On the lattice point
covering problem in dimension 2

Fei Xue*
Institut für Mathematik
Technische Universität Berlin
D-10623 Berlin, Germany
xue@math.tu-berlin.de

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Abstract
In this paper we study the lattice point covering property of some regular polygons in dimension 2.
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1 Introduction

Let \( K^n \) be the set of all convex bodies, i.e., compact convex sets, in the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) with non-empty interior. We denote by \( K^n_{(o)} \subset K^n \) the set of all convex bodies, having the origin as an interior point, i.e., \( 0 \in \text{int}(K) \), and by \( K^n_{(s)} \subset K^n_{(o)} \) those bodies which are symmetric with respect to \( 0 \), i.e., \( K = -K \).

We say that a convex body \( K \in K^n \) has the lattice point covering property, if \( K \) contains a lattice point of \( \mathbb{Z}^n \) in any position, i.e., in any translation and rotation of \( K \).

There are several beautiful results about the lattice point covering property.

**Theorem 1** (Niven & Zuckerman,[4]). A triangle with sides of lengths \( a, b, c \), with \( a \leq b \leq c \), has the lattice point covering property if and only if \( 2\Delta(c - 1) \geq c^2 \) where \( \Delta \) is the area of the triangle.

**Theorem 2** (Niven & Zuckerman,[4]). Let \( a \) and \( b \) be the distances between the pairs of opposite sides, say with \( a \leq b \), of a parallelogram \( ABCD \) with an interior angle \( \gamma \leq \pi/2 \). The parallelogram has the lattice point covering property if and only if \( a \geq 1 \) and one of the following conditions hold:

\[(i) \; b \geq \sqrt{2};\]
\[(ii) \; b \leq \sqrt{2} \text{ and } \alpha + \beta + \gamma \leq \pi/2, \text{ where } \alpha = \arccos(a/\sqrt{2}) \text{ and } \beta = \arccos(b/\sqrt{2}).\]

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Theorem 3 (Henk & Tsintsifas,[3]). Let \( E \subset \mathbb{R}^n \) be an ellipsoid with semi-axes \( \alpha_i, 1 \leq i \leq n \). The following statements are equivalent:

(i) \( E \) contains a lattice point of \( \mathbb{Z}^n \) in any position,

(ii) \( \sum_{i=1}^{n} \frac{1}{\alpha_i^2} \leq 4, \)

(iii) \( E \) contains a cube of edge length 1.

Let \( K \in \mathcal{K}^n \). Denote by \( Z(K) \) the lattice point covering radius of \( K \), i.e., the smallest positive number \( r \) such that \( rK \) has the lattice point covering property. We are here concerned with the lattice point covering properties of regular polygons. Let us denote by

\[
H_n = \text{conv} \left\{ \left( \cos \left( \frac{2k\pi}{n} \right), \sin \left( \frac{2k\pi}{n} \right) \right) : k = 0, 1, \ldots, n-1 \right\}
\]

the regular \( n \)-gon.

Our main result is:

Theorem 4. Let \( t_i > 0, i \in \mathbb{N} \).

(1) The following statements are equivalent:

(i) \( t_4n \cdot H_{4n} \) contains a lattice point of \( \mathbb{Z}^2 \) in any position,

(ii) \( t_4n \cdot H_{4n} \) contains a ball with radius \( \frac{\sqrt{2}}{2} \),

(iii) \( t_4n \geq \frac{\sqrt{2}}{\cos \frac{\pi}{2}} \).

(2) The following statements are equivalent for \( n = 1, 2 \):

(i) \( t_{4n+2} \cdot H_{4n+2} \) contains a lattice point of \( \mathbb{Z}^2 \) in any position,

(ii) \( t_{4n+2} \cdot H_{4n+2} \) contains \( [-\frac{1}{2}, \frac{1}{2}]^2 \),

(iii) \( t_6 \geq \frac{1}{3-\sqrt{3}} \approx 0.788675 \ldots \); \( t_{10} \geq \frac{\cos \frac{\pi}{5} - \sin \frac{\pi}{5} + \sin \frac{2\pi}{5} - \cos \frac{2\pi}{5}}{2\sin \frac{\pi}{5}} \approx 0.734342 \ldots \).

Corollary 5.

(i) \( Z(H_{4n}) = \frac{\sqrt{2}}{\cos \frac{\pi}{2n}} \), for \( n \in \mathbb{N} \).

