The second eigenvalue of some normal Cayley graphs of highly transitive groups

Xueyi Huang*
School of Mathematics and Statistics
Zhengzhou University
Zhengzhou, China
huangxymath@gmail.com

Qiongxiang Huang†
College of Mathematics and Systems Science
Xinjiang University
Urumqi, China
huangqx@xju.edu.cn

Sebastian M. Cioabă‡
Department of Mathematical Sciences
University of Delaware
Newark, USA
cioaba@udel.edu

Submitted: Aug 3, 2018; Accepted: May 21, 2019; Published: Jun 21, 2019

Abstract

Let G be a finite group acting transitively on $[n] = \{1, 2, \ldots, n\}$, and let $\Gamma = \mathrm{Cay}(G, T)$ be a Cayley graph of G. The graph Γ is called normal if T is closed under conjugation. In this paper, we obtain an upper bound for the second (largest) eigenvalue of the adjacency matrix of the graph Γ in terms of the second eigenvalues of certain subgraphs of Γ. Using this result, we develop a recursive method to determine the second eigenvalues of certain Cayley graphs of S_n, and we determine the second eigenvalues of a majority of the connected normal Cayley graphs (and some of their subgraphs) of S_n with $\max_{\tau\in T} |\mathrm{supp}(\tau)| \leqslant 5$, where $\mathrm{supp}(\tau)$ is the set of points in $[n]$ non-fixed by τ.

Mathematics Subject Classifications: 05C50

1 Introduction

Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a simple undirected graph of order n with adjacency matrix $A(\Gamma)$. The eigenvalues of $A(\Gamma)$, denoted by $\lambda_1(\Gamma) \geqslant \lambda_2(\Gamma) \geqslant \cdots \geqslant \lambda_n(\Gamma)$, are also called

*Supported by the China Postdoctoral Science Foundation under grant 2019M652556 and the Postdoctoral Research Sponsorship in Henan Province under grant 1902011.
†Corresponding author. Supported by the NSFC grants 11531011 and 11671344.
‡Supported by the NSF grants DMS-1600768 and CIF-1815922.

the *eigenvalues* of Γ. For a k-regular graph Γ, the spectral gap $\lambda_1(\Gamma) - \lambda_2(\Gamma) = k - \lambda_2(\Gamma)$ is closely related to the connectivity and expansion properties of Γ [2, 3, 16, 17, 30, 31, 23].

Let G be a finite group, and let T be a subset of G such that $e \notin T$ (e is the identity element of G) and $T = T^{-1}$. The *Cayley graph* $\mathrm{Cay}(G, T)$ of G with respect to T (called *connection set*) is defined as the undirected graph with vertex set G and edge set $\{\{g, \tau g\} \mid g \in G, \tau \in T\}$. Clearly, $\mathrm{Cay}(G, T)$ is a regular graph which is connected if and only if T is a generating subset of G. A Cayley graph $\mathrm{Cay}(G, T)$ is called *normal* if T is closed under conjugation.

Let S_n be the symmetric group on $[n] = \{1, 2, \dots, n\}$ with $n \geqslant 3$, and T a subset of S_n consisting of transpositions. The *transposition graph* $\mathrm{Tra}(T)$ of T is defined as the graph with vertex set $\{1, 2, \dots, n\}$ and with an edge connecting two vertices i and j if and only if $(i, j) \in T$. It is known that T can generate S_n if and only if $\mathrm{Tra}(T)$ is connected [21]. In 1992, Aldous [1] (see also [19, 9]) conjectured that the spectral gap of $\mathrm{Cay}(S_n, T)$ is equal to the algebraic connectivity (second least Laplacian eigenvalue) of $\mathrm{Tra}(T)$. Earlier efforts of several researchers solved various special cases of Aldous' conjecture. For instance, Diaconis and Shahshahani [15], and Flatto, Odlyzko and Wales [18] confirmed the conjecture for $\mathrm{Tra}(T)$ being a complete graph and a star, respectively; Handjani and Jungreis [22] confirmed the conjecture for $\mathrm{Tra}(T)$ being a tree; Friedman [19] proved that if $\mathrm{Tra}(T)$ is a bipartite graph then the spectral gap of $\mathrm{Cay}(S_n, T)$ is at most the algebraic connectivity of $\mathrm{Tra}(T)$; Cesi [9] confirmed the conjecture for $\mathrm{Tra}(T)$ being a complete multipartite graph. At last, Caputo, Liggett and Richthammer [7] completely confirmed the conjecture in 2010, their proof is an ingenious combination of two ingredients: a nonlinear mapping in the group algebra $\mathbb{C}S_n$ which permits a proof by induction on n, and a quite complicated estimate named the octopus inequality (see also [10] for a self-contained algebraic proof). Very recently, Cesi [11] proved an analogous result of Aldous' conjecture (now theorem) for the Weyl group $W(B_n)$. Most of the above results rely heavily on the representation theory of the symmetric group S_n.

The second eigenvalues of Cayley graphs of the symmetric group S_n or the alternating groups A_n have been determined also for some special generators that are not transpositions. For $1 \leqslant i < j \leqslant n$, let $r_{i,j} \in S_n$ be defined as

$$r_{i,j} = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j & j+1 & \cdots & n \\ 1 & \cdots & i-1 & j & j-1 & \cdots & i+1 & i & j+1 & \cdots & n \end{pmatrix}.$$

In [8], Cesi proved that the second eigenvalue of the pancake graph $\mathcal{P}_n = \mathrm{Cay}(S_n, \{r_{1,j} \mid 2 \leqslant j \leqslant n\})$ is equal to $n-2$. In [12], Chung and Tobin determined the second eigenvalues of the reversal graph $R_n = \mathrm{Cay}(S_n, \{r_{i,j} \mid 1 \leqslant i < j \leqslant n\})$ and a family of graphs that generalize the pancake graph $\mathcal{P}_n$. In [33], Parzanchevski and Puder proved that, for large enough n, if $S \subseteq S_n$ is a full conjugacy class generating S_n then the second eigenvalue of $\mathrm{Cay}(S_n, S)$ is always associated with one of eight low-dimensional representations of S_n. In [25], the authors determined the second eigenvalues of the alternating group graph $AG_n = \mathrm{Cay}(A_n, \{(1, 2, i), (1, i, 2) \mid 3 \leqslant i \leqslant n\})$ (introduced by Jwo, Lakshmivarahan and Dhall [28]), the extended alternating group graph $EAG_n = \mathrm{Cay}(A_n, \{(1, i, j), (1, j, i) \mid 2 \leqslant i < j \leqslant n\})$ and the complete alternating group graph $CAG_n = \mathrm{Cay}(A_n, \{(i, j, k), (i, k, j) \mid$

$1 \leqslant i < j < k \leqslant n\}$) (defined by Huang and Huang [24]).

Suppose that G is a finite group acting transitively on $[n]$ and let $\Gamma = \mathrm{Cay}(G, T)$. In the present paper, we first show that, for each $i \in [n]$, the left coset decomposition of G with respect to the stabilizer subgroup G_i is an equitable partition of Γ, and all these equitable partitions share the same quotient matrix B_Π. Based on this fact, we also prove that those eigenvalues of Γ not belonging to B_Π can be bounded above by the sum of second eigenvalues of some subgraphs of Γ. Now suppose further that Γ is connected and normal, and that the action of G on $[n]$ is of high transitivity. Using the previous result, we reduce the problem of proving $\lambda_2(\Gamma) = \lambda_2(B_\Pi)$ to that of verifying the result for some smaller graphs. This leads to a recursive procedure for determining the second eigenvalue of Γ. As applications, we determine the second eigenvalues of a majority of connected normal Cayley graphs of S_n with $\max_{\tau \in T} |\mathrm{supp}(\tau)| \leqslant 5$ (see Theorem 15 and Table 2), where $\mathrm{supp}(\tau)$ is the set of points in $[n]$ non-fixed by τ. There are 56 families of such graphs, and we determine the second eigenvalues for 41 families of them. In the process, we also determine the second eigenvalues of some subgraphs (over one hundred families) of these 41 families of normal Cayley graphs. From these results we can determine the spectral gap of $\mathrm{Cay}(S_n, \{(p, q) \mid 1 \leqslant p, q \leqslant n\})$ (previously done by Diaconis and Shahshahani [15]) and $\mathrm{Cay}(S_n, \{(1, q) \mid 2 \leqslant q \leqslant n\})$ (previously obtained by Flatto, Odlyzko and Wales [18, Theorem 3.7]). We show that a recent conjecture of Dai [14] is true as a consequence of Aldous' theorem and we discuss some related questions and open problems.

2 Main tools

Let Γ be a graph on n vertices. The vertex partition $\Pi : V(\Gamma) = V_1 \cup V_2 \cup \cdots \cup V_q$ is said to be an *equitable partition* of Γ if every vertex of V_i has the same number (denoted by b_{ij}) of neighbors in V_j, for all $i, j \in \{1, 2, \ldots, q\}$. The matrix $B_\Pi = (b_{ij})_{q \times q}$ is the *quotient matrix* of Γ with respect to Π, and the $n \times q$ matrix χ_Π whose columns are the characteristic vectors of $V_1, \ldots, V_q$ is the *characteristic matrix* of Π.

Lemma 1 (Brouwer and Haemers [5], p. 30; Godsil and Royle [21], pp. 196–198)**.** *Let Γ be a graph with adjacency matrix $A(\Gamma)$, and let $\Pi : V(\Gamma) = V_1 \cup V_2 \cup \cdots \cup V_q$ be an equitable partition of Γ with quotient matrix B_Π. Then the eigenvalues of B_Π are also eigenvalues of $A(\Gamma)$. Furthermore, $A(\Gamma)$ has the following two kinds of eigenvectors:*

(i) the eigenvectors in the column space of χ_Π, and the corresponding eigenvalues coincide with the eigenvalues of B_Π;

(ii) the eigenvectors orthogonal to the columns of χ_Π, i.e., those eigenvectors that sum to zero on each block V_i for $1 \leqslant i \leqslant q$.

If S is a subset of vertices of a graph Γ, let $\Gamma[S]$ denote the subgraph of Γ induced by S. For regular graphs, we have the following useful result.

Theorem 2. *Let Γ be a r-regular graph, and let λ ($\lambda \neq r$) be an eigenvalue of Γ. If Γ has an eigenvector f with respect to λ and a vertex partition $\Pi : V(\Gamma) = V_1 \cup V_2 \cup \cdots \cup V_q$ such that $\Gamma[V_i]$ is r_1-regular ($r_1 \leqslant r$) and f sums to zero on V_i for all $i \in \{1, 2, \ldots, q\}$, then*

$$\lambda \leqslant \max_{1\leqslant i\leqslant q} \lambda_2(\Gamma[V_i]) + \lambda_2(\Gamma_1),$$

where Γ_1 is the $(r-r_1)$-regular graph obtained from Γ by removing all edges in $\cup_{i=1}^{q} E(\Gamma[V_i])$.

Proof. By assumption, the induced subgraphs $\Gamma[V_i]$ share the same degree r_1, so Γ_1 is $(r - r_1)$-regular because Γ is r-regular. Also, the eigenvector f of λ sums to zero on V_i for each i. Set $E_1 = \cup_{i=1}^{q} E(\Gamma[V_i])$ and $E_2 = E(\Gamma) \setminus E_1 = E(\Gamma_1)$. By the Rayleigh quotient, we obtain

$$\begin{aligned} \lambda &= \frac{f^T A(\Gamma) f}{f^T f} \\ &= \frac{2 \sum\limits_{\{x,y\}\in E(\Gamma)} f(x)f(y)}{\sum\limits_{x\in V(\Gamma)} f(x)^2} \\ &= \frac{2 \sum\limits_{\{x,y\}\in E_1} f(x)f(y)}{\sum\limits_{x\in V(\Gamma)} f(x)^2} + \frac{2 \sum\limits_{\{x,y\}\in E_2} f(x)f(y)}{\sum\limits_{x\in V(\Gamma)} f(x)^2}. \end{aligned} \tag{1}$$

For the first term, we have

$$\begin{aligned} \frac{2 \sum\limits_{\{x,y\}\in E_1} f(x)f(y)}{\sum\limits_{x\in V(\Gamma)} f(x)^2} &= \frac{\sum\limits_{i=1}^{q} 2 \sum\limits_{\{x,y\}\in E(\Gamma[V_i])} f(x)f(y)}{\sum\limits_{i=1}^{q} \sum\limits_{x\in V_i} f(x)^2} \\ &\leqslant \max_{\substack{1\leqslant i\leqslant q \\ f|_{V_i}\neq 0}} \frac{2 \sum\limits_{\{x,y\}\in E(\Gamma[V_i])} f(x)f(y)}{\sum\limits_{x\in V_i} f(x)^2} \\ &= \max_{\substack{1\leqslant i\leqslant q \\ f|_{V_i}\neq 0}} \frac{f|_{V_i}^T A(\Gamma[V_i]) f|_{V_i}}{f|_{V_i}^T f|_{V_i}} \\ &\leqslant \max_{\substack{1\leqslant i\leqslant q \\ f|_{V_i}\neq 0}} \max_{g\perp \mathbf{1}_{V_i}} \frac{g^T A(\Gamma[V_i]) g}{g^T g} \\ &= \max_{\substack{1\leqslant i\leqslant q \\ f|_{V_i}\neq 0}} \lambda_2(\Gamma[V_i]) \\ &\leqslant \max_{1\leqslant i\leqslant q} \lambda_2(\Gamma[V_i]), \end{aligned} \tag{2}$$

where $f|_{V_i}$ is the restriction of f on V_i, $\mathbf{1}_{V_i}$ is the all ones vector on V_i, and the second inequality follows from $\sum_{x\in V_i} f(x) = 0$ $(1 \leqslant i \leqslant q)$. For the second term, since Γ_1 is regular and f is orthogonal to the all ones vector $\mathbf{1}$, we have

$$\frac{2\sum\limits_{\{x,y\}\in E_2} f(x)f(y)}{\sum\limits_{x\in V(\Gamma)} f(x)^2} = \frac{f^T A(\Gamma_1) f}{f^T f} \leqslant \max_{h\perp\mathbf{1}} \frac{h^T A(\Gamma_1) h}{h^T h} = \lambda_2(\Gamma_1). \tag{3}$$

Combining (1), (2) and (3), we conclude that

$$\lambda \leqslant \max_{1\leqslant i\leqslant q} \lambda_2(\Gamma[V_i]) + \lambda_2(\Gamma_1),$$

and the result follows. □

If the partition $\Pi : V(\Gamma) = V_1 \cup V_2 \cup \cdots \cup V_q$ is exactly an equitable partition of Γ with quotient matrix B_Π, then the eigenvectors of Γ with respect to those eigenvalues other than that of B_Π must sum to zero on each V_i by Lemma 1. From Theorem 2 one can immediately deduce the following result.

Corollary 3. *Let Γ be a r-regular graph. Assume that $\Pi : V(\Gamma) = V_1 \cup V_2 \cup \cdots \cup V_q$ is an equitable partition of Γ whose quotient matrix B_Π has constant diagonal entries. Then, for any eigenvalue λ of Γ that is not that of B_Π, we have*

$$\lambda \leqslant \max_{1\leqslant i\leqslant q} \lambda_2(\Gamma[V_i]) + \lambda_2(\Gamma_1),$$

where Γ_1 is the graph obtained from Γ by removing all edges in $\cup_{i=1}^q E(\Gamma[V_i])$.

