

A Geometric Interpretation of the Intertwining Number

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Abstract

We exhibit a connection between two statistics on set partitions, the intertwining number and the depth-index. In particular, results link the intertwining number to the algebraic geometry of Borel orbits. Furthermore, by studying the generating polynomials of our statistics, we determine the $q = -1$ specialization of a q -analogue of the Bell numbers. Finally, by using Renner's H -polynomial of an algebraic monoid, we introduce and study a t -analog of q -Stirling numbers.

Mathematics Subject Classifications: 05A15, 14M15

1 Introduction

This paper is concerned with the intertwining number of a set partition, which is a combinatorial statistic introduced by Ehrenborg and Readdy in [10]. This statistic is among the combinatorial parameters on set partitions whose generating function is an important q -analog of the Stirling numbers of the second kind:

$$S_q(n, k) = \begin{cases} q^{k-1}S_q(n-1, k-1) + [k]_q S_q(n-1, k) & \text{if } n, k \geq 1; \\ \delta_{n,k} & \text{if } n = 0 \text{ or } k = 0. \end{cases} \quad (1)$$

Here $\delta_{n,k}$ is the Kronecker's delta function. As far as we know, this recurrence has first appeared in a paper of Milne who showed that (1) has a combinatorial interpretation in

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terms of statistics on set partitions. After Milne’s work, many authors found interesting combinatorial statistics whose (bi)generating polynomials satisfy the recurrence in (1), see for example [23]. For more recent results on the q -Stirling numbers, and an exposition of the history of Stirling numbers, we recommend the articles [6, 8].

In the present paper we connect the intertwining number to another statistic on the set partitions, namely, the depth-index, which was recently introduced and studied in [3] by the first two authors. The depth-index, although defined in purely combinatorial terms, equals the dimension of the closure of a certain (doubled) Borel orbit, and thus, the intertwining number also receives a geometric interpretation. One purpose of this article is to show that the depth-index is related in an interesting way to other set partition statistics.

Let us briefly set up the notation that is necessary to state our main results. Let A be a set partition of $\{1, \dots, n\}$ into *blocks* A_1, \dots, A_k . The minimal elements of the blocks are the *openers*, and the maximal elements are the *closers* of the set partition. For example, $A = 18|2569|37|4$ in Π_9 has openers 1, 2, 3, 4 and closers 4, 7, 8, 9.

Let us assume that the elements of each block are listed in increasing order, that is, two elements i, j in a block are consecutive if j is the smallest element larger than i in the same block. Then the *arc diagram* of a set partition $A \in \Pi_n$ is obtained by placing labels $1, \dots, n$ in this order on a horizontal line, and connecting consecutive elements of each block by arcs, as in Figure 1.

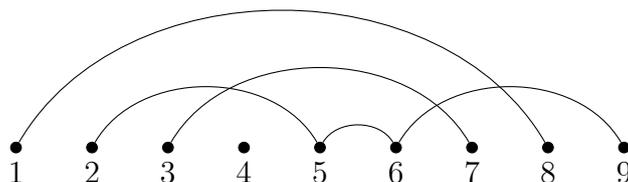


Figure 1: The arc-diagram of the set partition $A = 18|2569|37|4$.

The *extended arc diagram* is obtained from the arc diagram by adding a half-arc $(-\infty, i)$ from the far left to each opener i , and a half-arc (i, ∞) from each closer i to the far right. These arcs are drawn in such a way that half-arcs to the left do not cross, and half-arcs to the right do not cross either. An example is shown in Figure 2.

Two (generalized) arcs (i, j) and (k, ℓ) *cross* in A if $i < k < j < \ell$. The total number of crossings in A is the *intertwining number*¹ of A , denoted $i(A)$.

To indicate the geometric motive of our work, we will briefly mention the *Bruhat-Chevalley-Renner order* for set partitions: let \mathbb{B}_n be the group of invertible upper triangular $n \times n$ matrices and let \mathcal{B}_n be the monoid of upper triangular matrices with entries in $\{0, 1\}$, having at most one non-zero entry in each row and column. Then, for σ and τ in \mathcal{B}_n ,

$$\sigma \leq \tau \iff \mathbb{B}_n \sigma \mathbb{B}_n \subseteq \overline{\mathbb{B}_n \tau \mathbb{B}_n},$$

where \overline{X} denotes the Zariski closure of X . It was shown by Renner [18, sec.8] that this makes \mathcal{B}_n into a graded poset, the rank of σ being $\dim \mathbb{B}_n \sigma \mathbb{B}_n$. Let \mathcal{B}_n^{nil} be the semigroup

¹<http://www.findstat.org/St000490>

of nilpotent elements in \mathcal{B}_n . Then the following simple bijection between \mathcal{B}_n^{nil} and Π_n makes this order into an order of Π_n : the matrix corresponding to the set partition A has an entry equal to 1 in row i and column j if and only if (i, j) is an arc of A .