(ii) \( Z(H_6) = \frac{1}{3-\sqrt{3}} \).

(iii) \( Z(H_{10}) = \frac{\cos \frac{\pi}{5} - \sin \frac{\pi}{5} + \sin \frac{2\pi}{5} - \cos \frac{2\pi}{5}}{2\sin \frac{\pi}{5}} \).

This paper is organized as follows: in the Section 2 we will introduce the covering radius and the gauge function; then we will give a necessary and a sufficient condition of the lattice point covering property in the Section 3 and the Section 4, respectively; the content of the Section 5 will be the proof of the main theorem.
2 Covering Radius

The covering radius of $K \in \mathcal{K}^n_{(o)}$ with respect to $\mathbb{Z}^n$ is denoted by

$$c(K) = c(K, \mathbb{Z}^n) = \min\{\lambda > 0 : \lambda K + \mathbb{Z}^n = \mathbb{R}^n\}.$$ 

The gauge function $\|\cdot\|_K$ associated to a $K \in \mathcal{K}^n_{(o)}$ is the function

$$\|\cdot\|_K : \mathbb{R}^n \to [0, \infty)$$

defined by $\|v\|_K = \min\{t > 0 : v \in tK\}$.

**Theorem 6.** Let $K \in \mathcal{K}^n_{(o)}$. Then $K$ contains a lattice point of $\mathbb{Z}^n$ in any position if and only if, for any $o(K)$ rotation of $K$, $c(o(K)) \leq 1$.

**Proof.** If $c(o(K)) > 1$ for some rotation $o(K)$, then there exists a point $x \in \mathbb{R}^n$, such that for every $u \in \mathbb{Z}^2$,

$$\|x - u\|_{o(K)} > 1 \iff \|u - x\|_{-o(K)} > 1.$$ 

Therefore, $-o(K) + x$ does not contain a lattice point of $\mathbb{Z}^2$.

If $c(o(K)) \leq 1$ for any rotation $o(K)$, then for any point $x \in \mathbb{R}^n$, since $o(K) + \mathbb{Z}^n = \mathbb{R}^n$, there exists a lattice point $u \in \mathbb{Z}^2$, such that

$$\|x - u\|_{o(K)} \leq 1 \iff \|u - x\|_{-o(K)} \leq 1.$$ 

So, $-o(K) + x$ contain a lattice point $u$. \qed

Therefore, the lattice point covering property of a convex body depends on the covering radius of all rotations of this convex body.

3 Necessary Condition

According to the knowledge of the lattice covering for a centrally symmetric convex body, we have:

**Theorem 7** (I. Fáry,[1]). Let $K \in \mathcal{K}^2_{(s)}$, such that $K + \mathbb{Z}^2$ is a lattice covering. Then $K$ contains a spacefiller $L$, i.e., a parallelogram or a centrally symmetric hexagon, such that $L + \mathbb{Z}^2$ is a lattice tiling.

Since the lattice point covering property depends on the lattice covering of all rotations, we have:

**Theorem 8.** Let $K \in \mathcal{K}^2_{(s)}$. Then $K$ contains a lattice point of $\mathbb{Z}^2$ in any position, if and only if

1. $o(K) + \mathbb{Z}^2 = \mathbb{R}^2$,
2. $o(K)$ contains a spacefiller $L$, i.e., parallelogram or a centrally symmetric hexagon, such that $L + \mathbb{Z}^2$ is a lattice tiling,

for any rotation $o(K)$. 

Proof. Apply Theorem 6 and 7.

We here call it a necessary condition of the lattice point covering property, because
this condition has to hold, but is uneasy to check.

4 Sufficient Condition

For a planar convex body, it is possible to check some inscribed parallelograms, i.e.,
by checking the Steiner symmetrization of the convex body. We will give a sufficient condi-
tion of the lattice point covering property in this way.

The Steiner symmetrization of $K \in \mathcal{K}^2$ with respect to $\{ x \in \mathbb{R}^2 : x_2 = 0 \}$,
denoted by $\text{St}_1(K)$, is a convex body symmetric with respect to $\{ x \in \mathbb{R}^2 : x_2 = 0 \}$,
such that for each line $l$ vertical to $\{ x \in \mathbb{R}^2 : x_2 = 0 \}$,
$$\text{vol}_1(K \cap l) = \text{vol}_1(\text{St}_1(K) \cap l),$$
where $\text{vol}_1(L)$ denotes the length of a line segment $L$. For more information on the Steiner
symmetrization, we refer to [2, Section 9.1]. It is obvious that $\text{St}_1(K) \subset \text{St}_1(L)$ for two
convex bodies $K \subset L$.

