Non-bipartite distance-regular graphs with a small smallest eigenvalue

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Abstract

In 2017, Qiao and Koolen showed that for any fixed integer $D \geqslant 3$, there are only finitely many non-bipartite distance-regular graphs with $\theta_{\min} \leqslant -\alpha k$, where $0 < \alpha < 1$ is any fixed number. In this paper, we will study non-bipartite distance-regular graphs with relatively small $\theta_{\min}$ compared with k. In particular, we will show that if $\theta_{\min}$ is relatively close to $-k$, then the odd girth g must be large. Also we will classify the non-bipartite distance-regular graphs with $\theta_{\min} \leqslant -\frac{D-1}{D}k$ for $D = 4, 5$.

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1 Introduction

The *odd girth* of a non-bipartite graph is the length of its shortest odd cycle. Let Γ be a non-bipartite distance-regular graph with valency k, diameter D, odd girth g and smallest eigenvalue $\theta_{\min}$. In [6], Qiao and Koolen showed that for any fixed integer $D \geqslant 3$, there are only finitely many such graphs with $\theta_{\min} \leqslant -\alpha k$, where $0 < \alpha < 1$ is any fixed number. In this paper, we will study non-bipartite distance-regular graphs with relatively small $\theta_{\min}$ compared with k. In the next result, we will show that if $\theta_{\min}$ is relatively close to $-k$, then the odd girth g must be large.

Theorem 1. *Let Γ be a non-bipartite distance-regular graph with valency k and odd girth g, having smallest eigenvalue $\theta_{\min}$. Then there exists a constant $\varepsilon(g) > 0$ such that $\theta_{\min} \geqslant -(1 - \varepsilon(g))k$.*

Remark 2. The positive constant $\varepsilon(g)$ goes to 0 as the odd girth g goes to ∞. For example, the $(2t+1)$-gon has valency $k = 2$, odd girth $g = 2t+1$ and smallest eigenvalue $\theta_{\min} = 2\cos(\frac{2t\pi}{2t+1})$. Thus, $\varepsilon(g) \leqslant 1 + \frac{\theta_{\min}}{k} = 2\cos^2(\frac{t\pi}{2t+1})$.

In [6], Qiao and Koolen classified non-bipartite distance-regular graphs with valency k, diameter $D \leqslant 3$ and smallest eigenvalue $\theta_{\min} \leqslant -k/2$. Using Theorem 1, we will classify non-bipartite distance-regular graphs with valency k, diameter D and smallest eigenvalue $\theta_{\min} \leqslant -\frac{D-1}{D}k$, when $D = 4$ or 5.

Theorem 3. *Let Γ be a non-bipartite distance-regular graph with valency k, diameter D and smallest eigenvalue $\theta_{\min} \leqslant -\frac{D-1}{D}k$.*

i) If $D = 4$, then Γ is one of the following graph

a) the Coxeter graph with intersection array $\{3, 2, 2, 1; 1, 1, 1, 2\}$,

b) the 9-gon with intersection array $\{2, 1, 1, 1; 1, 1, 1, 1\}$,

c) the Odd graph O_5 with intersection array $\{5, 4, 4, 3; 1, 1, 2, 2\}$,

d) the folded 9-cube with intersection array $\{9, 8, 7, 6; 1, 2, 3, 4\}$.

ii) If $D = 5$, then Γ is one of the following graph

a) the 11-gon with intersection array $\{2, 1, 1, 1, 1; 1, 1, 1, 1, 1\}$,

b) the Odd graph O_6 with intersection array $\{6, 5, 5, 4, 4; 1, 1, 2, 2, 3\}$,

c) the folded 11-cube with intersection array $\{11, 10, 9, 8, 7; 1, 2, 3, 4, 5\}$.

This paper is organized as follows. In the next section, we give the definitions and some preliminary results. In Section 3, we give a proof of Theorem 1. In the last section, we give a proof of Theorem 3.

2 Preliminaries

For more background, see [4] and [7].