A purely combinatorial way to describe the above partial order on set partitions was recently introduced in [3], where the rank function of the poset was given by a certain combinatorial statistic (on arc-diagrams), called the depth-index of A and was denoted by $\mathfrak{t}(A)$. Now we are ready to outline the structure of this article and to mention our results which connect the depth-index to the other statistics.

First of all, in Section 2, we relate the intertwining number to the depth-index; this is our Theorem 3. On the one hand, the depth-index $\mathfrak{t}(A)$ is the rank function of the Bruhat-Chevalley-Renner order on the set partitions. On the other hand, the poset of doubled Borel orbits ordered by the containment relations on closures is ranked by the dimension function. In other words, the depth-index $\mathfrak{t}(A)$ gives the dimension of $\overline{\mathbb{B}_n \tau \mathbb{B}_n}$ where τ is the upper-triangular partial permutation matrix which corresponds to the set partition A . Thus, it follows from Theorem 3 that the intertwining number $\mathfrak{i}(A)$ is the rank function of the dual poset, $\mathfrak{i}(A) = \text{codim}(\overline{\mathbb{B}_n \tau \mathbb{B}_n})$. Notice that from a computational point of view the intertwining number is simpler than the depth index. In Section 3, we give another combinatorial interpretation of the depth-index and of the intertwining number using so-called rank-control matrices. In Section 4, we apply Theorem 3 to compute q -Bell numbers corresponding to the depth-index when $q = -1$. In Section 5, we use Renner's H -polynomial of an algebraic monoid to introduce and study a new t -analog of q -Stirling numbers.

2 The intertwining number and the depth-index

From now on, we identify a set partition with its arc-diagram. The blocks of a set partitions are chains in its arc-diagram. We will frequently use the following well-known and very important, albeit rather obvious fact: If A is an arc-diagram on n vertices with k arcs, then the number of chains of A is $n - k$, while singletons are also considered as chains. In this regard, we will denote by $\Pi_{n,k}$ ($k = 1, \dots, n$) the set of set partitions of $\{1, \dots, n\}$ with k blocks, or equivalently, the set of arc-diagrams on n vertices with k chains.

In this section we exhibit the relationship between Ehrenborg and Readdy's intertwining number, introduced in [10], and the depth index, introduced in [3].

Definition 1. Let A be a set partition from $\Pi_{n,k}$. Then the *intertwining number*² $\mathfrak{i}(A)$ of A is the total number of crossings in the extended arc diagram of A . More formally, for a pair of disjoint sets B and C of integers, the intertwining number is the cardinality of the set

$$\{(b, c) \in B \times C : \{\min(b, c) + 1, \dots, \max(b, c) - 1\} \cap (B \cup C) = \emptyset\},$$

and the intertwining number of a set partition A is the sum of intertwining numbers of all pairs of blocks of A .

²<http://www.findstat.org/St000490>

Let us denote by $\text{Arcs}(A)$ the set of arcs of A . The *depth index*³ $\mathfrak{t}(A)$ of A is

$$\sum_{i=1}^k (n-i) - \sum_{v=1}^n \text{depth}(v) + \sum_{\alpha \in \text{Arcs}(A)} \text{depth}(\alpha), \quad (2)$$

where $\text{depth}(v)$, which is called the depth of a vertex v , is the number of arcs $(i, j) \in \text{Arcs}(A)$ with $1 \leq i < v < j \leq n$, and $\text{depth}(\alpha)$, which is called the depth of an arc $\alpha = (u, v)$, is the number of arcs $(i, j) \in \text{Arcs}(A)$ with $1 \leq i < u < v < j \leq n$.

The intertwining number is best understood by visualizing it on the extended arc diagram of the set partition. An example is depicted in Figure 2.

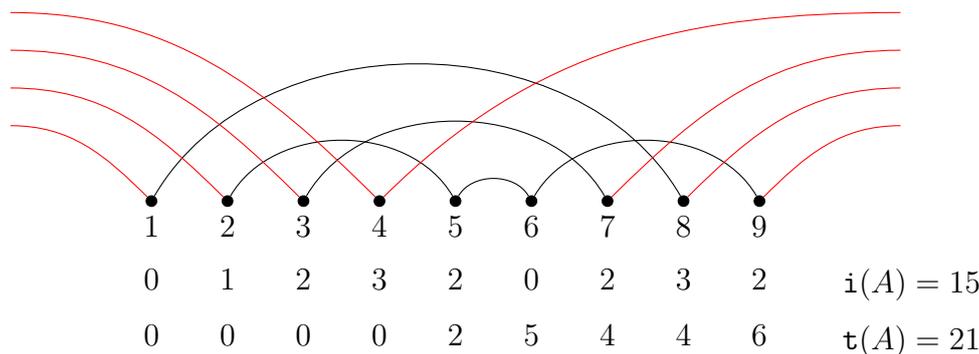


Figure 2: The extended arc diagram of the set partition $A = 18|2569|37|4$, together with the computation of the intertwining number and the depth index.