Here we give an example to show how to use the result of Corollary 3.

Example 4. Let Θ_1, Θ_2 be two connected k-regular graphs on n vertices. Let Γ be the graph (not unique) obtained from $\Theta_1 \cup \Theta_2$ by adding some new edges between Θ_1 and Θ_2 such that these edges form a r-regular bipartite graph Γ_1 (Γ_1 is easy to construct, cf. [26], Lemma 3.2). Clearly, Γ is a connected $(k+r)$-regular graph. Let V_1 and V_2 be the vertex subsets of Γ corresponding to Θ_1 and Θ_2, respectively. Then $V(\Gamma) = V_1 \cup V_2$ is clearly an equitable partition of Γ with quotient matrix

$$B_\Pi = \begin{bmatrix} k & r \\ r & k \end{bmatrix}.$$

Since $\lambda_2(\Gamma_1) \leqslant r$, each eigenvalue of Γ not belonging to B_Π is bounded above by

$$\max\{\lambda_2(\Theta_1), \lambda_2(\Theta_2)\} + r$$

according to Corollary 3. As $\lambda_2(B_\Pi) = k - r$, we conclude that

$$k - r \leqslant \lambda_2(\Gamma) \leqslant \max\{\max\{\lambda_2(\Theta_1), \lambda_2(\Theta_2)\} + r, k - r\}.$$

Note that the above bounds could be tight. Take $\Theta_1 = \Theta_2 = Q_n$, the n-dimensional hypercube, and let Γ be the graph (not unique) obtained from $\Theta_1 \cup \Theta_2$ by adding a perfect matching between Θ_1 and Θ_2 (such graphs contain the $(n+1)$-dimensional locally twisted cubes, cf. [34]). Since $\lambda_2(Q_n) = n - 2$ (cf. [5], p. 19), we have

$$n - 1 \leqslant \lambda_2(\Gamma) \leqslant \max\{\lambda_2(Q_n) + 1, n - 1\} = n - 1,$$

and thus $\lambda_2(\Gamma) = n - 1$, which attains the lower bound. Also, the Cartesian product $C_n \Box K_2$, which can be regarded as the graph obtained by adding a perfect matching between two copies of C_n, has second eigenvalue $2\cos\frac{2\pi}{n} + 1 = \lambda_2(C_n) + 1$, and so attains the upper bound.

By using Theorem 2, in what follows, we focus on providing upper bounds for some special eigenvalues of Cayley graphs. Before doing this, we need to do some preparatory work. First of all, we give the following useful result, which suggests that each Cayley graph has an equitable partition derived from left coset decomposition.

Lemma 5. *Let G be a finite group, and let $\Gamma = \mathrm{Cay}(G, T)$ be a Cayley graph of G. Then the set of left cosets of any subgroup H of G gives an equitable partition of Γ.*

Proof. Suppose that $\Pi : G = g_1H \cup g_2H \cup \cdots \cup g_kH$ is the left coset decomposition of G with respect to H, where $k = |G|/|H|$ and $g_1, \ldots, g_k$ are the representation elements. Clearly, Π is a vertex partition of Γ. For any $g \in g_iH$, we have $g = g_ih$ for some $h \in H$. Let $N(g)$ denote the set of neighbors of g in Γ. Then

$$|N(g) \cap g_jH| = |N(g_ih) \cap g_jH| = |(Tg_ih) \cap g_jH| = |T \cap (g_jHh^{-1}g_i^{-1})| = |T \cap (g_jHg_i^{-1})|,$$

which is independent on the choice of $g \in g_iH$. Thus Π is exactly an equitable partition of Γ, and the result follows. □

Let Ω be a nonempty set, and let G be a group acting on Ω. We say that the action of G on Ω ($|\Omega| \geqslant s$) is *s-transitive* if for all pairwise distinct $x_1, \ldots, x_s \in \Omega$ and pairwise distinct $y_1, \ldots, y_s \in \Omega$ there exists some $g \in G$ such that $x_i^g = y_i$ for $1 \leqslant i \leqslant s$. Clearly, a s-transitive action is always t-transitive for any $t < s$. In particular, we say that the action is *transitive* if it is 1-transitive. As usual, we denote by $G_x = \{g \in G \mid x^g = x\}$ the *stabilizer subgroup* of G with respect to $x \in \Omega$.

Now suppose that G is a finite group acting transitively on $[n] = \{1, 2, \ldots, n\}$. For each fixed $i \in [n]$, we have $|G|/|G_i| = n$ by the orbit-stabilizer theorem, and furthermore, we see that G has left coset decomposition

$$\Pi_i : G = g_{1,i}G_i \cup g_{2,i}G_i \cup \cdots \cup g_{n,i}G_i = G_{1,i} \cup G_{2,i} \cup \cdots \cup G_{n,i}, \tag{4}$$

where $g_{j,i}$ is an arbitrary element in G that maps j to i and

$$G_{j,i} = g_{j,i}G_i = \{g \in G \mid j^g = i\},$$

for all $j \in [n]$. Clearly, $|G_{j,i}| = |G_i| = |G|/n$.

Let $\Gamma = \mathrm{Cay}(G,T)$ be a Cayley graph of G. According to Lemma 5, for each $i \in [n]$, the left coset decomposition Π_i given in (4) is an equitable partition of Γ with quotient matrix $B_{\Pi_i} = (b_{st})_{n\times n}$, where

$$b_{st} = |T \cap g_{t,i}G_i g_{s,i}^{-1}| = |T \cap G_{t,s}| \tag{5}$$

is exactly the number of elements in T mapping t to s. Since $b_{st} = |T \cap G_{t,s}|$ is independent on the choice of i, all the equitable partitions Π_i share the same quotient matrix. For this reason, we use B_Π instead of B_{Π_i}. Also, by counting the edges between $G_{s,i}$ and $G_{t,i}$ in two ways, we obtain $b_{st} \cdot |G_{s,i}| = b_{ts} \cdot |G_{t,i}|$, which implies that $b_{st} = b_{ts}$ because $|G_{s,i}| = |G_{t,i}| = |G|/n$. Therefore, $B_\Pi = (b_{st})_{n\times n}$ is symmetric.

For any fixed $k \in [n]$, we also can partition the vertex set of Γ as another form

$$\Pi'_k : G = G_{k,1} \cup G_{k,2} \cup \cdots \cup G_{k,n}, \tag{6}$$

which is exactly the right coset decomposition of G with respect to G_k. In general, Π'_k is not an equitable partition of Γ. As in Theorem 2, we can decompose the edge set of Γ into $E(\Gamma) = E_1 \cup E_2$, where $E_1 = \cup_{i=1}^n E(\Gamma[G_{k,i}])$ and $E_2 = E(\Gamma) \setminus E_1$. Let Γ_1 denote the spanning subgraph of Γ with edge set E_2. The following lemma determines the structure of Γ_1 and $\Gamma[G_{k,i}]$ for all $i \in [n]$.

Lemma 6. *For any fixed $k \in [n]$, we have*

(i) $\Gamma[G_{k,i}] \cong \mathrm{Cay}(G_k, T \cap G_k)$ *for all $i \in [n]$;*

(ii) $\Gamma_1 = \mathrm{Cay}(G, T \setminus (T \cap G_k))$.

Proof. For (i), the corresponding isomorphism can be defined as

$$\begin{aligned} \phi : G_{k,i} = g_{k,i}G_i &\to g_{k,i}G_i g_{k,i}^{-1} = G_k \\ g_{k,i}g &\mapsto g_{k,i}g g_{k,i}^{-1}, \quad \forall g \in G_i. \end{aligned}$$

Clearly, ϕ is one-to-one and onto. Furthermore, we have

$$\begin{aligned} \{g_{k,i}g, g_{k,i}g'\} \in E(\Gamma[G_{k,i}]) &\Longleftrightarrow g_{k,i}g'(g_{k,i}g)^{-1} \in T \\ &\Longleftrightarrow g_{k,i}g'g^{-1}g_{k,i}^{-1} \in T \cap g_{k,i}G_i g_{k,i}^{-1} = T \cap G_k \\ &\Longleftrightarrow g_{k,i}g'g_{k,i}^{-1}(g_{k,i}g g_{k,i}^{-1})^{-1} \in T \cap G_k \\ &\Longleftrightarrow \{g_{k,i}g g_{k,i}^{-1}, g_{k,i}g' g_{k,i}^{-1}\} \in E(\mathrm{Cay}(G_k, T \cap G_k)), \end{aligned}$$

and so (i) follows. Now we consider (ii). Clearly, $\Gamma_1[G_{k,i}]$ is an empty graph for all $i \in [n]$. For any $g_{k,i}g \in G_{k,i} = g_{k,i}G_i$ and $g_{k,j}g' \in G_{k,j} = g_{k,j}G_j$ $(i \neq j)$, we have $\{g_{k,i}g, g_{k,j}g'\} \in E(\Gamma_1)$ if and only if $g_{k,j}g'(g_{k,i}g)^{-1} \in T$, which is the case if and only if $g_{k,j}g'(g_{k,i}g)^{-1} \in T \setminus (T \cap G_k)$ because $g_{k,j}g'(g_{k,i}g)^{-1} = g_{k,j}g'g^{-1}g_{k,i}^{-1} \notin G_k$ due to $i \neq j$. Therefore, each edge of Γ_1 comes from $T \setminus (T \cap G_k)$. Conversely, $T \setminus (T \cap G_k)$ can only be used to produce the edges in $E(\Gamma_1) = E_2$ because each edge in $E_1 = \cup_{i=1}^n E(\Gamma[G_{k,i}])$ comes from $T \cap G_k$. This proves (ii). □

Now we are in a position to give the main result of this section, which provides upper bounds for some special eigenvalues of Cayley graphs.

Theorem 7. *Let G be a finite group acting transitively on $[n] = \{1, 2, \dots, n\}$, and let $\Gamma = \mathrm{Cay}(G, T)$ be a Cayley graph of G. Then the left coset decomposition Π_i of G given in (4) leads to an equitable partition of Γ, and the corresponding quotient matrix $B_\Pi = B_{\Pi_i}$ is symmetric and independent on the choice of i. Moreover, if λ is an eigenvalue of Γ other than that of B_Π, then, for each $k \in [n]$, we have*

$$\lambda \leqslant \lambda_2(\mathrm{Cay}(G_k, T \cap G_k)) + \lambda_2(\mathrm{Cay}(G, T \setminus (T \cap G_k))),$$

where G_k is the stabilizer subgroup of G with respect to k.

Proof. From the above arguments, it suffices to prove the second part of the theorem. Let f be an arbitrary eigenvector of Γ with respect to λ. Since Π_i is an equitable partition of Γ for each i, we see that f must sum to zero on $G_{j,i}$ for all $i, j \in [n]$ by Lemma 1. For any fixed $k \in [n]$, let Π'_k be the vertex partition of Γ given in (6). In particular, we have that f sums to zero on $G_{k,i}$ for all $i \in [n]$. By Lemma 6, all these induced subgraphs $\Gamma[G_{k,i}]$ ($i \in [n]$) are isomorphic to $\mathrm{Cay}(G_k, T \cap G_k)$, and so share the same degree $|T \cap G_k|$. Let Γ_1 be the graph obtained from Γ by removing all edges in $\cup_{i=1}^n E(\Gamma[G_{k,i}])$. Note that $\Gamma_1 \cong \mathrm{Cay}(G, T \setminus (T \cap G_k))$ again by Lemma 6. Then, by applying Theorem 2 to the vertex partition Π'_k, we obtain

$$\begin{aligned}\lambda &\leqslant \max_{1 \leqslant i \leqslant n} \lambda_2(\Gamma[G_{k,i}]) + \lambda_2(\Gamma_1)\\ &= \lambda_2(\mathrm{Cay}(G_k, T \cap G_k)) + \lambda_2(\mathrm{Cay}(G, T \setminus (T \cap G_k))).\end{aligned}$$

By the arbitrariness of $k \in [n]$, our result follows. □

It is worth mentioning that Theorem 7 provides for us a recursive method to determine the second eigenvalue of the connected Cayley graph $\Gamma = \mathrm{Cay}(G, T)$. Indeed, by Lemma 1, all eigenvalues of B_Π are also that of Γ, so we have $\lambda_2(\Gamma) \geqslant \lambda_2(B_\Pi)$. Therefore, if there exists some $k \in [n]$ such that

$$\lambda_2(\mathrm{Cay}(G_k, T \cap G_k)) + \lambda_2(\mathrm{Cay}(G, T \setminus (T \cap G_k))) \leqslant \lambda_2(B_\Pi), \tag{7}$$

then we may conclude that $\lambda_2(\Gamma) = \lambda_2(B_\Pi)$ by Theorem 7. Thus the problem is reduced to determining the exact value of $\lambda_2(\mathrm{Cay}(G_k, T \cap G_k))$ and $\lambda_2(\mathrm{Cay}(G, T \setminus (T \cap G_k)))$, which reminds us that the way of induction could be applied.

In the next section, we shall see that if G and T satisfy some additional conditions then the problem of proving $\lambda_2(\Gamma) = \lambda_2(B_\Pi)$ can be reduced to that of verifying the result for some small graphs.

3 Normal Cayley graphs

For a finite group G, the *conjugacy class* of $g \in G$ is defined as the set $\mathcal{C}_g = \{h^{-1}gh \mid h \in G\}$. Recall that a Cayley graph $\mathrm{Cay}(G, T)$ is said to be normal if T is closed under conjugation, that is, T is the disjoint union of some conjugacy classes of G. It is well-known that the eigenvalues of a normal Cayley graph can be expressed in terms of the irreducible characters of G.

Theorem 8 ([4, 29, 32]). *The eigenvalues of a normal Cayley graph* $\mathrm{Cay}(G,T)$ *are given by*

$$\lambda_\chi = \frac{1}{\chi(1)} \sum_{\tau\in T} \chi(\tau),$$

where χ ranges over all the irreducible characters of G. Moreover, the multiplicity of λ_χ is $\chi(1)^2$.

However, it is often difficult to identify the second eigenvalues of normal Cayley graphs from Theorem 8. In this section, by using Theorem 7, we reduce the problem of determining the second eigenvalues of normal Cayley graphs of highly transitive groups to that of verifying the result for some smaller graphs.

From now on, we always assume that G acts transitively on $[n]$, and that $\Gamma = \mathrm{Cay}(G,T)$ is a connected normal Cayley graph of G, i.e., T is a generating subset of G which is also closed under conjugation. In order to use Theorem 7 recursively, we set $T_0 = T$, $\Gamma_0 = \mathrm{Cay}(G,T_0) = \Gamma$, and for $k = 1, 2, \ldots, n$, we define

$$\begin{aligned} \Gamma_k &= \mathrm{Cay}(G,T_k) \text{ with } T_k = T_{k-1} \setminus (T_{k-1}\cap G_k); \\ \Theta_k &= \mathrm{Cay}(G_k,R_k) \text{ with } R_k = T_{k-1}\cap G_k. \end{aligned} \tag{8}$$

We see that both Γ_k and Θ_k are subgraphs of Γ_{k-1}, and furthermore, by regarding T_{k-1} as T in Lemma 6, we have

Remark 9. The edge set of Γ_{k-1} ($k \geqslant 1$) can be decomposed into that of Γ_k and n copies of Θ_k.