All the graphs considered in this paper are finite, undirected and simple. Let Γ be a graph with vertex set $V = V(\Gamma)$ and edge set $E = E(\Gamma)$. Denote $x \sim y$ if the vertices $x, y \in V$ are adjacent. The *distance* $d(x, y) = d_\Gamma(x, y)$ between two vertices $x, y \in V(\Gamma)$ is the length of a shortest path connecting x and y. The maximum distance between two vertices in Γ is the *diameter* $D = D(\Gamma)$. We use $\Gamma_i(x)$ for the set of vertices at distance i from x and write, for the sake of simplicity, $\Gamma(x) := \Gamma_1(x)$. The *degree* of x is the number $|\Gamma(x)|$ of vertices adjacent to it. A graph is *regular with valency* k if the degree of each of its vertices is k. The *girth* and *odd girth* of a graph is the length of its shortest cycle, and shortest odd cycle, respectively. A graph Γ is called *bipartite* if it has no odd cycle.

A connected graph Γ with diameter D is called *distance-regular* if there are integers b_i, c_i $(i = 0, 1, \ldots, D)$ such that for any two vertices $x, y \in V(\Gamma)$ with $d(x, y) = i$, there are exactly c_i neighbors of y in $\Gamma_{i-1}(x)$ and b_i neighbors of y in $\Gamma_{i+1}(x)$, where we define $b_D = c_0 = 0$. In particular, Γ is a regular graph with valency $k := b_0$. We define $a_i := k - b_i - c_i$ $(i = 0, 1, \ldots, D)$ for notational convenience. Note that $a_i = |\Gamma(y) \cap \Gamma_i(x)|$ holds for any two vertices x, y with $d(x, y) = i$ $(i = 0, 1, \ldots, D)$.

For a distance-regular graph Γ and a vertex $x \in V(\Gamma)$, we denote $k_i := |\Gamma_i(x)|$ and $p^h_{ij} := |\{w \mid w \in \Gamma_i(x) \cap \Gamma_j(y)\}|$ for any $y \in \Gamma_h(x)$. It is easy to see that $k_i = b_0 b_1 \cdots b_{i-1}/(c_1 c_2 \cdots c_i)$ and hence it does not depend on x. The numbers a_i, b_i and c_i $(i = 0, 1, \ldots, D)$ are called the *intersection numbers*, and the array $\{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\}$ is called the *intersection array* of Γ. The matrix L is called the *intersection matrix* of Γ, where

$$L = \begin{pmatrix} a_0 & b_0 & 0 & & & \\ c_1 & a_1 & b_1 & & 0 & \\ & c_2 & a_2 & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & 0 & & \cdot & \cdot & b_{D-1} \\ & & & & c_D & a_D \end{pmatrix}.$$

Let Γ be a distance-regular graph with v vertices and diameter D. Let A_i $(i = 0, 1, \ldots, D)$ be the $(0, 1)$-matrix whose rows and columns are indexed by the vertices of Γ and the (x, y)-entry is 1 whenever $d(x, y) = i$ and 0 otherwise. We call A_i the *distance-i matrix* and $A := A_1$ the *adjacency matrix* of Γ. The *eigenvalues* $\theta_0 > \theta_1 > \cdots > \theta_D$ of the graph Γ are just the eigenvalues of its adjacency matrix A. We denote m_i the *multiplicity* of θ_i. Note that the $D + 1$ distinct eigenvalues of Γ are precisely the eigenvalues of L (see [7, Proposition 2.7]).

For each eigenvalue θ_i of Γ, let U_i be a matrix with its columns forming an orthonormal basis for the eigenspace associated with θ_i. And $E_i := U_i U_i^T$ is called the *minimal idempotent* associated with θ_i, satisfying $E_i E_j = \delta_{ij} E_j$ and $A E_i = \theta_i E_i$, where δ_{ij} is the Kronecker delta. Note that vE_0 is the all-ones matrix J.

The set of distance matrices $\{A_0 = I, A_1, A_2, \ldots, A_D\}$ forms a basis of a commutative $\mathbb{R}$-algebra $\mathcal{A}$, known as the *Bose-Mesner algebra*. The set of minimal idempotents $\{E_0 =$

$\frac{1}{v}J, E_1, E_2, \ldots, E_D\}$ is another basis of $\mathcal{A}$. There exist $(D+1)\times(D+1)$ matrices P and Q (see [4, p.45]), such that the following relations hold

$$A_i = \sum_{j=0}^{D} P_{ji}E_j \quad \text{and} \quad E_i = \frac{1}{v}\sum_{j=0}^{D} Q_{ji}A_j \quad (i = 0, 1, \ldots, D). \tag{1}$$

Note that $Q_{0i} = m_i$ (see [4, Lemma 2.2.1]).