Remark 2. The second sum appearing in (2), that is $\sum_{v=1}^n \text{depth}(v)$, coincides with the dimension exponent⁴,

$$\sum_{B \text{ is a block of } A} (\max B - \min B + 1) - n.$$

Moreover, the sum $\sum_{\alpha \in \text{Arcs}(A)} \text{depth}(\alpha)$ is just the number of nestings⁵ of A . Note that the number of nestings is equidistributed with the number of crossings⁶, see [12, Corollary 1.5]. This statistic and the dimension exponent both occur in the theory of supercharacters. More precisely, for any supercharacter indexed by a set partition A , the dimension is given by the dimension exponent of A , and the scalar product with a supercharacter indexed by the same set partition equals the number of crossings of A , see [1].

Our main result is the following:

Theorem 3. *For any set partition $A \in \Pi_n$, we have*

$$\mathfrak{t}(A) + i(A) = \binom{n}{2}.$$

³<http://www.findstat.org/St001094>

⁴<http://www.findstat.org/St000572>

⁵<http://www.findstat.org/St000233>

⁶<http://www.findstat.org/St000232>

For the proof it will be convenient to refine the intertwining number and the depth index as follows.

Definition 4. Let A be a set partition of $\{1, \dots, n\}$, and let $v \in \{1, \dots, n\}$.

The *partner* u of an element is 0 if v is the minimal element of its block, otherwise it is the largest element in the same block smaller than v , that is, (u, v) is an arc in the arc diagram.

The *partial intertwining number*, denoted by $i_v(A)$, is the number of crossings of the arc (or half-arc) ending in v with arcs or half-arcs whose smaller vertex i is between u and v .

The *partial depth index*, denoted by $t_v(A)$, is the sum of the number of (proper) arcs (i, j) with $u < i < j < v$ and the number u , which is the partner of v .

In Figure 2, the partial intertwining numbers are written on the second line, below the elements of the set partition. Since every crossing is counted precisely once, the sum of these numbers is the intertwining number of A . The partial depth indices are written below, on the third line.

It is clear that the sum of partial intertwining numbers is the intertwining number of the set partition. The corresponding statement for the depth index is also true:

Lemma 5. *The sum of the partial depth indices of a set partition is equal to its depth index.*

Before we prove this lemma, let us note a second useful fact.

Lemma 6. *For each $v \in \{1, \dots, n\}$, the sum of $t_v(A)$ and $i_v(A)$ equals $v - 1$, the total number of vertices before v .*

Clearly, Theorem 3 follows at once from Lemma 5 and Lemma 6.

Proof of Lemma 6. Let u be the partner of v . Then any arc (i, j) with $u < i < v$ either satisfies $u < i < j < v$, and thus contributes +1 to the partial depth index, or it contributes precisely one crossing to $i_v(A)$. \square

Proof of Lemma 5. Let A be a set partition of $\{1, \dots, n\}$ and let $v \in \{1, \dots, n\}$. Let A' be the set partition obtained from A by removing the last vertex with label n .

We will now use induction: we proceed with the assumption that $t(A') = \sum_{v=1}^{n-1} t_v(A')$. Moreover, by definition we have $t_v(A') = t_v(A)$ for $v < n$. (Clearly, if $n = 1$, then there is nothing to prove.)

If n is a singleton block of A , then $t(A)$ is obtained from $t(A')$ by adding the number of arcs of A . Otherwise, we assume that (m, n) is an arc in A , and there are ν arcs (i, j) in A with $m < i < j < n$. Then

$$t(A) = t(A') + (n - 1) - (n - 1 - m) + \nu,$$

since the new arc (m, n) contributes $n - 1$ to the first sum in the definition of the depth index, and the new arc increases the depth of each of the vertices between m and n by 1.

In both of these cases, $t(A) = t(A') + m + \nu = t(A') + t_n(A)$, hence the proof is finished. \square

3 The intertwining number and the rank control matrix

We start with setting up our notation.

Definition 7. For an $n \times n$ matrix X , let $X_{k,\ell}$ denote the lower-left $k \times \ell$ submatrix of X . Then the *rank control matrix* $R(X) = (r_{k,\ell})_{k,\ell=1}^n$ is the $n \times n$ matrix with entries defined by $r_{k,\ell} := \text{rank}(X_{k,\ell})$.

As far as we know, the rank control matrix $R(X)$ was introduced by Melnikov in [13]. A closely related version is used in [17] and in [14] for describing the Bruhat-Chevalley order on symmetric groups. Incitti [11] used it in his study of the Bruhat order on involutions. After Incitti's work, the rank control matrix is used in [2], [9], [5], [4] for studying Bruhat orders on partial involutions and partial fixed-point-free involutions. See [6] for related work on the rook monoid.

Next, we introduce the “inequalities statistic” of an arc-diagram.

Definition 8. Let A be a set partition from Π_n . We denote by $M(A) = (m_{i,j})_{i,j=1}^n$ the $n \times n$ matrix defined by

$$m_{i,j} = \begin{cases} 1 & \text{if } (i,j) \text{ is an arc in } A; \\ 0 & \text{otherwise.} \end{cases}$$

In this notation, the *inequalities statistic* of A , denoted by $D(A)$, is defined by

$$D(A) = |\{(i,j) \mid 2 \leq i \leq n, 1 \leq j \leq n-1 \text{ and } r_{i,j} \neq r_{i-1,j+1}\}|,$$

Here, $r_{i,j}$'s are the entries of rank control matrix $R(M(A)) = (r_{i,j})_{i,j=1}^n$.