**Theorem 9.** Let $K \in \mathcal{K}_{(s)}^2$. If for each rotation $o(K)$ of $K$, $\text{St}_1(o(K))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$, then $K$ contains a lattice point of $\mathbb{Z}^2$ in any position.

**Proof.** Notice that if $\text{St}_1(o(K))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$, then $o(K)$ contains a parallelogram in the form of $L = \text{conv} \{ (\frac{1}{2}, a), (\frac{1}{2}, a + 1), (-\frac{1}{2}, -a), (-\frac{1}{2}, -a - 1) \}$, which is a spacefiller with respect to $\mathbb{Z}^2$. Therefore $K$ has the lattice point covering property (cf. Theorem 6).

We also have the following proposition of lattice covering for sets symmetric with
respect to the axis $ox$ and $oy$.

**Proposition 10.** Let $K \in \mathcal{K}_{(s)}^2$. If $K$ is symmetric with respect to $\{ x \in \mathbb{R}^2 : x_1 = 0 \}$ and $\{ x \in \mathbb{R}^2 : x_2 = 0 \}$, then $K + \mathbb{Z}^2 = \mathbb{R}^2$ if and only if $K$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$.

**Proof.** If $K$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$, then $K + \mathbb{Z}^2$ is a lattice covering. Otherwise, if $K$ does not contain $[-\frac{1}{2}, \frac{1}{2}]^2$, i.e., $(\frac{1}{2}, \frac{1}{2}) \notin K$, then since $K$ is symmetric with respect to $\{ x \in \mathbb{R}^2 : x_1 = 0 \}$ and $\{ x \in \mathbb{R}^2 : x_2 = 0 \}$, $K$ does not contain any point of $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$, thus $K + \mathbb{Z}^2$ does not contain $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$.

5 Proof of the Main Theorem

In this section we discuss the lattice point covering property of some regular polygons.
The proofs have the following steps:

1. Prove that the Steiner symmetrizations of all rotations of the convex body contain $[-\frac{1}{2}, \frac{1}{2}]^2$ (using Theorem 9).
2. Prove that a smaller copy of the convex body does not have the lattice point covering property (using Theorem 8(1) and Proposition 10).

We first look at the regular 4n-gon.

**Proof of Theorem 4(1).** If \( t \cdot H_{4n} \) contains \( B_2(\sqrt{\frac{n}{2}}) \), where \( B_2(r) \) denotes the Euclidean disk of radius \( r \) centered at 0, then each rotation \( o(t \cdot H_{4n}) \) also contains \( B_2(\sqrt{\frac{n}{2}}) \). Notice that \( B_2(\sqrt{\frac{n}{2}}) \) contains \([-\frac{1}{2}, \frac{1}{2}]^2\), therefore \( o(t \cdot H_{4n}) + \mathbb{Z}^2 \) is always a lattice covering, thus \( t \cdot H_{4n} \) has the lattice point covering property (cf. Theorem 6).

If \( t \cdot H_{4n} \) does not contain \( B_2(\sqrt{\frac{n}{2}}) \), then \( o(t \cdot H'_{4n}, \frac{\pi}{4}) + (\frac{1}{2}, \frac{1}{2}) \), does not contain any lattice point of \( \mathbb{Z}^2 \), where \( o(t \cdot H'_{4n}, \frac{\pi}{4}) \) is the rotation of \( t \cdot H_{4n} \) by angle \( \frac{\pi}{4} \).

Denote by \( o(K, \theta) \) the counterclockwise rotation of \( K \) by angle \( \theta \), i.e.,

\[
o(H_n, \theta) = \text{conv} \left\{ \left( \cos \left( \frac{2k\pi}{n} + \theta \right), \sin \left( \frac{2k\pi}{n} + \theta \right) \right) : k = 0, 1, \cdots, n - 1 \right\}.
\]

Then we look at the regular hexagon.

