

Fractional L -intersecting families

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Abstract

Let $L = \{\frac{a_1}{b_1}, \dots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0, 1)$ is an irreducible fraction. Let $\mathcal{F} = \{A_1, \dots, A_m\}$ be a family of subsets of $[n]$. We say \mathcal{F} is a *fractional L -intersecting family* if for every distinct $i, j \in [m]$, there exists an $\frac{a}{b} \in L$ such that $|A_i \cap A_j| \in \{\frac{a}{b}|A_i|, \frac{a}{b}|A_j|\}$. In this paper, we introduce and study the notion of fractional L -intersecting families.

Mathematics Subject Classifications: 05D05, 05C50, 05C65

1 Introduction

Let $[n]$ denote $\{1, \dots, n\}$ and let $L = \{l_1, \dots, l_s\}$ be a set of s non-negative integers. A family $\mathcal{F} = \{A_1, \dots, A_m\}$ of subsets of $[n]$ is *L -intersecting* if for every $A_i, A_j \in \mathcal{F}$

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with $A_i \neq A_j$, we have $|A_i \cap A_j| \in L$. In 1975, it was shown by Ray-Chaudhuri and Wilson in [13] that if \mathcal{F} is t -uniform, then $|\mathcal{F}| \leq \binom{n}{s}$. Setting $L = \{0, \dots, s-1\}$, the family $\mathcal{F} = \binom{[n]}{s}$ is a tight example to the above bound, where $\binom{[n]}{s}$ denotes the set of all s -sized subsets of $[n]$. In the non-uniform case, it was shown by Frankl and Wilson in 1981 (see [7]) that if we don't put any restrictions on the cardinalities of the sets in \mathcal{F} , then $|\mathcal{F}| \leq \binom{n}{s} + \binom{n}{s-1} + \dots + \binom{n}{0}$. This bound is tight as demonstrated by the set of all subsets of $[n]$ of size at most s with $L = \{0, \dots, s-1\}$. The proof of this bound was using the method of higher incidence matrices. Later, in 1991, Alon, Babai, and Suzuki in [2] gave an elegant linear algebraic proof of this bound. They showed that if the cardinalities of the sets in \mathcal{F} belong to the set of integers $K = \{k_1, \dots, k_r\}$ with every $k_i > s-r$, then $|\mathcal{F}|$ is at most $\binom{n}{s} + \binom{n}{s-1} + \dots + \binom{n}{s-r+1}$. The collection of all the subsets of $[n]$ of size at least $s-r+1$ and at most s with $K = \{s-r+1, \dots, s\}$ and $L = \{0, \dots, s-1\}$ forms a tight example to this bound. In 2002, this result was extended by Grolmusz and Sudakov [8] to k -wise L -intersecting families. In 2003, Snevily showed in [14] that if L is a collection of s positive integers then $|\mathcal{F}| \leq \binom{n-1}{s} + \binom{n-1}{s-1} + \dots + \binom{n-1}{0}$. See [11] for a survey on L -intersecting families and their variants.

In this paper, we introduce a new variant of L -intersecting families called the fractional L -intersecting families. Let $L = \{\frac{a_1}{b_1}, \dots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0, 1)$ is an irreducible fraction. Let $\mathcal{F} = \{A_1, \dots, A_m\}$ be a family of subsets of $[n]$. We say \mathcal{F} is a *fractional L -intersecting family* if for every distinct $i, j \in [m]$, there exists an $\frac{a}{b} \in L$ such that $|A_i \cap A_j| \in \{\frac{a}{b}|A_i|, \frac{a}{b}|A_j|\}$. When \mathcal{F} is t -uniform, it is an L' -intersecting family where $L' = \{\lfloor \frac{a_1 t}{b_1} \rfloor, \dots, \lfloor \frac{a_s t}{b_s} \rfloor\}$ and therefore (using the result in [13]), $|\mathcal{F}| \leq \binom{n}{s}$. A tight example to this bound is given by the family $\mathcal{F} = \binom{[n]}{t}$ where $L = \{\frac{0}{t}, \dots, \frac{t-1}{t}\}$. So what is interesting is finding a good upper bound for $|\mathcal{F}|$ in the non-uniform case. Unlike in the case of the classical L -intersecting families, it is clear from the above definition that if A and B are two sets in a fractional L -intersecting family, then the cardinality of their intersection is a function of $|A|$ or $|B|$ (or both).

In Section 2.1, we prove the following theorem which gives an upper bound for the cardinality of a fractional L -intersecting family in the general case. We follow the convention that $\binom{a}{b}$ is 0, when $b > a$.

Theorem 1. *Let n be a positive integer. Let $L = \{\frac{a_1}{b_1}, \dots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0, 1)$ is an irreducible fraction. Let \mathcal{F} be a fractional L -intersecting family of subsets of $[n]$. Then, $|\mathcal{F}| \leq 2 \binom{n}{s} g^2(t, n) \ln(g(t, n)) + (\sum_{i=1}^{s-1} \binom{n}{i}) g(t, n)$, where $g(t, n) = \frac{2(2t+\ln n)}{\ln(2t+\ln n)}$ and $t = \max(s, \max(b_i : i \in [s]))$. Further,*

- (a) *if $s \leq n+1-2g(t, n) \ln(g(t, n))$, then $|\mathcal{F}| \leq 2 \binom{n}{s} g^2(t, n) \ln(g(t, n))$, and*
- (b) *if $t > n - c_1$, where c_1 is a positive integer constant, then*

$$|\mathcal{F}| \leq 2c_1 \binom{n}{s} g(t, n) \ln(g(t, n)) + c_1 \sum_{i=1}^{s-1} \binom{n}{i}.$$

Consider the following examples for a fractional L -intersecting family.

