On a Conjecture Regarding Identification in Hamming Graphs*

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Submitted: Apr 25, 2018; Accepted: May 29, 2019; Published: Jun 21, 2019

Abstract

In 2013, Goddard and Wash studied identifying codes in the Hamming graphs K_q^n. They stated, for instance, that $\gamma^{ID}(K_q^n) \leqslant q^{n-1}$ for any q and $n \geqslant 3$. Moreover, they conjectured that $\gamma^{ID}(K_q^3) = q^2$. In this article, we show that $\gamma^{ID}(K_q^3) \leqslant q^2 - q/4$ when q is a power of four, which disproves the conjecture. Goddard and Wash also gave the lower bound $\gamma^{ID}(K_q^3) \geqslant q^2 - q\sqrt{q}$. We improve this bound to $\gamma^{ID}(K_q^3) \geqslant q^2 - \frac{3}{2}q$. Moreover, we improve the above mentioned bound $\gamma^{ID}(K_q^n) \leqslant q^{n-1}$ to $\gamma^{ID}(K_q^n) \leqslant q^{n-k}$ for $n = 3\frac{q^k-1}{q-1}$ and to $\gamma^{ID}(K_q^n) \leqslant 3q^{n-k}$ for $n = \frac{q^k-1}{q-1}$, when q is a prime power. For these bounds, we utilize two classes of closely related codes, namely, the self-identifying and the self-locating-dominating codes. In addition, we show that the self-locating-dominating codes satisfy the result $\gamma^{SLD}(K_q^3) = q^2$ related to the above conjecture.

Mathematics Subject Classifications: 94B05, 94B25, 94B65, 05B15, 05C69

Keywords: Hamming graph; identifying code; linear codes over finite fields; Latin square; location-domination

1 Introduction

Sensor networks are systems consisting of sensors and links between them. As a monitoring tool they may be used for example in surveillance or to oversee arrays of processors. The basic idea is that a sensor is placed at some node of a network and then it monitors its

*A shortened version [15] of the paper has been presented at the 10th International Colloquium on Graph Theory and combinatorics, Lyon, 2018.

†Research supported by the University of Turku Graduate School (UTUGS) and the Vilho, Yrjö and Kalle Väisälä Foundation.

surroundings reporting on possible anomalies or irregularities. Based on these reports the central unit will deduce the location of the anomaly. The goal is to minimize the number of sensors in networks with certain structures. More on location in sensor networks can be found in [7, 18, 23].

A simple, undirected and connected graph $G = (V, E)$ is utilized to model the sensor network. The set of vertices adjacent to a vertex v is called the *open neighbourhood* of v, denoted by $N(v)$, and the set $N(v) \cup \{v\} = N[v]$ is called the *closed neighbourhood* of v. The vertices represent the possible locations of the sensors and the edge set determines the surrounding area of a sensor. In other words, a sensor placed on vertex u monitors locations $N[u]$ for irregularities.

A nonempty subset C of a vertex set V of a graph is called a *code* and its elements are called *codewords*. We define the *identifying set* or *I-set* of a vertex u as

$$I(u) = N[u] \cap C$$

or if the code or graph is unclear we may use the notation $I(C; u)$ or $I(G, C; u)$. The I-set can also be defined for a *set* of vertices. That is, for $U \subseteq V$, we define

$$I(U) = \bigcup_{v \in U} I(v).$$

In this paper, the code can be understood as the locations of the sensors within our sensor network and the I-set of u as the set of sensors which oversee the location u. The set of vertices C is called a *dominating set* if $I(v) \neq \varnothing$ for each vertex $v \in V$ and the minimum size of a dominating set in a graph G is called the *domination number* $\gamma(G)$. Hence, if sensors are placed at vertices which form a dominating set, then each location is monitored by a sensor and an irregularity is always detected. However, if the sensors report only that there is an irregularity within the area they monitor, then we need stronger condition than just a dominating set for locating the irregularity. For this purpose Karpovsky, Chakrabarty and Levitin defined *identifying codes* in [17]. More on identifying codes can be found at [20] and for recent development, see [1],[8] and[12].

Definition 1. A code $C \subseteq V$ is *identifying* in a graph G if C is a dominating set and

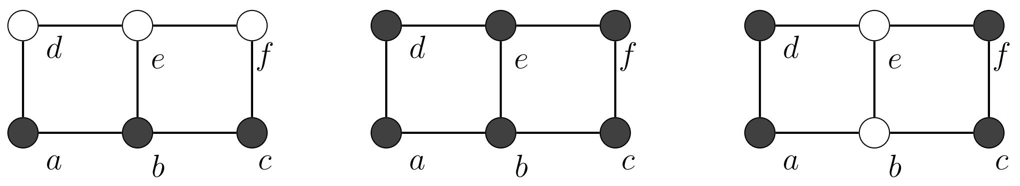
$$I(u) \neq I(v)$$

for each pair of distinct vertices $u, v \in V$. An identifying code C of minimum cardinality in a finite graph G is called *optimal* and its cardinality is denote with $\gamma^{ID}(G)$.

The previous definition of identifying code is illustrated in the following example.

Example 2. Let us consider graph G of Figure 1a and the code $C = \{a, b, c\}$. We have $I(d) = \{a\}$, $I(e) = \{b\}$, $I(f) = \{c\}$, $I(a) = \{a, b\}$, $I(b) = \{a, b, c\}$ and $I(c) = \{b, c\}$. Hence, each I-set is non-empty and unique and, therefore, the code C is an identifying code. Moreover, there are no smaller identifying codes in G since using at most two codewords we can form at most three different nonempty subsets of the code. Hence, C is an optimal identifying code in G and $\gamma^{ID}(G) = 3$.

Notice that there are some possible problems with identifying codes if more than one irregularity may occur in the sensor network. For instance, in the previous example, we have $I(b) = I(\{d, e, f\})$. Hence, if there were irregularities in the vertices d, e and f, then we would mistakenly deduce that an irregularity is in the vertex b. Moreover, we would not even notice that something went wrong. To overcome this problem, in [11], so called self-identifying codes, which are able to locate one irregularity and detect multiple ones, have been introduced. (Notice that in the original paper self-identifying codes are called 1^+-identifying.) The formal definition of self-identifying codes is given as follows.



(a) Identifying code in G.

(b) Self-identifying code in G.

(c) Self-locating-dominating code in G.

Figure 1: Optimal identifying, self-identifying and self-locating-dominating codes in G.

Definition 3. A code $C \subseteq V$ is called *self-identifying* in G if the code C is identifying in G and for all $u \in V$ and $U \subseteq V$ such that $|U| \geqslant 2$ we have

$$I(C; u) \neq I(C; U).$$

A self-identifying code C in a finite graph G with the smallest cardinality is called *optimal* and the number of codewords in an optimal self-identifying code is denoted by $\gamma^{SID}(G)$.

In addition to [11], self-identifying codes have also been previously discussed in [13, 14]. In these papers, two useful characterizations have been presented for self-identifying codes. These characterizations are presented in the following theorem.

Theorem 4 ([11, 13, 14])**.** *Let C be a code in G. Then the following statements are equivalent:*

(i) The code C is self-identifying in G.

(ii) For all distinct $u, v \in V$, we have $I(C; u) \setminus I(C; v) \neq \varnothing$.

(iii) For all $u \in V$, we have $I(C; u) \neq \varnothing$ and

$$\bigcap_{c \in I(C;u)} N[c] = \{u\}.$$

The previous definition of self-identifying codes is illustrated in the following example.

Example 5. Let G be the graph in Figure 1b and let C be a self-identifying code in G. If we now have $|I(a)| = 1$, then $\bigcap_{c \in I(a)} N[c] \neq \{a\}$ contradicting the fact that C is self-identifying due to Theorem 4(iii). Furthermore, if we have $I(a) = \{b, d\}$, then $I(a) \subseteq I(e)$ which contradicts with Theorem 4(ii). Finally, if $I(a) = \{a, b\}$ or $I(a) = \{a, d\}$, then respectively $I(a) \subseteq I(b)$ or $I(a) \subseteq I(d)$ (a contradiction). Hence, we must have $I(a) = \{a, b, d\}$ if C is self-identifying. Analogously, we get $I(f) = \{c, e, f\}$. Therefore, $C = V$ and, indeed, V is a self-identifying code in G.

For the situations when the sensor can distinguish whether the anomaly is in the open neighbourhood of the sensor or in the location of the sensor itself, we have *locating-dominating* codes which were introduced by Slater in [21, 24, 25] (for recent developments, see [2] and [20]). More precisely, a code $C \subseteq V$ is locating-dominating in G if the identifying sets $I(C; u)$ are nonempty and unique for all $u \in V \setminus C$. Inspired by self-identifying codes, we may analogously define so called self-locating-dominating codes, which have been introduced and motivated in [16].

Definition 6. A code $C \subseteq V$ is *self-locating-dominating* in G if for each vertex $u \in V \setminus C$ we have $I(u) \neq \varnothing$ and

$$\bigcap_{c \in I(u)} N[c] = \{u\}.$$

A self-locating-dominating code C in a finite graph G with the smallest cardinality is called *optimal* and the number of codewords in an optimal self-locating-dominating code is denoted by $\gamma^{SLD}(G)$.

In the following theorem, we show that self-locating-dominating codes have a characterization analogous to the one of self-identifying codes. By comparing Definition 6 and Theorem 7 to Theorem 4, we can see that they are almost the same except that only non-codewords are considered in the context of self-location-domination.

Theorem 7 ([16])**.** *A code $C \subseteq V$ is self-locating-dominating in G if and only if for each vertex $u \in V \setminus C$ and $v \in V$ $(u \neq v)$ we have*

$$I(u) \setminus I(v) \neq \varnothing.$$

The previous definition of self-locating-dominating codes is illustrated in the following example.