**Proof of Theorem 4(2),** \( n = 1 \). For the symmetric reason between \( H_6 \) and \( \mathbb{Z}^2 \), the case \( \frac{\pi}{12} \leq \theta \leq \frac{\pi}{6} \) is actually symmetric to the case \( 0 \leq \theta \leq \frac{\pi}{12} \) with respect to the line \( \{ x \in \mathbb{R}^2 : x_2 = x_1 \} \). We are going to prove that \( \text{St}_1(o(\frac{1}{3-\sqrt{3}}H_6, \theta)) \) contains \([-\frac{1}{2}, \frac{1}{2}]^2\) for \( 0 \leq \theta \leq \frac{\pi}{12} \) (cf. Theorem 9).

By calculation,

\[
\text{St}_1(o(H_6, \theta)) = \text{conv} \left\{ (\pm \cos \theta, 0), \left( \pm \cos(\theta - \frac{\pi}{3}), \pm \frac{\sin \frac{\theta}{3}}{2 \sin(\theta + \frac{\pi}{6})} \right) \right\},
\]

which is also symmetric with respect to \( \{ x \in \mathbb{R}^2 : x_1 = 0 \} \). In order to check whether \( \text{St}_1(o(\frac{1}{3-\sqrt{3}}H_6, \theta)) \) contains \([-\frac{1}{2}, \frac{1}{2}]^2\) for \( 0 \leq \theta \leq \frac{\pi}{12} \), notice that when \( 0 \leq \theta \leq \frac{\pi}{12} \), it holds

\[
\cos(\theta + \frac{\pi}{3}) \leq \frac{\sqrt{3}}{2 \cos \theta}
\]

(cf. Proposition 12), therefore the line \( \{ x \in \mathbb{R}^2 : x_2 = x_1 \} \) may intersect the boundary of \( \text{St}_1(o(H_6, \theta)) \) with the edge

\[
\text{conv} \left\{ (\cos \theta, 0), \left( \cos(\theta - \frac{\pi}{3}), \frac{\sin \frac{\pi}{3}}{2 \sin(\theta + \frac{\pi}{6})} \right) \right\}
\]

or the edge

\[
\text{conv} \left\{ \left( \cos \left( \theta - \frac{\pi}{3} \right), \frac{\sin \frac{\pi}{3}}{2 \sin \left( \theta + \frac{\pi}{6} \right)} \right), \left( \cos(\theta + \frac{\pi}{3}), \frac{\sqrt{3}}{2 \cos \theta} \right) \right\}.
\]
**Case 1:** \( \cos(\theta - \frac{\pi}{3}) \leq \frac{\sin \frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})} \), i.e., \( 0 \leq \theta \leq \arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6} \).

In this case, the line \( \{ x \in \mathbb{R}^2 : x_2 = x_1 \} \) intersects the edge
\[
\text{conv} \left\{ \left( \cos \theta, 0 \right), \left( \cos \left( \theta - \frac{\pi}{3} \right), \frac{\sin \frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})} \right) \right\}
\]
with \((s(\theta), s(\theta))\), where
\[
s(\theta) = \frac{\sin \frac{\pi}{3} \cos \theta}{\sin \frac{\pi}{3} - 2\sin^2(\theta + \frac{\pi}{6}) + 2\cos \theta \sin(\theta + \frac{\pi}{6})}.
\]
This function \(s(\theta)\) is increasing in \([0, \arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6}]\) (cf. Proposition 13), therefore,
\[
s(\theta) \geq s(0) = \frac{\sqrt{3}}{\sqrt{3} + 1},
\]
and
\[
\frac{1}{3 - \sqrt{3}} s(\theta) \geq \frac{1}{2}.
\]
So \(\text{St}_1(o(\frac{1}{3 - \sqrt{3}} H_6, \theta))\) always contains \([-\frac{1}{2}, \frac{1}{2}]^2\) when \(\theta \in [0, \arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6}]\).

**Case 2:** \( \cos(\theta - \frac{\pi}{3}) \geq \frac{\sin \frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})} \), i.e., \( \arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6} \leq \theta \leq \frac{\pi}{12} \).