In other words, $M(A)$ is the adjacency matrix of A , regarded as the directed graph with edges directed towards the vertices with bigger labels. The statistic $D(A)$ is the total number of inequalities along antidiagonals from south-west to north-east of the rank control matrix $R(M(A))$.

Proposition 9. Let A be a set partition of $\{1, \dots, n\}$ and let $(r_{i,j})_{i,j=1}^n$ denote the rank control matrix of $M(A)$. In this case, the entry $r_{i,j}$ ($i, j \in \{1, \dots, n\}$) is equal to the number of arcs (k, l) in A such that $i \leq k < l \leq j$.

The proof of Proposition 9 follows from the definition of rank control matrix of $M(A)$, so we omit writing it. Nevertheless, we give an example that explains it.

Example 10. For example, let A denote the arc-diagram



Then $M(A) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $R(M(A)) = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The antidiagonals from

south-west to north-east in the matrix $R(M(A))$ are: $(0, 1)$, $(0, 0, 2)$, $(0, 0, 1, 2)$, $(0, 0, 1)$

and $(0, 0)$. Note that we skipped the initial and the last antidiagonal sequences since they do not have any inequalities, hence they do not contribute to our statistic. Now, we have two inequalities in the third diagonal $(0, 0, 1, 2)$ ($0 \neq 1$ and $1 \neq 2$) and one inequality in the first, second and fourth diagonals. Thus, for this arc-diagram A we have $D(A) = 5$.

Let us point out that the depth index of A is equal to five as well:

$$\mathfrak{t}(A) = 3 + 2 - (0 + 0 + 0 + 0) + (0 + 0) = 5.$$

This is not a coincidence as we will show in our next result.

Theorem 11. *Let A be an arc-diagram from Π_n . By viewing A as a directed graph we let $R = (r_{i,j})_{i,j=1}^n$ denote the rank-control matrix of the adjacency matrix of A . In this case, the following relations hold true:*

1. $\mathfrak{t}(A) = D(A)$, where

$$D(A) = |\{(i, j) \mid 2 \leq i \leq n, 1 \leq j \leq n - 1 \text{ and } r_{i,j} \neq r_{i-1,j+1}\}|;$$

2. $\mathfrak{i}(A) = E(A)$, where

$$E(A) = |\{(i, j) \mid 2 \leq i \leq j + 1 \leq n \text{ and } r_{i,j} = r_{i-1,j+1}\}|,$$

We divide the proof into two propositions.

Proposition 12. *Let A be a set partition. Then*

$$\mathfrak{t}(A) = D(A).$$

Proof. For a fixed j with $2 \leq j \leq n$, we set $D_j(A) = |\{i \mid 2 \leq i \leq n, \text{ and } r_{i,j} \neq r_{i+1,j-1}\}|$. We will prove that $\mathfrak{t}_j(A) = D_j(A)$ for every j , $2 \leq j \leq n$, where $\mathfrak{t}_j(A)$ is the partial depth index defined as in Definition 4.

If the vertex j is not the endpoint of any arc, hence, the partner of j is 0, or, equivalently, the j -th column of the matrix $M(A)$ consists of 0's only, then $\mathfrak{t}_j(A)$ is the number of arcs (u, v) with $u < v < j$. In this case, $D_j(A)$ equals the number of 1's to the left of the j -th column in the matrix $M(A)$. Each such 1 in $M(A)$ corresponds to an arc in A , therefore, $\mathfrak{t}_j(A) = D_j(A)$.

If the j -th column of $M(A)$ has a 1 at the position (w, j) , then (w, j) is an arc in A for some w ($w < j$) and furthermore there are w inequalities of the form $r_{i,j} > r_{i+1,j-1}$ for each i such that $1 \leq i \leq w$. Also, it is easily seen that each arc (u, v) , where $w < u < v < j$, contributes an inequality of the form $r_{u,j} > r_{u+1,j-1}$. Therefore, $D_j(A) = w + |\{(u, v) \mid w < u < v < j, m_{u,v} = 1\}|$. Since w is the partner of the vertex j , we see that $\mathfrak{t}_j(A) = w + |\{(u, v) \mid w < u < v < j, (u, v) \text{ is an arc in } A\}|$. Therefore, in this case we have $\mathfrak{t}_j(A) = D_j(A)$ also.

Now, since by Lemma 5, $\mathfrak{t}(A) = \sum_{j=2}^n \mathfrak{t}_j(A)$, and $D(A) = \sum_{j=2}^n D_j(A)$, the proof is finished. \square

By combining Theorem 3 and Proposition 12 we are able to express the intertwining number in terms of the number of equalities in anti-diagonals of the rank control matrix. To this end, if A is an arc-diagram from Π_n , then let us denote by $E(A)$ the following statistic:

$$E(A) = |\{(i, j) \mid 2 \leq i \leq j + 1 \leq n \text{ and } r_{i,j} = r_{i-1,j+1}\}|,$$

where, as before, $r_{i,j}$'s are the entries of the rank control matrix $R(M(A))$.