Example 2. Let $L = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots, \frac{1}{n}, \dots, \frac{n-1}{n}\}$, where we omit fractions, like $\frac{2}{4}$, which are not irreducible. The collection of all the non-empty subsets of $[n]$ is a fractional

L -intersecting family of cardinality $2^n - 1$. Here, $|L| = s = \Theta(n^2)$. Since $t \geq s$, we can apply Statement (b) of Theorem 1 to get an upper bound of $c_1(2^n - 1)$ which is asymptotically tight. In general, when $L = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \dots, \frac{1}{n-c}, \dots, \frac{n-c-1}{n-c}\}$, where $c \geq 0$ is a constant, the set of all the non-empty subsets of $[n]$ of cardinality at most $n - c$ is an example which demonstrates that the bound given in Statement (b) of Theorem 1 is asymptotically tight.

Example 3. Let us now consider another example where $s (= |L|)$ is a constant. Let $L = \{\frac{0}{s}, \frac{1}{s}, \dots, \frac{s-1}{s}\}$. The collection of all the s -sized subsets of $[n]$ is a fractional L -intersecting family of cardinality $\binom{n}{s}$. In this case, the bound given by Theorem 1 is asymptotically tight up to a factor of $\frac{\ln^2 n}{\ln \ln n}$. We believe that if \mathcal{F} is a fractional L -intersecting family of maximum cardinality, where $s (= |L|)$ is a constant, then $|\mathcal{F}| \in \Theta(n^s)$.

Coming back to the classical L -intersecting families, it is known that when \mathcal{F} is an L -intersecting family where $|L| = s = 1$, the Fisher's Inequality (see Theorem 7.5 in [9]) yields $|\mathcal{F}| \leq n$. Study of such intersecting families was initiated by Ronald Fisher in 1940 (see [5]). This fundamental result of design theory is among the first results in the field of L -intersecting families. Analogously, consider the scenario when $L = \{\frac{a}{b}\}$ is a singleton set. Can we get a tighter bound (compared to Theorem 1) in this case? We show in Theorem 4 that if b is a constant prime we do have a tighter bound.

Theorem 4. *Let n be a positive integer. Let \mathcal{G} be a fractional L -intersecting families of subsets of $[n]$, where $L = \{\frac{a}{b}\}$, $\frac{a}{b} \in [0, 1)$, and b is a prime. Then, $|\mathcal{G}| \leq (b - 1)(n + 1) \lceil \frac{\ln n}{\ln b} \rceil + 1$.*

Assuming $L = \{\frac{1}{2}\}$, Examples 11 and 12 in Section 3 give fractional L -intersecting families on $[n]$ of cardinality $\frac{3n}{2} - 2$ thereby implying that the bound obtained in Theorem 4 is asymptotically tight up to a factor of $\ln n$ when b is a constant prime. We believe that the cardinality of such families is at most cn , where $c > 0$ is a constant.

The rest of the paper is organized in the following way: In Section 2.1, we give the proof of Theorem 1 after introducing some necessary lemmas in the beginning. In Theorem 9 in Section 2.2, we demonstrate that any fractional L -intersecting families on $[n]$ whose member sets are 'large enough' has size at most n . In Section 3, we consider the case when L is a singleton set and give the proof of Theorem 4. Later in this section, in Theorem 14, we consider the case when the cardinalities of the sets in the fractional L -intersecting family are restricted. Finally, we conclude with some remarks, some open questions, and a conjecture.

2 The general case

2.1 Proof of Theorem 1

Before we move to the proof of Theorem 1, we introduce a few lemmas that will be used in the proof.

2.1.1 A few auxiliary lemmas

The following lemma is popularly known as the ‘Independence Criterion’ or ‘Triangular Criterion’.

Lemma 5 (Lemma 13.11 in [9], Proposition 2.5 in [3]). *For $i = 1, \dots, m$ let $f_i : \Omega \rightarrow \mathbb{F}$ be functions and $v_i \in \Omega$ elements such that*

- (a) $f_i(v_i) \neq 0$ for all $1 \leq i \leq m$;
- (b) $f_i(v_j) = 0$ for all $1 \leq j < i \leq m$.

Then f_1, \dots, f_m are linearly independent members of the space \mathbb{F}^Ω .

Lemma 6. *Let p be a prime; $\Omega = \{0, 1\}^n$. Let $f \in \mathbb{F}_p^\Omega$ and let $i \in \mathbb{F}_p$. For any $A \subseteq [n]$, let $V_A \in \{0, 1\}^n$ denote its 0-1 incidence vector and let $x_A = \prod_{j \in A} x_j$. Assume $f(V_A) \neq 0$, for every $|A| \not\equiv i \pmod{p}$. Then, the set of functions $\{x_A f : |A| \not\equiv i \pmod{p} \text{ and } |A| < p\}$ is linearly independent in the vector space $\mathbb{F}_p^{\{0,1\}^n}$ over \mathbb{F}_p .*

Proof. Arrange every subset of $[n]$ of cardinality less than p in a linear order, say \prec , such that $A \prec B$ implies $|A| \leq |B|$. For any two distinct sets A and B with $|B| \leq |A|$, we know that $x_A(V_B)f(V_B) = 0$, where $x_A(V_B)$ denote the evaluation of the function x_A at V_B . Suppose $\sum_{A: |A| \not\equiv i \pmod{p}, |A| < p} \lambda_A x_A f = 0$ has a non-trivial solution. Then, identify the first set, say A_0 , in the linear order \prec for which λ_{A_0} is non-zero. Evaluate the functions on either side of the above equation at V_{A_0} to get $\lambda_{A_0} = 0$ which is a contradiction to our assumption. \square

The following lemma is from [3] (see Lemma 5.38).