Example 8. Let G be the graph in Figure 1c and let C be a self-locating-dominating code in G. Necessarily, the vertex a belongs to C since otherwise

$$I(a) \setminus I(e) = \varnothing.$$

Similar reasoning also applies to the vertices c, d and f. Hence, we have $\{a, c, d, f\} \subseteq C$. On the other hand, we have $N[a] \cap N[c] = \{b\}$ and $N[d] \cap N[f] = \{e\}$. Therefore, by the definition, $C = \{a, c, d, f\}$ is an optimal self-locating-dominating code in G.

A graph is called a *complete graph* on q vertices, denoted by K_q, if each pair of vertices of the graph is adjacent. The *Cartesian product* of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined as $G_1 \Box G_2 = (V_1 \times V_2, E)$, where E is a set of edges such that $(u_1, u_2)(v_1, v_2) \in E$ if and only if $u_1 = v_1$ and $u_2 v_2 \in E_2$, or $u_2 = v_2$ and $u_1 v_1 \in E_1$. The Cartesian product $K_q \Box K_q \Box \cdots \Box K_q$ of n copies of K_q is denoted by K_q^n.

Identifying codes have been extensively studied, for example, in the binary hypercubes K_2^n (see the many articles listed in [20]), and [10, 22] for other Cartesian products. In 2008, Gravier, Moncel and Semri [5] investigated identification in K_q^2. Goddard and Wash [4] studied identification in the more general case of K_q^n and they, in particular, gave a conjecture for the cardinality of an optimal identifying code in $K_q^3 = K_q \Box K_q \Box K_q$.

Conjecture 9 ([4])**.** For all $q \geqslant 1$, $\gamma^{ID}(K_q \Box K_q \Box K_q) = q^2$.

In [4], Goddard and Wash prove that $\gamma^{ID}(K_q^3) \leqslant q^2$ for all $q \geqslant 1$. Moreover, by an exhaustive computer search, they show that $\gamma^{ID}(K_q^3) = q^2$ when $q = 3$. Furthermore, they provide a lower bound stating that $\gamma^{ID}(K_q^3) \geqslant q^2 - q\sqrt{q}$. Recall that the domination number $\gamma(K_q^3) = \left\lceil \frac{q^2}{2} \right\rceil$ (see [3]).

In this paper, we first show a one-to-one correspondence between Latin squares and optimal self-locating-dominating codes in K_q^3 in Section 2. Then based on this observation we see that the bound q^2 in Conjecture 9 holds for *self-locating-dominating codes*. In Section 3, we show for identifying codes that $\gamma^{ID}(K_q^3) \leqslant q^2 - q/4$ when q is a power of four. The approach is based on the recursive use of suitable Latin squares. This result disproves the Conjecture 9. We also give constructions of identifying codes for values of q other than the powers of four. After that we improve the lower bound of identifying codes in K_q^3 from $q^2 - q\sqrt{q}$ to $q^2 - \frac{3}{2}q$. Finally, in Section 5, we consider identifying codes in K_q^n, $n > 3$. In [4], it has been shown that $\gamma^{ID}(K_q^n) \leqslant q^{n-1}$. In Section 5, we significantly improve this upper bound when q is a prime power using suitable linear codes over finite fields as well as self-identifying codes and self-locating-dominating codes.

2 Self-location-domination in K_q^3

In this section, we examine self-locating-dominating codes in K_q^3. The vertices of the graph are denoted by (x, y, z), where $1 \leqslant x, y, z \leqslant q$, i.e., the vertex set $V = \{(x, y, z) \mid (x, y, z) \in \mathbb{Z}^3, 1 \leqslant x, y, z \leqslant q\}$. Hence, K_q^3 can be viewed as a cube in $\mathbb{Z}^3$ consisting of coordinates (x, y, z). A *pipe* is defined as a set of vertices which fixes two of the three coordinates. For example, the set $\{(1, 2, z) \mid 1 \leqslant z \leqslant q\}$ is a pipe in K_q^3. A pipe that fixes y- and z-coordinates is called a *row*, a pipe fixing x- and z-coordinates is called a *column* and a pipe fixing x- and y-coordinates is called a *tower*. Two vertices are neighbours in K_q^3 if and only if they belong to the same pipe. Hence, we have $|N(v)| = 3q - 2$.

We can represent a code in $K_q \Box K_m \Box K_l$ by taking a two dimensional $q \times m$ grid and placing the z-coordinates of the codewords to the positions with their x- and y-coordinates. The tower $(1, 1, z)$ is considered to be at the top left corner. Some codes and their representations are illustrated in Example 17. Moreover, as we will see in Theorem

12, this representation of K_q^3 has connection to $q \times q$ *Latin square*, which is a $q \times q$ array filled with numbers from 1 to q in such a way that each number occurs exactly once in each row or column.

Lemma 10. *Let C be a code in K_q^3 and v be a vertex of K_q^3.*

(i) If a vertex v has two codewords in its I-set and they do not locate within a single pipe, then there is exactly one other vertex which has those two codewords in its I-set.

(ii) The I-set $I(v)$ is not a subset of any other I-set if and only if there are at least three codewords in $I(v)$ and they do not locate within a single pipe.

Proof. Let C be a code in K_q^3. (i) Let us have $I(u) = \{c_1, c_2\}$, where $u = (x, y, z), c_1 = (x_1, y_1, z_1)$ and $c_2 = (x_2, y_2, z_2)$. Since c_1 and c_2 do not belong to the same pipe, we can without loss of generality assume that $x_1 \neq x_2$, $y_1 \neq y_2$, $x = x_1$, $y = y_2$ and $z = z_1 = z_2$. Now we have $N[c_1] \cap N[c_2] = \{u, (x_2, y_1, z)\}$.

(ii) Let us first show that if we have less than three codewords in $I(u)$ or the codewords locate within a single pipe, then $I(u)$ is a subset of another I-set. Let us have $I(u) = \{c_1\}$ and $v \in N(c_1)$. Then $I(u) \subseteq I(v)$. If we have $|I(u)| = 2$ and the codewords do not locate within the same pipe, then the case is the same as in (i). If we have $I(u) = \{c_1, \ldots, c_n\}$ and $I(u) \subseteq P$ for some pipe P, then $I(u) \subseteq P \subseteq I(c_1)$. Let us then assume that $\{c_1, c_2, c_3\} \in I(u)$, c_1, c_2 do not belong to the same pipe and $N[c_1] \cap N[c_2] = \{w, u\}$. Hence, u and w do not belong to the same pipe and by (i) we have $N[w] \cap N[u] = \{c_1, c_2\}$. Therefore, $c_3 \notin I(w)$ and $I(u)$ is not a subset of any other I-set. □

In the following two theorems we show that the bound q^2 in Conjecture 9 is true for *self-locating-dominating codes*.

Theorem 11. *We have*

$$\gamma^{SLD}(K_q^3) \geqslant q^2.$$

Proof. Let C be a self-locating-dominating code in K_q^3. By Lemma 10, each non-codeword has to have at least three codewords in its I-set and each codeword has at least one codeword in its I-set. Hence, by double counting pairs (c, x) where $c \in C$ and $x \in N[c]$, we get

$$(3q - 2)|C| \geqslant 3(q^3 - |C|) + |C| \iff |C| \geqslant q^2.$$ □

In the following theorem, we show with the aid of Lemma 10 that each optimal self-locating-dominating code in K_q^3 can be represented as a Latin square (and vice versa).

Theorem 12. *There is a one-to-one correspondence between optimal self-locating-dominating codes in K_q^3 and $q \times q$ Latin squares.*

Proof. Let L be a $q \times q$ Latin square. Consider the Latin square L as a code C in K_q^3 as in if there is the value z in array slot (x, y), then $(x, y, z) \in C$. It is immediate that $|C| = q^2$ as L is a Latin square. Observe that now each non-codeword is covered by exactly three codewords not belonging to a same pipe since there is exactly one codeword in each tower, column and row intersecting with the non-codeword. Hence, by Lemma 10(ii), C is a self-locating-dominating code and we have $\gamma^{SLD}(K_q^3) = q^2$ due to Theorem 11.

Let C be an optimal self-locating-dominating code of cardinality q^2 in K_q^3. By Lemma 10, each non-codeword is covered by at least three codewords and each codeword is covered by at least one codeword. Hence, by double counting pairs (x, c) where $c \in C$ and $c \in N[x]$ we get the inequality

$$3q^3 - 2q^2 = (3q - 2)|C| \geqslant |C| + 3|V(K_q^3) \setminus C| = 3q^3 - 2q^2$$

and since both sides are equal, each codeword is covered by exactly one codeword (itself) and each non-codeword is covered by exactly three codewords. If there is a tower without codewords, then some tower has two codewords and there is a codeword which is covered by two codewords (a contradiction). So, there is exactly one codeword in each tower. Similarly, we may also show that there is exactly one codeword in each row and column. If we now represent this code using a $q \times q$ grid, we get a Latin square, since there is a number from 1 to q in each box of the Latin square and the same number never occurs twice in the same row or column. $\square$

Corollary 13. *We have*

$$\gamma^{SLD}(K_q^3) = q^2.$$

3 Constructions for identification in K_q^3

In this section, we consider identification in K_q^3 and present a bound $\gamma^{ID}(K_q^3) \leqslant q^2 - q/4$ when q is a power of four, giving an infinite family of counterexamples to Conjecture 9. First we give a construction for an identifying code in K_4^3 with cardinality 15 and then we use that identifying code and suitable Latin squares to recursively construct the infinite family.

Goddard and Wash [4] have shown the following results.

Theorem 14 ([4])**.** *For $m > 2l$ and $l > 2q$, we have*

$$\gamma^{ID}(K_q \square K_l \square K_m) = q(m - 1).$$

Moreover, for the complete graphs of equal order, we have

$$q^2 - q\sqrt{q} \leqslant \gamma^{ID}(K_q^3) \leqslant q^2.$$

Due to the recursive nature of our construction we first define an operation which combines two codes in K_q^3 and K_m^3 into a code in K_{qm}^3.