Note that $T_1 = T\setminus(T\cap G_1)$ consists of those elements in T moving 1, $T_2 = T_1\setminus(T_1\cap G_2)$ consists of those elements in T_1 moving 2, i.e., those elements in T moving both 1 and 2, and so on. Thus we have

Remark 10. For each $k \geqslant 1$, T_k is the set of $\tau \in T$ satisfying $\{1, 2, \ldots, k\} \subseteq \mathrm{supp}(\tau)$, i.e., $T_k = T \setminus (T \cap (\cup_{i=1}^{k} G_i))$, and thus $R_k = T_{k-1}\cap G_k$ is the set of elements in T moving $1, 2, \ldots, k-1$ but fixing k.

Note that G acts transitively on $[n]$. For $0 \leqslant k \leqslant n$, from Theorem 7 and (5) we see that the left coset decompositions Π_i ($i \in [n]$) of G given in (4) are equitable partitions of $\Gamma_k = \mathrm{Cay}(G,T_k)$ which share the same symmetric quotient matrix

$$B_\Pi^{(k)} = (b_{st}^{(k)})_{n\times n}, \text{ where } b_{st}^{(k)} = |T_k \cap G_{t,s}|. \tag{9}$$

In particular, $B_\Pi^{(0)} = B_\Pi$.

To achieve our goal, we need to determine the second eigenvalue of $B_\Pi^{(k)}$ ($k \geqslant 0$).

Lemma 11. *Let* $\Gamma_k = \mathrm{Cay}(G, T_k)$ $(k \geqslant 0)$ *be the graph defined in (8), and* $B_\Pi^{(k)}$ *the quotient matrix of* Γ_k *defined in (9). If* G *acts* $(k+2)$*-transitively on* $[n]$*, then* $\lambda_2(B_\Pi^{(k)}) = |T_k \cap G_{k+1}| - |T_k \cap G_{k+2,k+1}|$.

Proof. First suppose $k = 0$. According to (9), we have $B_\Pi^{(0)} = (b_{st}^{(0)})_{n\times n}$, where $b_{st}^{(0)} = |T_0 \cap G_{t,s}|$. Since G acts 2-transitively on $[n]$, for any $s \in [n]$, there exists some $g \in G$ such that g maps s to 1. Considering that $T_0 = T$ is closed under conjugation, we have $b_{ss}^{(0)} = |T_0 \cap G_{s,s}| = |T_0 \cap G_s| = |g^{-1}(T_0 \cap G_s)g| = |(g^{-1}T_0 g) \cap (g^{-1}G_s g)| = |T_0 \cap G_1| = b_{11}^{(0)}$. Similarly, for any two distinct $s, t \in [n]$, there exists some g in G mapping s to 1 and t to 2 by the 2-transitivity of G acting on $[n]$. Then $b_{st}^{(0)} = |T_0 \cap G_{t,s}| = |g^{-1}(T_0 \cap G_{t,s})g| = |(g^{-1}T_0 g) \cap (g^{-1}G_{t,s}g)| = |T_0 \cap G_{t^g,s^g}| = |T_0 \cap G_{2,1}| = b_{12}^{(0)}$. Combining these results, we have

$$B_\Pi^{(0)} = b_{11}^{(0)} \cdot I_n + b_{12}^{(0)} \cdot (J_n - I_n).$$

Thus the quotient matrix $B_\Pi^{(0)}$ has eigenvalues $|T| = b_{11}^{(0)} + (n-1)\cdot b_{12}^{(0)}$ of multiplicity one and $b_{11}^{(0)} - b_{12}^{(0)}$ of multiplicity $n-1$. Therefore, $\lambda_2(B_\Pi^{(0)}) = b_{11}^{(0)} - b_{12}^{(0)} = |T_0 \cap G_1| - |T_0 \cap G_{2,1}|$, and our result follows.

Now suppose $k \geqslant 1$. By definition, we see that $T_k = T \setminus (T \cap (\cup_{l=1}^k G_l))$. We claim that if g is an element in G fixing $\{1, 2, \ldots, k\}$ setwise then $g^{-1}T_k g = T_k$. Indeed, we have $g^{-1}T_k g = (g^{-1}Tg) \setminus ((g^{-1}Tg) \cap (\cup_{l=1}^k g^{-1}G_l g)) = T \setminus (T \cap (\cup_{l=1}^k G_{l^g})) = T \setminus (T \cap (\cup_{l=1}^k G_l)) = T_k$, as required.

We shall determine all eigenvalues of $B_\Pi^{(k)}$. According to (9), we see that $B_\Pi^{(k)} = (b_{st}^{(k)})$, where $b_{st}^{(k)} = |T_k \cap G_{t,s}|$. For $1 \leqslant s \leqslant k$, we have $b_{ss}^{(k)} = |T_k \cap G_{s,s}| = 0$ because T_k must move s but $G_{s,s} = G_s$ does not. For $k+1 \leqslant s \leqslant n$, by the $(k+2)$-transitivity of G acting on $[n]$, there is a $g \in G$ fixing $\{1, 2, \ldots, k\}$ setwise but moving s to $k+1$. Then $g^{-1}T_k g = T_k$ and $g^{-1}G_s g = G_{k+1}$ by above arguments, and thus $b_{ss}^{(k)} = |T_k \cap G_{s,s}| = |T_k \cap G_s| = |g^{-1}(T_k \cap G_s)g| = |(g^{-1}T_k g) \cap (g^{-1}G_s g)| = |T_k \cap G_{k+1}| = b_{k+1,k+1}^{(k)}$. For $1 \leqslant s < t \leqslant k$ (if $k \geqslant 2$), again by the $(k+2)$-transitivity, we can choose $g \in G$ such that g moves t to 2 and s to 1 but fixes $\{1, 2, \ldots, k\}$ setwise. Then we see that $b_{st}^{(k)} = |T_k \cap G_{t,s}| = |g^{-1}(T_k \cap G_{t,s})g| = |(g^{-1}T_k g) \cap (g^{-1}G_{t,s}g)| = |T_k \cap G_{2,1}| = b_{12}^{(k)}$. For $1 \leqslant s \leqslant k$ and $k+1 \leqslant t \leqslant n$, there also exists some g in G mapping s to 1, t to $k+1$ but fixing $\{1, 2, \ldots, k\}$ setwise, thus we get $b_{st}^{(k)} = |T_k \cap G_{t,s}| = |g^{-1}(T_k \cap G_{t,s})g| = |T_k \cap G_{k+1,1}| = b_{1,k+1}^{(k)}$. For $k+1 \leqslant s < t \leqslant n$, we take $g \in G$ such that g maps s to $k+1$ and t to $k+2$ but fixes $\{1, 2, \ldots, k\}$ setwise. Then $b_{st}^{(k)} = |T_k \cap G_{t,s}| = |g^{-1}(T_k \cap G_{t,s})g| = |T_k \cap G_{k+2,k+1}| = b_{k+1,k+2}^{(k)}$. Concluding these results, we have

$$b_{st}^{(k)} = b_{ts}^{(k)} = \begin{cases} 0, & \text{if } 1 \leqslant s = t \leqslant k; \\ |T_k \cap G_{k+1}| = b_{k+1,k+1}^{(k)}, & \text{if } k+1 \leqslant s = t \leqslant n; \\ |T_k \cap G_{2,1}| = b_{1,2}^{(k)}, & \text{if } 1 \leqslant s < t \leqslant k \text{ (for } k \geqslant 2); \\ |T_k \cap G_{k+1,1}| = b_{1,k+1}^{(k)}, & \text{if } 1 \leqslant s \leqslant k,\ k+1 \leqslant t \leqslant n; \\ |T_k \cap G_{k+2,k+1}| = b_{k+1,k+2}^{(k)}, & \text{if } k+1 \leqslant s < t \leqslant n. \end{cases}$$

Therefore, the quotient matrix $B_\Pi^{(k)}$ can be written as

$$B_\Pi^{(k)} = \begin{bmatrix} b_{1,2}^{(k)} \cdot (J_k - I_k) & b_{1,k+1}^{(k)} \cdot J_{k\times(n-k)} \\ b_{1,k+1}^{(k)} \cdot J_{(n-k)\times k} & b_{k+1,k+1}^{(k)} \cdot I_{n-k} + b_{k+1,k+2}^{(k)} \cdot (J_{n-k} - I_{n-k}) \end{bmatrix}.$$

Take $x_1 = (y_1^T, 0^T)^T \in \mathbb{R}^n$ and $x_2 = (0^T, y_2^T)^T \in \mathbb{R}^n$, where $y_1 \in \mathbb{R}^k$ and $y_2 \in \mathbb{R}^{n-k}$ are two arbitrary vectors orthogonal to the all ones vector, respectively. One can easily verify that $B_\Pi^{(k)} x_1 = -b_{1,2}^{(k)} \cdot x_1$ and $B_\Pi^{(k)} x_2 = (b_{k+1,k+1}^{(k)} - b_{k+1,k+2}^{(k)}) \cdot x_2$, so $-b_{1,2}^{(k)}$ and $b_{k+1,k+1}^{(k)} - b_{k+1,k+2}^{(k)}$ are eigenvalues of $B_\Pi^{(k)}$ with multiplicities at least $k-1$ and $n-k-1$, respectively. Also note that $|T_k|$ is always an eigenvalue of $B_\Pi^{(k)}$ with the all ones vector as its eigenvector because $\Gamma_k = \mathrm{Cay}(G, T_k)$ is $|T_k|$-regular. Thus there is just one eigenvalue, denoted by μ, that is not known. By computing the trace of $B_\Pi^{(k)}$ in two ways, we obtain

$$(n-k) \cdot b_{k+1,k+1}^{(k)} = |T_k| - (k-1) \cdot b_{1,2}^{(k)} + (n-k-1) \cdot (b_{k+1,k+1}^{(k)} - b_{k+1,k+2}^{(k)}) + \mu,$$

which gives that

$$\begin{aligned} \mu &= b_{k+1,k+1}^{(k)} + (n-k-1) \cdot b_{k+1,k+2}^{(k)} - (|T_k| - (k-1) \cdot b_{1,2}^{(k)}) \\ &= b_{k+1,k+1}^{(k)} + (n-k-1) \cdot b_{k+1,k+2}^{(k)} - (n-k) \cdot b_{1,k+1}^{(k)} \\ &= b_{k+1,k+1}^{(k)} + (n-k-1) \cdot b_{k+1,k+2}^{(k)} - (n-k) \cdot b_{k+1,1}^{(k)}. \end{aligned}$$

Thus the eigenvalues of $B_\Pi^{(k)}$ are $|T|$, $-b_{1,2}^{(k)}$ (with multiplicity $k-1$), $b_{k+1,k+1}^{(k)} - b_{k+1,k+2}^{(k)}$ (with multiplicity $n-k-1$) and $\mu = b_{k+1,k+1}^{(k)} + (n-k-1) \cdot b_{k+1,k+2}^{(k)} - (n-k) \cdot b_{k+1,1}^{(k)}$.

Now we prove that $\lambda_2(B_\Pi^{(k)}) = b_{k+1,k+1}^{(k)} - b_{k+1,k+2}^{(k)}$. Since $\lambda_1(B_\Pi^{(k)}) = |T_k|$, it remains to compare the remaining eigenvalues. To prove $b_{k+1,k+1}^{(k)} - b_{k+1,k+2}^{(k)} \geqslant \mu = b_{k+1,k+1}^{(k)} + (n-k-1) \cdot b_{k+1,k+2}^{(k)} - (n-k) \cdot b_{k+1,1}^{(k)}$, it suffices to show that $b_{k+1,1}^{(k)} \geqslant b_{k+1,k+2}^{(k)}$. Indeed, by the $(k+2)$-transitivity of G acting on $[n]$, there exists some $g \in G$ such that g moves 1 to $k+2$ but fixes $k+1$ and $\{2, \ldots, k\}$ setwise. Then $g^{-1} T_k g = (g^{-1} T g) \setminus ((g^{-1} T g) \cap (\cup_{l=1}^k g^{-1} G_l g)) = T \setminus (T \cap (\cup_{l=1}^k G_{l^g})) = T \setminus (T \cap (G_{k+2} \cup (\cup_{l=2}^k G_l)))$, and so we obtain

$$\begin{aligned} b_{k+1,1}^{(k)} &= |T_k \cap G_{1,k+1}| \\ &= |g^{-1}(T_k \cap G_{1,k+1}) g| \\ &= |(g^{-1} T_k g) \cap (g^{-1} G_{1,k+1} g)| \\ &= |(T \setminus (T \cap (G_{k+2} \cup (\cup_{l=2}^k G_l)))) \cap G_{k+2,k+1}| \\ &= |T \cap G_{k+2,k+1}| - |T \cap (G_{k+2} \cup (\cup_{l=2}^k G_l)) \cap G_{k+2,k+1}| \\ &= |T \cap G_{k+2,k+1}| - |T \cap (\cup_{l=2}^k G_l) \cap G_{k+2,k+1}|, \end{aligned} \tag{10}$$

where the last equality follows from $G_{k+2} \cap G_{k+2,k+1} = \varnothing$. Also, we see that

$$b_{k+1,k+2}^{(k)} = |T_k \cap G_{k+2,k+1}| = |T \cap G_{k+2,k+1}| - |T \cap (\cup_{l=1}^k G_l) \cap G_{k+2,k+1}|. \tag{11}$$

Combining (10) and (11) yields

$$b_{k+1,1}^{(k)} - b_{k+1,k+2}^{(k)} = |T \cap (\cup_{l=1}^{k} G_l) \cap G_{k+2,k+1}| - |T \cap (\cup_{l=2}^{k} G_l) \cap G_{k+2,k+1}| \geqslant 0,$$

as required. Now let us show that $b_{k+1,k+1}^{(k)} - b_{k+1,k+2}^{(k)} \geqslant -b_{1,2}^{(k)}$. Since $-b_{1,2}^{(k)}$ is not an eigenvalue of $B_{\Pi}^{(k)}$ when $k = 1$, we can suppose $k \geqslant 2$. If we can prove $b_{1,2}^{(k)} \geqslant b_{k+1,k+2}^{(k)}$, then the result follows because $b_{k+1,k+1}^{(k)} \geqslant 0$. As above, by taking $g \in G$ such that g maps 1 to $k+1$ and 2 to $k+2$ but fixes $\{3, \ldots, k\}$ setwise, we get

$$\begin{aligned} b_{1,2}^{(k)} &= |T_k \cap G_{2,1}| \\ &= |g^{-1}(T_k \cap G_{2,1})g| \\ &= |(g^{-1}T_k g) \cap G_{k+2,k+1}| \\ &= |T \cap G_{k+2,k+1}| - |T \cap (\cup_{l=3}^{k+2} G_l) \cap G_{k+2,k+1}| \\ &= |T \cap G_{k+2,k+1}| - |T \cap (\cup_{l=3}^{k} G_l) \cap G_{k+2,k+1}|. \end{aligned} \tag{12}$$

Combining (11) and (12), we have

$$b_{1,2}^{(k)} - b_{k+1,k+2}^{(k)} = |T \cap (\cup_{l=1}^{k} G_l) \cap G_{k+2,k+1}| - |T \cap (\cup_{l=3}^{k} G_l) \cap G_{k+2,k+1}| \geqslant 0,$$

and the result follows. Hence we conclude that

$$\lambda_2(B_{\Pi}^{k}) = b_{k+1,k+1}^{(k)} - b_{k+1,k+2}^{(k)} = |T_k \cap G_{k+1}| - |T_k \cap G_{k+2,k+1}|.$$

The proof is complete. □

Set

$$m = \max_{\tau \in T} |\mathrm{supp}(\tau)|.$$

If $m < n$, then we claim that $\Gamma_m = \mathrm{Cay}(G, T_m)$ is disconnected. Indeed, by the definition, T_m consists of those $\tau \in T$ such that $\{1, 2, \ldots, m\} \subseteq \mathrm{supp}(\tau)$. Since each element of T has at most m supports, we have $\mathrm{supp}(\tau) = \{1, 2, \ldots, m\}$ for any $\tau \in T_m$, which implies that T_m cannot generate G due to $m < n$.

