\in[n]$.

Since $N(s+k) = N(p+s+k) = N(n-t+k), N(t+k) = N(p+t+k) = N(n-s+k)$, we conclude that each vertex k belongs to exactly two maximal simplices, namely $N(s+k)$ and $N(t+k)$. Further, $N(i) \cap [p] \neq N(j) \cap [p] \ \forall\ 1 \leqslant i \neq j \leqslant p$.

Observe that, $|N(k) \cap \{1, \ldots, p\}| = 2$ for all $k \in [n]$. Since $N(k) = N(k+p)$ for all $k \in [n]$, $C_n(s,t)$ folded onto the induced subgraph $C_n(s,t)[\{1,\ldots,p\}]$. Hence $\mathcal{N}(C_n(s,t)) \simeq \mathcal{N}(C_n(s,t)[\{1,\ldots,p\}])$, by Proposition 10. Since each vertex k of $C_n(s,t)$ belongs to exactly two maximal simplices $N(s+k) = N(p+s+k)$ and $N(t+k) = N(p+t+k)$ of $\mathcal{N}(C_n(s,t))$, and $N(i) \cap [p] \neq N(j) \cap [p]$ $1 \leqslant i \neq j \leqslant p$, we see that each vertex $x \in [p]$ also belongs to exactly two maximal simplices of $\mathcal{N}(C_n(s,t)[\{1,\ldots,p\}])$. Further, since $\mathcal{N}(C_n(s,t)[\{1,\ldots,p\}])$ is a 1-dimensional complex, the connected components of $\mathcal{N}(C_n(s,t)[\{1,\ldots,p\}])$ cannot be contractible. The result follows from Remark 9. □

Proof of Theorem 4(A). Let $3s = t$ and $n = 10s$. For each $1 \leqslant i \leqslant s$, let G_i be the subgraph of $C_n(s,t)$ induced by the vertex set $\{i, i+s, i+2s, i+3s, i+4s, i+5s, i+6s, i+7s, i+8s, i+9s\}$. Observe that each G_i is isomorphic to $C_{10}(1,3)$ and $C_n(s,t) \cong \bigsqcup_{i=1}^{s} G_i$. Therefore $\mathcal{N}(C_n(s,t)) \cong \bigsqcup_{i=1}^{s} \mathcal{N}(C_{10}(1,3))$. It can be easily verified that $\mathcal{N}(C_{10}(1,3))$ is homeomorphic to disjoint union of two copies of simplicial boundary of a 4-simplex, namely the subcomplex $N(1) \cup N(3) \cup N(5) \cup N(7) \cup N(9) = Bd(\{2,4,6,8,10\})$ and the subcomplex $N(2) \cup N(4) \cup N(6) \cup N(8) \cup N(10) = Bd(\{1,3,5,7,9\})$.

We conclude that $\mathcal{N}(C_n(s,t)) \cong \bigsqcup_{2s-copies} S^3$. The case $3t = s$ and $n = 10t$, follows by symmetry. □

Proof of Theorem 4(B). Let $3s = t$ and $n = 12s$. For each $1 \leqslant i \leqslant s$, let G_i be the subgraph of $C_n(s,t)$ induced by the vertex set $\{i, i+s, i+2s, i+3s, i+4s, i+5s, i+6s, i+7s, i+8s, i+9s, i+10s, i+11s\}$. Observe that each G_i is isomorphic to $C_{12}(1,3)$ and $C_n(s,t) \cong \bigsqcup_{i=1}^{s} G_i$. We now compute $\mathcal{N}(C_{12}(1,3))$.

For each $i \in \{1,2\}$, let A_i be the subcomplex of $\mathcal{N}(C_{12}(1,3))$, where the set of maximal simplices $M(A_i) = \{N(i), N(i+2), N(i+4), N(i+6), N(i+8), N(i+10)\}$. Then $\mathcal{N}(C_{12}(1,3)) = A_1 \sqcup A_2$. It can be easily checked that for each k, $(\{k+1, k+3, k+9\}, N(k))$ is a collapsible pair in $\mathcal{N}(C_{12}(1,3))$ and therefore $N(k) \searrow \tau_k^1 = \{k+1, k+3, k+11\}, \tau_k^2 = \{k+3, k+9, k+11\}$ and $\tau_k^3 = \{k+1, k+9, k+11\}$. Since, $\tau_{k+10}^1 = \{k+11, k+1, k+9\} = \tau_k^3$ for all $k \in [12]$, we see that $\mathcal{N}(C_{12}(1,3))$ collapses to a subcomplex Δ, where the set of maximal simplices $M(\Delta) = \{\tau_k^i \mid i \in \{1,2\}, k \in [12]\}$. For, $i \in \{1,2\}$, let $\Delta_i = \Delta \cap A_i$. Then, $\Delta = \Delta_1 \sqcup \Delta_2$. It can be easily verified that each Δ_i is a shellable complex and the shelling order is given by

$$\tau_i^1, \tau_i^2, \tau_{i+2}^1, \tau_{i+2}^2, \tau_{i+4}^1, \tau_{i+4}^2, \tau_{i+6}^2, \tau_{i+6}^1, \tau_{i+8}^1, \tau_{i+8}^2, \tau_{i+10}^1, \tau_{i+10}^2. \tag{1}$$

Here, τ_{i+8}^2 and τ_{i+10}^2 are spanning simplices with respect to the shelling order 1. Hence, $\Delta_i \simeq S^2 \vee S^2$ and by Proposition 8. We conclude that $\mathcal{N}(C_n(s,t)) \cong \bigsqcup_{2s-copies} S^2 \vee S^2$.

If $3t = s$ and $n = 12t$, the result follows by symmetry. □

Proof of Theorem 4(C). Let $3s = t$ and $8s, 10s, 12s \neq n$. Observe that, $2s, 4s, 6s, 8s, 12s \not\equiv 0$. For each $k \in [n], N(k) = \{s+k, 3s+k, n-s+k, n-3s+k\}$. Since $4s, 6s, 8s, 10s, 12s \not\equiv 0$, we see that $N(3s+k) \cap N(n-3s+k) = \{k\}$ and therefore $(\{3s+k, n-3s+k\}, N(k))$ is a collapsible pair. Hence, $N(k) \searrow \tau_k^1 = \{3s+k, s+k, n-s+k\}$ and $\tau_k^2 = \{n-3s+k, s+k, n-s+k\} \; \forall \; k \in [n]$. Since $\tau_{n-2s+k}^1 = \{s+k, n-s+k, n-3s+k\} = \tau_k^2$, $\mathcal{N}(C_n(s,t))$ collapses to a subcomplex Δ, where $M(\Delta) = \{\tau_k^1 \mid k \in [n]\}$.