Let $E_i = U_iU_i^T$ be the minimal idempotent associated with θ_i, where the columns of U_i form an orthonormal basis of the eigenspace associated with θ_i. We denote the x-th row of $\sqrt{v/m_i}U_i$ by $\hat{x}$. Note that $E_i \circ A_j = \frac{1}{v}Q_{ji}A_j$, hence all the vectors $\hat{x}$ are unit vectors and the cosine of the angle between two vectors $\hat{x}$ and $\hat{y}$ is $u_j(\theta_i) := \frac{Q_{ji}}{Q_{0i}}$, where $d(x, y) = j$. The map $x \mapsto \hat{x}$ is called a *normalized representation* and the sequence $(u_j(\theta_i))_{j=0}^{D}$ is called the *standard sequence* of Γ, associated with θ_i. As $AU_i = \theta_iU_i$, we have $\theta_i\hat{x} = \sum_{y\sim x}\hat{y}$, and hence the following holds:

$$\begin{cases} c_ju_{j-1}(\theta_i) + a_ju_j(\theta_i) + b_ju_{j+1}(\theta_i) = \theta_iu_j(\theta_i) \quad (j = 1, 2, \ldots, D-1), \\ c_Du_{D-1}(\theta_i) + a_Du_D(\theta_i) = \theta_iu_D(\theta_i), \end{cases} \tag{2}$$

with $u_0(\theta_i) = 1$ and $u_1(\theta_i) = \frac{\theta_i}{k}$.

Lemma 4. (c.f. [7, Theorem 2.8]) *Let Γ be a distance-regular graph with diameter D and v vertices. Let θ be an eigenvalue of Γ and $(u_i)_{i=0}^{D}$ be the standard sequence associated with θ. Then the multiplicity $m(\theta)$ of θ as an eigenvalue of Γ satisfies*

$$m(\theta) = \frac{v}{\sum_{i=0}^{D} k_iu_i^2}, \tag{3}$$

$$\leqslant \max\left\{\frac{1}{u_1^2}, \ldots, \frac{1}{u_{j-1}^2}, \frac{\sum_{i=j}^{D} k_i}{k_ju_j^2}\right\} \quad (j = 1, 2, \ldots, D). \tag{4}$$

Proof. Equation (3) follows from [7, Theorem 2.8]. We only give a proof of Equation (4).

$$\frac{v}{\sum_{i=0}^{D} k_iu_i^2} = \frac{\sum_{i=0}^{D} k_i}{\sum_{i=0}^{D} k_iu_i^2} \leqslant \max\left\{\frac{\sum_{i=0}^{j-1} k_i}{\sum_{i=0}^{j-1} k_iu_i^2}, \frac{\sum_{i=j}^{D} k_i}{\sum_{i=j}^{D} k_iu_i^2}\right\},$$

$$\frac{\sum_{i=0}^{j-1} k_i}{\sum_{i=0}^{j-1} k_iu_i^2} \leqslant \max\left\{\frac{1}{u_1^2}, \ldots, \frac{1}{u_{j-1}^2}\right\},$$

$$\frac{\sum_{i=j}^{D} k_i}{\sum_{i=j}^{D} k_iu_i^2} \leqslant \frac{\sum_{i=j}^{D} k_i}{k_ju_j^2}.$$

□

Lemma 5. ([4, Proposition 4.1.6]) *Let Γ be a distance-regular graph with valency k and diameter D. Then the following conditions hold*

i) $1 = c_1 \leqslant c_2 \leqslant \cdots \leqslant c_D$,

ii) $k = b_0 \geqslant b_1 \geqslant \cdots \geqslant b_{D-1}$,

iii) k_i*'s* ($i = 1, 2, \ldots, D$) *are positive integers,*

iv) the multiplicities are positive integers.