In the following, we suppose further that the action of G on $[n]$ is $(m+a)$-transitive with $a \geqslant 1$. Under this assumption, it is clear that $n \geqslant m + a$, and so $m < n$, implying that Γ_m is disconnected. Denote by

$$G^{(0)} = G \text{ and } G^{(i)} = \cap_{j=1}^{i} G_{n-j+1} \text{ for } 1 \leqslant i \leqslant a-1. \tag{13}$$

Indeed, $G^{(i)}$ $(1 \leqslant i \leqslant a-1)$ is just the subgroup of G that fixes each point of $\{n-i+1, \ldots, n\}$. For this reason, we can also regard $G^{(i)}$ as a group acting on $[n-i] = \{1, 2, \ldots, n-i\}$. Moreover, this action is $(m+a-i)$-transitive because G acts $(m+a)$-transitively on $[n]$. For $0 \leqslant i \leqslant a-1$, we define

$$\begin{aligned} \Gamma_{k,i} &= \mathrm{Cay}(G^{(i)}, T_k \cap G^{(i)}) \text{ for } 0 \leqslant k \leqslant m; \\ \Theta_{k,i} &= \mathrm{Cay}(G^{(i)} \cap G_k, R_k \cap G^{(i)}) \text{ for } 1 \leqslant k \leqslant m, \end{aligned} \tag{14}$$

where $G^{(i)}$ is defined in (13), and T_k, R_k are given in (8). By definition, $\Gamma_{k,0} = \Gamma_k = \mathrm{Cay}(G, T_k)$, $\Theta_{k,0} = \Theta_k = \mathrm{Cay}(G_k, R_k)$, and $\Gamma_{k,i}$ is the subgraph of both $\Gamma_{k-1,i}$ and $\Gamma_{k,i-1}$. As in Remark 9, the edge set of $\Gamma_{k-1,i}$ can be decomposed into that of $\Gamma_{k,i}$ and $(n-i)$-copies of $\Theta_{k,i}$. Also, for each fixed i, we see that $T_0 \cap G^{(i)} = T \cap G^{(i)}$ is closed under conjugation in $G^{(i)}$, and $T_k \cap G^{(i)}$ is just the set of elements in $T \cap G^{(i)}$ moving each point of $\{1, 2, \dots, k\}$ (similar as Remark 10). Furthermore, since $n - i \geqslant m + a - i \geqslant m + 1$, we claim that $T_m \subseteq G^{(i)}$ and that $\Gamma_{m,i} = \mathrm{Cay}(G^{(i)}, T_m \cap G^{(i)}) = \mathrm{Cay}(G^{(i)}, T_m)$ is disconnected. In particular, we have $\lambda_2(\Gamma_{m,i}) = |T_m \cap G^{(i)}| = |T_m|$ for all $0 \leqslant i \leqslant a-1$. Recall that $G^{(i)}$ acts $(m+a-i)$-transitively $(m + a - i \geqslant m + 1)$ on $[n-i]$. According to Lemma 5 and the arguments in Section 2, every left coset decomposition of $G^{(i)}$ with respect to some stabilizer subgroup leads to an equitable partition of $\Gamma_{k,i}$, and all these equitable partitions share the same quotient matrix

$$B_{\Pi}^{(k,i)} = (b_{st}^{(k,i)})_{(n-i)\times(n-i)}, \text{ where } b_{st}^{(k,i)} = |T_k \cap G^{(i)} \cap G_{t,s}|.$$

Clearly, $B_{\Pi}^{(k,0)}$ coincides with $B_{\Pi}^{(k)}$. For $0 \leqslant k \leqslant m-1$, we have $k+2 \leqslant m+1 \leqslant m+a-i$, and so $G^{(i)}$ acts $(k+2)$-transitively on $[n-i]$. By applying Lemma 11 to $\Gamma_{k,i}$, we obtain

$$\lambda_2(B_{\Pi}^{(k,i)}) = |T_k \cap G^{(i)} \cap G_{k+1}| - |T_k \cap G^{(i)} \cap G_{k+2,k+1}|, \tag{15}$$

where $0 \leqslant k \leqslant m-1$ and $0 \leqslant i \leqslant a-1$.

Before giving the main result of this section, we need the following two lemmas.

Lemma 12. *Let m, a and $B_{\Pi}^{(k,i)}$ be defined as above. Assume that $a \geqslant 2$. For $0 \leqslant i \leqslant a-2$, we have*

$$\lambda_2(B_{\Pi}^{(k,i)}) - \lambda_2(B_{\Pi}^{(k,i+1)}) = \begin{cases} \lambda_2(B_{\Pi}^{(k+1,i)}), & \text{if } 0 \leqslant k \leqslant m-2; \\ |T_m|, & \text{if } k = m-1. \end{cases}$$

Proof. Since G acts $(m+a)$-transitively on $[n]$, there exists some $g_1, g_2 \in G$ such that g_1 moves $k+1$ to $k+2$, $n-i$ to $k+1$, g_2 moves $k+1$ to $k+2$, $k+2$ to $k+3$ and $n-i$ to $k+1$, and both of them fix $\{1, \dots, k\}$ and $\{n-i+1, \dots, n\}$ setwise. Then we have $g_j^{-1} T_k g_j = T_k$, $g_j^{-1} G^{(i)} g_j = G^{(i)}$ and $g_j^{-1} G^{(i+1)} g_j = g_j^{-1}(G_{n-i} \cap G^{(i)}) g_j = G_{k+1} \cap G^{(i)}$ for $j = 1, 2$, which gives that

$$\begin{cases} g_1^{-1}(T_k \cap G^{(i)} \cap G_{k+1}) g_1 = T_k \cap G^{(i)} \cap G_{k+2}; \\ g_1^{-1}(T_k \cap G^{(i+1)} \cap G_{k+1}) g_1 = T_k \cap G_{k+1} \cap G^{(i)} \cap G_{k+2}; \\ g_2^{-1}(T_k \cap G^{(i)} \cap G_{k+2,k+1}) g_2 = T_k \cap G^{(i)} \cap G_{k+3,k+2}; \\ g_2^{-1}(T_k \cap G^{(i+1)} \cap G_{k+2,k+1}) g_2 = T_k \cap G_{k+1} \cap G^{(i)} \cap G_{k+3,k+2}. \end{cases} \tag{16}$$

Also recall that $T_{k+1} = T_k \setminus (T_k \cap G_{k+1})$. According to (15) and (16), we deduce that

$$\begin{aligned}
\lambda_2(B_\Pi^{(k,i)}) - \lambda_2(B_\Pi^{(k,i+1)}) &= (|T_k \cap G^{(i)} \cap G_{k+1}| - |T_k \cap G^{(i)} \cap G_{k+2,k+1}|) - \\
&\quad (|T_k \cap G^{(i+1)} \cap G_{k+1}| - |T_k \cap G^{(i+1)} \cap G_{k+2,k+1}|) \\
&= (|T_k \cap G^{(i)} \cap G_{k+1}| - |T_k \cap G^{(i+1)} \cap G_{k+1}|) - \\
&\quad (|T_k \cap G^{(i)} \cap G_{k+2,k+1}| - |T_k \cap G^{(i+1)} \cap G_{k+2,k+1}|) \\
&= (|T_k \cap G^{(i)} \cap G_{k+2}| - |T_k \cap G_{k+1} \cap G^{(i)} \cap G_{k+2}|) - \\
&\quad (|T_k \cap G^{(i)} \cap G_{k+3,k+2}| - |T_k \cap G_{k+1} \cap G^{(i)} \cap G_{k+3,k+2}|) \\
&= |T_{k+1} \cap G^{(i)} \cap G_{k+2}| - |T_{k+1} \cap G^{(i)} \cap G_{k+3,k+2}|.
\end{aligned}$$

Therefore, if $0 \leqslant k \leqslant m-2$, we have $\lambda_2(B_\Pi^{(k,i)}) - \lambda_2(B_\Pi^{(k,i+1)}) = \lambda_2(B_\Pi^{(k+1,i)})$ again by (15); if $k = m-1$, we have $\lambda_2(B_\Pi^{(m-1,i)}) - \lambda_2(B_\Pi^{(m-1,i+1)}) = |T_m \cap G^{(i)} \cap G_{m+1}| - |T_m \cap G^{(i)} \cap G_{m+2,m+1}| = |T_m| - 0 = |T_m|$ because $\mathrm{supp}(\tau) = \{1, 2, \ldots, m\}$ for any $\tau \in T_m \cap G^{(i)} = T_m$. □

Lemma 13. *Let m, a, $\Gamma_{k,i}$ and $\Theta_{k,i}$ be defined as above. Assume that $a \geqslant 2$. For $0 \leqslant i \leqslant a-2$ and $0 \leqslant k \leqslant m-1$, we have $\Theta_{k+1,i} \cong \Gamma_{k,i+1}$.*

Proof. According to (14), we see that

$$\Theta_{k+1,i} = \mathrm{Cay}(G^{(i)} \cap G_{k+1}, R_{k+1} \cap G^{(i)}) = \mathrm{Cay}(G^{(i)} \cap G_{k+1}, T_k \cap G_{k+1} \cap G^{(i)})$$

and

$$\Gamma_{k,i+1} = \mathrm{Cay}(G^{(i+1)}, T_k \cap G^{(i+1)}).$$

By the $(m+a)$-transitivity of G acting on $[n]$, we can choose $g \in G$ such that g moves $k+1$ to $n-i$ but fixes $\{1, \ldots, k\}$ and $\{n-i+1, \ldots, n\}$ setwise. Then we see that $g^{-1}(G_{k+1} \cap G^{(i)})g = G_{n-i} \cap G^{(i)} = G^{(i+1)}$ and $g^{-1}(T_k \cap G_{k+1} \cap G^{(i)})g = T_k \cap G_{n-i} \cap G^{(i)} = T_k \cap G^{(i+1)}$. Thus g induces an isomorphism from $\Theta_{k+1,i}$ to $\Gamma_{k,i+1}$ naturally. □

Now we give the main result of this section, which indicates that the problem of proving $\lambda_2(\Gamma_k) = \lambda_2(B_\Pi^{(k)})$ $(0 \leqslant k \leqslant m-1)$ can be reduced to verifying the result for some small graphs.

Theorem 14. *Let G be a finite group acting on $[n]$, and let $\Gamma = \mathrm{Cay}(G, T)$ be a connected normal Cayley graph of G. Let $m = \max_{\tau \in T} |\mathrm{supp}(\tau)|$. If the action of G on $[n]$ is $(m+a)$-transitive with $a \geqslant 1$ and $\lambda_2(\Gamma_{k,a-1}) = \lambda_2(B_\Pi^{(k,a-1)})$ for all $k \in \{0, 1, \ldots, m-1\}$, then we have*

$$\lambda_2(\Gamma_k) = \lambda_2(\Gamma_{k,0}) = \lambda_2(B_\Pi^{(k,0)}) = \lambda_2(B_\Pi^{(k)}) = |T_k \cap G_{k+1}| - |T_k \cap G_{k+2,k+1}|,$$

where $0 \leqslant k \leqslant m-1$. In particular, $\lambda_2(\Gamma) = \lambda_2(\Gamma_0) = \lambda_2(B_\Pi^{(0)}) = |T \cap G_1| - |T \cap G_{2,1}|$.

Proof. If $a = 1$, there is nothing to prove. Thus we assume that $a \geqslant 2$. The main idea is to prove $\lambda_2(\Gamma_{k,i}) = \lambda_2(B_\Pi^{(k,i)})$ for all $0 \leqslant k \leqslant m-1$ and $0 \leqslant i \leqslant a-1$ by induction on k and i.

First of all, we shall verify the induction basis. By assumption, we have known that $\lambda_2(\Gamma_{k,a-1}) = \lambda_2(B_\Pi^{(k,a-1)})$ for all $0 \leqslant k \leqslant m-1$. Thus it suffices to verify $\lambda_2(\Gamma_{m-1,i}) = \lambda_2(B_\Pi^{(m-1,i)})$ for all $0 \leqslant i \leqslant a-1$. If $i = a-1$, we obtain the result again by assumption. Now suppose $0 \leqslant i < a-1$, and assume that the result holds for $i+1$, i.e., $\lambda_2(\Gamma_{m-1,i+1}) = \lambda_2(B_\Pi^{(m-1,i+1)})$. We shall prove $\lambda_2(\Gamma_{m-1,i}) = \lambda_2(B_\Pi^{(m-1,i)})$. According to the arguments below Theorem 7 and (7), we only need to show $\lambda_2(B_\Pi^{(m-1,i)}) \geqslant \lambda_2(\Theta_{m,i}) + \lambda_2(\Gamma_{m,i})$. From Lemma 13 we see that $\Theta_{m,i} \cong \Gamma_{m-1,i+1}$, so $\lambda_2(\Theta_{m,i}) = \lambda_2(\Gamma_{m-1,i+1}) = \lambda_2(B_\Pi^{(m-1,i+1)})$ by the induction hypothesis. Also, as mentioned above, we have $\lambda_2(\Gamma_{m,i}) = |T_m \cap G^{(i)}| = |T_m|$ because $\Gamma_{m,i}$ is disconnected. Therefore, from Lemma 12 we deduce that

$$\lambda_2(B_\Pi^{(m-1,i)}) - \lambda_2(\Theta_{m,i}) = \lambda_2(B_\Pi^{(m-1,i)}) - \lambda_2(B_\Pi^{(m-1,i+1)}) = |T_m| = \lambda_2(\Gamma_{m,i}),$$

as required. Thus we have built up the induction basis.

Now suppose $0 \leqslant k < m-1$ and $0 \leqslant i < a-1$, and assume that the result holds for $k+1, i$ and $k, i+1$, i.e., $\lambda_2(\Gamma_{k+1,i}) = \lambda_2(B_\Pi^{(k+1,i)})$ and $\lambda_2(\Gamma_{k,i+1}) = \lambda_2(B_\Pi^{(k,i+1)})$. We shall prove $\lambda_2(\Gamma_{k,i}) = \lambda_2(B_\Pi^{(k,i)})$. As above, it remains to show that $\lambda_2(B_\Pi^{(k,i)}) \geqslant \lambda_2(\Theta_{k+1,i}) + \lambda_2(\Gamma_{k+1,i})$. Again by Lemma 13 and the induction hypothesis, we have $\lambda_2(\Theta_{k+1,i}) = \lambda_2(\Gamma_{k,i+1}) = \lambda_2(B_\Pi^{(k,i+1)})$ and $\lambda_2(\Gamma_{k+1,i}) = \lambda_2(B_\Pi^{(k+1,i)})$. Then from Lemma 12 we obtain

$$\lambda_2(B_\Pi^{(k,i)}) - \lambda_2(\Theta_{k+1,i}) = \lambda_2(B_\Pi^{(k,i)}) - \lambda_2(B_\Pi^{(k,i+1)}) = \lambda_2(B_\Pi^{(k+1,i)}) = \lambda_2(\Gamma_{k+1,i}),$$

and the result follows.