Using the fact that $2s, 4s, 6s, 8s, 12s \not\equiv 0$, it can be easily check that $N(3s+k) \cap N(n-s+k) = \{2s+k, k\}$. Since $\{3s+k, n-s+k\} \nsubseteq \tau_{2s+k}^1 = \{5s+k, 3s+k, s+k\}$, we conclude that $(\{3s+k, n-s+k\}, \tau_k^1)$ is a collapsible pair. Hence, for each $k \in [n]$, $\tau_k^1 \searrow \delta_k^1 = \{3s+k, s+k\}$ and $\delta_k^2 = \{n-s+k, s+k\}$. Since $\delta_{n-2s+k}^1 = \{s+k, n-s+k\} = \delta_k^2$. Applying the collapsible pairs of the type $(\{3s+k, n-s+k\}, \tau_k^1)$ for each $k \in [n]$, Δ collapses to a 1-dimensional subcomplex Δ', where $M(\Delta') = \{\delta_k^1 \mid k \in [n]\}$.

Since each vertex $k \in [n]$ belongs to δ_{n-s+k}^1 and δ_{n-3s+k}^1, and $\delta_{n-s+k}^1 \neq \delta_{n-3s+k}^1$, connected components of Δ' cannot be contractible. Therefore, by Proposition 8, each connected component of Δ' is homotopy equivalent to wedge sum of circles. □

Proof of Theorem 5(A). Let $5s = 3t$ and $n = 4t$. Since $s \neq t, n \neq 4s$. Further, since $3s + t = n$ implies that $s = t$ and $3s - t = n$ implies that $3s = 5t = \frac{25}{3}s$, we see that $3s + t, 3s - t \neq n$. The result follows from Theorem 13. The case $5t = 3s$ and $n = 4s$, follows by symmetry. □

Proof of Theorem 5(B). Let $5s = 3t$ and $n = 4s$. Since 5 and 3 are relatively prime, there exists s' such that $s = 3s'$ and $t = 5s'$. Here, $n = 12s'$ and $N(k) = \{3s'+k, 5s'+k, 9s'+k, 7s'+k\} \; \forall \; k \in [n]$. For $i \in \{1, \ldots, 2s'\}$, let B_i be the subcomplex of $\mathcal{N}(C_n(s,t))$, where $M(B_i) = \{N(i + 2s'l) \mid l \in \{0,1,2,3,4,5\}\}$. It can be easily checked that the set of vertices of B_i is given by $V(B_i) = \{i+s', i+3s', i+5s', i+7s', i+9s', i+11s'\}$ and $V(B_l) \cap V(B_m) = \emptyset \; \forall \; 1 \leqslant l \neq m \leqslant 2s'$. Hence $B_l \cap B_m = \emptyset \; \forall \; 1 \leqslant l \neq m \leqslant 2s'$.

Let $k \in [n]$. Since $N(3s'+k) \cap N(5s'+k) \cap N(9s'+k) = \{k\}$, $(\{3s'+k, 5s'+k, 9s'+k\}, N(k))$ is a collapsible pair. Therefore $N(k) \searrow \tau_k^1 = \{3s'+k, 9s'+k, 7s'+k\}, \tau_k^2 = \{5s'+k, 9s'+k, 7s'+k\}$ and $\tau_k^3 = \{3s'+k, 5s'+k, 7s'+k\}$. Since, $\tau_{n-2s'+k}^2 = \{3s'+k, 7s'+k, 5s'+k\} = \tau_k^3$, we conclude that $\mathcal{N}(C_n(s,t))$ collapses to a subcomplex Δ, where $M(\Delta) = \{\tau_k^i \mid i \in \{1,2\}, k \in [n]\}$.

For $i \in \{1, \ldots, 2s'\}$, let $\Delta_i = \Delta \cap B_i$. Since $B_i \cap B_j = \emptyset, \Delta_i \cap \Delta_j = \emptyset \; \forall \; 1 \leqslant i \neq j \leqslant 2s'$. Further, $M(\Delta_i) = \{\tau_{i+2s'l}^j \mid j \in \{1,2\}, l \in \{0,1,2,3,4,5\}\} \; \forall \; 1 \leqslant i \leqslant 2s'$.

We now show that each Δ_i is a shellable complex and the shelling order is given by

$$\tau_i^1, \tau_i^2, \tau_{i+2s'}^1, \tau_{i+2s'}^2, \tau_{i+4s'}^1, \tau_{i+4s'}^2, \tau_{i+6s'}^1, \tau_{i+6s'}^2, \tau_{i+8s'}^1, \tau_{i+8s'}^2, \tau_{i+10s'}^1, \tau_{i+10s'}^2. \tag{2}$$

For each $i \in [n]$ and $0 \leqslant l \leqslant 5$, let $\tau_{i+2ls'} = \tau_i^1 \cup \tau_i^2 \cup \ldots \cup \tau_{i+2ls'}^1 \cup \tau_{i+2ls'}^2$. The following are easy to verify.

$\tau_i^1 = \{3s'+i, 9s'+i, 7s'+i\}, \tau_i^2 = \{5s'+i, 9s'+i, 7s'+i\}$ and $M(\tau_i^1 \cap \tau_i^2) = \{\{9s'+i, 7s'+i\}\}$.

$\tau_{i+2s'}^1 = \{5s'+i, 11s'+i, 9s'+i\}$ and $M(\tau_i \cap \tau_{i+2s'}^1) = \{\{5s'+i, 9s'+i\}\}$.

$\tau_{i+2s'}^2 = \{7s'+i, 11s'+i, 9s'+i\}$ and $M((\tau_i \cup \tau_{i+2s'}^1) \cap \tau_{i+2s'}^2) = \{\{11s'+i, 9s'+i\}, \{9s'+i, 7s'+i\}\}$.

$\tau^1_{i+4s'} = \{7s'+i, s'+i, 11s'+i\}$ and $M(\tau_{i+2s'} \cap \tau^1_{i+4s'}) = \{\{7s'+i, 11s'+i\}\}$.

$\tau^2_{i+4s'} = \{9s'+i, s'+i, 11s'+i\}$ and $M((\tau_{i+2s'} \cup \tau^1_{i+4s'}) \cap \tau^2_{i+4s'}) = \{\{s'+i, 11s'+i\}, \{9s'+i, 11s'+i\}\}$.

$\tau^1_{i+6s'} = \{9s'+i, 3s'+i, s'+i\}$ and $M(\tau_{i+4s'} \cap \tau^1_{i+6s'}) = \{\{3s'+i, 9s'+i\}, \{s'+i, 9s'+i\}\}$.

$\tau^2_{i+6s'} = \{11s'+i, 3s'+i, s'+i\}$ and $M((\tau_{i+4s'} \cup \tau^1_{i+6s'}) \cap \tau^2_{i+6s'}) = \{\{11s'+i, s'+i\}, \{3s'+i, s'+i\}\}$.

$\tau^1_{i+8s'} = \{11s'+i, 5s'+i, 3s'+i\}$ and $M(\tau_{i+6s'} \cap \tau^1_{i+8s'}) = \{\{11s'+i, 3s'+i\}, \{11s'+i, 5s'+i\}\}$.