Lemma 6. (c.f. [2, Proposition 3.1]) *Let Γ be a non-bipartite distance-regular graph with valency k and odd girth $g = 2t + 1$. Then*

$$\sum_{i=0}^{t} p_i(\eta) u_i \geqslant 0, \tag{5}$$

where $(u_i)_{i=0}^{D}$ is the standard sequence associated with the smallest eigenvalue $\theta_{\min}$, η is any eigenvalue of the g-gon, and $p_i(x)$ is defined as the following

$$\begin{aligned} p_0(x) &= 1, \\ p_1(x) &= x, \\ p_2(x) &= x^2 - 2, \\ p_i(x) &= x p_{i-1}(x) - p_{i-2}(x) \quad (i = 3, 4, \ldots, t). \end{aligned} \tag{6}$$

Proof. Let Δ be any g-gon in Γ. Let B_i ($i = 0, 1, \ldots, t$) be the matrix with rows and columns indexed by $V(\Delta)$, where the (v, w)-entry is 1 whenever $d_\Gamma(v, w) = i$ and 0 otherwise. Note that $d_\Delta(v, w) = d_\Gamma(v, w)$ for any two vertices $v, w \in V(\Delta)$, and we have $B_i = p_i(B_1)$ with $p_i(x)$ as Equation (6). By [2, Proposition 3.1], for any eigenvalue η of Δ, we have Equation (5). $\square$

Lemma 7. (c.f. [6, Lemma 5.2]) *Let Γ be a distance-regular graph with valency k and smallest eigenvalue $\theta_{\min}$. If $a_1 = 0$ and $\theta_{\min} < \frac{12-5k}{7}$, then $c_2 \leqslant 2$.*

Proof. Choose two vertices $x, y \in V(\Gamma)$ with $d(x, y) = 2$. As $a_1 = 0$, the subgraph induced on $\{x, y\} \cup (\Gamma(x) \cap \Gamma(y))$ is a K_{2,c_2}. Let $x \mapsto \hat{x}$ be a normalized representation associated with $\theta = \theta_{\min}$. Consider the Gram matrix of the image of the K_{2,c_2} with the bipartition, we see that

$$Q = \begin{pmatrix} \frac{1}{2}(1 + u_2) & u_1 \\ u_1 & \frac{1}{c_2}(1 + (c_2 - 1)u_2) \end{pmatrix}$$

is positive semidefinite, by [4, Proposition 3.7.1 (iii)]. Then $(1, 1)Q(1, 1)^t \geqslant 0$, which in turn implies $(u_1 + u_2)((2 + c_2)\frac{1-u_2}{u_1+u_2} + 4c_2) \geqslant 0$. As $a_1 = 0$, we see $u_1 + u_2 = \frac{(\theta+k)(\theta-1)}{k(k-1)} < 0$, that is $\frac{4c_2}{2+c_2} \leqslant -\frac{1-u_2}{u_1+u_2} = \frac{k-\theta}{1-\theta}$. When $k > 1$, we have $\theta < \frac{12-5k}{7} < \frac{4-k}{3}$ and $c_2 \leqslant \frac{2k-2\theta}{4-3\theta-k} < 3$. $\square$

Lemma 8. *Let Γ be a distance-regular graph with valency k and diameter D, having smallest eigenvalue $\theta_{\min}$ with associated standard sequence $(u_i)_{i=0}^{D}$. Then*

$$|u_{i+1}| \geqslant \frac{|(\theta_{\min} - a_i)u_i| - c_i|u_{i-1}|}{b_i} \quad (i = 1, \ldots, D-1). \tag{7}$$

Proof. By Equation (2) we see that $u_{i+1} = \frac{(\theta_{\min}-a_i)u_i - c_i u_{i-1}}{b_i}$ $(i = 1, \ldots, D-1)$. As $\theta_{\min} < 0$ is the smallest eigenvalue, by [4, Corollary 4.1.2], we see that u_{i+1}, $-u_i$ and u_{i-1} has the same sign. The result follows. □

3 Main Theorem

In this section we will prove our main result.

Proof of Theorem 1. If $g = 3$, then $\theta_{\min} \geqslant -\frac{k}{2}$ by [7, Proposition 2.11]. So we may assume $g \geqslant 5$. Let $t = \frac{g-1}{2}$ and Δ be a g-gon in Γ. Let $(u_i)_{i=0}^{D}$ be the standard sequence associated with the smallest eigenvalue $\theta = \theta_{\min}$.

Assume $c_t \leqslant \zeta k$ for some $\zeta \leqslant \frac{1}{2}$. By Lemma 6, we have $\sum_{i=0}^{t} p_i(\eta) u_i \geqslant 0$, where $p_i(x)$ is as Equation (6).