Definition 15. Let $C_1 \subseteq \{(x, y, z) \mid 1 \leqslant x, y, z \leqslant q\}$ and $C_2 \subseteq \{(x, y, z) \mid 1 \leqslant x, y, z \leqslant m\}$ be codes in K_q^3 and K_m^3 respectively. Define $\mathrm{Ext}(C_1, C_2) = \{(x, y, z, a, b, c) \mid (x, y, z) \in C_1, (a, b, c) \in C_2\}$, which is the Cartesian product of the sets C_1 and C_2.

The sextuple produced by $\mathrm{Ext}(C_1, C_2)$ (and also other sextuples) can be interpreted in the following way.

Observation 16. *We can interpret each vertex* $(v, u, w) \in K_{qm}^3 = \{(x', y', z') \mid 1 \leqslant x', y', z' \leqslant qm\}$ *as* (x, y, z, a, b, c) *where* $1 \leqslant x, y, z \leqslant q$ *and* $1 \leqslant a, b, c \leqslant m$ *by having* $v = x + q(a-1)$, $u = y + q(b-1)$ *and* $w = z + q(c-1)$. *In other words,* $\mathrm{Ext}(C_1, C_2)$ *can be interpreted as a code in* K_{qm}^3. *Furthermore, since each pipe fixes two out of three coordinates in a triple, a pipe fixes four out of six coordinates in a sextuple.*

Example 17. Let $C_1 = \{(1,1,1), (2,2,2)\}$ and $C_2 = \{(1,3,1), (2,2,2), (3,1,3)\}$ be codes in K_2^3 and K_3^3, respectively (see Figures 2 and 3). Recall that the vertex $(1,1,z)$ is represented by the top left box and the z coordinate corresponds to the number in that box. Then we have $\mathrm{Ext}(C_1, C_2) = \{(1,1,1,1,3,1),\ (1,1,1,2,2,2),\ (1,1,1,3,1,3),\ (2,2,2,1,3,1),\ (2,2,2,2,2,2),\ (2,2,2,3,1,3)\}$. Furthermore, we can consider this as a code in K_6^3 as seen in Figure 4.

1	
	2

Figure 2: Code C_1 in K_2^3.

		3
	2	
1		

Figure 3: Code C_2 in K_3^3.

				5	
					6
		3			
			4		
1					
	2				

Figure 4: Code $\mathrm{Ext}(C_1, C_2)$ in K_6^3.

Goddard and Wash [4] give the following construction for identifying codes of cardinality q^2 in the graph K_q^3.

Lemma 18. *The code* $C_q = \{(a, b, c) \mid a + b + c \equiv 0 \pmod{q}\}$ *is identifying in* K_q^3 *with the additional property that each pipe in* K_q^3 *has exactly one codeword.*

Proof. The code C_q is shown to be identifying in [4]. Furthermore, there is exactly one codeword in each pipe since if we fix two of the three coordinates, then for exactly one value of the third coordinate the equation $a + b + c \equiv 0 \pmod{q}$ is satisfied. □

By presenting the previous construction in a grid, we can consider it as a Latin square and hence, we can get the properties mentioned in the previous lemma also in that way. The identifying code in K_4^3 of the following theorem is of cardinality 15. The code is presented in Figure 5.

Theorem 19. *The code* $C^1 = \{(2,1,3),\ (2,1,4),\ (3,1,1),\ (4,1,2),\ (1,2,2),\ (1,2,4),\ (2,2,4),\ (3,2,2),\ (1,3,1),\ (2,3,2),\ (3,3,3),\ (4,3,3),\ (2,4,4),\ (4,4,1),\ (4,4,3)\}$ *is identifying in* K_4^3.

	3, 4	1	2
2, 4	4	2	
1	2	3	3
	4		1, 3

Figure 5: Identifying code C^1 in K_4^3.

	3	4	2
4		1	3
2	4		1
3	1	2	

Figure 6: Identifying code C_L in $K_4^3[V(K_4^3) \setminus Di]$.

1	3	4	2
4	**2**	1	3
2	4	**3**	1
3	1	2	**4**

Figure 7: Code C^t where bolded numbers represent cubes with the code C^{t-1} and other numbers represent cubes with the code $C_{4^{t-1}}$.

Proof. By examining Table 1 in the appendix, we notice that each I-set is nonempty and unique. □

In what follows, we call the set $Di = \{(j, j, j) \mid 1 \leqslant j \leqslant 4\}$ as the *diagonal*. We also need another code, C_L, to produce the infinite family of codes of the desired cardinality. The code is presented in Figure 6 for the graph $K_4^3[V(K_4^3) \setminus Di]$, that is, for the graph K_4^3 with its diagonal vertices Di deleted. Note that the empty squares of the array of Figure 6 is easy to fill in such a way that a Latin square is obtained.

Lemma 20. *The code* $C_L = \{(2,1,3), (3,1,4), (4,1,2), (1,2,4), (3,2,1), (4,2,3), (1,3,2), (2,3,4), (4,3,1), (1,4,3), (2,4,1), (3,4,2)\}$ *is identifying in* $K_4^3[V(K_4^3) \setminus Di]$ *and for each codeword* $c \in C_L$ *we have* $I(c) = \{c\}$.

Proof. By examining Table 2 in the appendix, we notice that each I-set is nonempty and unique. Furthermore, by checking the highlighted vertices we notice that we have $I(c) = \{c\}$ for each codeword $c \in C_L$. □

With the help of the codes C_q, C_L, C^1 and Di, we can construct a family of identifying codes of cardinality $q^2 - \frac{q}{4}$ in K_q^3 for $q = 4^t$, $t \in \mathbb{Z}$ and $t > 0$ as described in the following theorem.

Theorem 21. *The code* $C^t = \mathrm{Ext}(C_{q/4}, C_L) \cup \mathrm{Ext}(C^{t-1}, Di)$ *is identifying in* K_q^3*, where* $q = 4^t$ *and* $t \geqslant 2$*, of cardinality* $q^2 - \frac{q}{4}$.

Proof. Let C^t be a code recursively defined as $C^t = \mathrm{Ext}(C_{q/4}, C_L) \cup \mathrm{Ext}(C^{t-1}, Di)$. In other words, the code C^t can be intuitively interpreted as follows. The cube K_q^3 can be considered as K_4^3 where each vertex is replaced with a *subcube* $K_{q/4}^3$. More precisely, the last three digits of the sextuple notation give the vertex which has been replaced with a $K_{q/4}$ subcube and the first three coordinates give the location within the subcube. Furthermore, the code C^t can be considered as a union of codes $C_{q/4}$ placed into the

	3, 4	1	2	10	9	12	11	14	13	16	15	6	5	8	7
2, 4	4	2		9	12	11	10	13	16	15	14	5	8	7	6
1	2	3	3	12	11	10	9	16	15	14	13	8	7	6	5
	4		1, 3	11	10	9	12	15	14	13	16	7	6	5	8
14	13	16	15		7, 8	5	6	2	1	4	3	10	9	12	11
13	16	15	14	6, 8	8	6		1	4	3	2	9	12	11	10
16	15	14	13	5	6	7	7	4	3	2	1	12	11	10	9
15	14	13	16		8		5, 7	3	2	1	4	11	10	9	12
6	5	8	7	14	13	16	15		11, 12	9	10	2	1	4	3
5	8	7	6	13	16	15	14	10, 12	12	10		1	4	3	2
8	7	6	5	16	15	14	13	9	10	11	11	4	3	2	1
7	6	5	8	15	14	13	16		12		9, 11	3	2	1	4
10	9	12	11	2	1	4	3	6	5	8	7		15, 16	13	14
9	12	11	10	1	4	3	2	5	8	7	6	14, 16	16	14	
12	11	10	9	4	3	2	1	8	7	6	5	13	14	15	15
11	10	9	12	3	2	1	4	7	6	5	8		16		13, 15

Figure 8: An identifying code of size $16^2 - 4$ in K_{16}^3.

subcubes given by the code C_L and codes C^{t-1} placed into the subcubes given by the code Di (see Figure 7). Furthermore, the code C^2 is illustrated in Figure 8.

Since codes C_L and Di are separate in the graph K_4^3 and $|C^1| = 4^2 - \frac{4}{4}$, we can use induction on the cardinality $|C^t|$ and thus, have

$$|C^t| = |C_L| \cdot |C_{q/4}| + |Di| \cdot |C^{t-1}| = 12 \cdot \left(\frac{q}{4}\right)^2 + 4|C^{t-1}| = q^2 - \frac{q}{4}.$$

The basic idea of the code C^t is that codes C_L and Di identify the three latter coordinates of the sextuple (x, y, z, a, b, c) and the codes $C_{q/4}$ and C^{t-1} identify the first three coordinates. When $t = 2$, C^1 is an identifying code in K_4^3 (by Theorem 19). Let us now make an induction hypothesis that C^{t-1} is an identifying code in $K_{q/4}^3$ for $t \geqslant 2$.

Let us first show that if for $v = (x, y, z, a, b, c)$ and $w = (x', y', z', a', b', c')$ we have $(a, b, c) \neq (a', b', c')$, then $I(C^t; v) \neq I(C^t; w)$. Let us assume first that $(a, b, c), (a', b', c') \notin Di$. First notice that by Lemma 18 each pipe which goes through a subcube with the code $C_{q/4}$ intersects with exactly one codeword. Hence, if $c = (x'', y'', z'', a'', b'', c'') \in I(C^t; v)$ and $c \notin \mathrm{Ext}(C^{t-1}, Di)$, then $(a'', b'', c'') \in C_L$. Thus, if $I(C^t; v) = I(C^t; w)$, then $I(C_L; (a, b, c)) = I(C_L; (a', b', c'))$ which is not possible since C_L is an identifying code.

Moreover, if $(a,b,c) \in Di$, then $I(C^t; v) \subseteq \mathrm{Ext}(C^{t-1}, \{(a,b,c)\})$ since $N[Di] \cap C_L = \varnothing$ by Table 18 and $K_4^3[Di]$ is a discrete graph.