$\tau^2_{i+8s'} = \{s'+i, 5s'+i, 3s'+i\}$ and $M((\tau_{i+6s'} \cup \tau^1_{i+8s'}) \cap \tau^2_{i+8s'}) = \{\{s'+i, 3s'+i\}, \{5s'+i, 3s'+i\}\}$.

$\tau^1_{i+10s'} = \{s'+i, 7s'+i, 5s'+i\}$ and $M(\tau_{i+8s'} \cap \tau^1_{i+10s'}) = Bd(\tau^1_{i+10s'})$.

$\tau^2_{i+10s'} = \{3s'+i, 7s'+i, 5s'+i\}$ and $M((\tau_{i+8s'} \cup \tau^1_{i+10s'}) \cap \tau^2_{i+10s'}) = Bd(\tau^2_{i+10s'})$.

Thus the order given in (2) is a shelling order and the spanning simplices are $\tau^1_{i+10s'}$ and $\tau^2_{i+10s'}$. The result follows by Proposition 8. The case $5t = 3s$ and $n = 4t$, follows by symmetry. □

We need the following lemma to prove Theorem 5 (C).

Lemma 12. *Let $s, t \in \{1, \ldots, n-1\}$ such that $2s, 2(s+t), 3s+t, 3s-t, 4s \not\equiv 0$ (mod n). Then for each $k \in [n]$, $(\{s+k, n-s+k\}, N(k))$ is a collapsible pair in $\mathcal{N}(C_n(s,t))$.*

Proof. If there exists $x \in [n], x \neq k$ such that $\{s+k, n-s+k\} \subset N(x)$, then $x \in \{2s+k, s+t+k, n-t+s+k\} \cap \{n-s+t+k, n-2s+k, n-t-s+k\}$.

Since, $0 \not\equiv 3s-t, 2s+k \neq n-s+t+k$. Further, since $2s+k = n-2s+k$ implies that $4s \equiv 0$ and $2s+k = n-t-s+k$ implies that $3s+t \equiv 0$, we conclude that $x \neq 2s+k$.

$s+t+k = n-s+t+k \implies 2s \equiv 0$ and $s+t+k = n-2s+k \implies 3s+t \equiv 0$. Further, since $s+t+k = n-t-s+k \implies 0 \equiv 2(s+t)$, $x \neq s+t+k$.

Since, $s \neq t$ and $3s-t \not\equiv 0$, we conclude that $n-t+s+k \neq n-s+t+k, n-2s+k, n-t-s+k$.

Thus there exists no $x \in [n]$ different from k such that $\{s+k, n-s+k\} \subset N(x)$ and therefore $(\{s+k, n-s+k\}, N(k))$ is a collapsible pair. □

Proof of Theorem 5(C). Let $5s = 3t$ and one of $6s$ or $\frac{14s}{3}$ is equal to n. There exists an integer s' such that $s = 3s'$ and $t = 5s'$.

Case 1. $n = 6s$.

In this case $n = 18s'$ and $N(k) = \{3s'+k, 5s'+k, 13s'+k, 15s'+k\}$ for all $k \in [n]$. Since $3s-t = n \implies \frac{4s}{3} = n$ and $3s+t = n \implies \frac{14s}{3} = n$, we see that $3s-t, 3s+t \neq n$. Further, $s, t < \frac{n}{2}$ implies that $3s-t, 3s+t \not\equiv n$. Using Lemma 12, $N(k) \searrow \tau^1_k = \{3s'+k, 5s'+k, 13s'+k\}$ and $\tau^2_k = \{15s'+k, 5s'+k, 13s'+k\}$ for all $k \in [n]$. Since, $\tau^1_{k+10s'} = \{13s'+k, 15s'+k, 5s'+k\} = \tau^2_k$, $\mathcal{N}(C_n(s,t))$ collapses to a subcomplex Δ, where $M(\Delta) = \{\tau^1_k \mid k \in [n]\}$.

Let $k \in [n]$. Observe that $N(3s'+k) \cap N(5s'+k) = \{k, 8s'+k\}$ and $\{3s'+k, 5s'+k\} \not\subseteq \tau^1_{8s'+k}$. Hence $(\{3s'+k, 5s'+k\}, \tau^1_k)$ is a collapsible pair in Δ and therefore $\tau^1_k \searrow \delta^1_k = \{3s'+k, 13s'+k\}$ and $\delta^2_k = \{5s'+k, 13s'+k\}$. Since $\delta^1_{k+10s'} = \{13s'+k, 5s'+k\} = \delta^2_k$, we see that Δ collapses to a 1-dimensional subcomplex Δ', with $M(\Delta') = \{\delta_k^1 \mid k \in [n]\}$. Each vertex $x \in [n]$, belongs to $\delta^1_{5s'+x} = \{8s'+x, x\}$ and $\delta^1_{15s'+x} = \{x, 10s'+x\}$. Since $\delta^1_{5s'+x} \neq \delta^1_{15s'+x}$, connected components of Δ' cannot be contractible. Result follows from Remark 9.

Case 2. $n = \frac{14s}{3}$.

In this case $n = 14s'$ and $N(k) = \{3s'+k, 5s'+k, 11s'+k, 9s'+k\}$ $\forall$ $k \in [n]$. Since $3t - s = n \implies 4s = n$ and $3t + s = n \implies 6s = n$, we see that $3t - s, 3t + s \neq n$. Further, since $4t \neq n$, by Lemma 12, $(\{5s'+k, 9s'+k\}, N(k))$ is a collapsible pair and therefore $N(k) \searrow \tau_k^1 = \{5s'+k, 3s'+k, 11s'+k\}$ and $\tau_k^2 = \{9s'+k, 3s'+k, 11s'+k\}$. Since, $\tau^1_{k+6s'} = \{11s'+k, 9s'+k, 3s'+k\} = \tau_k^2$, we conclude that $\mathcal{N}(C_n(s,t))$ collapses to a subcomplex Δ, where $M(\Delta) = \{\tau_k^1 \mid k \in [n]\}$.

$N(3s'+k) \cap N(5s'+k) = \{k, 8s'+k\}$ and $\{3s'+k, 5s'+k\} \nsubseteq \tau^1_{8s'+k}$ implies that $(\{3s'+k, 5s'+k\}, \tau_k^1)$ is a collapsible pair in Δ and therefore $\tau_k^1 \searrow \delta_k^1 = \{3s'+k, 11s'+k\}$ and $\delta_k^2 = \{5s'+k, 11s'+k\}$. Further, since $\delta^1_{8s'+k} = \{11s'+k, 5s'+k\} = \delta_k^2$, we conclude that Δ collapses to a 1-dimensional subcomplex Δ', where $M(\Delta') = \{\delta_k^1 \mid k \in [n]\}$.

Each vertex $x \in [n]$, belongs to $\delta^1_{3s'+x} = \{6s'+x, x\}$ and $\delta^1_{11s'+x} = \{x, 8s'+x\}$. Since $\delta^1_{3s'+x} \neq \delta^1_{11s'+x}$, the result follows from Remark 9.