We claim that there exist constants N_i such that

$$\left| u_i - \left(\frac{\theta}{k}\right)^i \right| \leqslant N_i \zeta \quad (i = 0, 1, \ldots, t), \tag{8}$$

with

$$N_i = \begin{cases} 0, & i = 0, 1, \\ 2N_{i-1} + 4, & i = 2, 3 \ldots, t. \end{cases}$$

Note that $|u_0 - 1| = |u_1 - \frac{\theta}{k}| = 0$. Assume $|u_i - \left(\frac{\theta}{k}\right)^i| \leqslant N_i \zeta$ for some $1 \leqslant i \leqslant t-1$. As $c_t \leqslant \zeta k$, we see that $c_i \leqslant c_t \leqslant \zeta k$ and $b_i = k - c_i \geqslant (1-\zeta)k$. Then

$$\begin{aligned} \left| u_{i+1} - \left(\frac{\theta}{k}\right)^{i+1} \right| &= \left| \frac{\theta u_i - c_i u_{i-1}}{b_i} - \left(\frac{\theta}{k}\right)^{i+1} \right| \\ &\leqslant \left| \frac{\theta}{b_i} u_i - \frac{\theta}{b_i}\left(\frac{\theta}{k}\right)^i \right| + \left| \frac{\theta}{b_i}\left(\frac{\theta}{k}\right)^i - \left(\frac{\theta}{k}\right)^{i+1} \right| + \frac{c_i}{b_i} \cdot |u_{i-1}| \\ &= \left| \frac{\theta}{b_i} \right| \cdot \left| u_i - \left(\frac{\theta}{k}\right)^i \right| + \frac{c_i}{b_i} \cdot \left| \left(\frac{\theta}{k}\right)^{i+1} \right| + \frac{c_i}{b_i} \cdot |u_{i-1}| \\ &\leqslant \frac{k}{b_i} \cdot N_i \zeta + \frac{c_i}{b_i} + \frac{c_i}{b_i} \\ &\leqslant N_{i+1} \zeta, \end{aligned}$$

where $\frac{k}{b_i} \leqslant \frac{1}{1-\zeta} \leqslant 2$ and $\frac{c_i}{b_i} \leqslant \frac{\zeta}{1-\zeta} \leqslant 2\zeta$ $(\zeta \leqslant \frac{1}{2})$.

Note that $p_i(\eta)$ is an eigenvalue of the distance-i graph of Δ. Hence $|p_i(\eta)| \leqslant 2$ $(i = 0, 1, \dots, t)$, and by Equation (8), we have

$$\begin{aligned}\sum_{i=0}^{t} p_i(\eta)u_i &\leqslant \sum_{i=0}^{t} p_i(\eta)\left(\frac{\theta}{k}\right)^i + \sum_{i=0}^{t} |p_i(\eta)| \cdot |u_i - \left(\frac{\theta}{k}\right)^i| \\ &\leqslant \sum_{i=0}^{t} p_i(\eta)\left(\frac{\theta}{k}\right)^i + M_1\zeta, \end{aligned} \tag{9}$$

where $M_1 = \sum_{i=0}^{t} 2N_i$.

By Equation (6), we see that $p_i(x) = \lambda_1^i + \lambda_2^i$ $(i = 1, 2, \dots, t)$, with $\lambda_i = \frac{1}{2}(x \pm \sqrt{x^2 - 4})$ $(i = 1, 2)$. Define $f(x, y) = \sum_{i=0}^{t} p_i(x)y^i = \frac{1-(\lambda_1 y)^{t+1}}{1-\lambda_1 y} + \frac{1-(\lambda_2 y)^{t+1}}{1-\lambda_2 y} - 1$. Note that the eigenvalues of Δ are $2\cos\frac{2\pi j}{g}$ $(i = 0, 1, \dots, g-1)$. Take $\eta = 2\cos\frac{2\pi(t-1)}{g}$, then we see