We now divide the proof into cases based on the location of $v = (x,y,z,a,b,c)$. Let us consider the case where v is such that $(a,b,c) \in C_L$. Moreover, let $w = (x',y',z',a',b',c')$ and let us assume that $I(C^t; v) = I(C^t; w)$. By the previous deduction, we have $(a,b,c) = (a',b',c')$. Moreover, $I(C_{q/4}; (x,y,z)) = I(C_{q/4}; (x',y',z'))$ since $I(C^t; v) = I(C^t; w)$ and $(a,b,c) = (a',b',c')$. Since $C_{q/4}$ is an identifying code, we have $(x,y,z) = (x',y',z')$ and hence, $v = w$.

Let us then consider the case where $(a,b,c) \notin C_L \cup Di$. We have $|I(\mathrm{Ext}(C_{q/4}, C_L); v)| \geqslant 2$ since only codewords have one codeword in their I-sets in C_L. If $|I(\mathrm{Ext}(C_{q/4}, C_L); v)| \geqslant 3$, then everything is clear due to Lemma 10(ii) since in the codes $C_{q/4}$ and C_L there are no pipes with multiple codewords. Hence, we may assume that $|I(\mathrm{Ext}(C_{q/4}, C_L); v)| = 2$. Recall that no pipe in K_q^3 has two codewords which are in $\mathrm{Ext}(C_{q/4}, C_L)$ and by Lemma 10(i) there is exactly one vertex w such that $I(\mathrm{Ext}(C_{q/4}, C_L); v) \subseteq I(\mathrm{Ext}(C_{q/4}, C_L); w)$. We may assume that the codewords in $I(\mathrm{Ext}(C_{q/4}, C_L); v)$ locate in the subcubes which are placed at coordinates (a', b, c) and (a, b', c). However, now w locates in the subcube at coordinates (a', b', c) which is not possible.

Finally, we have the case $(a,b,c) \in Di$. We immediately notice that if $c \in I(C^t; v)$, then c is of the form $(x', y', z', a, b, c) \in \mathrm{Ext}(C^{t-1}, Di)$. Furthermore, the code C^{t-1} is an identifying code in $K_{q/4}^3$ and hence, the vertices within the same diagonal subcube have different I-sets than v and thus, the I-set of v is unique. □

So far, we have given constructions for identifying codes in K_q^3 with $q = 4^t$. However, we can further use these codes to construct new identifying codes for other values of q. For this purpose, we use Latin squares and Evans' Theorem.

Theorem 22 (Evans' Theorem [6])**.** *Any $q \times q$ Latin square can be extended into an $r \times r$ Latin square if $r \geqslant 2q$.*

	3, 4	1	2	9	5	6	7	8
2, 4	4	2		5	6	7	8	9
1	2	3	3	6	7	8	9	5
	4		1, 3	7	8	9	5	6
9	5	6	7	8	4	3	2	1
8	9	5	6	4	3	2	1	7
7	8	9	5	3	2	1	6	4
6	7	8	9	2	1	5	4	3
5	6	7	8	1	9	4	3	2

Figure 9: Identifying code of size 80 in K_9^3.

Theorem 23. *Let C be an identifying code in K_q^3 of cardinality m. If $r \geqslant 2q$, then we have an identifying code in K_r^3 of cardinality $r^2 - q^2 + m$.*

Proof. Let C be an identifying code in K_q^3 of cardinality m and $r \geqslant 2q$. Let us consider a $q \times q$ Latin square L' with values from 1 to q. According to Theorem 22, we can extend the Latin square L' into an $r \times r$ Latin square L. Let us assume that the Latin square L' locates in the coordinates (x, y, z), where $x, y, z \leqslant q$. Moreover, we use notation $(x, y, z) \in L$ if there is value z at the location (x, y) in the Latin square. Let us have

$$C' = C \cup \{(x, y, z) \mid \max\{x, y\} \geqslant q + 1 \text{ and } (x, y, z) \in L\}.$$

The code C' is illustrated in Figure 9 when $q = 4$, $r = 9$ and the original code $C = C^1$. We have $|C'| = r^2 + m - q^2$. Moreover, we have the following two observations on the structure of the code.

Observation 1: Each pipe P with at least one of the two fixed coordinates greater than q, has exactly one codeword in it. Indeed, since P is a pipe with at least one fixed coordinate greater than q, it does not intersect with L' and hence, it does intersect with $L \setminus L'$. Note that since L is a Latin square, each pipe intersects with exactly one vertex in L.

Observation 2: Vertex (x, y, z) does not belong to C' if exactly one of the three coordinates is greater than q. Indeed, if a vertex (x, y, z) with exactly one coordinate greater than q is a codeword, then the pipe which intersects with (x, y, z) and L' (there is such a pipe) contradicts against the structure of the Latin square L.

Let us show that C' is an identifying code by dividing the proof into cases based on the location of the vertex $v = (x, y, z) \in V(K_q^3)$ and whether v is a codeword or not. Let us first consider the case where at least two of the coordinates (x, y, z) of the vertex v are greater than q and v is a non-codeword. Hence, there is exactly one codeword in each pipe intersecting with v by Observation 1, $|I(v)| = 3$ and the codewords in $I(v)$ do not locate within a single pipe. Therefore, v is now uniquely identified by Lemma 10.

Let us then consider the case where exactly one of the coordinates (x, y, z) is greater than q. Now v is a non-codeword by Observation 2. We have $|I(v)| \geqslant 2$ since two of the pipes going through v fix the coordinate which is greater than q. Hence, if $|I(v)| > 2$, then the I-set is unique. On the other hand, if $|I(v)| = 2$, then there is another vertex w which has those two codewords in its I-set. Now, if $I(v) = \{c, c'\}$, then exactly one coordinate of c is less than $q + 1$ and the same is true for c' due to Observation 2 and since the codewords in $I(v)$ locate in the pipes with a fixed coordinate greater than q. Thus, it is straightforward to verify that each coordinate of w is greater than q. Hence, $|I(w)| \geqslant 3$ and $I(v)$ is unique.

Let us then consider the case where at least two of the coordinates (x, y, z) are greater than q and v is a codeword. We have $I(v) = \{v\}$ and each neighbour of v has at least two codewords in its I-set as we have seen in the two previous cases.

Finally, we have the case $x, y, z \leqslant q$. Now the vertex v is identified by the code C since C is an identifying code, each vertex with a coordinate greater than q has codewords in its I-set which do not belong to C by the previous considerations and $I(v) \subseteq C$ by Observation 2. □

By considering the identifying code C^t and Theorem 23, we get the following corollary which gives an identifying code in K_q^3 for all $q \geqslant 8$ of cardinality less than q^2.

Corollary 24. *If $2 \cdot 4^t \leqslant q \leqslant 2 \cdot 4^{t+1} - 1$, then we have*

$$\gamma^{ID}(K_q^3) \leqslant q^2 - 4^{t-1}.$$

4 Lower bound for identification in K_q^3

With our construction and the lower bound of Goddard and Wash, we now know that $q^2 - q\sqrt{q} \leqslant \gamma^{ID}(K_q^3) \leqslant q^2 - \frac{q}{4}$ when $q = 4^t$, $t \in \mathbb{Z}$, $t > 0$. In this section, we improve the lower bound to $\gamma^{ID}(K_q^3) \geqslant q^2 - \frac{3}{2}q$. The standard techniques for obtaining lower bounds for identifying codes in graphs are based on the covering properties of balls or symmetric differences (see [20]). For K_q^3 these methods are not powerful enough, so we provide a new approach, which builds on the method of Goddard and Wash [4].

Definition 25. Let C be a code in K_q^3 and let i be an integer such that $1 \leqslant i \leqslant q$. Define an *$x_i$-layer* of the graph K_q^3, denoted by D_i^1, as the set of vertices which fixes the coordinate $x = i$, i.e., $D_i^1 = \{(i, y, z) \mid 1 \leqslant y, z \leqslant q\}$. Analogously, we define a *y_i-layer* D_i^2 and a *z_i-layer* D_i^3. Let j be an integer such that $1 \leqslant j \leqslant 3$. Define then $X_i^j = \{v \in D_i^j \mid I(C; v) \cap D_i^j = \varnothing\}$ and $Y_i^j = \{v \in D_i^j \mid I(C; v) \cap D_i^j = \{v\}\}$. Furthermore, we use the following notation: $X^j = \bigcup_{i=1}^q X_i^j$, $Y^j = \bigcup_{i=1}^q Y_i^j$ and $X = \bigcup_{j=1}^3 X^j$, $Y = \bigcup_{j=1}^3 Y^j$ and $C_i^j = C \cap D_i^j$. A codeword which does not belong to Y is called a *corner*, and a *fellow* is a codeword belonging to Y such that it has another codeword in its open neighbourhood.

Lemma 26. *Let C be an identifying code in K_q^3. For a pipe P, the following statements hold:*

(i) The pipe P does not contain multiple fellows.

(ii) The pipe P does not contain a corner, a fellow and a vertex $v \in X$.

(iii) The pipe P does not contain a codeword and vertices $v, v' \in X$.

Proof. (i) Let c and c' be fellows in P. Therefore, as $c' \in I(c)$ and $c \in I(c')$, we have a contradiction; $I(c) = I(c')$. (ii) Let c be a corner, c' a fellow and v a vertex of X in P. Hence, since $c \in I(c')$ and $c \in I(v)$, we have $I(c') \subseteq P$ and $I(v) \subseteq P$. This implies that $I(c) = I(v)$. (iii) Let c be a codeword and $v, v' \in X$ in P. Similarly, we again have $I(v) \subseteq P$ and $I(v') \subseteq P$ and, thus, $I(v) = I(v')$. □

We can show that for each vertex $x \in X$, there is a corner that is linked to the vertex x. Later, in the proof of Lemma 29, we show that each corner is linked to at most three vertices of X.