If $5t = 3s$ and one of $6t$ or $\frac{14t}{3}$ is equal to n, the result follows by symmetry. ☐

Proof of Theorem 5(D). Let $5s = 3t$ and $n \neq 4s, 4t, 6s, \frac{14s}{3}$. Now, $3s - t = n \implies \frac{4s}{3} = n$, which is not possible as $s < \frac{n}{2}$. Further, since $3s + t = n \implies \frac{14s}{3} = n, 3t + s = n \implies 6s = n$ and $3t - s = n \implies 4s = n$, we see that $3s - t, 3s + t, 3t + s, 3t - s \neq n$. The result follows from Theorem 14. The case $5t = 3s$ and $n \neq 4s, 4t, 6t, \frac{14t}{3}$, follows by symmetry. ☐

Proof of Theorem 6(A). Let us first assume that one of the $3s - t, 3t - s, 3s + t$ or $3t + s$ is equal to n.

Case 1. $3t - s = n$ or $3s - t = n$.

Assume that $3t - s = n$. We consider the following two cases.

(i) $2s \neq t$.

In this case, $3s+t, 3s-t, 4s \not\equiv n$. By Lemma 12, $N(k) \searrow \tau_k^1 = \{s+k, t+k, n-t+k\}$ and $\tau_k^2 = \{n-s+k, t+k, n-t+k\}$ for all $k \in [n]$. Further, since $\tau_k^2 = \tau^1_{t-s+k}$, $\mathcal{N}(C_n(s,t))$ collapses to a subcomplex Δ, with $M(\Delta) = \{\tau_i^1 \mid i \in [n]\}$. It can be easily checked that $N(s+k) \cap N(n-t+k) = \{k, 2t+k\}$. Since, $\{s+k, n-t+k\} \nsubseteq \tau^1_{2t+k}$, $\tau_k^1 \searrow \delta_k^1 = \{s+k, t+k\}$ and $\delta_k^2 = \{n-t+k, t+k\}$ for all $k \in [n]$. Now, $\delta^1_{t-s+k} = \{t+k, n-t+k\} = \delta_k^2$ implies that Δ collapses to a 1-dimensional subcomplex Δ', where $M(\Delta') = \{\delta_i^1 \mid i \in [n]\}$.

Each vertex $k \in [n]$ can belongs to only δ_j^1 for $j \in \{s+k, t+k, n-s+k, n-t+k\}$. Since, $\delta^1_{s+k} = \{2s+k, s+t+k\}, \delta^1_{t+k} = \{s+t+k, 2t+k\}, \delta^1_{n-s+k} = \{k, n-s+t+k\}, \delta^1_{n-t+k} = \{n+s-t+k, k\}$, we observe that k belongs to only δ^1_{n-s+k} and δ^1_{n-t+k}. Further, since $\delta^1_{n-s+k} \neq \delta^1_{n-t+k}$ and Δ' is a 1-dimensional complex, each connected component of Δ' is homotopy equivalent to S^1.

(ii) $2s = t$.

In this case, $n = 5s$ and $N(k) = \{s+k, 2s+k, 3s+k, 4s+k\}$ $\forall$ $k \in [n]$. For $i \in [n]$, let $\Gamma_i := \{i, s+i \pmod n, 2s+i \pmod n, 3s+i \pmod n, 4s+i \pmod n\}$ be a 4-simplex. It can be easily checked that $\Gamma_i \cap \Gamma_j = \emptyset$ $\forall$ $1 \leqslant i \neq j \leqslant s$. The subcomplex of $\mathcal{N}(C_n(s,t))$ induced by the vertices $k, k+s, k+2s, k+3s$ and $k+4s$, which is equal to $N(k) \cup N(k+s) \cup N(k+2s) \cup N(k+3s) \cup N(k+4s)$ is $\text{Bd}(\Gamma_k)$. We conclude that $\mathcal{N}(C_n(s,t)) \cong \bigsqcup\limits_{k \in \{1,2,\ldots,s\}} Bd(\Gamma_k)$. Since $Bd(\Gamma_k)$ is homeomorphic to S^3, the result follows.

The case $3s - t = n$ follows from symmetry.

Case 2. $3s-t, 3t-s \neq n$, *i.e.*, $3t+s = n$ or $3s+t = n$.
Assume that $3t + s = n$.

(i) $2t \neq s$.

In this case, $3s+t, 3s-t, 4s \not\equiv n$. From Lemma 12, $N(k) \searrow \tau_k^1 = \{s+k, t+k, n-t+k\}$ and $\tau_k^2 = \{n-s+k, t+k, n-t+k\}$ for all $k \in [n]$. Further, since $\tau_k^2 = \tau_{2t+k}^1$, $\mathcal{N}(C_n(s,t))$ collapses to a subcomplex Δ, where $M(\Delta) = \{\tau_k^1 \mid k \in [n]\}$.

It can be easily verified that $N(s+k) \cap N(t+k) = \{k, s+t+k\}$. Since, $\{s+k, t+k\} \nsubseteq \tau_{s+t+k}^1 = \{2s+t+k, s+2t+k, s+k\}$, $\tau_k^1 \searrow \delta_k^1 = \{s+k, n-t+k\}$ and $\delta_k^2 = \{t+k, n-t+k\}$ for all $k \in [n]$. Further, since $\delta_{2t+k}^1 = \{2t+s+k, t+k\} = \{n-t+k, t+k\} = \delta_k^2$, Δ collapses to a 1-dimensional subcomplex Δ', where $M(\Delta') = \{\delta_i^1 \mid i \in [n]\}$.

Each vertex $k \in [n]$ can belong to only δ_j^1 for $j \in \{s+k, t+k, n-s+k, n-t+k\}$. Since, $\delta_{s+k}^1 = \{2s+k, n-t+s+k\}, \delta_{t+k}^1 = \{s+t+k, k\}, \delta_{n-s+k}^1 = \{k, 2t+k\}$ and $\delta_{n-t+k}^1 = \{n-t+s+k, t+s+k\}$, we conclude that k belongs to only δ_{t+k}^1 and δ_{n-s+k}^1. Since Δ' is a 1-dimensional complex, each connected component of Δ is homotopy equivalent to S^1.

(ii) $2t = s$.

Since $n = 3t+s$ and $s = 2t, n = 5t$. For each $k \in [n]$, let $\Gamma_k := \{k, t+k \pmod n, 2t+k \pmod n, 3t+k \pmod n, 4t+k \pmod n\}$ be a 4-simplex. It can be easily verified that $\Gamma_i \cap \Gamma_j = \emptyset$ $\forall$ $1 \leqslant i \neq j \leqslant t$ and the subcomplex of $\mathcal{N}(C_n(s,t))$ induced by the vertices $k, k+t, k+2t, k+3t$ and $k+4t$, is equal to $N(k) \cup N(k+t) \cup N(k+3t) \cup N(k+3t) \cup N(k+4t) = Bd(\Gamma_k)$. We conclude that $\mathcal{N}(C_n(s,t)) \cong \bigsqcup\limits_{k \in \{1,2,\ldots,t\}} Bd(\Gamma_k)$.