$$f(\eta, -1) = -M_2, \tag{10}$$

where $M_2 = 1/\cos\frac{(t-1)\pi}{g}$. In fact,

$$\begin{aligned} f\left(2\cos\frac{2\pi j}{g}, -1\right) &= \frac{1 - \left(-e^{2\pi \mathrm{i}\cdot\frac{j}{g}}\right)^{t+1}}{1 - \left(-e^{2\pi \mathrm{i}\cdot\frac{j}{g}}\right)} + \frac{1 - \left(-e^{-2\pi \mathrm{i}\cdot\frac{j}{g}}\right)^{t+1}}{1 - \left(-e^{-2\pi \mathrm{i}\cdot\frac{j}{g}}\right)} - 1 \\ &= (-1)^t \cdot \frac{e^{2\pi \mathrm{i}\cdot\frac{j(t+1)}{g}} + e^{-2\pi \mathrm{i}\cdot\frac{jt}{g}}}{1 + e^{2\pi \mathrm{i}\cdot\frac{j}{g}}} \\ &= (-1)^{t+j}/\cos\frac{j\pi}{g}. \end{aligned}$$

Take $\zeta = \min\{\frac{M_2}{2M_1}, \frac{1}{2}\}$. Note that M_1, M_2, and hence ζ is determined by g. By Equation (10), we see $f(\eta, -1) + M_1\zeta \leqslant -\frac{M_2}{2} < 0$. We also have $f(\eta, 0) + M_1\zeta = 1 + M_1\zeta > 0$. By Equation (5) and (9), we have $0 \leqslant f(\eta, \frac{\theta}{k}) + M_1\zeta$. Take $-(1 - \varepsilon_1(\zeta))$ as the smallest root y of the equation $f(\eta, y) + M_1\zeta = 0$ in the interval $(-1, 0)$. It follows that $\theta \geqslant -(1 - \varepsilon_1(\zeta))k$.

Now we consider the case $c_t > \zeta k$.

If $c_t > 1$, then we claim that the diameter $D \leqslant \frac{4t}{\zeta^2}$ and $\theta_{\min} \geqslant -(1 - \varepsilon_2(\zeta))k$ for some constant $\varepsilon_2(\zeta) > 0$. Without loss of generality, we may assume $4^i t \leqslant D \leqslant 4^{i+1}t$ for some integer $i \geqslant 1$. If $c_{2t-1+j} = c_{2t-1} \neq 1$, by [7, Theorem 7.1], we see $j \leqslant 2t - 1$, that is $c_{4t-1} > c_{2t-1}$. Then $c_{4t-1} = c_{2t+j} > c_{2t-1+j}$ for some $0 \leqslant j \leqslant 2t - 1$, and $c_{4t-1} \geqslant 2c_t$ by [7, Proposition 7.2]. This implies $k \geqslant c_{4^i t} \geqslant 2^i c_t$, that is $D \leqslant 4t(\frac{k}{c_t})^2 \leqslant \frac{4t}{\zeta^2}$. Then by [6, Theorem 1.1], the set S of distance-regular graphs with valency k, diameter $D \leqslant \frac{4t}{\zeta^2}$, smallest eigenvalue $\theta_{\min} \leqslant -(1 - \varepsilon_1(\zeta))k$ and odd girth g is finite. Take

$$\varepsilon_2(\zeta) = \begin{cases} \min\limits_{\Gamma \in S} \dfrac{k + \theta_{\min}}{k}, & \text{if } S \neq \varnothing, \\ \varepsilon_1(\zeta), & \text{otherwise.} \end{cases}$$

If $c_t = 1$, then $k < \frac{1}{\zeta}$. The set S' of distance-regular graphs with valency $k < \frac{1}{\zeta}$ and odd girth g is finite, by [1, Theorem 1.1]. Take

$$\varepsilon_3(\zeta) = \begin{cases} \min\limits_{\Gamma \in S'} \dfrac{k + \theta_{\min}}{k}, & \text{if } S' \neq \varnothing, \\ \varepsilon_1(\zeta), & \text{otherwise.} \end{cases}$$

Take $\varepsilon = \min\{\varepsilon_1(\zeta), \varepsilon_2(\zeta), \varepsilon_3(\zeta)\}$ and the result follows. □

Remark 9. When the odd girth $g = 5$ and $c_2 \leqslant \zeta k$, we may take $N_2 = \frac{2}{1-\zeta}$. Then $f(x, y) + M_1\zeta = 1 + xy + (x^2 - 2)y^2 + \frac{4\zeta}{1-\zeta}$. By substituting $\eta = 2\cos\frac{2\pi}{5}$ into $f(\eta, \frac{\theta}{k}) + M_1\zeta \geqslant 0$, we find an inequality between ζ and $\frac{\theta}{k}$. For example, if $\zeta = 0.1$, then $\theta \geqslant -0.78k$.