Therefore, we may conclude that $\lambda_2(\Gamma_{k,i}) = \lambda_2(B_\Pi^{(k,i)})$ for all $0 \leqslant k \leqslant m-1$ and $0 \leqslant i \leqslant a-1$. In particular, for $0 \leqslant k \leqslant m-1$, we have $\lambda_2(\Gamma_k) = \lambda_2(\Gamma_{k,0}) = \lambda_2(B_\Pi^{(k,0)}) = |T_k \cap G_{k+1}| - |T_k \cap G_{k+2,k+1}|$. □

According to Theorem 14, to prove $\lambda_2(\Gamma) = \lambda_2(\Gamma_0) = \lambda_2(B_\Pi^{(0)}) = |T \cap G_1| - |T \cap G_{2,1}|$ (and as by-products, $\lambda_2(\Gamma_k) = \lambda_2(B_\Pi^{(k)})$ for $1 \leqslant k \leqslant m-1$), it suffices to verify $\lambda_2(\Gamma_{k,a-1}) = \lambda_2(B_\Pi^{(k,a-1)})$ for all $k \in \{0, 1, \ldots, m-1\}$. Note that if a is relatively large, i.e., the action of G on $[n]$ is of high transitivity, then the graph $\Gamma_{k,a-1}$ will be of small order. This makes it easier to verify the equalities. It is well-known that the symmetric group S_n acts n-transitively on $[n]$, so Theorem 14 is particularly effective for normal Cayley graphs of S_n. In the next section, we consider to determine the second eigenvalues of connected normal Cayley graphs of S_n with $m \leqslant 5$.

4 The second eigenvalues of normal Cayley graphs of symmetric groups

Let $G = S_n$ be the symmetric group on $[n]$ with $n \geqslant 3$. As mentioned earlier, S_n acts n-transitively on $[n]$, and also, two elements in S_n are conjugated if and only if they share the same cycle type (see [27, Theorem 6.5]). Let $\Gamma = \mathrm{Cay}(S_n, T)$ be a normal Cayley graph of S_n, that is, T is the disjoint union of some conjugacy classes of S_n. Then Γ is connected if and only if T contains some odd permutation. This is because T generates a non-identity normal subgroup of S_n while A_n is the unique nontrivial normal subgroup of S_n for $n \neq 4$, and A_4 and $\{e, (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\} \leqslant A_4$ are the only nontrivial normal subgroups of S_4.

In this section, as applications of Theorem 14, we consider the second eigenvalues of connected normal Cayley graphs of S_n for which each element of the connection set has at most five supports.

For convenience, we first list all the nontrivial conjugacy classes of S_n with each element having at most five supports:

$$\begin{cases} \mathcal{C}^{(1)} = \{(p,q) \mid 1 \leqslant p,q \leqslant n\}; \\ \mathcal{C}^{(2)} = \{(p,q,r) \mid 1 \leqslant p,q,r \leqslant n\}; \\ \mathcal{C}^{(3)} = \{(p,q)(r,s) \mid 1 \leqslant p,q,r,s \leqslant n\}; \\ \mathcal{C}^{(4)} = \{(p,q,r,s) \mid 1 \leqslant p,q,r,s \leqslant n\}; \\ \mathcal{C}^{(5)} = \{(p,q,r)(s,t) \mid 1 \leqslant p,q,r,s,t \leqslant n\}; \\ \mathcal{C}^{(6)} = \{(p,q,r,s,t) \mid 1 \leqslant p,q,r,s,t \leqslant n\}, \end{cases} \tag{17}$$

where p,q,r,s,t are pairwise distinct. For $k \in [n]$, we denote by $\mathcal{C}_k^{(i)}$ (see Table 1) the set of elements in $\mathcal{C}^{(i)}$ that moves each point of $\{1,2,\ldots,k\}$, where $1 \leqslant i \leqslant 6$.

Table 1: The structure of $\mathcal{C}_k^{(i)}$ for $1 \leqslant i \leqslant 6$ and $k \in [n]$.

i	k	$\mathcal{C}_k^{(i)}$
1	1	$\{(1,q) \mid 2 \leqslant q \leqslant n\}$
1	2	$\{(1,2)\}$
1	$\geqslant 3$	$\varnothing$
2	1	$\{(1,q,r) \mid 2 \leqslant q,r \leqslant n\}$
2	2	$\{(1,2,r),(1,r,2) \mid 3 \leqslant r \leqslant n\}$
2	3	$\{(1,2,3),(1,3,2)\}$
2	$\geqslant 4$	$\varnothing$
3	1	$\{(1,q)(r,s) \mid 2 \leqslant q,r,s \leqslant n\}$
3	2	$\{(1,2)(r,s),(1,r)(2,s) \mid 3 \leqslant r,s \leqslant n\}$
3	3	$\{(1,2)(3,s),(1,3)(2,s),(1,s)(2,3) \mid 4 \leqslant s \leqslant n\}$
3	4	$\{(1,2)(3,4),(1,3)(2,4),(1,4)(2,3)\}$
3	$\geqslant 5$	$\varnothing$
4	1	$\{(1,q,r,s) \mid 2 \leqslant q,r,s \leqslant n\}$
4	2	$\{(1,2,r,s),(1,r,2,s),(1,r,s,2) \mid 3 \leqslant r,s \leqslant n\}$
4	3	$\{(1,2,3,s),(1,2,s,3),(1,3,2,s),(1,3,s,2),(1,s,2,3),(1,s,3,2) \mid 4 \leqslant s \leqslant n\}$
4	4	$\{(1,2,3,4),(1,2,4,3),(1,3,2,4),(1,3,4,2),(1,4,2,3),(1,4,3,2)\}$
4	$\geqslant 5$	$\varnothing$
5	1	$\{(1,p,q)(r,s),(p,q,r)(1,s) \mid 2 \leqslant p,q,r,s \leqslant n\}$
5	2	$\{(p,q,r)(1,2),(1,p,q)(2,r),(2,p,q)(1,r),(1,2,p)(q,r),(1,p,2)(q,r) \mid 3 \leqslant p,q,r \leqslant n\}$

continued on next page

continued from previous page

i	k	$\mathcal{C}_k^{(i)}$
5	3	$\left\{ \begin{array}{l} (1,2,3)(p,q),(1,3,2)(p,q),(1,2,p)(3,q),(1,p,2)(3,q),(1,3,p)(2,q),(1,p,3)(2,q), \\ (2,3,p)(1,q),(2,p,3)(1,q),(1,p,q)(2,3),(2,p,q)(1,3),(3,p,q)(1,2) \end{array} \middle\vert\ 4 \leqslant p,q \leqslant n \right\}$
5	4	$\left\{ \begin{array}{l} (1,2,3)(4,p),(1,3,2)(4,p),(1,2,4)(3,p),(1,4,2)(3,p),(1,2,p)(3,4), \\ (1,p,2)(3,4),(1,3,4)(2,p),(1,4,3)(2,p),(1,3,p)(2,4),(1,p,3)(2,4), \\ (1,4,p)(2,3),(1,p,4)(2,3),(2,3,4)(1,p),(2,4,3)(1,p),(2,3,p)(1,4), \\ (2,p,3)(1,4),(2,4,p)(1,3),(2,p,4)(1,3),(3,4,p)(1,2),(3,p,4)(1,2) \end{array} \middle\vert\ 5 \leqslant p \leqslant n \right\}$
5	5	$\left\{ \begin{array}{l} (1,2,3)(4,5),(1,3,2)(4,5),(1,2,4)(3,5),(1,4,2)(3,5),(1,2,5)(3,4), \\ (1,5,2)(3,4),(1,3,4)(2,5),(1,4,3)(2,5),(1,3,5)(2,4),(1,5,3)(2,4), \\ (1,4,5)(2,3),(1,5,4)(2,3),(2,3,4)(1,5),(2,4,3)(1,5),(2,3,5)(1,4), \\ (2,5,3)(1,4),(2,4,5)(1,3),(2,5,4)(1,3),(3,4,5)(1,2),(3,5,4)(1,2) \end{array} \right\}$
5	$\geqslant 6$	$\varnothing$
6	1	$\{(1,q,r,s,t) \mid 2 \leqslant q,r,s,t \leqslant n\}$
6	2	$\{(1,2,r,s,t),(1,r,2,s,t),(1,r,s,2,t),(1,r,s,t,2) \mid 3 \leqslant r,s,t \leqslant n\}$
6	3	$\left\{ \begin{array}{l} (1,2,3,s,t),(1,3,2,s,t),(1,2,s,3,t),(1,3,s,2,t),(1,2,s,t,3),(1,3,s,t,2), \\ (1,s,2,3,t),(1,s,3,2,t),(1,s,2,t,3),(1,s,3,t,2),(1,s,t,2,3),(1,s,t,3,2) \end{array} \middle\vert\ 4 \leqslant s,t \leqslant n \right\}$
6	4	$\left\{ \begin{array}{l} (1,2,3,4,t),(1,2,3,t,4),(1,2,4,3,t),(1,2,4,t,3),(1,2,t,3,4),(1,2,t,4,3), \\ (1,3,2,4,t),(1,3,2,t,4),(1,3,4,2,t),(1,3,4,t,2),(1,3,t,2,4),(1,3,t,4,2), \\ (1,4,2,3,t),(1,4,2,t,3),(1,4,3,2,t),(1,4,3,t,2),(1,4,t,2,3),(1,4,t,3,2), \\ (1,t,2,3,4),(1,t,2,4,3),(1,t,3,2,4),(1,t,3,4,2),(1,t,4,2,3),(1,t,4,3,2) \end{array} \middle\vert\ 5 \leqslant t \leqslant n \right\}$
6	5	$\left\{ \begin{array}{l} (1,2,3,4,5),(1,2,3,5,4),(1,2,4,3,5),(1,2,4,5,3),(1,2,5,3,4),(1,2,5,4,3), \\ (1,3,2,4,5),(1,3,2,5,4),(1,3,4,2,5),(1,3,4,5,2),(1,3,5,2,4),(1,3,5,4,2), \\ (1,4,2,3,5),(1,4,2,5,3),(1,4,3,2,5),(1,4,3,5,2),(1,4,5,2,3),(1,4,5,3,2), \\ (1,5,2,3,4),(1,5,2,4,3),(1,5,3,2,4),(1,5,3,4,2),(1,5,4,2,3),(1,5,4,3,2) \end{array} \right\}$
6	$\geqslant 6$	$\varnothing$

Now suppose that $\Gamma = \mathrm{Cay}(S_n, T)$ $(= \Gamma_0)$ is a normal Cayley graph of S_n with $m = \max_{\tau\in T} |\mathrm{supp}(\tau)| \leqslant 5$. For $k \in [n]$, let $T_k = T \setminus (T \cap (\cup_{i=1}^{k}(S_n)_i))$ (see Remark 10) and $\Gamma_k = \mathrm{Cay}(S_n, T_k)$ be defined as in (8). Then T $(= T_0)$ and T_k $(k \in [n])$ can be respectively written as $T = \cup_{i\in\mathcal{I}_T}\mathcal{C}^{(i)}$ (see (17)) and $T_k = \cup_{i\in\mathcal{I}_T}\mathcal{C}_k^{(i)}$ (see Table 1), where $\mathcal{I}_T$ is some nonempty subset of $\{1,2,3,4,5,6\}$. Moreover, by the arguments at the beginning of this section, we obtain that $\Gamma = \mathrm{Cay}(S_n, T)$ is connected if and only if $T = \cup_{i\in\mathcal{I}_T}\mathcal{C}^{(i)}$ with

$$\mathcal{I}_T \in \mathcal{P} \setminus \{\varnothing, \{2\}, \{3\}, \{6\}, \{2,3\}, \{2,6\}, \{3,6\}, \{2,3,6\}\} \tag{18}$$

where $\mathcal{P}$ is the power set of $\{1,2,\ldots,6\}$.

Now we give the main result of this section, which determines the second eigenvalues of a majority of connected normal Cayley graphs (and some subgraphs of these graphs) on S_n satisfying $m = \max_{\tau\in T} |\mathrm{supp}(\tau)| \leqslant 5$.

Theorem 15. *Let $\Gamma = \mathrm{Cay}(S_n, T)$ $(= \Gamma_0)$ be a connected normal Cayley graph of S_n $(n \geqslant 7)$ with $m = \max_{\tau\in T} |\mathrm{supp}(\tau)| \leqslant 5$ (that is, $T = \cup_{i\in\mathcal{I}_T}\mathcal{C}^{(i)}$ with $\mathcal{I}_T$ given in (18)). Let Γ_k and T_k be defined as in (8). If $\mathcal{I}_T \neq \{1,3\}, \{1,6\}, \{4,6\}, \{1,2,3\}, \{1,2,6\}, \{1,3,6\}, \{1,4,6\}, \{2,4,6\}, \{3,4,6\}, \{1,2,3,6\}, \{1,2,4,6\}, \{1,3,4,6\}, \{2,3,4,6\}, \{2,3,5,6\}, \{1,2,3,4,6\}$, then for $0 \leqslant k \leqslant m-1$, the graph Γ_k is connected and has second eigenvalue*

$$\lambda_2(\Gamma_k) = \lambda_2(B_\Pi^{(k)}) = |T_k \cap (S_n)_{k+1}| - |T_k \cap (S_n)_{k+2,k+1}|.$$

Proof. Take $a = n - 6$ $(\geqslant 1)$. Since $n \geqslant 7$ and $m \leqslant 5$, we see that S_n acts $(m+a)$-transitively on $[n]$ due to $m + a < n$. By Theorem 14, to prove $\lambda_2(\Gamma_k) = \lambda_2(B_\Pi^{(k)})$ for

$0 \leqslant k \leqslant m-1$, it remains to verify $\lambda_2(\Gamma_{k,a-1}) = \lambda_2(B_\Pi^{(k,a-1)})$ for $0 \leqslant k \leqslant m-1$. Since $S_n^{(a-1)} = S_n^{(n-7)} = \cap_{i=1}^{n-7}(S_n)_{n-i+1} \cong S_7$, we have $\Gamma_{k,a-1} = \mathrm{Cay}(S_n^{(n-7)}, T_k \cap S_n^{(n-7)}) \cong \mathrm{Cay}(S_7, T_k \cap S_7)$ according to (14). Also note that $\lambda_2(B_\Pi^{(k,a-1)}) = |T_k \cap S_n^{(a-1)} \cap (S_n)_{k+1}| - |T_k \cap S_n^{(a-1)} \cap (S_n)_{k+2,k+1}| = |T_k \cap (S_7)_{k+1}| - |T_k \cap (S_7)_{k+2,k+1}|$ by (15). Thus the problem is reduced to verify

$$\lambda_2(\mathrm{Cay}(S_7, T_k \cap S_7)) = |T_k \cap (S_7)_{k+1}| - |T_k \cap (S_7)_{k+2,k+1}| \tag{19}$$

for $0 \leqslant k \leqslant m-1$. Recall that $T_0 = T = \cup_{i\in\mathcal{I}_T}\mathcal{C}^{(i)}$ with $\mathcal{I}_T$ given in (18), and $T_k = \cup_{i\in\mathcal{I}_T}\mathcal{C}_k^{(i)}$ is just the set of $\tau \in T$ such that $\{1,2,\ldots,k\} \subseteq \mathrm{supp}(\tau)$ for $1 \leqslant k \leqslant m-1$. Using computer, we can check that (19) is true except for those T's with $\mathcal{I}_T = \{1,3\},\{1,6\},\{4,6\},\{1,2,3\},\{1,2,6\},\{1,3,6\},\{1,4,6\},\{2,4,6\},\{3,4,6\},\{1,2,3,6\},\{1,2,4,6\},\{1,3,4,6\},\{2,3,4,6\},\{2,3,5,6\}$ or $\{1,2,3,4,6\}$. Therefore, for the remaining T's, we may conclude that

$$\lambda_2(\Gamma_k) = \lambda_2(B_\Pi^{(k)}) = |T_k \cap (S_n)_{k+1}| - |T_k \cap (S_n)_{k+2,k+1}|,$$

where $0 \leqslant k \leqslant m-1$ (in Table 2, we list the exact values of the first two largest eigenvalues of these Γ_k's); and furthermore, we observe that $\lambda_2(\Gamma_k) = \lambda_2(B_\Pi^{(k)}) < |T_k| = \lambda_1(\Gamma_k)$, so Γ_k is also connected for $1 \leqslant k \leqslant m-1$.