In this case, the line \( \{ x \in \mathbb{R}^2 : x_2 = x_1 \} \) intersects the edge
\[
\text{conv} \left\{ \left( \cos \left( \theta - \frac{\pi}{3} \right), \frac{\sin \frac{\pi}{3}}{2\sin(\theta + \frac{\pi}{6})} \right), \left( \cos \left( \theta + \frac{\pi}{3} \right), \frac{\sqrt{3}}{2\cos \theta} \right) \right\}
\]
with \((t(\theta), t(\theta))\), where
\[
t(\theta) = \frac{2\sqrt{3} \sin \left( \theta + \frac{\pi}{6} \right) + \sqrt{3} \cos \left( \theta + \frac{\pi}{3} \right)}{4 \cos \theta \sin \left( \theta + \frac{\pi}{6} \right) + \sqrt{3}}.
\]
This function \(t(\theta)\) is decreasing in \([\arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6}, \frac{\pi}{12}]\), and in fact it is decreasing in \([\arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6}, \frac{\pi}{12}]\) (cf. Proposition 14), therefore
\[
t(\theta) \geq t \left( \frac{\pi}{12} \right) > t \left( \frac{\pi}{6} \right) = \frac{\sqrt{3}}{\sqrt{3} + 1}.
\]
Therefore, the line $\frac{1}{3-\sqrt{3}}t(\theta) > \frac{1}{2}$.

Thus $St_1(o(H_6, \theta))$ always contains $[-\frac{1}{2}, \frac{1}{2}]^2$ when $\theta \in \arcsin\left(\frac{\sqrt{2}}{2}\right) - \frac{\pi}{6}, \frac{\pi}{12}$.

To see that $\frac{1}{3-\sqrt{3}}$ is the minimum number, we refer to Proposition 10 applied to $\rho H_6$ for any $\rho < \frac{1}{3-\sqrt{3}}$. \qed

Now we look at the regular 10-gon.

**Proof of Theorem 4(2), $n = 2$.** For the symmetric reason between $H_{10}$ and $\mathbb{Z}^2$, the case $\frac{\pi}{20} \leq \theta \leq \frac{\pi}{12}$ is actually symmetric to the case $0 \leq \theta \leq \frac{\pi}{20}$ with respect to the line $\{x \in \mathbb{R}^2 : x_2 = x_1\}$. We are going to prove that $St_1(o(\cos(\frac{5}{2} - \sin(\frac{\pi}{20} + \cos(\frac{5}{2}) H_{10}, \theta)))$ contains $[-\frac{1}{2}, \frac{1}{2}]^2$ for $0 \leq \theta \leq \frac{\pi}{20}$ (cf. Theorem 9).

By calculation,

$$St_1(o(H_{10}, \theta)) = \text{conv}\left\{ \pm \cos(\theta, 0), \right. \left. \left(\pm \cos(\theta - \frac{\pi}{5}), \pm \sin\frac{\pi}{10} \sin\frac{\pi}{5}\sin(\frac{\pi}{10} + \theta) \right), \left(\pm \cos(\theta + \frac{\pi}{5}), \pm \sin\frac{\pi}{10} \sin\frac{3\pi}{5}\sin(\frac{3\pi}{10} - \theta) \right), \left(\pm \cos(\theta - \frac{2\pi}{5}), \pm \sin\frac{3\pi}{10} \sin\frac{2\pi}{5}\sin(\frac{3\pi}{10} + \theta) \right), \left(\pm \cos(\theta + \frac{2\pi}{5}), \pm \sin\frac{2\pi}{5}\cos\theta \right) \right\}.$$ 

While $\theta \in \left[0, \frac{\pi}{20}\right]$, it holds

$$\cos\left(\theta - \frac{\pi}{5}\right) > \frac{\sin\frac{\pi}{10} \sin\frac{\pi}{5}}{\sin(\frac{\pi}{10} + \theta)} \quad \text{(cf. Proposition 16)},$$

$$\cos\left(\theta + \frac{\pi}{5}\right) > \frac{\sin\frac{\pi}{5} \sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10} - \theta)} \quad \text{(cf. Proposition 17)},$$

$$\cos\left(\theta - \frac{2\pi}{5}\right) < \frac{\sin\frac{3\pi}{10} \sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10} + \theta)} \quad \text{(cf. Proposition 18)},$$

$$\cos\left(\theta + \frac{2\pi}{5}\right) < \frac{\sin\frac{2\pi}{5}}{\cos\theta} \quad \text{(cf. Proposition 19)}.$$