The case $3s + t = n$ follows by symmetry. □

Proof of Theorem 6 (B) follows from Theorem 13 and Proof of Theorem 6 (C) follows from Theorem 14.

Theorem 13. *Let $s, t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ such that $2s, 2t, 2(s+t), 3s+t, 3s-t, 4s \neq n$, $3s \neq t, 3t \neq s$. If $4t = n$, then each connected component of $\mathcal{N}(C_n(s,t))$ is homotopy equivalent to a garland of the 2-dimensional spheres S^2.*

Proof. By Lemma 12, $\mathcal{N}(C_n(s,t))$ collapses to a 2-dimensional subcomplex Δ, whose maximal simplices $M(\Delta) = \{\tau_k^i \mid i \in \{1,2\}, k \in [n]\}$, where $\tau_k^1 = \{s+k, t+k, n-t+k\}$ and $\tau_k^2 = \{n-s+k, t+k, n-t+k\}$. For any integer i, let $\Gamma_i := \{t+i \pmod n, n-t+i \pmod n, s+i \pmod n, s+2t+i \pmod n\}$ be a 3-simplex. Clearly, the simplicial complexes $\tau_k^1 \cup \tau_{s+t+k}^2 \cup \tau_{2t+k}^1 \cup \tau_{n-t+s+k}^2 = Bd(\Gamma_k)$ and $\tau_{n-s-t+k}^1 \cup \tau_k^2 \cup \tau_{n-s+t+k}^1 \cup \tau_{2t+k}^2 = Bd(\Gamma_{n-s-t+k})$. Hence, each 1-simplex of Δ is part of $\text{Bd}(\Gamma_i)$ for some $i \in [n]$. Thus $\Delta = \bigcup\limits_{1 \leqslant i \leqslant n} Bd(\Gamma_i)$.

In the rest of the proof, if we write an integer x as a vertex of Γ_i for some i, then it is understood that we are taking $x (\text{mod } n)$. Now $Bd(\Gamma_{n-s-t+k}) \cap Bd(\Gamma_k) = \{t+k, n-t+k\}$ and $Bd(\Gamma_k) \cap Bd(\Gamma_{k+s+t}) = \{s+k, s+2t+k\}$, *i.e.*, $Bd(\Gamma_k)$ shares a common 1-simplex $\{t+k, n-t+k\}$ with $Bd(\Gamma_{n-s-t+k})$ and a common 1-simplex $\{s+k, s+2t+k\}$ with Γ_{s+t+k}.

To show that each component of Δ is homotopy equivalent to a garland of the 2-dimensional spheres, it is enough to show that each vertex k belongs to exactly the boundaries of two 3-simplices. Let $x \in [n]$. There exists $k \in [n]$ such that $x \in Bd(\Gamma_k)$, *i.e.*, $x \in \{t+k, n-t+k, s+k, s+2t+k\}$. It is clear from the above discussion that there exists $k' \neq k$ such that $x \in Bd(\Gamma_{k'})$, *i.e.*, $x \in \{t+k', n-t+k', s+k', s+2t+k'\}$. Since $n = 4t$, $\Gamma_{k+2t} = \Gamma_k \ \forall \ k \in [n]$. Let $x = t+k$. If $x \equiv t+k'$, then $k \equiv k'$, a contradiction. If $x \equiv n-t+k'$, then $k' \equiv k-2t$. But $\Gamma_{k-2t} = \Gamma_k$. Since, $t+k \equiv s+k'$ implies that $k' \equiv t-s+k$ and $t+k \equiv s+2t+k'$ implies that $k' \equiv k-t-s$, we conclude that $x \in Bd(\Gamma_k)$ and $Bd(\Gamma_{n-s-t+k})$ only. By a similar argument as the one above, we can easily verify that, $n-t+k \in Bd(\Gamma_k) \cap Bd(\Gamma_{n-s-t+k})$, $s+k \in Bd(\Gamma_k) \cap Bd(\Gamma_{s+t+k})$ and $s+2t+k \in Bd(\Gamma_k) \cap Bd(\Gamma_{s+t+k})$ only.

Thus x belongs to either $Bd(\Gamma_k) \cap Bd(\Gamma_{n-s-t+k})$ or $Bd(\Gamma_k) \cap Bd(\Gamma_{s+t+k})$ only. ☐

A *d-dimensional pseudo manifold* is a pure d-dimensional simplicial complex such that every $(d-1)$-simplex is a face of exactly two d-simplices. A (topological) *n-manifold* is a hausdorff space X such that every point $x \in X$, has a neighborhood which is homeomorphic to $\mathbb{R}^n$. A 2-manifold is called a *surface.*

Theorem 14. *Let $s, t \in \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$ such that $2s, 2t, 2(s+t), 3s+t, 3t+s, 3s-t, 3t-s, 4s, 4t \neq n, 3s \neq t$ and $3t \neq s$. Then each connected component of $\mathcal{N}(C_n(s,t))$ is homotopy equivalent to connected sum of tori.*

We recall the following result to prove Theorem 14.

Proposition 15. *(Theorem 3A.3, [5])*
If C is a chain complex of free abelian groups, then there exist short exact sequences

$$0 \longrightarrow H_n(C;\mathbb{Z}) \otimes \mathbb{Z}_2 \longrightarrow H_n(C;\mathbb{Z}_2) \longrightarrow Tor(H_{n-1}(C;\mathbb{Z}), \mathbb{Z}_2) \longrightarrow 0$$

for all n and these sequences split.

Proof of Theorem 14. By Lemma 12, $\mathcal{N}(C_n(s,t))$ collapses to a subcomplex X, with $M(X) = \{\tau_k^i \mid i \in \{1,2\}, k \in [n]\}$, where $\tau_k^1 = \{s+k, t+k, n-t+k\}$ and $\tau_k^2 = \{n-s+k, t+k, n-t+k\}$.