4 Distance-regular graphs with relatively small $\theta_{\min}$

In this section we study distance-regular graphs with relatively small $\theta_{\min}$. In the rest of this section we will give a proof of Theorem 3.

Proof of Theorem 3. Assume Γ has odd girth $g = 2t + 1$. Let $(u_i)_{i=0}^{D}$ be the standard sequence associated with the smallest eigenvalue $\theta = \theta_{\min}$.

We first consider the case $D = 4$. We may assume $k \geqslant 5$, otherwise Γ is the 9-gon or the Coxeter graph by [3] and [5, Theorem 1.1].

As $\theta < -\frac{k}{2}$, by [7, Proposition 2.11], we have $a_1 = 0$. If $a_2 \neq 0$, that is $t = 2$, then substitute $\eta = 2\cos\frac{2\pi(t-1)}{g}$ into Equation (5) and we get $\frac{(k-t)(2k+\sqrt{5}t+t+\sqrt{5}-1)}{2k(k-1)} \geqslant 0$, which implies that $\theta \geqslant \frac{-2k-\sqrt{5}+1}{\sqrt{5}+1}$. Combine it with $\theta \leqslant -\frac{3}{4}k$, we see that $k \leqslant 2$. Hence $a_2 = 0$. Note that $\theta \leqslant -\frac{3}{4}k < \frac{12-5k}{7}$, by Lemma 7, we see $c_2 \leqslant 2$.

If $a_3 \neq 0$, then consider

$$\begin{cases} \displaystyle\sum_{i=0}^{t} p_i(\eta) u_i \geqslant 0 \\ -\dfrac{D-1}{D} k \geqslant \theta \end{cases} \tag{11}$$

with $\eta = 2$, we obtain that $k \leqslant 4$ if $c_2 = 1$, and $k \leqslant 8$ if $c_2 = 2$. No intersection arrays satisfy Lemma 5, with $5 \leqslant k \leqslant 8$, $D = 4$, $a_1 = a_2 = 0 \neq a_3$, $c_2 = 2$ and $\theta_{\min} \leqslant -\frac{3}{4}k$. Hence $a_3 = 0$.

Assume $k \geqslant 36$. Since $k \geqslant 36$, $c_2 \leqslant 2$ and $\theta \leqslant -\frac{3}{4}k$, by Equation (7), we obtain $|u_2| \geqslant 0.5500$ and $|u_3| \geqslant 0.3926$. Now we consider the intersection matrix L of Γ, where

$$L = \begin{pmatrix} 0 & k & 0 & 0 & 0 \\ 1 & 0 & k-1 & 0 & 0 \\ 0 & c_2 & 0 & k-c_2 & 0 \\ 0 & 0 & c_3 & 0 & k-c_3 \\ 0 & 0 & 0 & c_4 & k-c_4 \end{pmatrix}.$$

We see that $k^2+\theta^2 \leqslant tr(L^2) \leqslant k^2+6k+c_4(2k-c_4)$, where $c_2 \leqslant 2$ and $c_3 \leqslant c_4$. Since $k \geqslant 36$ and $\theta \leqslant -\frac{3}{4}k$, we obtain that $\frac{c_4}{k} \geqslant 0.2227$. By Lemma 4, we see that $m \leqslant \max\{\frac{1}{u_1^2}, \frac{1}{u_2^2}, \frac{k_3+k_4}{k_3u_3^2}\}$. Since $k_3b_3 = k_4c_4$, we see $\frac{k_3+k_4}{k_3u_3^2} \leqslant \frac{1}{u_3^2}(1+\frac{k}{c_4})$. With $|u_1| \geqslant \frac{3}{4}$, $|u_2| \geqslant 0.5500$, $|u_3| \geqslant 0.3926$ and $\frac{c_4}{k} \geqslant 0.2227$, we obtain $m < 36$. By [4, Theorem 4.4.4], we see that $k \leqslant m < 36$, a contradiction. It follows that $k \leqslant 35$. Then we check the intersection arrays satisfy Lemma 5, with $5 \leqslant k \leqslant 35$, $D = 4$, $a_1 = a_2 = a_3 = 0 \neq a_4$, $c_2 = 1$ or 2 and $\theta_{\min} \leqslant -\frac{3}{4}k$, and we get the folded 9-cube and odd graph O_5. This shows the case $D = 4$.