Lemma 7 (Lemma 5.38 in [3]). *Let p be a prime; $\Omega = \{0, 1\}^n$. Let $f \in \mathbb{F}_p^\Omega$ be defined as $f(x) = \sum_{i=1}^n x_i - k$. For any $A \subseteq [n]$, let $V_A \in \{0, 1\}^n$ denote its 0-1 incidence vector and let $x_A = \prod_{j \in A} x_j$. Assume $0 \leq s, k \leq p-1$ and $s+k \leq n$. Then, the set of functions $\{x_A f : |A| \leq s-1\}$ is linearly independent in the vector space \mathbb{F}_p^Ω over \mathbb{F}_p .*

2.1.2 The proof

Proof of Theorem 1. Let p be a prime with $p > t$. We partition \mathcal{F} into p parts, namely $\mathcal{F}_0, \dots, \mathcal{F}_{p-1}$, where $\mathcal{F}_i = \{A \in \mathcal{F} : |A| \equiv i \pmod{p}\}$.

Estimating $|\mathcal{F}_i|$, when $i > 0$.

Let $\mathcal{F}_i = \{A_1, \dots, A_m\}$ and let V_1, \dots, V_m denote their corresponding 0-1 incidence vectors. Define m functions f_1 to f_m , where each $f_j \in \mathbb{F}_p^{\{0,1\}^n}$, in the following way.

$$f_j(x) = (\langle V_j, x \rangle - \frac{a_1}{b_1}i)(\langle V_j, x \rangle - \frac{a_2}{b_2}i) \cdots (\langle V_j, x \rangle - \frac{a_s}{b_s}i).$$

Note that since $|A_j| \equiv i \pmod{p}$, $\langle V_j, V_j \rangle \equiv i \pmod{p}$. Since $p > t$, for every $l \in [s]$, $i \not\equiv \frac{a_l}{b_l} i \pmod{p}$ unless $i \equiv 0 \pmod{p}$. So,

$$f_j(x) \begin{cases} \neq 0, & \text{if } x = V_j \\ = 0, & \text{otherwise.} \end{cases} \quad (1)$$

So, f_j 's are linearly independent in the vector space $\mathbb{F}_p^{\{0,1\}^n}$ over \mathbb{F}_p (by Lemma 5). Since $x = (x_1, x_2, \dots, x_n) \in \{0, 1\}^n$, $x_i^r = x_i$, for any positive integer r . Each f_j is thus an appropriate linear combination of distinct monomials of degree at most s . Therefore, $|\mathcal{F}_i| = m \leq \sum_{j=0}^s \binom{n}{j}$. We can improve this bound by using the “swallowing trick” in a way similar to the way it is used in the proof of Theorem 1.1 in [2]. Let $f \in \mathbb{F}_p^{\{0,1\}^n}$ be defined as $f(x) = \sum_{j \in [n]} x_j - i$. From Lemma 6, we know that the set of functions $\{x_A f : |A| \not\equiv i \pmod{p} \text{ and } |A| < s\}$ is linearly independent in the vector space $\mathbb{F}_p^{\{0,1\}^n}$ over \mathbb{F}_p .

Claim 8. $\{f_j : 1 \leq j \leq m\} \cup \{x_A f : |A| \not\equiv i \pmod{p} \text{ and } |A| < s\}$ is a collection of functions that is linearly independent in the vector space $\mathbb{F}_p^{\{0,1\}^n}$ over \mathbb{F}_p .

In order to prove the claim, assume $\sum_{j=1}^m \lambda_j f_j + \sum_{A: |A| \leq s-1, |A| \not\equiv i \pmod{p}} \mu_A x_A f = 0$ for some $\lambda_j, \mu_A \in \mathbb{F}_p$. Evaluating at V_j , all terms in the second sum vanish (since $f(V_j) = 0$) and by Equation 1, only the term with subscript j remains of the first sum. We infer that $\lambda_j = 0$, for every j . It then follows from Lemma 6 that every μ_A is zero thus proving the claim.