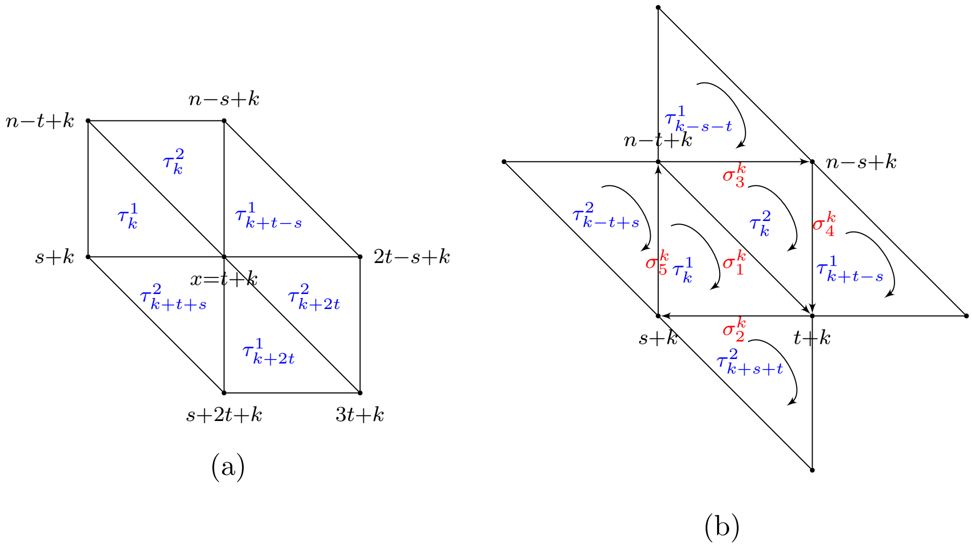


Figure 2

Claim 16. *X is a 2-dimensional pseudo manifold.*

Proof of Claim 16. It is enough to show that any 1-dimensional face of τ_k^1 and τ_k^2, *i.e.*, $\sigma_1^k = \{t+k, n-t+k\}, \sigma_2^k = \{s+k, t+k\}, \sigma_3^k = \{n-t+k, n-s+k\}, \sigma_4^k = \{t+k, n-s+k\}$ and $\sigma_5^k = \{s+k, n-t+k\}$ are faces of exactly two 2-dimensional simplices of X.

Clearly, σ_1^k is a face of τ_k^1 and τ_k^2. By Lemma 12, $(\sigma_1^k, N(k))$ is a collapsible pair and therefore σ_1^k is not a face of any $\tau_{k'}^1$ or $\tau_{k'}^2$ for all $k' \neq k$.

$\sigma_2^k \subset \tau_{k+s+t}^2 = \{k+t, s+2t+k, s+k\}$. Since $s, t < \frac{n}{2}, s+2t+k \equiv n-t+k$ implies that $3t+s = n$, which is not possible. Hence $\tau_{k+s+t}^2 \neq \tau_k^1$. Further, $3s-t$ and $3t-s \neq n$ implies that $N(s+k) \cap N(t+k) = \{k, s+t+k\}$. Since, $\sigma_2^k \not\subseteq \tau_{s+t+k}^1$, σ_2^k is a face of τ_k^1 and τ_{k+s+t}^2 only. Since $n-t+k \equiv s+(k-s-t)$ and $n-s+k \equiv t+(k-s-t)$, we see that, σ_3^k is a face of τ_{k-s-t}^1 and τ_k^2 only.

$\sigma_4^k \subset \tau_{k+t-s}^1 = \{t+k, 2t-s+k, n-s+k\}$ and $\sigma_4^k \not\subseteq \tau_{k+t-s}^2$. Since $2t-s+k \equiv n-t+k$ implies that $3t-s = n$, we see that $\tau_{k+t-s}^1 \neq \tau_k^2$. Further, $2(s+t), 3s+t, 3t+s \neq n$ implies that $N(t+k) \cap N(n-s+k) = \{k, k-t+s\}$. Thus σ_4^k is a face of τ_k^2 and τ_{k+t-s}^1 only. Since $s+k = t+(k-t+s)$ and $n-t+k = n-s+(k-t+s)$, we see that, σ_5^k is a face of τ_{k-t+s}^2 and τ_k^1 only. ☐

Claim 17. *$||X||$, the geometric realization of X, is a surface, i.e., a 2-manifold.*

Proof of Claim 17. Let $x \in ||X||$. Since, X is a pseudomanifold, if x belongs to interior of some 1-simplex or 2-simplex, then we can easily construct an open neighborhood of x, homeomorphic to $\mathbb{R}^2$. Assume x is a 0-simplex of X. Without loss of generality, we can assume that $x = t+k$, for some $k \in [n]$. Then $x \in \tau_k^1, \tau_k^2, \tau_{k+t-s}^2, \tau_{k+2t}^2, \tau_{k+2t}^1$ and τ_{k+t+s}^2(see Figure 2a). Using the fact that $2(s+t), 3t-s, 3t+s, 4t \not\equiv 0$, it can be easily checked that $|\{s+k, n-t+k, n-s+k, 2t-s+k, 3t+k, s+2t+k\}| = 6$. Since

$N(x) = \{k, k+t-s, k+2t, k+s+t\}$, $x \notin \tau_m^1$ or τ_m^2 for all $m \neq k, k+t-s, k+2t, k+s+t$. Hence we can easily construct a neighborhood of x in $||X||$, which is homeomorphic to $\mathbb{R}^2$. Thus X is a surface. $\square$

We now define the orientations for simplices of X. For any oriented simplex σ we write $+\sigma$, if it is positively oriented and $-\sigma$, otherwise. We denote any positively oriented 2-simplex with vertex set $\{a, b, c\}$ by $\langle a, b, c\rangle$ and 1-simplex with vertex set $\{a, b\}$ by $\langle a, b\rangle$.

For any $k \in [n]$, we define $+\tau_k^1 = \langle s+k, n-t+k, t+k\rangle$ and $+\tau_k^2 = \langle n-t+k, n-s+k, t+k\rangle$. Further, we define $+\sigma_1^k = \langle n-t+k, t+k\rangle, +\sigma_2^k = \langle t+k, s+k\rangle, +\sigma_3^k = \langle n-t+k, n-s+k\rangle, +\sigma_4^k = \langle n-s+k, t+k\rangle$ and $+\sigma_5^k = \langle s+k, n-t+k\rangle$. Observe that any 1-simplex of X is equal to σ_i^k for some $k \in [n]$ and $i \in \{1, 2, 3, 4, 5\}$.

Let $C = (C_i, \partial_i)$ be the simplicial chain complex of X with coefficients in $\mathbb{Z}_2$. Since, there are n 0-simplices in $X, C_0 \cong \mathbb{Z}_2^n$. Further, since there are $2n$ simplices of dimension 2 and X is a pseudo manifold, $C_1 \cong \mathbb{Z}_2^{3n}$ and $C_2 \cong \mathbb{Z}_2^{2n}$. Since there is no simplex of any other dimension in X, $C_i = 0$ for all $i \neq 1, 2, 3$. Thus

$$C = 0 \longrightarrow \mathbb{Z}_2^{2n} \xrightarrow{\partial_2} \mathbb{Z}_2^{3n} \xrightarrow{\partial_1} \mathbb{Z}_2^{n} \xrightarrow{\partial_0} 0$$

Let p be the number of connected components of X. It is well known that $H_0(X;\mathbb{Z}_2) \cong \mathbb{Z}_2^p$. Since $p \geqslant 1$, $\text{Rank}(\partial_1) \leqslant n-1$. If $\text{Ker } \partial_1 \cong \mathbb{Z}_2^r$, then $r \geqslant 3n-n+1 = 2n+1$. Since $\text{Rank}(\partial_2)$ can be at most $2n$, $H_1(X;\mathbb{Z}_2) \neq 0$. From Proposition 15, $H_1(X;\mathbb{Z}_2) \cong H_1(X;\mathbb{Z}) \otimes \mathbb{Z}_2$ $\oplus \text{Tor}(H_0(X;\mathbb{Z}), \mathbb{Z}_2)$. Since $H_0(X;\mathbb{Z}) \cong \mathbb{Z}^p$, $\text{Tor}(H_0(X);\mathbb{Z}_2) = 0$. So $H_1(X;\mathbb{Z}) = 0$, implies that $H_1(X;\mathbb{Z}_2) = 0$, which is a contradiction. Hence $H_1(X;\mathbb{Z}) \neq 0$.