Therefore, the line $\{x \in \mathbb{R}^2 : x_2 = x_1\}$ intersects $St_1(o(H_{10}, \theta))$ with the edge

$$\text{conv}\left\{ \left(\cos(\theta + \frac{\pi}{5}), \frac{\sin\frac{\pi}{5} \sin\frac{3\pi}{10}}{\sin(\frac{3\pi}{10} - \theta)} \right), \left(\cos(\theta - \frac{2\pi}{5}), \frac{\sin\frac{3\pi}{10} \sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10} + \theta)} \right) \right\}$$

at the point $(t(\theta), t(\theta))$, where

$$\frac{t(\theta) - \frac{\sin\frac{3\pi}{10} \sin\frac{2\pi}{5}}{\sin(\frac{3\pi}{10} + \theta)}}{t(\theta) - \cos(\theta - \frac{2\pi}{5})} = \frac{\sin\frac{\pi}{5} \sin\frac{3\pi}{10}}{\cos(\theta + \frac{\pi}{5})} - \frac{\sin\frac{3\pi}{10} \sin\frac{2\pi}{5}}{\cos(\theta - \frac{2\pi}{5})}. \quad (4)$$
The function \( t(\theta) \) is increasing in \([0, \frac{\pi}{20}]\) (cf. Proposition 15), therefore

\[
t(\theta) \geq t(0) = \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{5} - \sin \frac{\pi}{5} + \sin \frac{2\pi}{5} - \cos \frac{2\pi}{5}},
\]

and

\[
\frac{\cos \frac{\pi}{5} - \sin \frac{\pi}{5} + \sin \frac{2\pi}{5} - \cos \frac{2\pi}{5}}{2 \sin \frac{\pi}{5}} t(\theta) \geq \frac{1}{2}.
\]

So \( St_1(o(\frac{\cos \frac{\pi}{5} - \sin \frac{\pi}{5} + \frac{3\pi}{10} - \cos \frac{2\pi}{5}}{2 \sin \frac{\pi}{5}} H_{10}, \theta)) \) always contains \([-\frac{1}{2}, \frac{1}{2}]\) when \( \theta \in [0, \frac{\pi}{20}] \).

To see that \( \frac{\cos \frac{\pi}{5} - \sin \frac{\pi}{5} + \frac{3\pi}{10} - \cos \frac{2\pi}{5}}{2 \sin \frac{\pi}{5}} \) is the minimum number, we refer to Proposition 10 applied to \( \rho H_{10} \) for any \( \rho < \frac{\cos \frac{\pi}{5} - \sin \frac{\pi}{5} + \frac{3\pi}{10} - \cos \frac{2\pi}{5}}{2 \sin \frac{\pi}{5}} \).

\[ \square \]

Remark 11. \( St_1(o(H_{2n}, \theta)) \), \( \theta \in [0, \frac{\pi}{4n}] \), has the vertices

\[
(\cos \theta, 0),
\]

\[
\left( \cos(\theta - \frac{2\pi}{n}), \frac{\sin \frac{\pi}{n} \sin \frac{3\pi}{n}}{\sin(\frac{\pi}{n} + \theta)} \right),
\]

\[
\left( \cos(\theta + \frac{2\pi}{n}), \frac{\sin \frac{2\pi}{n} \sin \frac{3\pi}{n}}{\sin(\frac{3\pi}{n} - \theta)} \right),
\]

\[
\left( \cos(\theta - \frac{4\pi}{n}), \frac{\sin \frac{3\pi}{n} \sin \frac{4\pi}{n}}{\sin(\frac{3\pi}{n} + \theta)} \right),
\]

One can deal with all regular \((4n + 2)\)-gons similarly.

\section*{A Some Inequalities}

\begin{proposition}
\end{proposition}

\[ \cos(\theta + \frac{\pi}{3}) \leq \frac{\sqrt{3}}{2 \cos \theta} \]

for \( 0 \leq \theta \leq \frac{\pi}{12} \).

\begin{proof}
It is equivalent to

\[
2 \cos \theta \cos(\theta + \frac{\pi}{3}) \leq \sqrt{3}
\]

\[
\iff \cos \frac{\pi}{3} + \cos(2\theta + \frac{\pi}{3}) \leq \sqrt{3}
\]

\[
\iff \cos(2\theta + \frac{\pi}{3}) \leq \sqrt{3} - \frac{1}{2},
\]

which holds true since \( \cos(2\theta + \frac{\pi}{3}) \) is decreasing in \( \theta \in [0, \frac{\pi}{12}] \) and \( \cos \frac{\pi}{3} < \sqrt{3} - \frac{1}{2} \). \[ \square \]