Now we consider the case $D = 5$. Similar to the case $D = 4$, we may assume $k \geqslant 5$, otherwise Γ is the 11-gon, by [3] and [5, Theorem 1.1]. As $\theta < -\frac{k}{2}$, by [7, Proposition 2.11], we have $a_1 = 0$. Substitute $\eta = 2\cos\frac{2\pi(t-1)}{g}$ with $t = 2$ into Equation (5), we obtain $\theta \geqslant \frac{-2k-\sqrt{5}+1}{\sqrt{5}+1}$. Together with $\theta \leqslant -\frac{4}{5}k$, we see $k \leqslant 2$, and hence $a_2 = 0$. Since $\theta \leqslant -\frac{4}{5}k < \frac{12-5k}{7}$, by Lemma 7, we have and $c_2 \leqslant 2$.

If $a_3 \neq 0$, then consider Equation (11) with $\eta = 2$, we obtain that $k \leqslant 3$ if $c_2 = 1$, and $k \leqslant 5$ if $c_2 = 2$. By [4, Theorem 1.13.2], no such graphs exist with $k = 5$ and $c_2 = 2$. Hence $a_3 = 0$.

We consider $a_4 \neq 0$. If $c_3 \leqslant 0.3750k$, combine it with Equation (11), where $\eta = -1$ ($g = 9$), we see that $k \leqslant 24$. Assume $k \geqslant 24$, then $c_3 \geqslant 0.3750k$. By Equation (7), we obtain $|u_2| \geqslant 0.6243$ and $|u_3| \geqslant 0.4721$. Note $\frac{k_4}{k_3} = \frac{b_3}{c_4} \leqslant \frac{1-c_3}{c_3}$ and $\frac{k_5}{k_3} = \frac{b_3b_4}{c_4c_5} \leqslant (\frac{1-c_3}{c_3})^2$. By Lemma 4, we see that

$$m \leqslant \max\left\{\frac{1}{u_1^2}, \frac{1}{u_2^2}, \frac{1}{u_3^2}(1+\frac{1-c_3}{c_3}+(\frac{1-c_3}{c_3})^2)\right\}, \tag{12}$$

that is $k \leqslant m \leqslant 24$ by [4, Theorem 4.4.4]. No intersection arrays satisfy Lemma 5 with $5 \leqslant k \leqslant 24$, $D = 5$, $a_1 = a_2 = a_3 = 0 \neq a_4$, $c_2 = 1$ or 2 and $\theta \leqslant -\frac{4}{5}k$. Hence $a_4 = 0$.

Assume $k \geqslant 71$. Then by Equation (7), we see that $|u_2| \geqslant 0.6348$ and $|u_3| \geqslant 0.4994$, where $\theta_1 \leqslant -\frac{4}{5}k$, $c_2 = 1$ or 2. Then as $m \geqslant k \geqslant 71$, by Equation (12), we obtain $c_3 \leqslant 0.2166k$. It implies $|u_4| \geqslant 0.3344$ by Equation (7). Consider the intersection matrix L, and we see that $k^2+\theta^2 \leqslant tr(L^2) \leqslant k^2+6k+4c_5k-c_5^2$, which implies $c_5 \geqslant 0.1440k$. And we see $k \leqslant m \leqslant \min\{\frac{1}{u_1^2}, \frac{1}{u_2^2}, \frac{1}{u_3^2}, \frac{1}{u_4^2}(1+\frac{k}{c_5})\} \leqslant 71$. It follows that $k \leqslant 71$. Then we check all intersection arrays satisfying Lemma 5 with $5 \leqslant k \leqslant 71$, $D = 5$, $a_1 = a_2 = a_3 = a_4 = 0 \neq a_5$, $c_2 = 1$ or 2 and $\theta \leqslant -\frac{4}{5}k$ and we obtain the odd graph O_6 and the folded 11-cube. This shows the case $D = 5$. □

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