Proposition 13. *Let A be an arc-diagram. Then*

$$\mathbf{i}(A) = \mathbf{E}(A).$$

Proof. Let n be the number of vertices of A . By Theorem 3, we have $\mathbf{i}(A) = \binom{n}{2} - \mathbf{t}(A)$. Using Theorem 12 we now have $\mathbf{i}(A) = \binom{n}{2} - \mathbf{D}(A)$. To finish the proof we will show that $\mathbf{E}(A) = \binom{n}{2} - \mathbf{D}(A)$.

Now, notice that $i > j$ implies $r_{i,j} = r_{i-1,j+1}$ since the adjacency matrix $M(A)$ of A is strictly upper triangular. (The edges are directed towards vertices with bigger indices.) Therefore, the pairs (i, j) with $i > j$ do not contribute to $\mathbf{D}(A)$. By definition, pairs (i, j) with $i > j$ do not contribute to $\mathbf{E}(A)$ either. By arguing in a similar manner we see that a pair (i, j) with $2 \leq i \leq j \leq n$ contributes either to $\mathbf{E}(A)$ or to $\mathbf{D}(A)$ but not to both. Clearly, the number of such pairs equals $\binom{n}{2}$. This shows that $\mathbf{E}(A) + \mathbf{D}(A) = \binom{n}{2}$, and the proof is finished. \square

Proof of Theorem 11. This is a combination of Propositions 12 and 13. \square

Before proceeding to the next section, we briefly discuss the algebraic geometric significance of the equalities $\mathbf{t}(A) = \mathbf{D}(A)$ and $\mathbf{i}(A) = \mathbf{E}(A)$.

We already mentioned that the doubled Borel group $\mathbb{B}_n \times \mathbb{B}_n$ acts on matrices via

$$(B_1, B_2) \cdot X = B_1 X B_2^{-1} \quad \text{for } (B_1, B_2) \in \mathbb{B}_n \times \mathbb{B}_n, X \in \text{Mat}_n, \quad (3)$$

and the action (3) restricts to give an action on $\overline{\mathbb{B}}_n$.

Proposition 14. *Let X and Y be two matrices from Mat_n such that $Y = BXC$, where B and C are from \mathbb{B}_n . If $X_{k\ell}$ and $Y_{k\ell}$ denotes the lower-left $k \times \ell$ submatrices of X and Y , respectively, then*

$$\text{rank}(X_{k\ell}) = \text{rank}(Y_{k\ell}) \quad \text{for all } 1 \leq k, \ell \leq n.$$

Proof. Let us write the matrix BXC in block form:

$$\begin{pmatrix} * & * \\ 0_{k \times (n-k)} & B' \end{pmatrix} \begin{pmatrix} * & * \\ X_{k\ell} & * \end{pmatrix} \begin{pmatrix} C' & * \\ 0_{(n-\ell) \times \ell} & * \end{pmatrix} = \begin{pmatrix} * & * \\ B' X_{k\ell} C' & * \end{pmatrix},$$

and therefore, $Y_{k\ell} = B' X_{k\ell} C'$. Matrices B' and C' are invertible (upper-triangular $k \times k$ and $\ell \times \ell$ submatrices of B and C respectively), which implies that $Y_{k\ell}$ and $X_{k\ell}$ have equal ranks. \square

Proposition 14 shows that the rank control matrix R is an invariant of a $\mathbb{B}_n \times \mathbb{B}_n$ -orbit. Let \mathcal{N}_n denote the $\binom{n}{2}$ dimensional affine space of upper triangular $n \times n$ matrices which are nilpotent. If Z is from \mathcal{N}_n , then the closure $\overline{\mathbb{B}_n Z \mathbb{B}_n}$ in $\overline{\mathbb{B}}_n$ is an affine algebraic subvariety of \mathcal{N}_n . As we mentioned in the introduction, such orbit closures are parametrized by the arc-diagrams, and furthermore, the Bruhat-Chevalley-Renner order can be interpreted in a combinatorial way on the arc-diagrams. All of this is recorded in [3].

Remark 15. There is another combinatorial method to determine the relation between two elements in the Bruhat-Chevalley-Renner order. Namely, it is sufficient to compare the corresponding rank control matrices componentwise. More precisely, for two $k \times \ell$ matrices $X = (x_{i,j})$, $Y = (y_{i,j})$ let us define $X \leq Y$ if $x_{i,j} \leq y_{i,j}$ for all i, j , $1 \leq i \leq k$, $1 \leq j \leq \ell$. It can be shown (see [2]) that $\overline{\mathbb{B}_n Z_1 \mathbb{B}_n} \subseteq \overline{\mathbb{B}_n Z_2 \mathbb{B}_n}$ if and only if $R(Z_1) \leq R(Z_2)$. Now, if A and B are two arc-diagrams, then we write that $A \leq B$ if the corresponding nilpotent partial permutation matrices σ_A and σ_B satisfy $\sigma_A \leq \sigma_B$. In this notation, $A \leq B$ if and only if $R(M(A)) \leq R(M(B))$.