Lemma 27. *Let C be an identifying code in K_q^3. If $x \in X$, then there exists a codeword $c \in I(x)$ such that c is a corner or a fellow with a corner in $I(c)$.*

Proof. Let C be an identifying code in K_q^3 and let $x \in X$. Since C is an identifying code, we have $I(x) \neq \varnothing$. Let us say that $c \in I(x)$. We can assume that c is not a corner since otherwise we are immediately done. Furthermore, if $I(c) = \{c\}$, then $I(c) = I(x)$. We can now assume that there exists another codeword $c' \in I(c)$ $(c' \neq c)$. Now since $c \in Y$ and $|I(c)| \geqslant 2$, c is a fellow. Moreover, by Lemma 26(i), c' is not a fellow and therefore it is not in Y. Thus, c' is a corner. □

Definition 28. Let C be an identifying code in K_q^3 and $c \in C$ be a corner. If there are two codewords $c', c'' \in I(c) \cap D_i^j$ which do not belong to the same pipe, then we say that c is a corner of the layer D_i^j. Furthermore, we denote by k_i^j the total number of corners of the layer D_i^j.

We have shown that there is a corner for each vertex in X. We will further show that each corner can be associated to at most three vertices of X.

Lemma 29. *Let C be an identifying code in K_q^3. Then we have*

$$|X| \leqslant 3 \sum_{j=1}^{3} \sum_{i=1}^{q} k_i^j.$$

Proof. Let $c \in C$ be a corner. By Lemma 26, we have in total at most three fellows and vertices of X in $N[c]$. Moreover, if c' is a fellow, then $|N(c') \cap X| \leqslant 1$ since if $v, v' \in N(c') \cap X$, then $I(v) = I(v')$. Hence, for each corner c there are at most three vertices in X such that they are in the neighbourhood of c or in the neighbourhood of a fellow $c' \in I(c)$.

With the aid of Lemma 27, we notice that for each vertex $x \in X$ there exists in $N(x)$ a corner or a fellow with a corner in its neighbourhood. Thus, we have $|X| \leqslant 3\,|\{c \mid c \text{ is a corner}\}|$. Moreover, each corner is counted in the sum $\sum_{j=1}^{3} \sum_{i=1}^{q} k_i^j$ and hence, we have $|X| \leqslant 3 \sum_{j=1}^{3} \sum_{i=1}^{q} k_i^j$. □

To approximate the cardinality of each X_i^j, we need the domination number of $K_q \square K_q$. Later, this result is used to approximate the number of vertices of X in a layer. The following lemma is Exercise 1.12 in [9].

Lemma 30 ([9])**.** *For each positive integer q, we have*

$$\gamma(K_q \square K_q) = q.$$

Definition 31. Let C be a code in K_q^3 and $M_i^j \subseteq D_i^j$ be a minimum dominating set of D_i^j such that $C_i^j \subseteq M_i^j$. Then we denote $f_i^j = |M_i^j| - q$. Note that q is the domination number of $K_q \square K_q$. Let us further denote $a_i^j = q - |C_i^j|$.

Note that we need $q + f_i^j$ codewords to dominate the layer D_i^j. Hence, value f_i^j can be understood as a measurement of how much the structure of the code within a layer increases the cardinality of X_i^j. Indeed, observe that if a non-codeword in the layer D_i^j is not dominated by C_i^j, then it belongs to X_i^j and there is a row and a column within the layer D_i^j without codewords. Moreover, observe that we have only $q - a_i^j$ codewords in the layer D_i^j. Hence, there are at least $q + f_i^j - (q - a_i^j) = f_i^j + a_i^j$ rows and columns which do not have a codeword when $f_i^j + a_i^j \geqslant 0$. Thus, we have $|X_i^j| \geqslant (a_i^j + f_i^j)^2$. The previous observations are illustrated in Figure 10. Furthermore, the number of corners in a layer is connected with f_i^j as explained in the following lemma.

Lemma 32. *Let C be an identifying code in K_q^3. Then $2f_i^j \geqslant k_i^j$ for each i and j.*

Proof. Let C be an identifying code in K_q^3, and consider a layer D_i^j for some $1 \leqslant i \leqslant q$ and $1 \leqslant j \leqslant 3$. Since the vertices of D_i^j can be viewed as a graph K_q^2, we can consider pipes locating within it as rows and columns. There are q rows and q columns in D_i^j and a subset of D_i^j is dominating if and only if it intersects with all rows or all columns. Indeed, if there are a row R and a column S which do not contain any codewords, then the vertex in their intersection is not dominated. Let us now assume that C_i^j intersects with n rows and h columns ($n \geqslant h$) and that M_i^j is a minimal dominating set of D_i^j such that $C_i^j \subseteq M_i^j$.

We have $|M_i^j| = |C_i^j| + (q - n)$ since C_i^j dominates n out of q rows and $n \geqslant h$. Thus, we have

$$f_i^j = |M_i^j| - q = |C_i^j| - n.$$

If we have one or two corners of the layer D_i^j in a row, then that row has at least two codewords in it and deleting a corner still preserves at least one codeword in that row. If a row has $m \geqslant 3$ corners in it, then it has at least m codewords and hence, deleting $m - 1$ corners still maintains a codeword in the row. Therefore, we may delete at least half of the corners in such a way that there still are codewords in n rows. Hence, we have $n \leqslant |C_i^j| - \frac{1}{2}k_i^j$ and thus, we have

$$f_i^j \geqslant \frac{1}{2}k_i^j. \qquad \square$$

From Lemmas 29 and 32 we get following corollary.

Corollary 33. *If C is an identifying code in K_q^3, then we have*

$$|X| \leqslant 6 \sum_{j=1}^{3} \sum_{i=1}^{q} f_i^j.$$

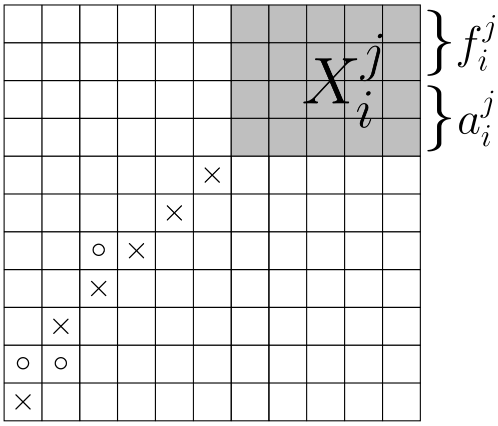


Figure 10: A code within a layer of K_{12}^3 with 3 corners marked by $\circ$, 7 other codewords marked by $\times$, $f = 2$ and $|X| = 20$.

Now we can prove the new lower bound for $\gamma^{ID}(K_q^3)$. The proof is based on the idea that each vertex in X requires corners, corners increase the values f_i^j and the values f_i^j increase the cardinality of X. Figure 10 shows how corners and the number of codewords affect the size of X.

Theorem 34. *We have*

$$\gamma^{ID}(K_q^3) \geqslant q^2 - \frac{3}{2}q.$$

Proof. Let C be an identifying code in K_q^3 of an optimal size $\gamma^{ID}(K_q^3)$. We have,

$$|C| = \frac{1}{3}\sum_{j=1}^{3}\sum_{i=1}^{q}|C_i^j| = \frac{1}{3}\sum_{j=1}^{3}\sum_{i=1}^{q}(q - a_i^j) = q^2 - \frac{1}{3}\sum_{j=1}^{3}\sum_{i=1}^{q}a_i^j. \tag{1}$$

Since there exists an identifying code with size q^2 by [4], we have $\frac{1}{3}\sum_{j=1}^{3}\sum_{i=1}^{q}a_i^j \geqslant 0$. Let M_i^j be a minimum dominating set in D_i^j such that $C_i^j \subseteq M_i^j$. Notice that $f_i^j \geqslant 0$ and a_i^j can be negative but $a_i^j \geqslant -f_i^j$ since $f_i^j = |M_i^j| - q \geqslant |C_i^j| - q = -a_i^j$. Hence, we have $f_i^j + a_i^j \geqslant 0$.

Now we can give an approximation for X:

$$|X| \geqslant \sum_{j=1}^{3}\sum_{i=1}^{q}(a_i^j + f_i^j)^2. \tag{2}$$

We can do this approximation since there are at least $|M_i^j| - |C_i^j|$ rows and columns without codewords in D_i^j. Hence, we have $|X_i^j| \geqslant \left(|M_i^j| - |C_i^j|\right)^2 = \left((q + f_i^j) - (q - a_i^j)\right)^2 = \left(f_i^j + a_i^j\right)^2$. Furthermore, by Corollary 33, we have

$$6\sum_{j=1}^{3}\sum_{i=1}^{q}f_i^j \geqslant |X| \geqslant \sum_{j=1}^{3}\sum_{i=1}^{q}(a_i^j + f_i^j)^2. \tag{3}$$

Now we can give a lower bound for $|C|$:

$$\begin{aligned}
|C| &\overset{(1)}{=} q^2 - \frac{1}{3}\sum_{j=1}^{3}\sum_{i=1}^{q} a_i^j \\
&= q^2 - \frac{1}{3}\left(\sum_{j=1}^{3}\sum_{i=1}^{q} a_i^j + \sum_{j=1}^{3}\sum_{i=1}^{q} f_i^j - \sum_{j=1}^{3}\sum_{i=1}^{q} f_i^j\right) \\
&= q^2 - \frac{1}{3}\left(\sum_{j=1}^{3}\sum_{i=1}^{q} \left(a_i^j + f_i^j\right) - \sum_{j=1}^{3}\sum_{i=1}^{q} f_i^j\right) \\
&\overset{(3)}{\geqslant} q^2 - \frac{1}{3}\left(\sum_{j=1}^{3}\sum_{i=1}^{q} \left(a_i^j + f_i^j\right) - \frac{\sum_{j=1}^{3}\sum_{i=1}^{q}\left(a_i^j + f_i^j\right)^2}{6}\right) \\
&\overset{(*)}{\geqslant} q^2 - \frac{1}{3}\left(3qA - \frac{3qA^2}{6}\right) \\
&= q^2 - q\left(A - \frac{A^2}{6}\right) \\
&\overset{(**)}{\geqslant} q^2 - \frac{3}{2}q.
\end{aligned}$$

We get the inequality $(*)$ by using Lagrange's method. We can minimize the value of sum $\sum_{j=1}^{3}\sum_{i=1}^{q}\left(a_i^j + f_i^j\right)^2$ while retaining the value of sum $\sum_{j=1}^{3}\sum_{i=1}^{q}\left(a_i^j + f_i^j\right)$. The minimum of sum $\sum_{j=1}^{3}\sum_{i=1}^{q}\left(a_i^j + f_i^j\right)^2$ equals to $3qA^2$ where $A = \frac{\sum_{j=1}^{3}\sum_{i=1}^{q}\left(a_i^j+f_i^j\right)}{3q}$ that is the average value of $a_i^j + f_i^j$. For inequality $(**)$ we notice that $A - \frac{A^2}{6}$ gains its maximum value at $A = 3$. □

5 Results in K_q^n when $n > 3$

In this section, we consider identifying, self-identifying and self-locating-dominating codes in K_q^n, when $n > 3$ and $q > 2$. Goddard and Wash [4] showed that $\gamma^{ID}(K_q^n) \leqslant q^{n-1}$. In what follows, we first give optimal self-identifying and self-locating-dominating codes in K_q^n for certain values of n and q. Then based on these codes we are able to significantly improve the bound $\gamma^{ID}(K_q^n) \leqslant q^{n-1}$ when q is a *prime power*.