\begin{proposition}
\end{proposition}

\[ s(\theta) = \frac{\sin \frac{\pi}{3} \cos \theta}{\sin \frac{\pi}{3} - 2 \sin^2(\theta + \frac{\pi}{6}) + 2 \cos \theta \sin(\theta + \frac{\pi}{6})} \geq s(0) \]

for \( 0 \leq \theta \leq \arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6} \).
Proof. Notice that
\[ s(\theta) = \frac{\sqrt{3} \cos \theta}{-3 + \sqrt{3} + 4 \cos^2 \theta}. \]
Since \( \frac{x}{-3 + \sqrt{3} + 4 \cos^2 x} \) is decreasing in \( x \in [\cos \left( \arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6} \right), 1] \) and \( \cos \theta \) is decreasing in \( \theta \in [0, \arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6}] \) then \( s(\theta) \) is increasing.

**Proposition 14.**

\[ t(\theta) = \frac{2\sqrt{3} \sin \left( \theta + \frac{\pi}{6} \right) + \sqrt{3} \cos \left( \theta + \frac{\pi}{3} \right)}{4 \cos \theta \sin \left( \theta + \frac{\pi}{6} \right) + \sqrt{3}} \geq t\left( \frac{\pi}{6} \right) \]

for \( \arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6} \leq \theta \leq \frac{\pi}{6} \).

Proof. Notice that
\[ t(\theta) = \frac{3 \sin(\theta + \frac{\pi}{3})}{4 \sin^2(\theta + \frac{\pi}{3}) + \sqrt{3} - 1}. \]
Since \( \frac{x}{4x^2 + \sqrt{3} - 1} \) is decreasing in \( x \in [\sin \left( \arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6} \right), \frac{1}{2}] \) and \( \sin(\theta + \frac{\pi}{3}) \) is increasing in \( \theta \in [\arcsin \left( \frac{\sqrt{3}}{2} \right) - \frac{\pi}{6}, \frac{\pi}{6}] \) then \( t(\theta) \) is decreasing.

**Proposition 15.**

\[ t(\theta) \geq t(0) \]

for \( 0 \leq \theta \leq \frac{\pi}{20} \) where
\[ t(\theta) - \cos(\theta - \frac{2\pi}{5}) = \frac{\sin \left( \frac{3\pi}{10} \right) \sin \frac{2\pi}{5}}{\sin(\frac{3\pi}{10} + \theta)} - \frac{\sin \left( \frac{3\pi}{10} \right) \sin \frac{2\pi}{5}}{\cos(\theta + \frac{\pi}{5}) - \cos(\theta - \frac{2\pi}{5})}. \]

Proof. Notice that
\[ t(\theta) = \frac{2 \sin \frac{3\pi}{10} \cos \frac{\pi}{10} \cos \theta}{2 \cos^2 \theta + \sin \frac{3\pi}{5} - \cos \frac{3\pi}{5} - 1}. \]
Since \( \frac{2x^2 + \sin \left( \frac{\sqrt{3}}{2} \right) - \cos \left( \frac{\sqrt{3}}{2} \right) - 1}{\cos \theta} \) is decreasing in \( x \in [\cos \left( \frac{\pi}{20} \right), 1] \) and \( \cos \theta \) is decreasing in \( \theta \in [0, \frac{\pi}{20}] \) thus \( t(\theta) \) is increasing.

**Proposition 16.**

\[ \cos(\theta - \frac{\pi}{5}) > \frac{\sin \frac{\pi}{10} \sin \frac{\pi}{5}}{\sin \left( \frac{\pi}{10} + \theta \right)} \]

for \( 0 \leq \theta \leq \frac{\pi}{20} \).
Proof. The statement is equivalent to
\[
\sin\left(\frac{\pi}{10} + \theta\right) \cos\left(\theta - \frac{\pi}{5}\right) > \sin \frac{\pi}{10} \sin \frac{\pi}{5}
\]
\[\Leftrightarrow \sin(2\theta - \frac{\pi}{10}) + \sin \frac{3\pi}{10} > 2 \sin \frac{\pi}{10} \sin \frac{\pi}{5}
\]
\[\Leftrightarrow \sin(2\theta - \frac{\pi}{10}) > 2 \sin \frac{\pi}{10} \sin \frac{\pi}{5} - \sin \frac{3\pi}{10}.
\]
Since \(\sin(2\theta - \frac{\pi}{10})\) is increasing for \(0 \leq \theta \leq \frac{\pi}{20}\), and
\[
\sin \frac{3\pi}{10} - \sin \frac{\pi}{10} - 2 \sin \frac{\pi}{10} \sin \frac{\pi}{5} > 0,
\]
the inequality holds for \(0 \leq \theta \leq \frac{\pi}{20}\). \qed