Let $D = (D_i, d_i)$ be the simplicial chain complex of X with $\mathbb{Z}$ coefficients. Then

$$D = 0 \longrightarrow \mathbb{Z}^{2n} \xrightarrow{d_2} \mathbb{Z}^{3n} \xrightarrow{d_1} \mathbb{Z}^{n} \xrightarrow{d_0} 0$$

Let $c = \sum_{k \in [n]} ((+\tau_k^1) + (+\tau_k^2))$ be a 2-chain. It can be easily verified (see Figure 2b) that σ_1^k has +ve orientation in τ_k^1 and −ve orientation in τ_k^2. σ_2^k has +ve orientation in τ_k^1 and −ve in τ_{k+s+t}^2. Similarly σ_3^k has +ve orientation in τ_k^2 and −ve in τ_{k-t-s}^1. The simplex σ_4^k having +ve orientation in τ_k^2 and −ve in τ_{k+t-s}^1, and σ_5^k has +ve orientation in τ_k^1 and −ve in τ_{k-t+s}^2. Since X is a pseudo manifold, each 1-simplex of X will occur twice in $d_2(c)$, once with +ve sign and once with −ve sign. Hence $d_2(c) = 0$. Since $\tau_k^i \neq 0$ in $D_2 = \mathbb{Z}^{2n}, 0 \neq c \in \text{Ker } d_2$. Hence $H_2(X;\mathbb{Z}) \neq 0$.

From Claim 17, each component of X is a compact surface. From the classification of surfaces, we know that any connected compact surface is homeomorphic either to S^2, to a connected sum of tori or to a connected sum of projective planes.

Since $H_1(X;\mathbb{Z}) \neq 0$, the connected components of X cannot be homeomorphic to S^2. Further, since $H_2(S;\mathbb{Z}) = 0$ for any non orientable surface S, the connected components of X cannot be homeomorphic to a connected sum of projective planes. Hence each connected component of X is homeomorphic to a connected sum of tori. $\square$

Let Δ be a simplicial complex. An *m-path* in Δ is a sequence $\sigma_1 \ldots \sigma_t, t \geqslant 2$ of m-simplices such that σ_i and σ_{i+1} have a common $(m-1)$-dimensional face, for all $1 \leqslant i \leqslant$

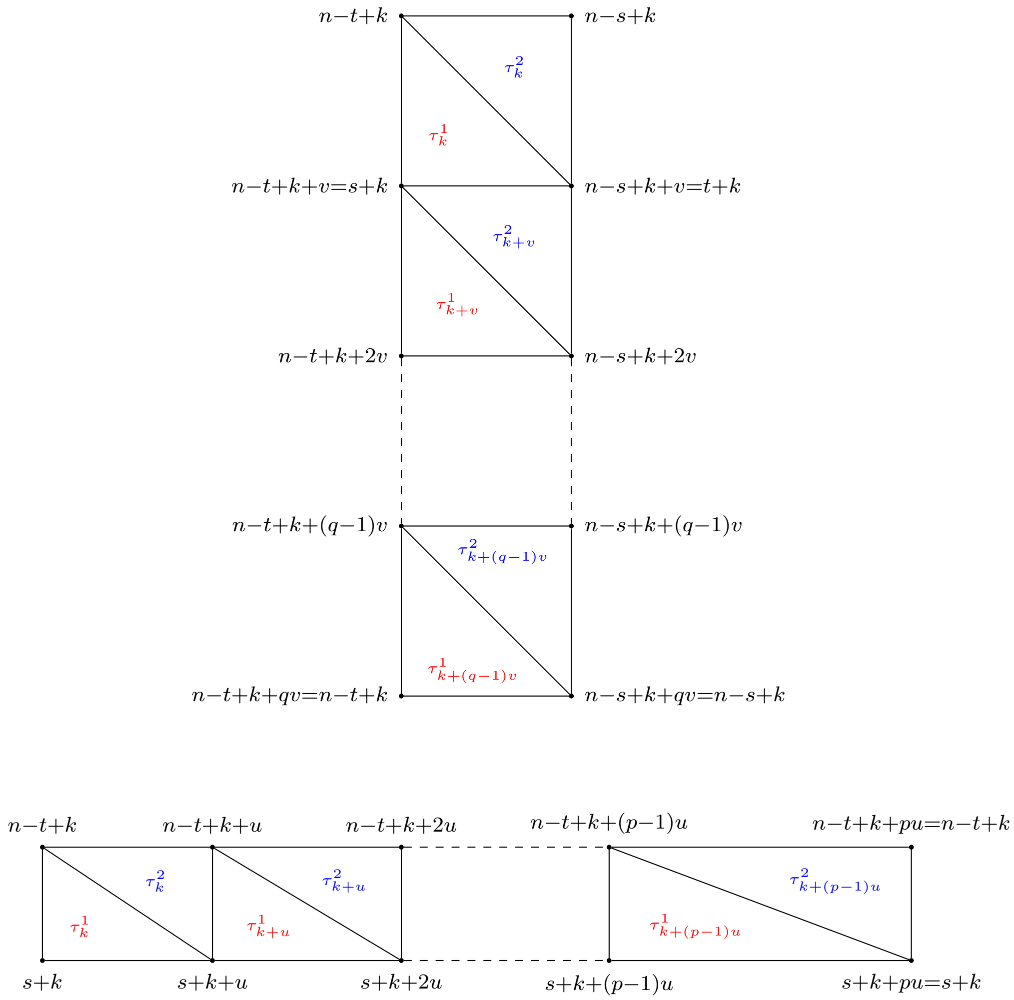


Figure 3

$t-1$. Further, if σ_1 and σ_t have a common $(m-1)$-dimensional face, then it is said to be closed m-path.

Proof of Theorem 7. Using Lemma 12, $\mathcal{N}(C_n(s,t))$ collapses to a subcomplex X with $M(X) = \{\tau_k^i \mid i \in \{1,2\}, k \in [n]\}$, where $\tau_k^1 = \{s+k, t+k, n-t+k\}$ and $\tau_k^2 = \{n-s+k, t+k, n-t+k\}$. Let $u = t-s$ and $v = t+s$. Let $k \in [n]$. Since, $s+u+k = t+k$ and $n-t+u+k = n-s+k$, we see that τ_k^2 and τ_{k+u}^1 have a common 1-simplex $\{n-s+k, t+k\}$. Further, $n-s+k+v = t+k$ and $n-t+k+v = s+k$ implies that τ_k^1 and τ_{k+v}^2 have a common 1-simplex $\{s+k, t+k,\}$. Since τ_k^1 and τ_k^2 have a common 1-simplex $\{t+k, n-t+k\}$, we conclude that $\tau_k^1\tau_k^2$, $\tau_k^2\tau_{k+u}^1$ and $\tau_k^1\tau_{k+v}^2$ are 2-paths in X.