This completes the proof. □

Table 2: The first two eigenvalues of $\Gamma_k = \mathrm{Cay}(S_n, T_k)$, where $T_k = \cup_{i\in\mathcal{I}_T}\mathcal{C}_k^{(i)}$.

$\mathcal{I}_T$	m	k	$\lambda_1(\Gamma_k)$	$\lambda_2(\Gamma_k)$
$\{1\}$	2	0	$(n(n-1))/2$	$(n(n-3))/2$
		1	$n-1$	$n-2$
$\{4\}$	4	0	$(n(n-1)(n-2)(n-3))/4$	$(n(n-2)(n-3)(n-5))/4$
		1	$(n-1)(n-2)(n-3)$	$(n-3)(n^2-6n+6)$
		2	$3(n-2)(n-3)$	$3n^2-21n+34$
		3	$6(n-3)$	$6(n-4)$
$\{5\}$	5	0	$(n(n-1)(n-2)(n-3)(n-4))/6$	$(n(n-2)(n-3)(n-4)(n-6))/6$
		1	$(5(n-1)(n-2)(n-3)(n-4))/6$	$(5(n-3)(n-4)(n^2-7n+7))/6$
		2	$(10(n-2)(n-3)(n-4))/3$	$(5(n-4)(2n^2-16n+27))/3$
		3	$10(n-3)(n-4)$	$5(2n^2-18n+39)$
		4	$20(n-4)$	$20(n-5)$
$\{1,2\}$	3	0	$(n(2n-1)(n-1))/6$	$(n(n-1)(2n-7))/6$
		1	$(n-1)^2$	$(n-1)(n-3)$
		2	$2n-3$	$2n-5$
$\{1,4\}$	4	0	$(n(n-1)(n^2-5n+8))/4$	$(n(n-4)(n-3)^2)/4$
		1	$(n-1)(n^2-5n+7)$	$(n-4)(n^2-5n+5)$
		2	$3n^2-15n+19$	$3n^2-21n+35$
		3	$6(n-3)$	$6(n-4)$
$\{1,5\}$	5	0	$(n(n-1)(n^3-9n^2+26n-21))/6$	$(n(n-5)(n-3)(n^2-7n+9))/6$
		1	$((n-1)(5n^3-45n^2+130n-114))/6$	$(5n^4-70n^3+340n^2-659n+408)/6$
		2	$(10n^3-90n^2+260n-237)/3$	$(10n^3-120n^2+455n-537)/3$
		3	$10(n-3)(n-4)$	$5(2n^2-18n+39)$
		4	$20(n-4)$	$20(n-5)$
$\{2,4\}$	4	0	$(n(3n-5)(n-1)(n-2))/12$	$(n(n-2)(3n^2-20n+29))/12$
		1	$(n-1)(n-2)^2$	$n^3-8n^2+19n-13$

continued on next page

continued from previous page

$\mathcal{I}_T$	m	k	$\lambda_1(\Gamma_k)$	$\lambda_2(\Gamma_k)$
		2	$(n-2)(3n-7)$	$(3n-7)(n-4)$
		3	$2(3n-8)$	$2(3n-11)$
$\{2,5\}$	5	0	$(n(n-1)(n-2)(n^2-7n+14))/6$	$(n(n-2)(n-5)(n-4)^2)/6$
		1	$((n-1)(n-2)(5n^2-35n+66))/6$	$((n-5)(5n^3-45n^2+121n-90))/6$
		2	$(2(n-2)(5n^2-35n+63))/3$	$(10n^3-120n^2+461n-558)/3$
		3	$2(5n^2-35n+61)$	$10n^2-90n+197$
		4	$20(n-4)$	$20(n-5)$
$\{3,4\}$	4	0	$(3n(n-1)(n-2)(n-3))/8$	$(3n(n-2)(n-3)(n-5))/8$
		1	$(3(n-1)(n-2)(n-3))/2$	$(3(n-3)(n^2-6n+6))/2$
		2	$(9(n-2)(n-3))/2$	$(3(3n^2-21n+34))/2$
		3	$9(n-3)$	$9(n-4)$
$\{3,5\}$	5	0	$(n(n-1)(n-2)(n-3)(4n-13))/24$	$(n(n-2)(n-3)(4n^2-37n+81))/24$
		1	$((n-1)(n-2)(n-3)(5n-17))/6$	$((n-3)(5n^3-52n^2+157n-122))/6$
		2	$((n-3)(20n-71)(n-2))/6$	$(20n^3-231n^2+847n-978)/6$
		3	$(n-3)(10n-37)$	$10n^2-87n+183$
		4	$20n-77$	$20n-97$
$\{4,5\}$	5	0	$(n(2n-5)(n-1)(n-2)(n-3))/12$	$(n(n-2)(2n-11)(n-3)^2)/12$
		1	$((5n-14)(n-1)(n-2)(n-3))/6$	$((n-3)(5n^3-49n^2+139n-104))/6$
		2	$((n-3)(10n-31)(n-2))/3$	$(10n^3-111n^2+392n-438)/3$
		3	$2(n-3)(5n-17)$	$10n^2-84n+171$
		4	$2(10n-37)$	$2(10n-47)$
$\{5,6\}$	5	0	$(11n(n-1)(n-2)(n-3)(n-4))/30$	$(11n(n-2)(n-3)(n-4)(n-6))/30$
		1	$(11(n-1)(n-2)(n-3)(n-4))/6$	$(11(n-4)(n-3)(n^2-7n+7))/6$
		2	$(22(n-3)(n-4)(n-2))/3$	$(11(n-4)(2n^2-16n+27))/3$
		3	$22(n-3)(n-4)$	$11(2n^2-18n+39)$
		4	$44(n-4)$	$44(n-5)$
$\{1,2,4\}$	4	0	$(n(n-1)(3n^2-11n+16))/12$	$(n(n-4)(3n^2-14n+19))/12$
		1	$(n-1)(n^2-4n+5)$	$(n-3)(n^2-5n+5)$
		2	$3n^2-13n+15$	$3n^2-19n+29$
		3	$2(3n-8)$	$2(3n-11)$
$\{1,2,5\}$	5	0	$(n(n-1)(n^3-9n^2+28n-25))/6$	$(n(n^4-15n^3+82n^2-189n+151))/6$
		1	$((n-1)(5n^3-45n^2+136n-126))/6$	$((n-3)(5n^3-55n^2+181n-146))/6$
		2	$(10n^3-90n^2+266n-249)/3$	$((n-5)(10n^2-70n+111))/3$
		3	$2(5n^2-35n+61)$	$10n^2-90n+197$
		4	$20(n-4)$	$20(n-5)$
$\{1,3,4\}$	4	0	$(n(n-1)(3n^2-15n+22))/8$	$(n(n-3)(3n^2-21n+34))/8$
		1	$((n-1)(3n^2-15n+20))/2$	$(3n^3-27n^2+74n-58)/2$
		2	$((3n-7)(3n-8))/2$	$((3n-8)(3n-13))/2$
		3	$9(n-3)$	$9(n-4)$
$\{1,3,5\}$	5	0	$(n(n-1)(4n^3-33n^2+89n-66))/24$	$(n(n-3)(n-5)(4n^2-25n+30))/24$
		1	$((n-1)(5n^3-42n^2+115n-96))/6$	$(5n^4-67n^3+313n^2-587n+354)/6$
		2	$(20n^3-171n^2+475n-420)/6$	$((n-4)(20n^2-151n+243))/6$
		3	$(n-3)(10n-37)$	$10n^2-87n+183$
		4	$20n-77$	$20n-97$
$\{1,4,5\}$	5	0	$(n(2n^2-13n+24)(n-1)^2)/12$	$(n(n-3)(n-4)(n-5)(2n-3))/12$
		1	$((n-1)(5n^3-39n^2+100n-78))/6$	$((n-4)(n-5)(5n^2-19n+15))/6$
		2	$(10n^3-81n^2+215n-183)/3$	$((n-5)(10n^2-61n+87))/3$
		3	$2(n-3)(5n-17)$	$10n^2-84n+171$
		4	$2(10n-37)$	$2(10n-47)$
$\{1,5,6\}$	5	0	$(n(n-1)(11n^3-99n^2+286n-249))/30$	$(n(n-3)(11n^3-132n^2+484n-513))/30$
		1	$((n-1)(11n^3-99n^2+286n-258))/6$	$(11n^4-154n^3+748n^2-1457n+912)/6$
		2	$(22n^3-198n^2+572n-525)/3$	$(22n^3-264n^2+1001n-1185)/3$
		3	$22(n-3)(n-4)$	$11(2n^2-18n+39)$
		4	$44(n-4)$	$44(n-5)$
$\{2,3,4\}$	4	0	$(n(n-1)(n-2)(9n-19))/24$	$(n(n-2)(9n^2-64n+103))/24$
		1	$((n-1)(n-2)(3n-7))/2$	$(3n^3-25n^2+62n-44)/2$
		2	$((n-2)(9n-23))/2$	$(9n^2-59n+90)/2$

continued on next page

continued from previous page

$\mathcal{I}_T$	m	k	$\lambda_1(\Gamma_k)$	$\lambda_2(\Gamma_k)$
		3	$9n-25$	$9n-34$
$\{2,3,5\}$	5	0	$(n(n-1)(n-2)(4n^2-25n+47))/24$	$(n(n-2)(n-5)(4n^2-29n+55))/24$
		1	$((n-1)(n-2)(5n^2-32n+57))/6$	$((n-4)(5n^3-47n^2+131n-99))/6$
		2	$((n-2)(20n^2-131n+225))/6$	$(20n^3-231n^2+859n-1014)/6$
		3	$10n^2-67n+113$	$(10n-37)(n-5)$
		4	$20n-77$	$20n-97$
$\{2,4,5\}$	5	0	$(n(n-1)(n-2)(2n^2-11n+19))/12$	$(n(n-2)(n-5)(2n^2-13n+23))/12$
		1	$((n-1)(n-2)(5n^2-29n+48))/6$	$(5n^4-64n^3+292n^2-551n+342)/6$
		2	$((n-2)(10n^2-61n+99))/3$	$((n-4)(10n^2-71n+114))/3$
		3	$2(5n^2-32n+52)$	$10n^2-84n+173$
		4	$2(10n-37)$	$2(10n-47)$
$\{2,5,6\}$	5	0	$(n(n-1)(n-2)(11n^2-77n+142))/30$	$(n(n-2)(n-4)(11n^2-99n+208))/30$
		1	$((n-1)(n-2)(11n^2-77n+138))/6$	$(11n^4-154n^3+754n^2-1493n+954)/6$
		2	$(2(n-2)(11n^2-77n+135))/3$	$(22n^3-264n^2+1007n-1206)/3$
		3	$2(11n^2-77n+133)$	$22n^2-198n+431$
		4	$44(n-4)$	$44(n-5)$
$\{3,4,5\}$	5	0	$(n(4n-7)(n-1)(n-2)(n-3))/24$	$(n(n-2)(n-3)(4n^2-31n+51))/24$
		1	$((n-1)(n-2)(n-3)(5n-11))/6$	$((n-3)(5n^3-46n^2+121n-86))/6$
		2	$((n-3)(20n-53)(n-2))/6$	$(20n^3-213n^2+721n-774)/6$
		3	$(n-3)(10n-31)$	$10n^2-81n+159$
		4	$20n-71$	$20n-91$
$\{3,5,6\}$	5	0	$(n(n-1)(n-2)(n-3)(44n-161))/120$	$(n(n-2)(n-3)(44n^2-425n+981))/120$
		1	$((n-1)(n-2)(n-3)(11n-41))/6$	$((n-3)(11n^3-118n^2+367n-290))/6$
		2	$((n-3)(44n-167)(n-2))/6$	$(44n^3-519n^2+1939n-2274)/6$
		3	$(n-3)(22n-85)$	$22n^2-195n+417$
		4	$44n-173$	$44n-217$
$\{4,5,6\}$	5	0	$(n(n-1)(n-2)(n-3)(22n-73))/60$	$(n(n-2)(n-3)(22n^2-205n+453))/60$
		1	$((n-1)(n-2)(n-3)(11n-38))/6$	$((n-3)(11n^3-115n^2+349n-272))/6$
		2	$((n-3)(22n-79)(n-2))/3$	$(22n^3-255n^2+938n-1086)/3$
		3	$2(n-3)(11n-41)$	$22n^2-192n+405$
		4	$2(22n-85)$	$2(22n-107)$
$\{1,2,3,4\}$	4	0	$(n(n-1)(9n^2-37n+50))/24$	$(n(9n^3-82n^2+243n-242))/24$
		1	$((n-1)(3n^2-13n+16))/2$	$((n-3)(n-4)(3n-4))/2$
		2	$(9n^2-41n+48)/2$	$((9n-23)(n-4))/2$
		3	$9n-25$	$9n-34$
$\{1,2,3,5\}$	5	0	$(n(n-1)(4n^3-33n^2+97n-82))/24$	$(n(4n^4-57n^3+298n^2-663n+514))/24$
		1	$((n-1)(5n^3-42n^2+121n-108))/6$	$((n-3)(5n^3-52n^2+163n-128))/6$
		2	$(20n^3-171n^2+487n-444)/6$	$(20n^3-231n^2+859n-1008)/6$
		3	$10n^2-67n+113$	$(10n-37)(n-5)$
		4	$20n-77$	$20n-97$
$\{1,2,4,5\}$	5	0	$(n(n-1)(2n^3-15n^2+41n-32))/12$	$(n(n-4)(2n^3-19n^2+58n-53))/12$
		1	$((n-1)(5n^3-39n^2+106n-90))/6$	$((n-3)(5n^3-49n^2+145n-110))/6$
		2	$(10n^3-81n^2+221n-195)/3$	$(10n^3-111n^2+398n-453)/3$
		3	$2(5n^2-32n+52)$	$10n^2-84n+173$
		4	$2(10n-37)$	$2(10n-47)$
$\{1,2,5,6\}$	5	0	$(n(n-1)(11n^3-99n^2+296n-269))/30$	$(n(11n^4-165n^3+890n^2-2025n+1619))/30$
		1	$((n-1)(11n^3-99n^2+292n-270))/6$	$((n-3)(11n^3-121n^2+391n-314))/6$
		2	$(22n^3-198n^2+578n-537)/3$	$(22n^3-264n^2+1007n-1203)/3$
		3	$2(11n^2-77n+133)$	$22n^2-198n+431$
		4	$44(n-4)$	$44(n-5)$
$\{1,3,4,5\}$	5	0	$(n(n-1)(4n^3-27n^2+59n-30))/24$	$(n(n-5)(n-3)(4n^2-19n+18))/24$
		1	$((n-1)(5n^3-36n^2+85n-60))/6$	$(5n^4-61n^3+259n^2-443n+246)/6$
		2	$(20n^3-153n^2+385n-312)/6$	$(20n^3-213n^2+721n-768)/6$
		3	$(n-3)(10n-31)$	$10n^2-81n+159$
		4	$20n-71$	$20n-91$
$\{1,3,5,6\}$	5	0	$(n(n-1)(44n^3-381n^2+1069n-906))/120$	$(n(n-3)(44n^3-513n^2+1831n-1902))/120$
		1	$((n-1)(11n^3-96n^2+271n-240))/6$	$(11n^4-151n^3+721n^2-1385n+858)/6$