Recall that the depth index statistic gives us the dimensions of the $\mathbb{B}_n \times \mathbb{B}_n$ -orbits, therefore, by Theorem 3, the intertwining number of an arc-diagram gives the codimension of the corresponding orbit closure in \mathcal{N}_n . Note also that the dimension of the variety $\overline{\mathbb{B}_n Z \mathbb{B}_n}$ equals $\dim \mathcal{N}_n$ minus the number of algebraically independent polynomial relations that defines the affine variety $\overline{\mathbb{B}_n Z \mathbb{B}_n}$. Therefore, by using Proposition 13, we obtain the following result.

Proposition 16. *Let $\overline{\mathbb{B}_n Z \mathbb{B}_n}$ ($Z \in \mathcal{N}_n$) be a doubled Borel subgroup orbit. Then the parameter $\mathbf{E}(R(Z))$ which counts the number of equalities in the anti-diagonals in the upper triangle of $R(Z)$ actually counts the number of algebraically independent polynomial equations that define the variety $\overline{\mathbb{B}_n Z \mathbb{B}_n}$.*

There is a proof of Proposition 16 that does not use Theorem 3. However, this direct proof is somewhat long and it is very similar to the proof of Theorem 7.6 of [2], so, we omit it.

4 $q = -1$ specialization

The next result of our article builds on the connections between various set partition statistics that we established so far. The n -th Bell number, denoted by B_n is the sum $B_n := \sum_k S(n, k)$. It follows from Theorem 3 that the following polynomial is a q -analog of the Bell numbers:

$$B_n(q) := q^{\binom{n}{2}} X_n \left(\frac{1}{q} \right), \quad \text{where } X_n(q) = \sum_{\sigma \in \Pi_n} q^{\mathbf{t}(\sigma)}. \quad (4)$$

Theorem 17. *Let $X_n(q)$ denote the rank generating series of the depth-index statistic as in (4). Then*

$$X_n(-1) = (-1)^{\binom{n}{2}} B_n(-1) = \begin{cases} 1 & \text{if } n \equiv 0 \text{ or } 1 \text{ or } 3 \text{ or } 10 \pmod{12}; \\ -1 & \text{if } n \equiv 4 \text{ or } 6 \text{ or } 7 \text{ or } 9 \pmod{12}; \\ 0 & \text{if } n \equiv 2 \pmod{3}. \end{cases} \quad (5)$$

Proof. As we hinted at in the introduction, the intertwining number on $\Pi_{n,k}$ has the q -Stirling numbers as its generating series,

$$S_q(n, k) = \sum_{A \in \Pi_{n,k}} q^{\mathbf{i}(A)} \quad (k = 0, 1, \dots, n-1). \quad (6)$$

Let us define $H_n(q, t)$ as the generating function $H_n(q, t) := \sum_{k=0}^n S_q(n, k)t^k$. Then, by the definition of $B_n(q)$, we have $H_n(q, 1) = B_n(q)$. The following specialization of $\sum_{n=0}^{\infty} H_n(q, t)x^n$ follows from [24, Theorem 1 eqn. (1.5)]:

$$\sum_{n \geq 0} H_n(-1, 1)x^n = \frac{1}{1 - x + x^2} = \frac{1 + x}{1 + x^3}. \quad (7)$$

The power series expansion of (7) yields (5). □

Remark 18. The above proof of Theorem 17, which is pointed to us by the referee, is different (and more insightful) than our original proof, which uses a simple counting argument.

Remark 19. Another statistic with the same generating function is introduced by Milne in [15].

Definition 20. Let $A = A_1|A_2| \dots |A_k$ be a set partition from $\Pi_{n,k}$. The *dual major index*⁷ of A is defined by

$$\widehat{\text{maj}}(A) = \sum_{i=1}^k (i-1) |A_i|.$$

This nomenclature is due to Sagan [20]. Although it is not referred to as the ‘dual major index’ in [22], $\widehat{\text{maj}}$ is the second of the three statistics that are studied by Wagner. In [21], Steingrímsson refers to $\widehat{\text{maj}}$ as “LOS”, the left opener statistic. A bijection $\phi : \Pi_{n,k} \rightarrow \Pi_{n,k}$ satisfying

$$\widehat{\text{maj}}(\phi(A)) = \mathbf{i}(A),$$

and thus proving the equidistribution of the intertwining number and the dual major index was found by Parviainen in [16]. It is defined as follows. As before, let $\mathbf{i}_v(A)$ be the number of crossings of the arc (or line) ending in v with arcs or lines whose smaller vertex i is between the partner of v and v . Then the blocks of the set partition $\phi(A)$ are the sets of elements sharing the same number $\mathbf{i}_v(A)$. For example, the image of the set partition $A = 18|2569|37|4$ from Figure 2 is $\phi(A) = 16|2|3579|48$ and the dual major index of $\phi(A)$ is given by $\widehat{\text{maj}}(\phi(A)) = 0 \cdot 2 + 1 \cdot 1 + 2 \cdot 4 + 3 \cdot 2 = 15$.