For later use, we first begin by introducing some notation and preliminary results based on the classical book [19] of coding theory. For the rest of the section, we assume that q is a prime power. Then there exists a finite field with q elements, and we denote this field by $\mathbb{F}_q$. The set of all n-tuples of $\mathbb{F}_q$ forms a vector space $\mathbb{F}_q^n$. The vector space $\mathbb{F}_q^n$ can be considered as a graph by defining two vertices of $\mathbb{F}_q^n$ to be adjacent if they differ in exactly one coordinate. This graph is called the q-ary hypercube or the q-ary Hamming space. A vertex of a q-ary Hamming space is called a *word*. For two words u and v of $\mathbb{F}_q^n$, the *Hamming distance* is defined as the usual (geodesic) distance $d(u, v)$ of the graph, i.e., the (Hamming) distance is the number of coordinate places in which u and v differ. It is easy to see that the q-ary hypercube is isomorphic to the Cartesian product of n copies of K_q. Denote the all-zero word of $\mathbb{F}_q^n$ by $\mathbf{0}$. A word with one at the ith coordinate place and zero in other coordinates is denoted by e_i.

A linear subspace of $\mathbb{F}_q^n$ is called a *q-ary linear code*. Let C be a linear code in $\mathbb{F}_q^n$ with dimension d. Then there exists an $(n-d)\times n$ matrix H such that $Hu^T = \mathbf{0}$ if and only if $u \in C$. Now H is called the *parity check matrix* of C. Observe that an equivalence relation in $\mathbb{F}_q^n \setminus \{\mathbf{0}\}$ is obtained by defining for all $u, v \in \mathbb{F}_q^n \setminus \{\mathbf{0}\}$, $u \sim v$ if and only if $u = \lambda v$ for some $\lambda \in \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$. Now each equivalence class consists of $q-1$ words of $\mathbb{F}_q^n \setminus \{\mathbf{0}\}$. Assuming k is a positive integer, we form a $k \times (q^k-1)/(q-1)$ matrix H over $\mathbb{F}_q$ by taking as its columns one element from each equivalence class of $\mathbb{F}_q^k \setminus \{\mathbf{0}\}$. Concerning H as a parity check matrix, we obtain a linear code C of length n and dimension $n-k$ such that $|I(C;u)| = 1$ for all $u \in \mathbb{F}_q^n$, i.e., the Hamming distance between any two codewords of C is at least three. The linear code C is called the *Hamming code* of length n and it consists of q^{n-k} codewords.

Let us first begin by presenting a lemma which proves useful in later discussions.

Lemma 35. *Let C be a code in $\mathbb{F}_q^n$.*

(i) For two distinct codewords c_1 and c_2 of C, we have

$$|N[c_1] \cap N[c_2]| = \begin{cases} q, & \text{if } d(c_1,c_2) = 1; \\ 2, & \text{if } d(c_1,c_2) = 2; \\ 0, & \text{if } d(c_1,c_2) > 2. \end{cases}$$

(ii) For three distinct codewords c_1, c_2 and c_3 of C such that there exists a pair of them with the distance equal to 2 and there exists $u \in \mathbb{F}_q^n$ satisfying $c_1, c_2, c_3 \in N[u]$, we obtain that $N[c_1] \cap N[c_2] \cap N[c_3] = \{u\}$.

Proof. (i) Let $c_1, c_2 \in C$ be such that $c_1 \neq c_2$. If $d(c_1,c_2) > 2$, then it is immediate that $N[c_1] \cap N[c_2] = \varnothing$. Furthermore, if $d(c_1,c_2) = 1$, then c_1 and c_2 differ in exactly one coordinate place and, hence, the intersection $N[c_1] \cap N[c_2]$ consists of all the words having same values in the rest of the $n-1$ coordinate. Therefore, we have $|N[c_1] \cap N[c_2]| = q$. Finally, suppose that $d(c_1,c_2) = 2$. Without loss of generality, we may assume that $c_1 = c_2 + \lambda_1 e_1 + \lambda_2 e_2$ for some $\lambda_i \in \mathbb{F}_q (i = 1,2)$. Hence, we have $N[c_1] \cap N[c_2] = \{c_1 + \lambda_1 e_1, c_1 + \lambda_2 e_2\}$ and $|N[c_1] \cap N[c_2]| = 2$.

(ii) Let c_1, c_2 and c_3 be distinct codewords of C such that the distance between two of them is equal to two and there exists $u \in \mathbb{F}_q^n$ satisfying $c_1, c_2, c_3 \in N[u]$. Without loss of generality, we may assume that $d(c_1,c_2) = 2$. By the first case, we obtain that $N[c_1] \cap N[c_2] = \{u, v\}$ and $d(u,v) = 2$ for some $v \in \mathbb{F}_q^n$. Now, without loss of generality, we may assume that $c_1 = u + \lambda_1 e_1$ and $c_2 = u + \lambda_2 e_2$, where $\lambda_1, \lambda_2 \in \mathbb{F}_q$. Therefore, we have $v = u + \lambda_1 e_1 + \lambda_2 e_2$. Hence, we are immediately done if $u = c_3$ as $d(c_3, v) \geqslant 2$. Otherwise, $c_3 = u + \lambda_i' e_i$ ($\lambda_i' \in \mathbb{F}_q$) with c_3 being distinct from c_1 and c_2 and the claim follows as $d(c_3,v) \geqslant 2$. Thus, in all cases, we obtain that $N[c_1] \cap N[c_2] \cap N[c_3] = \{u\}$. □

In what follows, we introduce an approach to construct identifying codes based on self-identifying codes in $\mathbb{F}_q^n$. We first begin by presenting a characterization for self-identifying codes in $\mathbb{F}_q^n$.

Theorem 36. *A code C is self-identifying in $\mathbb{F}_q^n$ if and only if for each word $u \in \mathbb{F}_q^n$ we have $|I(C;u)| \geqslant 3$ and there exist $c_1, c_2 \in I(C;u)$ such that $d(c_1, c_2) = 2$.*

Proof. Assume that C is a self-identifying code in $\mathbb{F}_q^n$. Suppose first that there exists $u \in \mathbb{F}_q^n$ such that $I(C;u) = \{c_1, c_2\}$, where $c_1, c_2 \in C$. By Lemma 35, we obtain that the intersection $N[c_1] \cap N[c_2]$ contains at least two vertices. This contradicts with the characterization of Theorem 4. Similarly, there does not exist a word $u \in \mathbb{F}_q^n$ which is covered by exactly one codeword of C. Hence, we may assume that $I(C;u)$ contains at least 3 codewords. Suppose then that there does not exist a pair of codewords c and c' in $I(C;u)$ such that $d(c,c') = 2$. Hence, the pairwise distance of any two codewords in $I(C;u)$ is one, i.e., the codewords differ in only one coordinate. This implies that the intersection

$$\bigcap_{c \in I(C;u)} N[c]$$

contains q words contradicting with the assumption that C is a self-identifying code. Thus, the claim follows.

Assume then that $C \subseteq \mathbb{F}_q^n$ is a code such that for any $u \in \mathbb{F}_q^n$ we have $|I(C;u)| \geqslant 3$ and there exist $c_1, c_2 \in I(C;u)$ with $d(c_1, c_2) = 2$. By Lemma 35(ii), we immediately obtain that for any $v \in \mathbb{F}_q^n$ we have

$$\bigcap_{c \in I(C;v)} N[c] = \{v\}.$$

Thus, C is a self-locating-dominating code in $\mathbb{F}_q^n$. □

For the next theorem, we recall the following notation: for any word $u \in \mathbb{F}_q^n$ and code $C \subseteq \mathbb{F}_q^n$,

$$u + C = \{u + c \mid c \in C\}.$$

In the following theorem, we present an infinite family of optimal self-identifying codes in $\mathbb{F}_q^n$.

Theorem 37. *Let q be a prime power and let n and k be integers such that $n = (q^k - 1)/(q-1)$. If C is a Hamming code in $\mathbb{F}_q^n$, then $C \cup (e_1 + C) \cup (e_2 + C)$ is an optimal self-identifying code in $\mathbb{F}_q^n$ with cardinality $3q^{n-k}$.*

Proof. Let C be a Hamming code in $\mathbb{F}_q^n$ and denote the code $C \cup (e_1 + C) \cup (e_2 + C)$ by C'. Since e_1, e_2 and $e_2 - e_1$ do not belong to C, the code C' is formed by the Hamming code C and two of its distinct cosets. Hence, it is immediate that each word of $\mathbb{F}_q^n$ is covered by exactly three codewords. Therefore, the claim clearly follows if for all $u \in \mathbb{F}_q^n$ there exist $c_1, c_2 \in I(C';u)$ such that $d(c_1, c_2) = 2$. Suppose to the contrary that there exists a word $v \in \mathbb{F}_q^n$ such that the pairwise distance of any two codewords of $I(C;v)$ is one. By the construction of C', we have $I(C';v) = \{c_1, c_2, c_3\}$, where $c_1 \in C$, $c_2 \in e_1 + C$ and

$c_3 \in e_2 + C$. Therefore, as $d(c_1, c_2) = 1$ and $d(c_1, c_3) = 1$, we respectively have $c_2 = c_1 + e_1$ and $c_3 = c_1 + e_2$. However, then a contradiction follows since $d(c_2, c_3) = 2$. Thus, the code C' is self-identifying in $\mathbb{F}_q^n$ with $|C'| = 3q^{n-k}$ (as $|C| = q^{n-k}$).