continued on next page

continued from previous page

$\mathcal{I}_T$	m	k	$\lambda_1(\Gamma_k)$	$\lambda_2(\Gamma_k)$
		2	$(44n^3-387n^2+1099n-996)/6$	$((n-4)(44n^2-343n+567))/6$
		3	$(n-3)(22n-85)$	$22n^2-195n+417$
		4	$44n-173$	$44n-217$
$\{1,4,5,6\}$	5	0	$(n(n-1)(2n-3)(11n^2-75n+136))/60$	$(n(n-3)(n-4)(22n^2-161n+219))/60$
		1	$((n-1)(11n^3-93n^2+256n-222))/6$	$((n-4)(11n^3-104n^2+278n-201))/6$
		2	$(22n^3-189n^2+527n-471)/3$	$(22n^3-255n^2+938n-1083)/3$
		3	$2(n-3)(11n-41)$	$22n^2-192n+405$
		4	$2(22n-85)$	$2(22n-107)$
$\{2,3,4,5\}$	5	0	$(n(n-1)(n-2)(4n^2-19n+29))/24$	$(n(n-5)(n-2)(4n^2-23n+37))/24$
		1	$((n-1)(n-2)(5n^2-26n+39))/6$	$(5n^4-61n^3+265n^2-479n+288)/6$
		2	$((n-2)(20n^2-113n+171))/6$	$(20n^3-213n^2+733n-810)/6$
		3	$10n^2-61n+95$	$(2n-7)(5n-23)$
		4	$20n-71$	$20n-91$
$\{2,4,5,6\}$	5	0	$(n(n-1)(n-2)(22n^2-139n+239))/60$	$(n(n-2)(22n^3-271n^2+1088n-1439))/60$
		1	$((n-1)(n-2)(11n^2-71n+120))/6$	$(11n^4-148n^3+700n^2-1349n+846)/6$
		2	$((n-2)(22n^2-145n+243))/3$	$((n-4)(22n^2-167n+276))/3$
		3	$2(11n^2-74n+124)$	$22n^2-192n+407$
		4	$2(22n-85)$	$2(22n-107)$
$\{3,4,5,6\}$	5	0	$(n(n-1)(n-2)(n-3)(44n-131))/120$	$(n(n-2)(n-3)(44n^2-395n+831))/120$
		1	$((11n-35)(n-1)(n-2)(n-3))/6$	$((n-3)(11n^3-112n^2+331n-254))/6$
		2	$((n-3)(44n-149)(n-2))/6$	$(44n^3-501n^2+1813n-2070)/6$
		3	$(n-3)(22n-79)$	$22n^2-189n+393$
		4	$44n-167$	$44n-211$
$\{1,2,3,4,5\}$	5	0	$(n(n-1)(4n^3-27n^2+67n-46))/24$	$(n(4n^4-51n^3+238n^2-477n+334))/24$
		1	$((n-1)(5n^3-36n^2+91n-72))/6$	$((n-3)(n-4)(5n^2-26n+23))/6$
		2	$(20n^3-153n^2+397n-336)/6$	$((n-4)(20n^2-133n+201))/6$
		3	$10n^2-61n+95$	$(2n-7)(5n-23)$
		4	$20n-71$	$20n-91$
$\{1,2,3,5,6\}$	5	0	$(n(n-1)(44n^3-381n^2+1109n-986))/120$	$(n(44n^4-645n^3+3410n^2-7635n+6026))/120$
		1	$((n-1)(11n^3-96n^2+277n-252))/6$	$((n-3)(11n^3-118n^2+373n-296))/6$
		2	$(44n^3-387n^2+1111n-1020)/6$	$(44n^3-519n^2+1951n-2304)/6$
		3	$22n^2-151n+257$	$22n^2-195n+419$
		4	$44n-173$	$44n-217$
$\{1,2,4,5,6\}$	5	0	$(n(n-1)(22n^3-183n^2+517n-448))/60$	$(n(n-4)(22n^3-227n^2+722n-697))/60$
		1	$((n-1)(11n^3-93n^2+262n-234))/6$	$((n-3)(11n^3-115n^2+355n-278))/6$
		2	$(22n^3-189n^2+533n-483)/3$	$(22n^3-255n^2+944n-1101)/3$
		3	$2(11n^2-74n+124)$	$22n^2-192n+407$
		4	$2(22n-85)$	$2(22n-107)$
$\{1,3,4,5,6\}$	5	0	$(n(n-1)(44n^3-351n^2+919n-726))/120$	$(n(n-3)(44n^3-483n^2+1621n-1602))/120$
		1	$((n-1)(11n^3-90n^2+241n-204))/6$	$(11n^4-145n^3+667n^2-1241n+750)/6$
		2	$(44n^3-369n^2+1009n-888)/6$	$(44n^3-501n^2+1813n-2064)/6$
		3	$(n-3)(22n-79)$	$22n^2-189n+393$
		4	$44n-167$	$44n-211$
$\{2,3,4,5,6\}$	5	0	$(n(n-1)(n-2)(44n^2-263n+433))/120$	$(n(n-2)(44n^3-527n^2+2056n-2653))/120$
		1	$((n-1)(n-2)(11n^2-68n+111))/6$	$(11n^4-145n^3+673n^2-1277n+792)/6$
		2	$((n-2)(44n^2-281n+459))/6$	$(44n^3-501n^2+1825n-2106)/6$
		3	$22n^2-145n+239$	$(22n-79)(n-5)$
		4	$44n-167$	$44n-211$
$\{1,2,3,4,5,6\}$	5	0	$(n(n-1)(44n^3-351n^2+959n-806))/120$	$(n(44n^4-615n^3+3110n^2-6705n+5126))/120$
		1	$((n-1)(11n^3-90n^2+247n-216))/6$	$((11n-13)(n-3)(n-4)(n-5))/6$
		2	$(44n^3-369n^2+1021n-912)/6$	$((44n-105)(n-4)(n-5))/6$
		3	$22n^2-145n+239$	$(22n-79)(n-5)$
		4	$44n-167$	$44n-211$

Note that the method in Theorem 15 is invalid for those $T = \cup_{i\in\mathcal{I}_T}\mathcal{C}^{(i)}$ with

$$\mathcal{I}_T \in \left\{ \begin{array}{l} \{1,3\},\{1,6\},\{4,6\},\{1,2,3\},\{1,2,6\},\{1,3,6\},\{1,4,6\}, \\ \{2,4,6\},\{3,4,6\},\{1,2,3,6\},\{1,2,4,6\},\{1,3,4,6\}, \\ \{2,3,4,6\},\{2,3,5,6\},\{1,2,3,4,6\} \end{array} \right\}. \tag{20}$$

Thus we have the following problem:

Problem 16. For $T = \cup_{i\in\mathcal{I}_T}\mathcal{C}^{(i)}$ with $\mathcal{I}_T$ shown in (20), what is the second eigenvalue of the normal Cayley graph $\Gamma = \mathrm{Cay}(S_n, T)$?

Remark 17. It is worth mentioning that for small m (for example, $m = 6$ or 7), as in Theorem 15, one can also determine the second eigenvalues of some connected normal Cayley graphs (and some subgraphs of these graphs) of S_n as long as the computer can verify the conditions of Theorem 14.

Remark 18. It is well-known that the alternating group A_n ($n \geqslant 3$) acts ($n-2$)-transitively on $[n]$. Thus the method used in Theoerm 15 is still valid for determining the second eigenvalues of those connected normal Cayley graphs (and some subgraphs of these graphs) of A_n when m is relatively small.

Let $T = \mathcal{C}^{(1)}$ (see (17)) be the set of all transpositions in S_n ($n \geqslant 3$). Then $m = 2$ and $T_1 = T_{m-1} = \mathcal{C}_1^{(1)} = \{(1, q) \mid 2 \leqslant q \leqslant n\}$. If $n \geqslant 7$, by Theorem 15 (see also Table 2), the spectral gap of $\Gamma = \mathrm{Cay}(S_n, T)$ and $\Gamma_1 = \mathrm{Cay}(S_n, T_1)$ are $|T| - |T \cap (S_n)_1| + |T \cap (S_n)_{2,1}| = \frac{1}{2}n(n-1) - \frac{1}{2}(n-1)(n-2) + 1 = n$ and $|T_1| - |T_1 \cap (S_n)_2| + |T_1 \cap (S_n)_{3,2}| = n-1-(n-2)+0 = 1$, respectively. If $3 \leqslant n \leqslant 6$, one can easily verify that the result also holds. Thus, the two results below are consequences of our work.

Corollary 19 (Diaconis and Shahshahani [15])**.** *For $n \geqslant 3$, the spectral gap of* $\mathrm{Cay}(S_n, \{(p, q) \mid 1 \leqslant p, q \leqslant n\})$ *is n.*

Corollary 20 (Flatto, Odlyzko and Wales [18])**.** *For $n \geqslant 3$, the spectral gap of* $\mathrm{Cay}(S_n, \{(1, q) \mid 2 \leqslant q \leqslant n\})$ *is* 1.

5 Further research

Let G be finite group acts transitively on $[n]$ (for example, $G = S_n$ or A_n), and let $\mathrm{Cay}(G, T)$ be a Cayley graph of G. By Theorem 7, the left coset decomposition given in (4) is always an equitable partition of $\mathrm{Cay}(G, T)$, and the corresponding quotient matrix $B_\Pi = (b_{s,t})_{n\times n}$ (see (5)) is symmetric, where $b_{s,t}$ ($=b_{t,s}$) is the number of elements in T moving t to s. Since the eigenvalues of B_Π are also eigenvalues of $\mathrm{Cay}(G, T)$, we have $\lambda_2(B_\Pi) \leqslant \lambda_2(\mathrm{Cay}(G, T))$. Inspired by the main result of Section 4, we pose the following problem.

Problem 21. Let G be finite group acts transitively on $[n]$. For which connected Cayley graphs of G, the equality $\lambda_2(B_\Pi) = \lambda_2(\mathrm{Cay}(G, T))$ holds?

Let T be a symmetric generating subset of G. We define the *permutation graph* $\mathrm{Per}(T)$ as the edge-weighted graph with vertex set $\{1, 2, \ldots, n\}$ in which each edge $e = st$ ($s \neq t$) has weight $w(e) = b_{s,t}$, the number of elements in T moving t to s as mentioned above. If $G = S_n$ and T contains only transpositions, it is clear that the permutation graph $\mathrm{Per}(T)$ coincides with the transposition graph $\mathrm{Tra}(T)$ defined in Section 1. Since $\mathrm{Cay}(G, T)$ is $|T|$-regular, the sum of each row of the quotient matrix B_Π is equal to $|T|$. We can verify that $B_\Pi = |T| \cdot I_n - L(\mathrm{Per}(T))$, where $L(\mathrm{Per}(T))$ is the Laplacian matrix of the permutation graph $\mathrm{Per}(T)$. This implies that $\lambda_2(B_\Pi) = |T| \cdot I_n - \mu_{n-1}(L(\mathrm{Per}(T)))$, where $\mu_{n-1}(L(\mathrm{Per}(T)))$ denotes the second least eigenvalue of $L(\mathrm{Per}(T))$, i.e., the algebraic connectivity of $\mathrm{Per}(T)$. Therefore, the spectral gap of $\mathrm{Cay}(G, T)$ satisfies the inequality

$$|T| - \lambda_2(\mathrm{Cay}(G,T)) \leqslant |T| - \lambda_2(B_\Pi) = \mu_{n-1}(L(\mathrm{Per}(T))).$$

Then we can restate Problem 21 as below.

Problem 22. Let G be finite group acts transitively on $[n]$. For which connected Cayley graphs of G, the spectral gap of $\mathrm{Cay}(G, T)$ equals to the algebraic connectivity of the permutation graph $\mathrm{Per}(T)$?

In fact, Aldous' theorem give a positive answer of Problem 21 (or Problem 22) in the case that $G = S_n$ and T consists of transpositions. Also, the result of Theorem 15 in this paper gives a partial answer of Problem 21 (or Problem 22) for the connected normal Cayley graphs (and some of their subgraphs) of S_n with $\max_{\tau \in T} |\mathrm{supp}(\tau)| \leqslant 5$.

For any $\sigma \in S_n$, there exists a unique partition $[n] = I_1 \cup \cdots \cup I_m$ of $[n]$ into contiguous blocks such that $\sigma(I_i) = I_i$ for each $i \in [m]$. Here, each I_i consists of consecutive elements in $[n]$, so that $I_i = \{a, a+1, \ldots, b\}$ for some pair of natural numbers $a \leqslant b$. If this partition is of cardinality m, then we call σ an *m-reducible permutation.* In [13, 14], Dai introduced and discussed some combinatorial properties of a new variant of the family of Johnson graphs, the Full-Flag Johnson graphs. He showed that the Full-Flag Johnson graph $FJ(n, r)$ $(r < n)$ is isomorphic to the Cayley graph $\mathrm{Cay}(S_n, RP^{(r)})$, where $RP^{(r)}$ is the set of all $(n-r)$-reducible permutations of S_n. For a positive integer n, the Cayley graph $\mathrm{Cay}(S_n, \{(i, i+1) \mid 1 \leqslant i \leqslant n-1\})$ is called the *permutahedron* of order n, which is a well-known combinatorial graph. Observe that each $(n-1)$-reducible permutation of S_n must be of the form $(i, i+1)$ for some $i \in [n-1]$, we have $RP^{(1)} = \{(i, i+1) \mid 1 \leqslant i \leqslant n-1\}$, and so the permutahedron of order n is just the Full-Flag Johnson graph $FJ(n, 1)$. Thus the Full-Flag Johnson graphs can be also viewed as the generalizations of permutahedra [14].