5 H -polynomials

In the previous section we considered the generating functions $B_n(q)$, $X_n(q)$ of the statistics $\widehat{\text{maj}}$, \mathbf{t} , respectively. We noted that $B_n(q) = q^{\binom{n}{2}} X_n(1/q)$. We considered also the following generating polynomial of q -Stirling numbers: $H_n(q, t) := \sum_{k=0}^n S_q(n, k)t^k$. For $n = 0, 1, 2$, the values of the polynomial $H_n(q, t)$ are given by

$$\begin{aligned} H_0(q, t) &= 1, \\ H_1(q, t) &= S_q(1, 0) + S_q(1, 1)t = t, \\ H_2(q, t) &= S_q(2, 0) + S_q(2, 1)t + S_q(2, 2)t^2 = t + qt^2. \end{aligned}$$

In this section, by relating the depth index and the intertwining number to the numbers of rational points of \mathbb{B}_{n-1} over finite fields, we prove the following result.

⁷<http://www.findstat.org/St000493>

Theorem 21. *We have*

$$H_n\left(q, \frac{1}{1-q}\right) = \frac{1}{(1-q)^n},$$

for all $n \geq 0$.

Let M be an algebraic monoid with a reductive group of units denoted by G . Let B denote the Borel subgroup in G , T denote the maximal torus contained in B , and let R denote the Renner monoid defined as $R := \overline{N_G(T)}/T$, where $N_G(T)$ is the normalizer of T in G and the bar indicates that we are taking the closure in Zariski topology. The H -polynomial of the Renner monoid of R is defined by

$$H(R, q) := \sum_{\sigma \in R} q^{a(\sigma)}(q-1)^{b(\sigma)},$$

where $a(\sigma)$ is the dimension of the unipotent part of the orbit $B\sigma B$ and $b(\sigma)$ is the dimension of the diagonalizable part of $B\sigma B$.

One can think of the H -polynomial of R as a transformed ‘‘Hasse-Weil motivic zeta function’’ of the projectivization $\mathbb{P}(M - \{0\})$ of M . Indeed, by treating q as a power of a prime number, for all sufficiently large q , $(H(R, q) - 1)/(q - 1)$ is equal to the number of rational points over \mathbb{F}_q (the finite field with q elements) of $\mathbb{P}(M - \{0\})$. See [19, Remark 3.2]. For an application of this idea to the rook theory, we recommend [7].

We will consider the simplest example, where M is Mat_n , the monoid of all $n \times n$ matrices. Clearly, the projectivization of Mat_n is the $n^2 - 1$ dimensional projective space, hence its number of \mathbb{F}_q -rational points is given by $(q^{n^2} - 1)/(q - 1)$. The Renner monoid of Mat_n is the rook monoid \mathcal{R}_n . Furthermore, if $\sigma \in \mathcal{R}_n$, then $a(\sigma) = \ell(\sigma) - \text{rank}(\sigma)$, and $b(\sigma) = \text{rank}(\sigma)$, where $\text{rank}(\sigma)$ is the rank of σ as a matrix. Therefore, the H -polynomial of \mathcal{R}_n becomes

$$q^{n^2} = H(\mathcal{R}_n, q) = \sum_{\sigma \in \mathcal{R}_n} q^{\ell(\sigma) - \text{rank}(\sigma)}(q-1)^{\text{rank}(\sigma)}. \quad (8)$$

An important consequence of the formula in (8) is that if X denotes any $\mathbb{B}_n \times \mathbb{B}_n$ -stable smooth and irreducible subvariety of Mat_n , then $H(X, q)$ gives the number of \mathbb{F}_q -rational points of X . In particular, we apply this observation to \mathcal{N}_n , the semigroup of nilpotent elements in \mathbb{B}_n . Note that, there is a natural algebraic variety isomorphism between \mathcal{N}_n and $\overline{\mathbb{B}}_{n-1}$. The isomorphism is given by the deleting of the first column and the last row of the elements of \mathcal{N}_n . Moreover, under this isomorphism, $\mathbb{B}_n \times \mathbb{B}_n$ -orbits in \mathcal{N}_n are isomorphically mapped onto $\mathbb{B}_{n-1} \times \mathbb{B}_{n-1}$ -orbits in $\overline{\mathbb{B}}_{n-1}$. Therefore, there is no loss of information to work with $\overline{\mathbb{B}}_{n-1}$ instead of \mathcal{N}_n . Now, on one hand we have the number of points of $\overline{\mathbb{B}}_{n-1}$ over \mathbb{F}_q ,

$$|\overline{\mathbb{B}}_{n-1}| = q^{\binom{n}{2}}. \quad (9)$$

On the other hand, we have that

$$H(\overline{\mathbb{B}}_{n-1}, q) = \sum_{\sigma \in \mathcal{R}_{n-1}} q^{\ell(\sigma) - \text{rank}(\sigma)}(q-1)^{\text{rank}(\sigma)} = \sum_{\sigma \in \mathcal{R}_{n-1}} q^{\text{t}(\sigma) - \text{rank}(\sigma)}(q-1)^{\text{rank}(\sigma)}. \quad (10)$$