Since each function in the collection of functions in Claim 8 can be obtained as a linear combination of distinct monomials of degree at most s , we can infer that $m + \sum_{j \neq i, j=0}^{s-1} \binom{n}{j} \leq \sum_{j=0}^s \binom{n}{j}$. We thus have

$$|\mathcal{F}_i| \leq \begin{cases} \binom{n}{s} + \binom{n}{i}, & \text{if } i < s \\ \binom{n}{s}, & \text{otherwise} \end{cases} \quad (2)$$

Observe that $i \leq p-1$. We will shortly see that the prime p we choose is always at most $2g(t, n) \ln(g(t, n))$, where $g(t, n) = \frac{(2t+\ln n)}{\ln(2t+\ln n)}$. So if $s \leq n+1 - 2g(t, n) \ln(g(t, n))$, the condition $s+i \leq n$ (here i stands for the symbol k in Lemma 7) given in Lemma 7 is satisfied and therefore the more powerful Lemma 7 can be used instead of Lemma 6 while applying the swallowing trick. We can then claim that (proof of this claim is similar to the proof of Claim 8 and is therefore omitted) $\{f_j : 1 \leq j \leq m\} \cup \{x_A f : |A| < s\}$ (where $f(x) = \sum_{j=1}^n x_j - i$) is a collection of functions that is linearly independent in the vector space $\mathbb{F}_p^{\{0,1\}^n}$ over \mathbb{F}_p which can be obtained as a linear combination of distinct monomials of degree at most s . It then follows that $|\mathcal{F}_i| \leq \binom{n}{s}$.

In the rest of the proof, we shall assume the general bound for $|\mathcal{F}_i|$ given by Inequality 2. (Using the $\binom{n}{s}$ upper bound for $|\mathcal{F}_i|$ in place of Inequality 2 when $s \leq n+1 - 2g(t, n) \ln(g(t, n))$ in the rest of the proof will yield the tighter bound for $|\mathcal{F}|$ given in Statement (a) in the theorem.)

Observe that we still do not have an estimate on $|\mathcal{A}_0|$ since $i \equiv \frac{a_i}{b_i}i \pmod{p}$ when $i \equiv 0 \pmod{p}$. To overcome this problem, consider the collection $P = \{p_{q+1}, \dots, p_r\}$ of $r - q$ smallest primes with $p_q \leq t < p_{q+1} < \dots < p_r$ (p_j denotes the j -th prime; $p_1 = 2, p_2 = 3$, and so on) such that for every $A \in \mathcal{F}$, there exists a prime $p \in P$ with $p \nmid |A|$. Note that if we repeat the steps done above for each $p \in P$, we obtain the following upper bound.

$$\begin{aligned} |\mathcal{F}| &\leq (p_{q+1} + \dots + p_r - (r - q)) \binom{n}{s} + (r - q) \sum_{j=1}^{s-1} \binom{n}{j} \\ &< (r - q) \left(p_r \binom{n}{s} + \sum_{j=1}^{s-1} \binom{n}{j} \right) \end{aligned}$$

To obtain a small cardinality set P of the desired requirement, we choose the minimum r such that $p_{q+1}p_{q+2}\dots p_r > n$. If $t > n - c_1$, for some positive integer constant c_1 , then $P = \{p_{q+1}, \dots, p_{q+c_1}\}$ satisfies the desired requirements of P . We thus have,

$$|\mathcal{F}| < \begin{cases} c_1 \left(p_r \binom{n}{s} + \sum_{j=1}^{s-1} \binom{n}{j} \right), & \text{if } t > n - c_1 \text{ (here } c_1 \text{ is a +ve integer constant)} \\ r \left(p_r \binom{n}{s} + \sum_{j=1}^{s-1} \binom{n}{j} \right), & \text{otherwise} \end{cases} \quad (3)$$

The product of the first k primes is the *primorial function* $p_k\#$ and it is known that $p_k\# = e^{(1+o(1))k \ln k}$. Given a natural number N , let $N\#$ denote the product of all the primes less than or equal to N (some call this the primorial function). It is known that $N\# = e^{(1+o(1))N}$. Since $\frac{p_r\#}{t\#} = p_{k+1}p_{k+2}\dots p_r$, setting $\frac{e^{(1+o(1))r \ln r}}{e^{(1+o(1))t}}$ $> n$, we get, $r \leq \frac{2(2t + \ln n)}{\ln(2t + \ln n)} = g(t, n)$. Using the prime number theorem (see Section 5.1 of [15]), the r th prime p_r is at most $2r \ln r$. Thus, we have $p_r \leq 2g(t, n) \ln(g(t, n))$. Substituting for r and p_r in Inequality 3 gives the theorem. \square

2.2 When the sets in \mathcal{F} are ‘large enough’

In the following theorem, we show that when the sets in a fractional L -intersecting \mathcal{F} are ‘large enough’, then $|\mathcal{F}|$ is at most n .

Theorem 9. *Let n be a positive integer. Let $L = \{\frac{a_1}{b_1}, \dots, \frac{a_s}{b_s}\}$, where for every $i \in [s]$, $\frac{a_i}{b_i} \in [0, 1)$ is an irreducible fraction. Let $\frac{a}{b} = \max(\frac{a_1}{b_1}, \dots, \frac{a_s}{b_s})$. Let \mathcal{F} be a fractional L -intersecting family of subsets of $[n]$ such that for every $A \in \mathcal{F}$, $|A| > \alpha n$, where $\alpha = \max(\frac{1}{2}, \frac{4a-b}{2b})$. Then, $|\mathcal{F}| \leq n$.*

Proof. Let $\mathcal{F} = \{A_1, A_2, \dots, A_m\}$. For every $A_i \in \mathcal{F}$, we define its $(+1, -1)$ -incidence vector as:

$$X_{A_i}(j) = \begin{cases} +1, & \text{if } j \in A_i \\ -1, & \text{if } j \notin A_i. \end{cases} \quad (4)$$

We prove the theorem by proving the following claim.

Claim 10. X_{A_1}, \dots, X_{A_m} are linearly independent in the vector space \mathbb{R}^n over \mathbb{R} .