Proposition 17.
\[
\cos\left(\theta + \frac{\pi}{5}\right) > \frac{\sin \frac{\pi}{5} \sin \frac{3\pi}{10}}{\sin(\frac{3\pi}{10} - \theta)}
\]
for \(0 \leq \theta \leq \frac{\pi}{20}\).

Proof. The statement is equivalent to
\[
\sin\left(\frac{3\pi}{10} - \theta\right) \cos\left(\theta + \frac{\pi}{5}\right) > \sin \frac{\pi}{5} \sin \frac{3\pi}{10}
\]
\[\Leftrightarrow \sin \frac{\pi}{2} + \sin\left(\frac{\pi}{10} - 2\theta\right) > 2 \sin \frac{\pi}{5} \sin \frac{3\pi}{10}
\]
\[\Leftrightarrow \sin\left(\frac{\pi}{10} - 2\theta\right) > 2 \sin \frac{\pi}{5} \sin \frac{3\pi}{10} - \sin \frac{\pi}{2}.
\]
Since \(\sin\left(\frac{\pi}{10} - 2\theta\right)\) is decreasing for \(0 \leq \theta \leq \frac{\pi}{20}\), and
\[
0 > 2 \sin \frac{\pi}{5} \sin \frac{3\pi}{10} - \sin \frac{\pi}{2},
\]
the inequality holds for \(0 \leq \theta \leq \frac{\pi}{20}\). \qed

Proposition 18.
\[
\cos\left(\theta - \frac{2\pi}{5}\right) < \frac{\sin \frac{3\pi}{10} \sin \frac{2\pi}{5}}{\sin\left(\frac{3\pi}{10} + \theta\right)}
\]
for \(0 \leq \theta \leq \frac{\pi}{20}\).

Proof. The statement is equivalent to
\[
\sin\left(\frac{3\pi}{10} + \theta\right) \cos\left(\theta - \frac{2\pi}{5}\right) < \sin \frac{3\pi}{10} \sin \frac{2\pi}{5}
\]
\[\Leftrightarrow \sin(2\theta - \frac{\pi}{10}) + \sin \frac{7\pi}{10} < 2 \sin \frac{3\pi}{10} \sin \frac{2\pi}{5}
\]
\[\Leftrightarrow \sin(2\theta - \frac{\pi}{10}) < 2 \sin \frac{3\pi}{10} \sin \frac{2\pi}{5} - \sin \frac{7\pi}{10}.
\]
Since \( \sin(2\theta - \frac{\pi}{10}) \) is increasing for \( 0 \leq \theta \leq \frac{\pi}{20} \), and
\[
0 < 2 \sin \frac{3\pi}{10} \sin \frac{2\pi}{5} - \sin \frac{7\pi}{10},
\]
the inequality holds for \( 0 \leq \theta \leq \frac{\pi}{20} \). \hfill \Box

**Proposition 19.**
\[
\cos(\theta + \frac{2\pi}{5}) < \frac{\sin \frac{2\pi}{5}}{\cos \theta}
\]
for \( 0 \leq \theta \leq \frac{\pi}{20} \).

**Proof.** The statement is equivalent to
\[
\cos \theta \cos(\theta + \frac{2\pi}{5}) < \sin \frac{2\pi}{5}
\]
\[
\iff \cos(2\theta + \frac{2\pi}{5}) + \cos \frac{2\pi}{5} < 2 \sin \frac{2\pi}{5}
\]
\[
\iff \cos(2\theta + \frac{2\pi}{5}) < 2 \sin \frac{2\pi}{5} - \cos \frac{2\pi}{5}.
\]
Since \( \cos(2\theta + \frac{2\pi}{5}) \) is decreasing for \( 0 \leq \theta \leq \frac{\pi}{20} \), and
\[
\cos \frac{2\pi}{5} < 2 \sin \frac{2\pi}{5} - \cos \frac{2\pi}{5},
\]
the inequality holds for \( 0 \leq \theta \leq \frac{\pi}{20} \). \hfill \Box

**References**