(i) $s = \frac{p-q}{2}$ and $t = \frac{p+q}{2}$.

In this case $q = t - s = u$ and $p = t + s = v$. Now, $\omega = \tau^1_k \tau^2_k \tau^1_{k+u} \tau^2_{k+u} \cdots \tau^1_{k+(p-1)u} \tau^2_{k+(p-1)u}$ and $\Gamma = \tau^2_k \tau^1_k \tau^2_{k+v} \tau^1_{k+v} \cdots \tau^2_{k+(q-1)v} \tau^1_{k+(q-1)v}$ are 2-paths in X. Since $k+pu$ and $k+qv \equiv k \pmod{n}$, ω (see the horizontal rectangular strip of Figure 3) and Γ (see the vertical rectangular strip of Figure 3) are closed paths in X. Thus the simplices of X can be arranged in a rectangular grid of order $(q+1) \times (p+1)$ as depicted in Figure 4. Since X has $2n$ 1-dimensional simplices, to prove that this rectangular grid gives a triangulation of a torus, it is enough to show that there is no identification among the vertices other than that shown in Figure 4. Any vertex of $C_n(s,t)$ can be written as $n-t+k$ for some $k \in [n]$ and therefore can be made as the left uppermost corner vertex $a_{1,1}$ of this grid. Hence, it is enough to show that $n - t + k = a_{1,1} \neq a_{i,j}$ unless $i \in \{1, q+1\}$ and $j \in \{1, p+1\}$, *i.e.*, $n-t+k$ lies only on the four corners of the grid. But, since $a_{i,j} = (i-1)p + n - t + k + (j-1)q$, $a_{i,j} = n - t + k$ implies that $(i-1)p + (j-1)q \equiv 0 \pmod{n}$. Further, since $gcd(p,q) = 1$, this is only possible if $i \in \{1, q+1\}$ and $j \in \{1, p+1\}$.

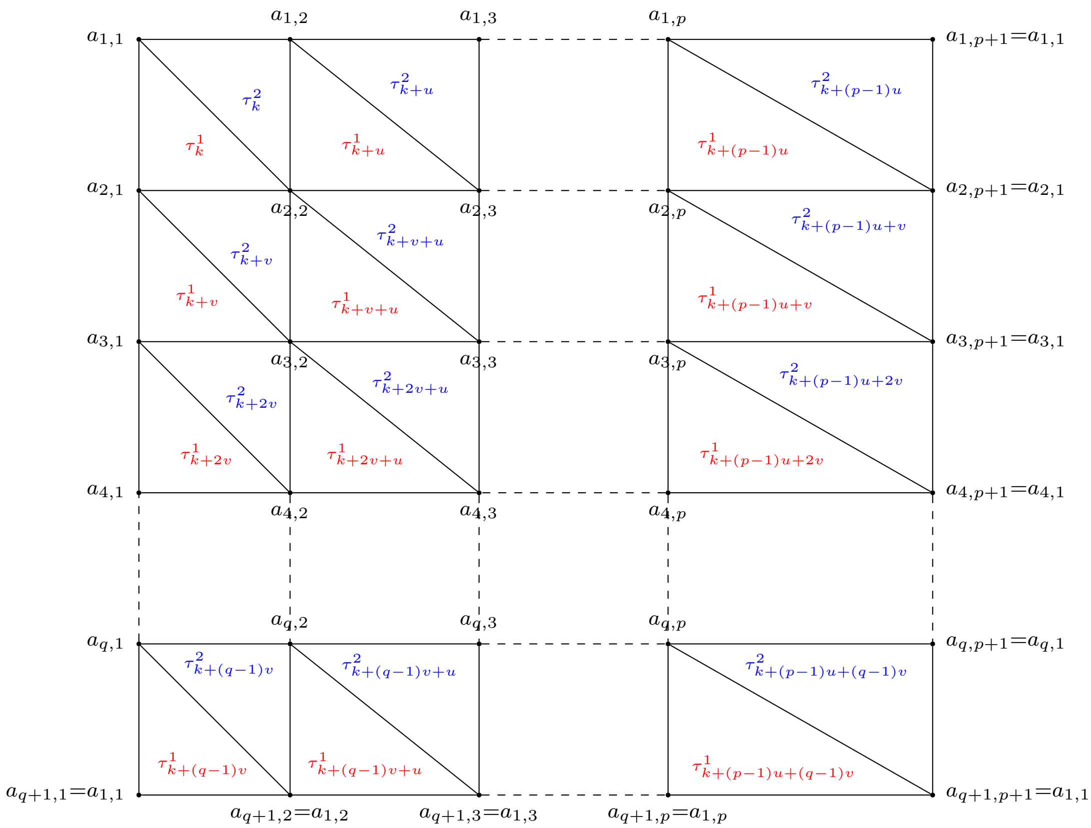


Figure 4: $u{=}t{-}s, v{=}t{+}s, a_{i,j}{=}(i{-}1)v{+}n{-}t{+}k{+}(j{-}1)u, 1{\leqslant}i{\leqslant}q{+}1, 1{\leqslant}j{\leqslant}p{+}1.$

(ii) $s = \frac{p^2-q}{2}$ and $t = \frac{p^2+q}{2}$.

In this case, $v = s + t = p^2$ and $u = t - s = q$. Since $pq = n$, we have the closed 2-paths $\tau^1_k \tau^2_k \tau^1_{k+u} \tau^2_{k+u} \cdots \tau^1_{k+(p-1)u} \tau^2_{k+(p-1)u}$ and $\tau^2_k \tau^1_k \tau^2_{k+v} \tau^1_{k+v} \cdots \tau^1_{k+(q-1)v} \tau^1_{k+(q-1)v}$.

In this case also, we can arrange the simplices of X in a rectangular grid of order $(q+1) \times (p+1)$, as depicted in Figure 4. By an argument similar as of the case (i), to prove that this rectangular grid gives a triangulation of a torus, it is enough to show that $n - t + k = a_{1,1} \neq a_{i,j}$ unless $i \in \{1, q+1\}$ and $j \in \{1, p+1\}$. But, since $a_{i,j} = (i-1)v + n - t + k + (j-1)u$, $a_{i,j} = n - t + k$ implies that $(i-1)v + (j-1)u \equiv 0 \pmod{n}$. Since $v = p^2, u = q, (i-1)p^2 + (j-1)q \equiv 0 \pmod{n}$. This is only possible when $i \in \{1, q+1\}$ and $j \in \{1, p+1\}$ as $gcd(p,q) = 1$. □

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