Assume for contradiction that X_{A_1}, \dots, X_{A_m} are linearly dependent in the vector space \mathbb{R}^n over \mathbb{R} . Then, we have some reals $\lambda_{A_1}, \dots, \lambda_{A_m}$ where not all of them are zeroes such that

$$\lambda_{A_1}X_{A_1} + \dots + \lambda_{A_m}X_{A_m} = 0. \quad (5)$$

It is given that, for every $A_i \in \mathcal{F}$, $|A_i| > \frac{n}{2}$. Let $u = (1, 1, \dots, 1) \in \mathbb{R}^n$ be the all ones vector. Then, $\langle X_{A_i}, u \rangle > 0$, for every $A_i \in \mathcal{F}$. Therefore, if all non-zero λ_{A_i} s in Equation (5) are of the same sign, say positive, then the inner product of u with the L.H.S of Equation (5) would be non-zero which is a contradiction. Hence, we can assume that not all λ_{A_i} s are of the same sign. We rewrite Equation (5) by moving all negative λ_{A_i} s to the R.H.S. Without loss of generality, assume $\lambda_{A_1}, \dots, \lambda_{A_k}$ are non-negative and the rest are negative. Thus, we have

$$v = \lambda_{A_1}X_{A_1} + \dots + \lambda_{A_k}X_{A_k} = -(\lambda_{A_{k+1}}X_{A_{k+1}} + \dots + \lambda_{A_m}X_{A_m}),$$

where v is a non-zero vector.

For any two distinct sets $A, B \in \mathcal{F}$, $\exists \frac{a_i}{b_i} \in L$ such that

$$\langle X_A, X_B \rangle = \begin{cases} n - 2|A| + \frac{4a_i - 2b_i}{b_i}|B|, & \text{if } |A \cap B| = \frac{a_i}{b_i}|B|, \\ n - 2|B| + \frac{4a_i - 2b_i}{b_i}|A|, & \text{otherwise (that is, if } |A \cap B| = \frac{a_i}{b_i}|A|). \end{cases} \quad (6)$$

Since $\frac{a}{b} = \max(\frac{a_1}{b_1}, \dots, \frac{a_s}{b_s})$, we have $\langle X_A, X_B \rangle \leq n - 2|A| + \frac{4a - 2b}{b}|B|$ or $\langle X_A, X_B \rangle \leq n - 2|B| + \frac{4a - 2b}{b}|A|$. Applying the fact that the cardinality of every set S in \mathcal{F} satisfies $\alpha n < |S| \leq n$, where $\alpha = \max(\frac{1}{2}, \frac{4a - b}{2b})$, we get $\langle X_A, X_B \rangle < 0$. This implies that $\langle v, v \rangle = \langle \lambda_{A_1}X_{A_1} + \dots + \lambda_{A_k}X_{A_k}, -(\lambda_{A_{k+1}}X_{A_{k+1}} + \dots + \lambda_{A_m}X_{A_m}) \rangle < 0$ which is a contradiction. This proves the claim and thereby the theorem. \square

3 L is a singleton set

As explained in Section 1, Fisher's Inequality is a special case of the classical L -intersecting families, where $|L| = 1$. In this section, we study fractional L -intersecting families with $|L| = 1$; a fractional variant of the Fisher's inequality.

3.1 Proof of Theorem 4

Statement of Theorem 4: Let n be a positive integer. Let \mathcal{G} be a fractional L -intersecting families of subsets of $[n]$, where $L = \{\frac{a}{b}\}$, $\frac{a}{b} \in [0, 1)$, and b is a prime. Then, $|\mathcal{G}| \leq (b - 1)(n + 1)\lceil \frac{\ln n}{\ln b} \rceil + 1$.

Proof. It is easy to see that if $a = 0$, then $|\mathcal{G}| \leq n$ with the set of all singleton subsets of $[n]$ forming a tight example to this bound. So assume $a \neq 0$. Let $\mathcal{F} = \mathcal{G} \setminus \mathcal{H}$, where $\mathcal{H} = \{A \in \mathcal{G} : b \nmid |A|\}$. From the definition of a fractional $\frac{a}{b}$ -intersecting family it is clear that $|\mathcal{H}| \leq 1$. The rest of the proof is to show that $|\mathcal{F}| \leq (b - 1)(n + 1)\lceil \frac{\ln n}{\ln b} \rceil$. We

do this by partitioning \mathcal{F} into $(b-1)\lceil \log_b n \rceil$ parts and then showing that each part is of size at most $n+1$. We define \mathcal{F}_i^j as

$$\mathcal{F}_i^j = \{A \in \mathcal{F} \mid |A| \equiv j \pmod{i}\}.$$

Since b divides $|A|$, for every $A \in \mathcal{F}$, under this definition \mathcal{F} can be partitioned into families $\mathcal{F}_{b^k}^{ib^{k-1}}$, where $2 \leq k \leq \lceil \log_b n \rceil$ and $1 \leq i \leq b-1$. We show that, for every $i \in [b-1]$ and for every $2 \leq k \leq \lceil \log_b n \rceil$, $|\mathcal{F}_{b^k}^{ib^{k-1}}| \leq n+1$.