On the other hand, if D is an arbitrary self-identifying code in $\mathbb{F}_q^n$, then each word of $\mathbb{F}_q^n$ is covered by at least three codewords of D by Theorem 36. Therefore, using a double counting argument similar to the proof of Theorem 11, we obtain that $(n(q-1)+1)|D| \geqslant 3q^n$ which further implies $|D| \geqslant 3q^{n-k}$. Therefore, the self-identifying code C' in $\mathbb{F}_q^n$ is optimal as $|C'| = 3q^{n-k}$. This concludes the proof of the claim. □

For the following theorem, we recall the notation of the *direct sum*: for any codes $C_1 \subseteq \mathbb{F}_q^{n_1}$ and $C_2 \subseteq \mathbb{F}_q^{n_2}$, where n_1 and n_2 are positive integers, we denote

$$C_1 \oplus C_2 = \{(c_1, c_2) \mid c_1 \in C_1, c_2 \in C_2\}.$$

In the following theorem, we present a simple method of constructing self-identifying codes in $\mathbb{F}_q^{n+1}$ based on the ones in $\mathbb{F}_q^n$.

Theorem 38. *If C is a self-identifying code in $\mathbb{F}_q^n$, then $C \oplus \mathbb{F}_q$ is a self-identifying code in $\mathbb{F}_q^{n+1}$.*

Proof. Let C be a self-identifying code in $\mathbb{F}_q^n$ and $u = (u_1, u_2)$ be a word of $\mathbb{F}_q^{n+1}$ such that $u_1 \in \mathbb{F}_q^n$ and $u_2 \in \mathbb{F}_q$. By Theorem 36, the word u_1 is covered by at least three codewords c_1, c_2 and c_3 of C with the additional property that the distance between two of them is equal to two. Hence, the word $u \in \mathbb{F}_q^{n+1}$ is covered at least by the codewords (c_1, u_2), (c_2, u_2) and (c_3, u_2) of $C \oplus \mathbb{F}_q$, and the codewords satisfy the additional property that the distance between two of them is equal to two. Therefore, by Theorem 36, the code $C \oplus \mathbb{F}_q$ is self-identifying in $\mathbb{F}_q^{n+1}$. □

Recall that each self-identifying code is always identifying. Therefore, by the previous theorems, we have $\gamma^{ID}(\mathbb{F}_q^n) \leqslant 3q^{n-k}$ for integers n, k and ℓ such that $n = (q^k-1)/(q-1)+\ell$, where q is a prime power. This significantly improves over the previous upper bound $\gamma^{ID}(\mathbb{F}_q^n) \leqslant q^{n-1}$ by Goddard and Wash [4]; however, recall that they do not require that q is a prime power. In what follows, we introduce another way to construct identifying codes based on self-locating-dominating codes in $\mathbb{F}_q^n$. We first begin by presenting a characterization for self-locating-dominating codes in $\mathbb{F}_q^n$.

Theorem 39. *A code C is self-locating-dominating in $\mathbb{F}_q^n$ if and only if for each word $u \in \mathbb{F}_q^n \setminus C$ we have $|I(C; u)| \geqslant 3$ and there exist $c_1, c_2 \in I(C; u)$ such that $d(c_1, c_2) = 2$.*

Proof. Recall that a code C is self-locating-dominating if for all $u \in \mathbb{F}_q^n \setminus C$

$$\bigcap_{c \in I(C;u)} N[c] = \{u\}.$$

By Theorem 4, a code is self-identifying if and only if the same condition is satisfied for all words $u \in \mathbb{F}_q^n$. Hence, the claim follows by an argument analogous to the proof of Theorem 36. □

In the following theorem, we present an infinite family of optimal self-locating-dominating codes in $\mathbb{F}_q^n$.

Theorem 40. *Let q be a prime power and let n and k be integers such that $n = 3(q^k - 1)/(q-1)$. Assume that H is a $k \times n$ parity check matrix formed from the $k \times (n/3)$ parity check matrix of the Hamming code by repeating each column three times. Now the code C corresponding to the parity check matrix H is an optimal self-locating-dominating code in $\mathbb{F}_q^n$ with cardinality q^{n-k}.*

Proof. Let u be a word of $\mathbb{F}_q^n$. Now we obtain the following observations:

- Suppose that $Hu^T = x \in \mathbb{F}_q^k$ and $x \neq \mathbf{0}$. Due to the construction of the parity check matrix H, there exist exactly three columns h_{i_1}, h_{i_2} and h_{i_3} of H such that $x = \lambda h_{i_1} = \lambda h_{i_2} = \lambda h_{i_3}$ for some $\lambda \in \mathbb{F}_q$. Hence, there exist exactly three words λe_{i_1}, λe_{i_2} and λe_{i_3} of weight one in $\mathbb{F}_q^n$ such that the indices i_j are all different and $H(u + \lambda e_{i_j})^T = \mathbf{0}$, i.e., $u + \lambda e_{i_j}$ belongs to C. Therefore, the word u is covered by exactly three codewords of C in $\mathbb{F}_q^n$. Moreover, the distance between any of these codewords is equal to two.

- If $Hu^T = \mathbf{0} \in \mathbb{F}_q^k$, then analogously to the previous case we can observe that $u \in C$ is covered by exactly one codeword of C in $\mathbb{F}_q^n$; namely, by itself.

Thus, by the previous observations, we know that $I(C; u) = \{u\}$ if $u \in C$ and for non-codewords $u \in \mathbb{F}_q^n \setminus C$ we have $|I(C; u)| = 3$ with the additional property that the distance between any two codewords of $I(C; u)$ is equal to two. Therefore, by Theorem 39, the code C is self-locating-dominating in $\mathbb{F}_q^n$. Moreover, it is easy to calculate that $|C| = q^{n-k}$.

On the other hand, if D is an arbitrary self-locating-dominating code in $\mathbb{F}_q^n$, then each word of $\mathbb{F}_q^n \setminus D$ is covered by at least three codewords of D by Theorem 39. Therefore, using a double counting argument similar to the proof of Theorem 37, we obtain that $(n(q-1)+1)|D| \geqslant 3(q^n - |D|) + |D|$ which further implies $|D| \geqslant q^{n-k}$. Therefore, the self-locating-dominating code C in $\mathbb{F}_q^n$ is optimal as $|C| = q^{n-k}$. This concludes the proof of the claim. □

In a similar way, we can also construct self-locating-dominating codes (albeit not optimal) for other lengths n.

Theorem 41. *Let q be a prime power and let n, k and ℓ be integers such that $n = 3(q^k - 1)/(q-1) + \ell$. Assume that H is a $k \times n$ parity check matrix formed from the $k \times ((n-\ell)/3)$ parity check matrix of the Hamming code by repeating the first column $\ell + 3$ times and each other column three times. Now the code C corresponding to the parity check matrix H is self-locating-dominating in $\mathbb{F}_q^n$ with cardinality q^{n-k}.*

Proof. The proof of the claim is similar to the one of Theorem 40. □

Observe that the codes $C \subseteq \mathbb{F}_q^n$ constructed in Theorems 40 and 41 are such that for each codeword $c \in V$ we have $I(c) = \{c\}$ and for non-codewords $u \in \mathbb{F}_q^n \setminus C$ we have

$$\bigcap_{c \in I(u)} N[c] = \{u\}.$$

Therefore, all the constructed self-locating-dominating codes are also identifying in $\mathbb{F}_q^n$. Hence, we have $\gamma^{ID}(\mathbb{F}_q^n) \leqslant q^{n-k}$ for all integers n, k and ℓ such that $n = 3(q^k-1)/(q-1)+\ell$, where q is a prime power. Thus, using the constructions based on the self-locating-dominating codes, we are able to significantly improve the previous upper bound $\gamma^{ID}(\mathbb{F}_q^n) \leqslant q^{n-1}$ by [4] (recall again that in [4] it is not required that q is a prime power).

In [4], it is stated that the best known lower bound for identifying codes in $\mathbb{F}_q^n$ is the following one by Karpovsky *et al.* [17].

Theorem 42 ([17])**.** *We have*

$$\gamma^{ID}(\mathbb{F}_q^n) \geqslant \frac{2q^n}{nq-n+2}.$$

Assume that q is a prime power and n and k are integers such that $n = 3(q^k-1)/(q-1)$. As stated above, we now have $\gamma^{ID}(\mathbb{F}_q^n) \leqslant q^{n-k}$. Now the previous lower bound can be written as follows:

$$\gamma^{ID}(\mathbb{F}_q^n) \geqslant \frac{2q^n}{nq-n+2} = \frac{2q^n}{3q^k-1} \geqslant \frac{2q^n}{3q^k} = \frac{2}{3}q^{n-k}.$$

Hence, comparing the previous lower and upper bounds, it can be seen that they are of the same order $\Theta(q^{n-k})$. Analogously, it can be shown that the (self-)identifying codes obtained in Theorem 37 for lengths $n = (q^k-1)/(q-1)$ are also rather small compared to the lower bound above.

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Appendix

Below we give the tables which show that the code C^1 in Theorem 19 and the code C_L in Theorem 20 are identifying codes.