By combining (9) with (10) and noting that this is a polynomial identity (since it holds true for all sufficiently large prime powers), we have

$$q^{\binom{n}{2}} = \sum_{\sigma \in \mathcal{B}_{n-1}} q^{\mathfrak{t}(\sigma) - \text{rank}(\sigma)} (q-1)^{\text{rank}(\sigma)} = \sum_{\sigma \in \mathcal{B}_{n-1}} q^{\mathfrak{t}(\sigma)} (1-1/q)^{\text{rank}(\sigma)}. \quad (11)$$

Eqn.(11) suggests a variation of the generating series of the statistic \mathfrak{t} and it provides a way to evaluate this generating series.

Proof of Theorem 21. Recall that

$$q^{\binom{n}{2}} \sum_{\sigma \in \mathcal{B}_{n-1}, \text{rank}(\sigma)=k} q^{-\mathfrak{t}(\sigma)} = \sum_{A \in \Pi_{n,n-k}} q^{\mathfrak{i}(A)} = S_q(n, n-k), \quad (12)$$

where $k \in \{1, \dots, n\}$.

We multiply both sides of (12) by t^{n-k} and then sum over $k \in \{1, \dots, n\}$ to get

$$H_n(q, t) = q^{\binom{n}{2}} \sum_{k=1}^n \sum_{\sigma \in \mathcal{B}_{n-1}, \text{rank}(\sigma)=k} q^{-\mathfrak{t}(\sigma)} t^{n-k},$$

which is equivalent to

$$H_n(q, t) = q^{\binom{n}{2}} t^n \sum_{\sigma \in \mathcal{B}_{n-1}} q^{-\mathfrak{t}(\sigma)} t^{-\text{rank}(\sigma)}. \quad (13)$$

At the same time, since eqn. (11) is equivalent to $q^{-\binom{n}{2}} = \sum_{\sigma \in \mathcal{B}_{n-1}} q^{-\mathfrak{t}(\sigma)} (1-q)^{\text{rank}(\sigma)}$, by replacing t with $(1-q)^{-1}$ in (13), we obtain

$$H_n\left(q, \frac{1}{1-q}\right) = q^{\binom{n}{2}} \frac{q^{-\binom{n}{2}}}{(1-q)^n} = \frac{1}{(1-q)^n}.$$

This finishes the proof of our theorem for $n \geq 1$. The case of $n = 0$ has already been computed before, hence, the proof is complete. \square

Remark 22. Clearly, by Theorem 21, we have $\sum_{n \geq 0} H_n(q, 1/(1-q)) x^n = \frac{1}{1-x/(1-q)}$. It is pointed to us by the referee that, once interpreted in terms of q -Stirling numbers, this equality follows from [24, Theorem 1 eqn. (1.5)] also.

6 Final Remarks

We close our paper by listing our new formulas for the intertwining number. Let A be an arc diagram on n vertices with k arcs.

1. $\mathfrak{i}(A) = \binom{n}{2} - \mathfrak{t}(A)$: This is the main result of the present paper.
2. $\mathfrak{i}(A) = \binom{n}{2} - \ell(\sigma)$, where $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_n)$ is the partial permutation corresponding to A . Recall that σ is defined so that $\sigma_j = i$ whenever (i, j) is an arc in A . $\ell(\sigma)$ is the length of the partial permutation σ . See the formulas in the introduction.

3. $i(A) = \binom{n}{2} - \dim \mathbb{B}_n \sigma \mathbb{B}_n$, where σ is as in the previous item.
4. $i(A) = E(A)$, where $E(A)$ is the number of equalities in the antidiagonals of the upper-triangle of the rank control matrix of A . See Section 3.
5. $i(A) = \binom{n}{2} - c(A)$. The statistic $c(A)$ is defined in [3]. We recall its definition here for completeness. Let α be an arc in A and let $cross(\alpha)$ denote the total number of chains that are crossed by α in A . Note that a chain can be crossed at most twice. In this case, we consider it as a single crossing. Let A be an arc-diagram on n vertices with k arcs denoted by $\alpha_1, \alpha_2, \dots, \alpha_k$ and $n - k$ chains denoted $\beta_1, \beta_2, \dots, \beta_{n-k}$. The crossing-index of A is defined by the formula

$$c(A) = \sum_{i=1}^k (n - i) - \sum_{j=1}^{n-k} depth(\beta_j) - \sum_{m=1}^k cross(\alpha_m),$$

where $depth(\beta_j)$ ($j = 1, \dots, n - k$) is the number of arcs that are above β_j . It is easy to show (by induction) that $c(A) = \tau(A)$. For details, see [3].

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