In order to estimate $|\mathcal{F}_{b^k}^{ib^{k-1}}|$, for each $A \in \mathcal{F}_{b^k}^{ib^{k-1}}$, create a vector X_A as follows:

$$X_A(j) = \begin{cases} \frac{1}{\sqrt{b^{k-2}}}, & \text{if } j \in A; \\ 0, & \text{otherwise.} \end{cases}$$

Note that, for $A, B \in \mathcal{F}_{b^k}^{ib^{k-1}}$

$$\langle X_A, X_B \rangle \equiv \begin{cases} b \pmod{b^2}, & \text{if } A = B, \\ ai \pmod{b}, & \text{if } A \neq B, \end{cases} \quad (7)$$

Let $|\mathcal{F}_{b^k}^{ib^{k-1}}| = m$. Let $M_{k,i}$ denote the $m \times n$ matrix formed by taking X_{AS} as rows for each $A \in \mathcal{F}_{b^k}^{ib^{k-1}}$. Then, $|\mathcal{F}_{b^k}^{ib^{k-1}}| \leq n+1$ can be proved by considering $B = M_{k,i} \times M_{k,i}^T$ and showing that $B - aiJ$ (, where J is the $m \times m$ all 1 matrix,) has full rank; determinant of $B - aiJ$ is non-zero since the only term not divisible by the prime b in the expansion of its determinant comes from the product of all the diagonals (note that $a < b$, $i < b$, and since b is a prime, we have $b \nmid ai$). \square

We shall call \mathcal{F} a *bisection closed family* if \mathcal{F} is a fractional L -intersecting family where $L = \{\frac{1}{2}\}$. We have two different constructions of families that are bisection closed and are of cardinality $\frac{3n}{2} - 2$ on $[n]$.

Example 11. Let n be an even positive integer. Let \mathcal{B} denote the collection of 2-sized sets that contain only 1 as a common element in any two sets, i.e. $\{1, 2\}, \{1, 3\}, \dots, \{1, n\}$; and let \mathcal{C} denote collection of 4-sized sets that contain only $\{1, 2\}$ as common elements, i.e. $\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \dots, \{1, 2, n-1, n\}$. It is not hard to see that $\mathcal{B} \cup \mathcal{C}$ is indeed bisection closed.

Example 12. The second example of a bisection closed family of cardinality $\frac{3n}{2} - 2$ comes from *Recursive Hadamard matrices*. A Recursive Hadamard matrix $H(k)$ of size $2^k \times 2^k$ can be obtained from $H(k-1)$ of size $2^{k-1} \times 2^{k-1}$ as follows

$$H(k) = \begin{bmatrix} H(k-1) & H(k-1) \\ H(k-1) & -H(k-1) \end{bmatrix},$$

where $H(0) = 1$. Now consider the matrix:

$$M(k) = \begin{bmatrix} H(k-1) & H(k-1) \\ H(k-1) & -H(k-1) \\ H(k-1) & J(k-1) \end{bmatrix},$$

where $J(k-1)$ denotes the $2^{k-1} \times 2^{k-1}$ all 1s' matrix. Let $M'(k)$ be the matrix obtained from $M(k)$ by removing the first and the (2^k+1) th rows and replacing the -1's by 1's and 1's by 0's. $M'(k)$ is clearly bisection closed and has cardinality $\frac{3n}{2} - 2$, where $n = 2^k$.

3.2 Restricting the cardinalities of the sets in \mathcal{F}

When $L = \{\frac{a}{b}\}$, where b is a prime, Theorem 4 yields an upper bound of $O(\frac{b}{\log b} n \log n)$ for $|\mathcal{F}|$. However, we believe that when $|L| = 1$, the cardinality of any fractional L -intersecting family on $[n]$ would be at most cn , where $c > 0$ is a constant. To this end, we show in Theorem 14 that when the sizes of the sets in \mathcal{F} are restricted, we can achieve this.

The following lemma is crucial to the proof of Theorem 14.

Lemma 13. [1, 4] Let A be an $m \times m$ real symmetric matrix with $a_{i,i} = 1$ and $|a_{i,j}| \leq \epsilon$ for all $i \neq j$. Let $\text{tr}(A)$ denote the trace of A , i.e., the sum of the diagonal entries of A . Let $\text{rk}(A)$ denote the rank of A . Then,

$$\text{rk}(A) \geq \frac{(\text{tr}(A))^2}{\text{tr}(A^2)} \geq \frac{m}{1 + (m-1)\epsilon^2}.$$

Proof. Let $\lambda_1, \dots, \lambda_m$ denote the eigenvalues of A . Since only $\text{rk}(A)$ eigenvalues of A are non-zero, $(\text{tr}(A))^2 = (\sum_{i=1}^m \lambda_i)^2 = (\sum_{i=1}^{\text{rk}(A)} \lambda_i)^2 \leq \text{rk}(A) \sum_{i=1}^{\text{rk}(A)} \lambda_i^2 = \text{rk}(A) \text{tr}(A^2)$, where the inequality follows from the Cauchy-Schwartz Inequality. Thus, $\text{rk}(A) \geq \frac{(\text{tr}(A))^2}{\text{tr}(A^2)}$. Substituting $\text{tr}(A) = m$ and $\text{tr}(A^2) = m + m(m-1)\epsilon^2$ in the above inequality proves the theorem. \square

Theorem 14. Let n be a positive integer and let $\delta > 1$. Let \mathcal{F} be a fractional L -intersecting family of subsets of $[n]$, where $L = \{\frac{a}{b}\}$, $\frac{a}{b} \in [0, 1)$ is an irreducible fraction and for every $A \in \mathcal{F}$, $|A|$ is an integer in the range $\left[\frac{b}{4(b-a)}n - \frac{b}{4a\delta}\sqrt{n}, \frac{b}{4(b-a)}n + \frac{b}{4a\delta}\sqrt{n}\right]$. Then, $|\mathcal{F}| < \frac{\delta^2}{\delta^2-1}n$.