$I(1,1,1)$	$\{(3,1,1),(1,3,1)\}$	$I(2,1,1)$	$\{(3,1,1),(2,1,3),(2,1,4)\}$
$I(3,1,1)$	$\{(3,1,1)\}$	$I(4,1,1)$	$\{(3,1,1),(4,4,1),(4,1,2)\}$
$I(1,2,1)$	$\{(1,3,1),(1,2,2),(1,2,4)\}$	$I(2,2,1)$	$\{(2,2,4)\}$
$I(3,2,1)$	$\{(3,1,1),(3,2,2)\}$	$I(4,2,1)$	$\{(4,4,1)\}$
$I(1,3,1)$	$\{(1,3,1)\}$	$I(2,3,1)$	$\{(1,3,1),(2,3,2)\}$
$I(3,3,1)$	$\{(1,3,1),(3,3,3),(3,1,1)\}$	$I(4,3,1)$	$\{(1,3,1),(4,3,3),(4,4,1)\}$
$I(1,4,1)$	$\{(1,3,1),(4,4,1)\}$	$I(2,4,1)$	$\{(2,4,4),(4,4,1)\}$
$I(3,4,1)$	$\{(3,1,1),(4,4,1)\}$	$I(4,4,1)$	$\{(4,4,1),(4,4,3)\}$
$I(1,1,2)$	$\{(1,2,2),(4,1,2)\}$	$I(2,1,2)$	$\{(2,1,3),(2,1,4),(2,3,2),(4,1,2)\}$
$I(3,1,2)$	$\{(3,1,1),(3,2,2),(4,1,2)\}$	$I(4,1,2)$	$\{(4,1,2)\}$
$I(1,2,2)$	$\{(1,2,2),(1,2,4),(3,2,2)\}$	$I(2,2,2)$	$\{(1,2,2),(2,2,4),(2,3,2),(3,2,2)\}$
$I(3,2,2)$	$\{(1,2,2),(3,2,2)\}$	$I(4,2,2)$	$\{(1,2,2),(3,2,2),(4,1,2)\}$
$I(1,3,2)$	$\{(1,2,2),(1,3,1),(2,3,2)\}$	$I(2,3,2)$	$\{(2,3,2)\}$
$I(3,3,2)$	$\{(2,3,2),(3,2,2),(3,3,3)\}$	$I(4,3,2)$	$\{(2,3,2),(4,1,2),(4,3,3)\}$
$I(1,4,2)$	$\{(1,2,2)\}$	$I(2,4,2)$	$\{(2,3,2),(2,4,4)\}$
$I(3,4,2)$	$\{(3,2,2)\}$	$I(4,4,2)$	$\{(4,1,2),(4,4,1),(4,4,3)\}$
$I(1,1,3)$	$\{(2,1,3)\}$	$I(2,1,3)$	$\{(2,1,3),(2,1,4)\}$
$I(3,1,3)$	$\{(2,1,3),(3,1,1),(3,3,3)\}$	$I(4,1,3)$	$\{(2,1,3),(4,1,2),(4,3,3),(4,4,3)\}$
$I(1,2,3)$	$\{(1,2,2),(1,2,4)\}$	$I(2,2,3)$	$\{(2,1,3),(2,2,4)\}$
$I(3,2,3)$	$\{(3,2,2),(3,3,3)\}$	$I(4,2,3)$	$\{(4,3,3),(4,4,3)\}$
$I(1,3,3)$	$\{(1,3,1),(3,3,3),(4,3,3)\}$	$I(2,3,3)$	$\{(2,1,3),(2,3,2),(3,3,3),(4,3,3)\}$
$I(3,3,3)$	$\{(3,3,3),(4,3,3)\}$	$I(4,3,3)$	$\{(3,3,3),(4,3,3),(4,4,3)\}$
$I(1,4,3)$	$\{(4,4,3)\}$	$I(2,4,3)$	$\{(2,1,3),(2,4,4),(4,4,3)\}$
$I(3,4,3)$	$\{(3,3,3),(4,4,3)\}$	$I(4,4,3)$	$\{(4,3,3),(4,4,1),(4,4,3)\}$
$I(1,1,4)$	$\{(1,2,4),(2,1,4)\}$	$I(2,1,4)$	$\{(2,1,3),(2,1,4),(2,2,4),(2,4,4)\}$
$I(3,1,4)$	$\{(2,1,4),(3,1,1)\}$	$I(4,1,4)$	$\{(2,1,4),(4,1,2)\}$
$I(1,2,4)$	$\{(1,2,2),(1,2,4),(2,2,4)\}$	$I(2,2,4)$	$\{(1,2,4),(2,1,4),(2,2,4),(2,4,4)\}$
$I(3,2,4)$	$\{(1,2,4),(2,2,4),(3,2,2)\}$	$I(4,2,4)$	$\{(1,2,4),(2,2,4)\}$
$I(1,3,4)$	$\{(1,2,4),(1,3,1)\}$	$I(2,3,4)$	$\{(2,1,4),(2,2,4),(2,3,2),(2,4,4)\}$
$I(3,3,4)$	$\{(3,3,3)\}$	$I(4,3,4)$	$\{(4,3,3)\}$
$I(1,4,4)$	$\{(1,2,4),(2,4,4)\}$	$I(2,4,4)$	$\{(2,1,4),(2,2,4),(2,4,4)\}$
$I(3,4,4)$	$\{(2,4,4)\}$	$I(4,4,4)$	$\{(2,4,4),(4,4,1),(4,4,3)\}$

Table 1: I-sets of code C^1.

$I(1,1,1)$	$\varnothing$	$I(2,1,1)$	$\{(2,1,3),(2,4,1)\}$
$I(3,1,1)$	$\{(3,1,4),(3,2,1)\}$	$I(4,1,1)$	$\{(4,1,2),(4,3,1)\}$
$I(1,2,1)$	$\{(1,2,4),(3,2,1)\}$	$I(2,2,1)$	$\{(2,4,1),(3,2,1)\}$
$I(3,2,1)$	$\{(3,2,1)\}$	$I(4,2,1)$	$\{(3,2,1),(4,2,3),(4,3,1)\}$
$I(1,3,1)$	$\{(1,3,2),(4,3,1)\}$	$I(2,3,1)$	$\{(2,3,4),(2,4,1),(4,3,1)\}$
$I(3,3,1)$	$\{(3,2,1),(4,3,1)\}$	$I(4,3,1)$	$\{(4,3,1)\}$
$I(1,4,1)$	$\{(1,4,3),(2,4,1)\}$	$I(2,4,1)$	$\{(2,4,1)\}$
$I(3,4,1)$	$\{(2,4,1),(3,2,1),(3,4,2)\}$	$I(4,4,1)$	$\{(2,4,1),(4,3,1)\}$
$I(1,1,2)$	$\{(1,3,2),(4,1,2)\}$	$I(2,1,2)$	$\{(2,1,3),(4,1,2)\}$
$I(3,1,2)$	$\{(3,1,4),(3,4,2),(4,1,2)\}$	$I(4,1,2)$	$\{(4,1,2)\}$
$I(1,2,2)$	$\{(1,2,4),(1,3,2)\}$	$I(2,2,2)$	$\varnothing$
$I(3,2,2)$	$\{(3,2,1),(3,4,2)\}$	$I(4,2,2)$	$\{(4,1,2),(4,2,3)\}$
$I(1,3,2)$	$\{(1,3,2)\}$	$I(2,3,2)$	$\{(1,3,2),(2,3,4)\}$
$I(3,3,2)$	$\{(1,3,2),(3,4,2)\}$	$I(4,3,2)$	$\{(1,3,2),(4,1,2),(4,3,1)\}$
$I(1,4,2)$	$\{(1,3,2),(1,4,3),(3,4,2)\}$	$I(2,4,2)$	$\{(2,4,1),(3,4,2)\}$
$I(3,4,2)$	$\{(3,4,2)\}$	$I(4,4,2)$	$\{(3,4,2),(4,1,2)\}$
$I(1,1,3)$	$\{(1,4,3),(2,1,3)\}$	$I(2,1,3)$	$\{(2,1,3)\}$
$I(3,1,3)$	$\{(2,1,3),(3,1,4)\}$	$I(4,1,3)$	$\{(2,1,3),(4,1,2),(4,2,3)\}$
$I(1,2,3)$	$\{(1,2,4),(1,4,3),(4,2,3)\}$	$I(2,2,3)$	$\{(2,1,3),(4,2,3)\}$
$I(3,2,3)$	$\{(3,2,1),(4,2,3)\}$	$I(4,2,3)$	$\{(4,2,3)\}$
$I(1,3,3)$	$\{(1,3,2),(1,4,3)\}$	$I(2,3,3)$	$\{(2,1,3),(2,3,4)\}$
$I(3,3,3)$	$\varnothing$	$I(4,3,3)$	$\{(4,2,3),(4,3,1)\}$
$I(1,4,3)$	$\{(1,4,3)\}$	$I(2,4,3)$	$\{(1,4,3),(2,1,3),(2,4,1)\}$
$I(3,4,3)$	$\{(1,4,3),(3,4,2)\}$	$I(4,4,3)$	$\{(1,4,3),(4,2,3)\}$
$I(1,1,4)$	$\{(1,2,4),(3,1,4)\}$	$I(2,1,4)$	$\{(2,1,3),(2,3,4),(3,1,4)\}$
$I(3,1,4)$	$\{(3,1,4)\}$	$I(4,1,4)$	$\{(3,1,4),(4,1,2)\}$
$I(1,2,4)$	$\{(1,2,4)\}$	$I(2,2,4)$	$\{(1,2,4),(2,3,4)\}$
$I(3,2,4)$	$\{(1,2,4),(3,1,4),(3,2,1)\}$	$I(4,2,4)$	$\{(1,2,4),(4,2,3)\}$
$I(1,3,4)$	$\{(1,2,4),(1,3,2),(2,3,4)\}$	$I(2,3,4)$	$\{(2,3,4)\}$
$I(3,3,4)$	$\{(2,3,4),(3,1,4)\}$	$I(4,3,4)$	$\{(2,3,4),(4,3,1)\}$
$I(1,4,4)$	$\{(1,2,4),(1,4,3)\}$	$I(2,4,4)$	$\{(2,3,4),(2,4,1)\}$
$I(3,4,4)$	$\{(3,1,4),(3,4,2)\}$	$I(4,4,4)$	$\varnothing$

Table 2: I-sets of code C_L, the codewords are highlighted. The empty I-sets belong to the diagonal vertices.