Let M_n be the tridiagonal matrix of order n defined as below:

$$M_n = \begin{bmatrix} n-2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & n-3 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & n-3 & 1 & \cdots & 0 & 0 & 0 & 0 \\ & & & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & n-3 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & n-3 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & n-2 \end{bmatrix}.$$

At the end of the paper [14], Dai proved that the eigenvalues of M_n are also eigenvalues of the permutahedron $FJ(n, 1)$, and conjectured that $\lambda_2(M_n) = \lambda_2(FJ(n, 1))$. In fact, since $FJ(n, 1) = \mathrm{Cay}(S_n, RP^{(1)})$ with $RP^{(1)} = \{(i, i+1) \mid 1 \leqslant i \leqslant n-1\}$, M_n is just the quotient matrix of $FJ(n, 1)$ shown in (5). Thus we may conclude that Dai's conjecture follows from Aldous' theorem immediately by the arguments at the beginning of this section.

Now consider the graph $FJ(n, 2) = \mathrm{Cay}(S_n, RP^{(2)})$ where $RP^{(2)}$ consists of all $(n-2)$-reducible permutations of S_n. By definition, we can check that each $(n-2)$-reducible permutation of S_n belongs to one of the following three classes:

$$\begin{cases} Q^{(1)} = \{(i, i+1, i+2), (i, i+2, i+1) \mid 1 \leqslant i \leqslant n-2\}; \\ Q^{(2)} = \{(i, i+2) \mid 1 \leqslant i \leqslant n-2\}; \\ Q^{(3)} = \{(i, i+1)(j, j+1) \mid 1 \leqslant i \leqslant n-3, 3 \leqslant j \leqslant n-1, i < j-1\}. \end{cases}$$

Therefore, we have $RP^{(2)} = Q^{(1)} \cup Q^{(2)} \cup Q^{(3)}$. Furthermore, by Theorem 7 and (5), the graph $FJ(n,2) = \mathrm{Cay}(S_n, RP^{(2)})$ has the quotient matrix

$$B_n = \begin{bmatrix} \frac{n^2-n-6}{2} & n-2 & 2 & 0 & 0 & 0 \cdots 0 & 0 & 0 & 0 & 0 & 0 \\ n-2 & \frac{n^2-3n-2}{2} & n-2 & 2 & 0 & 0 \cdots 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & n-2 & \frac{n^2-3n-6}{2} & n-2 & 2 & 0 \cdots 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & n-2 & \frac{n^2-3n-6}{2} & n-2 & 2 \cdots 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 \cdots 2 & n-2 & \frac{n^2-3n-6}{2} & n-2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \cdots 0 & 2 & n-2 & \frac{n^2-3n-6}{2} & n-2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \cdots 0 & 0 & 2 & n-2 & \frac{n^2-3n-2}{2} & n-2 \\ 0 & 0 & 0 & 0 & 0 & 0 \cdots 0 & 0 & 0 & 2 & n-2 & \frac{n^2-n-6}{2} \end{bmatrix}_{n \times n}.$$

In accordance with Problem 21, we ask if $\lambda_2(FJ(n,2)) = \lambda_2(B_n)$? Using computer, we can verify that the equality holds for $4 \leqslant n \leqslant 7$ and we make the following conjecture.

Conjecture 23. For $n \geqslant 4$, $\lambda_2(FJ(n,2)) = \lambda_2(B_n)$.

Theorem 7 indicates a possible method to prove Conjecture 23. Now we describe the detail of the method. For $k = 1, 2$, we define

$$FJ_k(n,2) = \mathrm{Cay}(S_n, RP_k^{(2)}),$$

where $RP_1^{(2)} = \{(1,2,3), (1,3,2), (1,3), (1,2)(3,4), (1,2)(4,5), \ldots, (1,2)(n-1,n)\}$ and $RP_2^{(2)} = \{(1,2)(n-1,n)\}$. Note that $RP_1^{(2)}$ is the set of elements in $RP^{(2)} = Q^{(1)} \cup Q^{(2)} \cup Q^{(3)}$ moving 1 while $RP_2^{(2)}$ is the set of elements in $RP_1^{(2)}$ moving n. Clearly, $FJ_1(n,2)$ is connected and $FJ_2(n,2)$ is just the disjoint union of $\frac{n!}{2}$ K_2's. Again by Theorem 7, the graph $FJ_1(n,2)$ has the quotient matrix

$$B_n^{(1)} = \begin{bmatrix} 0 & n-2 & 2 & 0 & 0 \cdots 0 & 0 & 0 & 0 \\ n-2 & 1 & 1 & 0 & 0 \cdots 0 & 0 & 0 & 0 \\ 2 & 1 & n-4 & 1 & 0 \cdots 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & n-2 & 1 \cdots 0 & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 \cdots 1 & n-2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \cdots 0 & 1 & n-2 & 1 \\ 0 & 0 & 0 & 0 & 0 \cdots 0 & 0 & 1 & n-1 \end{bmatrix}_{n \times n}.$$

Using computer, we can check that $\lambda_2(FJ_1(n,2)) = \lambda_2(B_n^{(1)})$ holds for $4 \leqslant n \leqslant 7$, and so we propose the following conjecture.

Conjecture 24. For $n \geqslant 4$, $\lambda_2(FJ_1(n,2)) = \lambda_2(B_n^{(1)})$.

In order to prove Conjecture 23 by induction on n, we can assume that the result holds for $n-1$, i.e., $\lambda_2(FJ(n-1,2)) = \lambda_2(B_{n-1})$. By the arguments below Theorem 7 and (7), it suffices to show that

$$\lambda_2(B_n) \geqslant \lambda_2(\mathrm{Cay}((S_n)_1, RP^{(2)} \cap (S_n)_1)) + \lambda_2(\mathrm{Cay}(S_n, RP^{(2)} \setminus (RP^{(2)} \cap (S_n)_1))).$$

Note that $\mathrm{Cay}((S_n)_1, RP^{(2)} \cap (S_n)_1) \cong FJ(n-1,2)$ and $\mathrm{Cay}(S_n, RP^{(2)} \setminus (RP^{(2)} \cap (S_n)_1)) = \mathrm{Cay}(S_n, RP_1^{(2)}) = FJ_1(n,2)$. Thus, if Conjecture 24 is true, it remains to verify the following inequality:

$$\lambda_2(B_n) \geqslant \lambda_2(B_{n-1}) + \lambda_2(B_n^{(1)}). \tag{21}$$

Thus we also need to prove Conjecture 24. As above, we can assume $\lambda_2(FJ_1(n-1,2)) = \lambda_2(B_{n-1}^{(1)})$, and it suffices to show that

$$\begin{aligned} \lambda_2(B_n^{(1)}) &\geqslant \lambda_2(\mathrm{Cay}((S_n)_n, RP_1^{(2)} \cap (S_n)_n)) + \lambda_2(\mathrm{Cay}(S_n, RP_1^{(2)} \setminus (RP_1^{(2)} \cap (S_n)_n))) \\ &= \lambda_2(FJ_1(n-1,2)) + \lambda_2(FJ_2(n,2)) \\ &= \lambda_2(B_{n-1}^{(1)}) + 1, \end{aligned} \tag{22}$$

here we use the facts $\mathrm{Cay}((S_n)_n, RP_1^{(2)} \cap (S_n)_n) \cong FJ_1(n-1,2)$ and $\mathrm{Cay}(S_n, RP_1^{(2)} \setminus (RP_1^{(2)} \cap (S_n)_n)) = FJ_2(n,2) \cong \frac{n!}{2}K_2$. Therefore, if one can prove (21) and (22), then Conjecture 23 and Conjecture 24 follow immediately. However, it is not easy to identify the second eigenvalues of B_n and $B_n^{(1)}$, so we leave it as an open problem.

In accordance with Problem 21, for $r \geqslant 3$, we pose the following problem.

Problem 25. For $3 \leqslant r < n$, does the quotient matrix given in (5) always contain the second eigenvalue of the Full-Flag graph $FJ(n,r) = \mathrm{Cay}(S_n, RP^{(r)})$?

On the other hand, for regular graphs, the smallest eigenvalue is closely related to the independent number. Let Γ be a k-regular graph with smallest eigenvalue τ and independent number $\alpha(\Gamma)$, the well-known Hoffman ratio bound asserts that

$$\alpha(\Gamma) \leqslant \frac{|V(\Gamma)|}{1 - k/\tau},$$

and that if the equality holds for some independent set S with characteristic vector v_S, then $v_S - \frac{|S|}{|V(\Gamma)|}\mathbf{1}$ is an eigenvector of the eigenvalue τ. By applying the Hoffman ratio bound to several important families of graphs belonging to classical P- or Q-polynomial association schemes (such as Johnson scheme, Hamming scheme, Grassmann scheme) and some famous Cayley graphs (such as the derangement graph) on the symmetric group S_n, variants of Erdős-Ko-Rado Theorems for sets, vector spaces, integer sequences and permutations have been obtained by various researchers (see Godsil and Meagher [20] for the detail). Recently, Brouwer, Cioabă, Ihringer and McGinnis [6] determine the smallest eigenvalues of (distance-j) Hamming graphs, (distance-j) Johnson graphs, and the graphs of the relations of classical P- and Q-polynomial association schemes. Motivated by these works, it is interesting to consider the smallest eigenvalues of normal

Cayley graphs of S_n. A natural question is that whether the method developed in this paper is valid for the smallest eigenvalues. However, it is not the case. According to the proof of Lemma 11, the quotient matrix B_Π $(= B_\Pi^0)$ of the normal Cayley graph $\Gamma_0 = \mathrm{Cay}(S_n, T_0 = T)$ has eigenvalues $|T|$ and $|T \cap G_1| - |T \cap G_{2,1}|$ (with multiplicity $n-1$). Thus we have $\lambda_n(B_\Pi) = \lambda_2(B_\Pi) = |T \cap G_1| - |T \cap G_{2,1}|$. If $n \geqslant 7$, we can verify that $\lambda_n(B_\Pi) = \lambda_2(B_\Pi) \geqslant 0$ holds for all connected normal Cayley graphs of S_n with $\max_{\tau \in T} |\mathrm{supp}(\tau)| \leqslant 5$, which implies that $\lambda_n(B_\Pi)$ cannot be the smallest eigenvalue. Thus we pose the following problem.

Problem 26. For normal Cayley graphs of S_n, are there some good general methods to determine the smallest eigenvalues?

Acknowledgements

The authors are grateful to the anonymous referees for their useful and constructive comments, which have considerably improved the presentation of this paper.

References

[1] D. Aldous. https://www.stat.berkeley.edu/users/aldous/Research/OP/sgap.html.

[2] N. Alon. Eigenvalues and expanders. *Combinatorica*, 6(2):83–96, 1986.

[3] N. Alon, V. D. Milman. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *J. Combin. Theory, Ser. B*, 38(1):73–88, 1985.

[4] L. Babai. Spectra of Cayley graphs. *J. Combin. Theory, Ser. B*, 27(2):180–189, 1979.

[5] A. E. Brouwer, W. H. Haemers. *Spectra of Graphs*. Springer, 2011.

[6] A. E. Brouwer, S. M. Cioabă, F. Ihringer and M. McGinnis. The smallest eigenvalues of Hamming graphs, Johnson graphs and other distance-regular graphs with classical parameters. *J. Combin. Theory, Ser. B*, 133:88–121, 2018.

[7] P. Caputo, T. M. Liggett and T. Richthammer. Proof of Aldous' spectral gap conjecture. *J. Amer. Math. Soc.*, 23(3):831–851, 2010.

[8] F. Cesi. Cayley graphs on the symmetric group generated by initial reversals have unit spectral gap. *Electron. J. Combin.*, 16(1):#N29, 2009.

[9] F. Cesi. On the eigenvalues of Cayley graphs on the symmetric group generated by a complete multipartite set of transpositions. *J. Algebraic Combin.*, 32(2):155–185, 2010.

[10] F. Cesi. A few remarks on the octopus inequality and Aldous' spectral gap conjecture. *Commun. Algebra*, 44(1):279–302, 2016.

[11] F. Cesi. On the spectral gap of some Cayley graphs on the Weyl group $W(B_n)$. arXiv:1807.11833, 2018.

[12] F. Chung, J. Tobin. The spectral gap of graphs arising from substring reversals. *Electron. J. Combin.*, 23(3):#P3.4, 2017.

[13] I. Dai. Combinatorial Properties of Full-Flag Johnson Graphs. In *Combinatorial Algorithms*, vol. 9538 of *Lecture Notes in Comput. Sci.*, pages 112–123. Springer, 2016.

[14] I. Dai. Diameter bounds and recursive properties of Full-Flag Johnson graphs. *Discrete Math.*, 341:1932–1944, 2018.

[15] P. Diaconis, M. Shahshahani. Generating a random permutation with random transpositions. *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, 57(2):159–179, 1981.

[16] J. Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Amer. Math. Soc.*, 284(2):787–794, 1984.

[17] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Math. J.*, 23(2):298–305, 1973.

[18] L. Flatto, A. M. Odlyzko and D. B. Wales. Random shuffles and group representations. *Ann. Probab.*, 13(1):154–178, 1985.

[19] J. Friedman. On Cayley graphs on the symmetric group generated by transpositions. *Combinatorica*, 20(4):505–519, 2000.

[20] C. Godsil, K. Meagher. *Erdős-Ko-Rado Theorems: Algebraic Approaches.* Cambridge University Press, 2016.

[21] C. Godsil, G. Royle. *Algebraic Graph Theory*, Graduate Texts in Mathematics, vol. 207. Springer, 2001.

[22] S. Handjani, D. Jungreis. Rate of convergence for shuffling cards by transpositions. *J. Theoret. Probab.*, 9(4): 983–993, 1996.

[23] S. Hoory, N. Linial and A. Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc.*, 43(4):439–561, 2006.

[24] X. Huang, Q. Huang. Automorphism group of the complete alternating group graph. *Appl. Math. Comput.*, 314:58–64, 2017.

[25] X. Huang, Q. Huang. The adjacency spectral gap of some Cayley graphs on alternating groups. *J. Algebraic Combin.*, in press, 2018. doi:10.1007/s10801-018-0843-1.

[26] X. Huang, Q. Huang and L. Lu. Construction of graphs with exactly k main eigenvalues. *Linear Algebra Appl.*, 486:204–218, 2015.

[27] I. M. Isaacs. *Algebra: A Graduate Course*, Graduate Studies in Mathematics, vol. 100 (reprint of the 1994 original). Amer. Math. Soc., 2009.

[28] J. S. Jwo, S. Lakshmivarahan and S. K. Dhall. A new class of interconnection networks based on the alternating group. *Networks*, 23:315–326, 1993.

[29] A. Lubotzky. *Discrete Groups, Expanding Graphs, and Invariant Measures.* Birkhäuser Verlag, 1994.

[30] A. Lubotzky. Expander graphs in pure and applied mathematics. *Bull. Amer. Math. Soc.*, 49(1):113–162, 2012.

[31] B. Mohar. Isoperimetric numbers of graphs. *J. Combin. Theory, Ser. B*, 47(3):274–291, 1989.

[32] M. R. Murty. Ramanujan graphs. *J. Ramanujan Math. Soc.*, 18:1–20, 2003.

[33] O. Parzanchevski, D. Puder. Aldous' spectral gap conjecture for normal sets. arXiv:1804.02776, 2018.

[34] X. Yang, D. J. Evans and G. M. Megson. The locally twisted cubes. *Int. J. Comput. Math.*, 82:401–413, 2005.