Proof. For any $A \in \mathcal{F}$, let $Y_A \in \mathbb{R}^n$ be a vector defined as:

$$Y_A(j) = \begin{cases} +\frac{1}{\sqrt{n}}, & \text{if } j \in A \\ -\frac{1}{\sqrt{n}}, & \text{if } j \notin A. \end{cases}$$

Clearly, $\langle Y_A, Y_A \rangle = 1$. For any two distinct sets $A, B \in \mathcal{F}$, we have

$$\langle Y_A, Y_B \rangle = \begin{cases} \frac{n-2|A|+\frac{4a-2b}{b}|B|}{n}, & \text{if } |A \cap B| = \frac{a}{b}|B|, \\ \frac{n-2|B|+\frac{4a-2b}{b}|A|}{n}, & \text{otherwise (that is, if } |A \cap B| = \frac{a}{b}|A|). \end{cases} \quad (8)$$

Suppose $\mathcal{F} = \{A_1, \dots, A_m\}$. Let B be the $m \times n$ matrix with Y_{A_1}, \dots, Y_{A_m} as its rows. Then, from Equation 8, it follows that BB^T is an $m \times m$ real symmetric matrix with the diagonal entries being 1 and the absolute value of any other entry being at most $\frac{1}{\delta\sqrt{n}}$. Applying Lemma 13, we have $n \geq \text{rk}(BB^T) \geq \frac{m}{1+\frac{m-1}{\delta^2 n}} > \frac{m}{1+\frac{m}{\delta^2 n}}$. Thus, $n + \frac{m}{\delta^2} > m$ or $m < \frac{\delta^2}{\delta^2-1}n$. \square

4 Discussion

In Theorem 1, we gave a general upper bound for $|\mathcal{F}|$, where \mathcal{F} is a fractional L -intersecting family. In Section 1, we also gave an example to show that this bound is asymptotically tight up to a factor of $\frac{\ln^2 n}{\ln \ln n}$, when $s (= |L|)$ is a constant. However, when s is a constant, we believe that $|\mathcal{F}| \in \Theta(n^s)$.

Consider the following special case for a fractional L -intersecting family \mathcal{F} , where $L = \{\frac{1}{2}\}$. We call such a family a bisection-closed family (see definition in Section 3).

Conjecture 15. If \mathcal{F} is a bisection-closed family, then $|\mathcal{F}| \leq cn$, where $c > 0$ is a constant.

We have not been able to find an example of a bisection-closed family of size $2n$ or more.

The problem of determining a linear sized upper bound for the size of any bisection-closed family leads us to pose the following question:

Open Problem 16. Suppose $0 < a_1 \leq \dots \leq a_n$ are n distinct reals. Let $\mathcal{M}_n(a_1, \dots, a_n)$ denote the set of all symmetric matrices M satisfying $m_{ij} \in \{a_i, a_j\}$ for $i \neq j$ and $m_{ii} = 0$ for all i . Then, does there exist an absolute constant $c > 0$ such that $rk(M) \geq cn$, for all $M \in \mathcal{M}_n(a_1, \dots, a_n)$?

To see how this question ties in with our problem, suppose that a family $\mathcal{F} \subset \mathcal{P}([n])$ is a bisection closed family, i.e., for $A, B \in \mathcal{F}$ and $A \neq B$ then $|A \cap B| \in \{|A|/2, |B|/2\}$. For simplicity, let us write $\mathcal{F} = \{A_1, \dots, A_m\}$ and denote $|A_i| = a_i$ where the a_i are arranged in ascending order. We say A bisects B if $|A \cap B| = |B|/2$. For each $A \in \mathcal{F}$, let $\mathbf{u}_A \in \mathbb{R}^n$ where $\mathbf{u}_A(i) = 1$ if $i \in A$ and -1 if $i \notin A$. Then note that

$$\begin{aligned} \langle \mathbf{u}_A, \mathbf{u}_B \rangle &= n - 2|A| && \text{if } A \text{ bisects } B, \\ &= n - 2|B| && \text{if } B \text{ bisects } A, \\ \|\mathbf{u}_A\|^2 &= n. \end{aligned}$$

Consider the $m \times m$ matrix M whose rows and columns are indexed by the members of \mathcal{F} , with $M_{A,B} = \langle \mathbf{u}_A, \mathbf{u}_B \rangle$. Then, since M is a Gram matrix of vectors in \mathbb{R}^n , it follows that $rk(M) \leq n$. If $\mathcal{X} = \frac{1}{2}(nJ - M)$, where J is the all ones matrix of order m , then $rk(\mathcal{X}) \leq n + 1$. But note that $\mathcal{X} \in \mathcal{M}(a_1, \dots, a_m)$. So, if the answer to the aforementioned open problem is ‘yes’, then $rk(\mathcal{X}) \geq cm$. This gives $cm \leq r(\mathcal{X}) \leq n + 1$ which in turn gives $m \leq c^{-1}(n + 1)$.

The problem of determining the maximum size of a fractional L -intersecting family is far from robust in the following sense. Suppose $L = \{1/2\}$ and we consider the problem of determining the size of an ‘ ε -approximately fractional L -intersecting family,’ i.e., for any $A \neq B$ we have that at least one of $\frac{|A \cap B|}{|A|}, \frac{|A \cap B|}{|B|} \in (1/2 - \varepsilon, 1/2 + \varepsilon)$ for small $\varepsilon > 0$, then such families can in fact be exponentially large in size. Let each set A_i be chosen uniformly and independently at random from $\mathcal{P}([n])$. Then since each $|A_i|$ and $|A_i \cap A_j|$ are independent binomial $B(n, 1/2)$ and $B(n, 1/4)$ respectively, by standard Chernoff

bounds (see [12], chapter 5), it follows (by straightforward computations) that one can get such a family of cardinality at least $e^{2\varepsilon^2 n/75}$. In fact this same construction gives super-polynomial sized families even if $\varepsilon = n^{-1/2+\delta}$ for any fixed $\delta > 0$.

Another interesting facet of the fractional intersection notion is the following extension of l -avoiding families [6, 10]¹. A set B *bisects* another set A if $|A \cap B| = \frac{|A|}{2}$. A family \mathcal{F} of even subsets of $[n]$ is called *fractional* $(\frac{1}{2})$ -*avoiding* (or *bisection-free*) if for every $A, B \in \mathcal{F}$, neither B bisects A nor A bisects B (if we allow odd subsets in the definition of a fractional $(\frac{1}{2})$ -avoiding family, then the set of all the odd-sized subsets on $[n]$ is an example of one such family). Let $\bar{\vartheta}(n)$ denote the maximum cardinality of a fractional $(\frac{1}{2})$ -avoiding family on $[n]$. Let $A, B \subseteq [n]$ such that $|A| > \frac{2n}{3}$ and $|B| > \frac{2n}{3}$. It is not very hard to see that $|A \cap B| > n/3$ whereas $|A \cap ([n] \setminus B)| < n/3$. So, neither A can bisect B nor B can bisect A . Therefore, if we construct a family $\mathcal{F} = \{A \subseteq [n] \mid |A| > \frac{2n}{3}, |A| \text{ is even}\}$, \mathcal{F} is fractional $(\frac{1}{2})$ -avoiding. Moreover, $|\mathcal{F}| = \sum_{2 \mid i, i=0}^{\frac{n}{3}-1} \binom{n}{i} > 1.88^n$, for sufficiently large n (using Stirling's formula). Let us now try to find an upper bound to the cardinality of a fractional $(\frac{1}{2})$ -avoiding family. An application of a result of Frankl and Rödl [6, Corollary 1.6] gives the following theorem for the cardinalities of l -avoiding families as a corollary (see [10, Theorem 1.1]).

Theorem 17. [6, 10] Let $\alpha, \epsilon \in (0, 1)$ with $\epsilon \leq \frac{\alpha}{2}$. Let $k = \lfloor \alpha n \rfloor$ and $l \in [\max(0, 2k - n) + \epsilon n, k - \epsilon n]$. Then any l -avoiding family $\mathcal{A} \subseteq \binom{[n]}{k}$ satisfies $|\mathcal{A}| \leq (1 - \delta)^n \binom{n}{k}$ where $\delta = \delta(\alpha, \epsilon) > 0$.

For any fractional $(\frac{1}{2})$ -avoiding family \mathcal{F} , any $\mathcal{F}' \subseteq \mathcal{F}$ consisting of sets of cardinality l is $\frac{l}{2}$ -avoiding. So, given any fractional $(\frac{1}{2})$ -avoiding family \mathcal{F} , split \mathcal{F} into families $\mathcal{F}_{\leq \frac{n}{3}-1}, \mathcal{F}_{\frac{n}{3}}, \dots, \mathcal{F}_{2\frac{n}{3}}, \mathcal{F}_{\geq 2\frac{n}{3}+1}$. From Theorem 17, we know that each \mathcal{F}_i has a cardinality at most $(1 - \delta_i)^n \binom{n}{i}$ for $\frac{n}{3} \leq i \leq \frac{2n}{3}$. Let $\delta = \min(\delta_{\frac{n}{3}}, \dots, \delta_{2\frac{n}{3}})$. Then $\sum_{i=\frac{n}{3}}^{\frac{2n}{3}} |\mathcal{F}_i| \leq ((1 - \delta)2)^n$. Further, $|\mathcal{F}_{\leq \frac{n}{3}-1}| \leq \sum_{i=0}^{\frac{n}{3}-1} \binom{n}{i}$ and $|\mathcal{F}_{\geq 2\frac{n}{3}+1}| \leq \sum_{i=0}^{\frac{n}{3}-1} \binom{n}{i} < 2^{nH(\frac{1}{3})} < 1.89^n$, where $H(\nu) = -\nu \log_2 \nu - (1 - \nu) \log_2 (1 - \nu)$ is the binary entropy function. Thus, for sufficiently large values of n , $1.88^n \leq \bar{\vartheta}(n) \leq ((1 - \epsilon)2)^n$, for some $0 < \epsilon \leq 0.06$.

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¹A family \mathcal{F} is called l -avoiding if for each $A, B \in \mathcal{F}$, $|A \cap B| \neq l$ for some $l \in [n]$.

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